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# RECURRENT PERTURBATIONS OF CERTAIN TRANSIENT RADIALLY SYMMETRIC DIFFUSIONS

BY D. IOFFE

*University of California, Davis*

If $L$ generates a transient diffusion, then the corresponding exterior Dirichlet problem (EP) has in general many bounded solutions. We consider perturbations of $L$ by a first-order term and assume that EP can be solved uniquely for each perturbed operator. Then as the perturbation tends to 0, the sequence of perturbed solutions may converge to a solution of the original EP. Using a skew-product representation of diffusions, we give an integral criterion for the uniqueness of this limit and show that it takes place iff the Kuramochi boundary of $L$ at $\infty$ is a singleton. In the case when uniqueness fails, we provide a description of a subclass of limiting solutions in terms of boundary conditions for the original process in the natural scale.

**1. Introduction.** Let $L$ be a differential operator on $\mathbb{R}^d$ of the form:

$$L = \frac{1}{2}\frac{\partial^2}{\partial\rho^2} + b(\rho)\frac{\partial}{\partial\rho} + \frac{V(\rho)}{2}\Delta_s, \tag{1.1}$$

where $\Delta_s$ denotes the Laplace–Beltrami operator on $S^{d-1}$. We will follow [1] and say that $L$ belongs to $\mathscr{L}(\mathbb{R}^d)$ if it is locally uniformly elliptic and the coefficients $b$ and $V$ are locally Hölder [6]:

$$L \in \mathscr{L}(\mathbb{R}^d). \tag{A}$$

The main import of (A) is the use of Schauder a priori estimates and the validity of probabilistic representations of solutions, which we will use from now on without further comment.

Assume that $L$ generates a transient diffusion. As it well known, this is equivalent to

$$\int^{\infty} \exp\left[-2\int^{\rho} b(z)\,dz\right] d\rho < \infty. \tag{1.2}$$

Consider now the exterior Dirichlet problem

$$(\mathrm{EP})_L \qquad \begin{aligned} Lu &= 0 \quad \text{in } D, \quad u \text{ is bounded}, \\ u|_{\partial D} &= \phi, \qquad \phi \in C(\partial D), \end{aligned}$$

where $D$ is an exterior domain ($D^c$ is compact and contains some neighborhood of the origin) with smooth boundary $\partial D$. Since $L$ generates a transient diffusion, in order to obtain a unique solution to $(\mathrm{EP})_L$, one has to specify

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some additional conditions at $\infty$ such as, for example,

$$\lim_{|x|\to\infty} u(x) = u_0 = \text{const.} \tag{1.3}$$

Although $(\mathrm{EP})_L$ and (1.3) yield *a* unique bounded solution for each $u_0$, there can be bounded solutions to $(\mathrm{EP})_L$ which do not satisfy condition (1.3). We note that no general condition at $\infty$ is known that can be used to define every bounded solution to $(\mathrm{EP})_L$ in a unique way.

Suppose now that $\{L_n\}$ is a sequence of first-order perturbations of $L$, that is, $L_n \in \mathscr{L}(\mathbb{R}^d)$, and is given by

$$L_n = \frac{1}{2}\frac{\partial^2}{\partial\rho^2} + b(\rho)\frac{\partial}{\partial\rho} + \langle\beta_n(x), \nabla\rangle + \frac{V(\rho)}{2}\Delta_s, \tag{1.1$_n$}$$

where $\beta_n$ converges to 0 uniformly on compacts of $\mathbb{R}^d$. Assume that each $L_n$ generates a recurrent diffusion. We will call such a sequence a *perturbation* of $L$. Because $L_n$ is recurrent, the problem

$$(\mathrm{EP})_n \qquad \begin{aligned} &L_n u = 0 \quad \text{in } D, \quad u \text{ is bounded,}\\ &u|_{\partial D} = \phi, \qquad \phi \in C(\partial D)\end{aligned}$$

yields a unique bounded solution $u_n^\phi$ [9]. We will call $\{L_n\}$ a *$\phi$-admissible perturbation* if it is a perturbation and $u^\phi \overset{\text{def}}{=} \lim_{n\to\infty} u_n^\phi$ exists. By the De Giorgi–Nash theorem (see [6]), it follows that the convergence above is locally uniform. Furthermore, the results of [15], Chapter 11, and condition (A) imply that $u^\phi$ solves $(\mathrm{EP})_L$. We will refer to $u^\phi$ as an *admissible solution* to $(\mathrm{EP})_L$. Now for any perturbation $\{L_n\}$, the sequence $\{u_n^\phi\}$ is bounded. Using once more the De Giorgi–Nash result, we obtain that $\{L_n\}$ contains some $\phi$-admissible subsequence. By the separability of $C(\partial D)$, one can apply a diagonalization argument to extract a subsequence $\{L_{n_k}\}$, which is $\phi$-admissible for each $\phi \in C(\partial D)$. A sequence $\{L_n\}$ which is $\phi$-admissible for every $\phi \in C(\partial D)$ will be called an *admissible perturbation*. Finally, we will say that two admissible perturbations $\{L_n\}$ and $\{\tilde{L}_n\}$ are equivalent if $\lim_{n\to\infty} u_n^\phi = \lim_{n\to\infty} \tilde{u}_n^\phi$ for each $\phi \in C(\partial D)$.

DEFINITION 1.4. The exterior Dirichlet problem $(\mathrm{EP})_L$ is said to be stable if there is a unique admissible solution for each $\phi \in C(\partial D)$.

Our goal in this paper is to investigate stability properties of $(\mathrm{EP})_L$ and to describe the class of admissible solutions. Let us point out for further convenience that the following are equivalent:

(i) $(\mathrm{EP})_L$ is stable.
(ii) Every perturbation is an admissible one.
(iii) All admissible perturbations are equivalent.

To understand better the motivation and the questions we should pose, let us briefly survey the work of Pinsky in [11], where $(\mathrm{EP})_{1/2\Delta}$ was studied. First of all, it was shown that $(\mathrm{EP})_{1/2\Delta}$ is stable.

In the case of the operator $L$ given by (1.1) we will prove the following theorem.

THEOREM 1.5. $(\mathrm{EP})_L$ *is stable iff* $\int^\infty V(\rho)(\exp[2\int^\rho b(z)\,dz])\,d\rho = \infty$.

REMARK. It is interesting to note that $(\mathrm{EP})_L$ can be stable even if the Martin boundary at $\infty$ for $L$ is isomorphic to $S^{d-1}$. Indeed, by the results of [12], the latter happens iff

$$\int^\infty V(\rho)\left\{\exp\left[2\int^\rho b(z)\,dz\right]\right\}\left\{\int_\rho^\infty \exp\left[-2\int^z b(s)\,ds\right]dz\right\}dp < \infty.$$

Thus the diffusions for which $(\mathrm{EP})_L$ is unstable have to be "moderately transient" with respect to $V$ in the sense that the drift $b(\cdot)$ has to be large enough to ensure the transience condition (1.2) but small enough so that the integral in Theorem 1.5 is finite. In fact, $(\mathrm{EP})_L$ is stable iff the so-called Kuramochi boundary at $\infty$ for $L$ is a singleton (cf. [8] and Theorem 1.9).

If $(\mathrm{EP})_L$ is stable, then there is a unique admissible solution $\bar{u}^\phi$ to $(\mathrm{EP})_L$ for each $\phi \in C(\partial D)$. In the case of $L = 1/2\Delta$, Pinsky [11] gives two different ways to characterize $\bar{u}^\phi$.

Consider first the following variational problem:

PROBLEM 1.6. Solve

$$\inf \int_D |\nabla u|^2,$$

$$u \in W_{\mathrm{loc}}^{1,\phi}(D), \quad u \text{ is bdd},$$

where $W_{\mathrm{loc}}^{1,\phi}(D)$ denotes the set of functions with one locally integrable generalized derivative and trace $\phi$ on $\partial D$.

Pinsky's result [11] states that $\bar{u}^\phi$ is the unique solution to Problem 1.6 for each $\phi \in C(\partial D)$.

The second way to describe $\bar{u}^\phi$ is a little bit more involved and is based on the notion of the harmonic measure boundary at $\infty$, introduced in [12]. Namely, let $h(x)$ denote the unique solution of $(\mathrm{EP})_L$ (with $L = 1/2\Delta$) and (1.3) in the special case of $\phi \equiv 1$ and $u_0 = 0$. Of course, $h(x) = P_x\{\tau_D < \infty\}$. Following Doob [4], Brownian motion in the exterior domain $D$, conditioned on $\{\tau_D < \infty\}$, may be realized as a Markov diffusion process on $D$ with generator $1/2\Delta^h$, where $\Delta^h$ is defined by $\Delta^h f = 1/h\,\Delta(hf)$. Letting $X^h$ denote the conditioned process, set $\mu_x^h(dz) = P_x\{X^h(\tau_D) \in dz\}$ for each $x \in D$. Then $\mu_x^h(dz)$ is, of course, a probability measure on $\partial D$. Pinsky [11] showed that $\mu_\infty^h = w - \lim_{|x|\to\infty} \mu_x^h$ exists and that the unique admissible solution $\bar{u}^\phi$ of $(\mathrm{EP})^\phi$ satisfies (and therefore is uniquely defined by) condition (1.3) with $u_0 = \int_{\partial D}\phi(z)\mu_\infty^h(dz)$.

Returning to the case of a general radially symmetric operator given by (1.1), in view of Theorem 1.5 and the results of [11] cited above, it is natural to pose the following questions:

Q1. If $(\mathrm{EP})_L$ is stable, then what is the unique admissible solution $\bar{u}^{\phi}$?
Q2. If $(\mathrm{EP})_L$ is unstable, then how can one characterize the class of admissible solutions for $(\mathrm{EP})_L$?

To answer Q1, we start by adjusting the variational problem (Problem 1.6) to the case of $L$ given by (1.1). Set $H(r) = \exp[2\int^r b(s)\,ds]$ and define the functional $J$ by

$$J(u) = \int_D H(r)\left[(u'_r)^2 + V(r)\|\nabla_s u\|^2\right] dr\,d\sigma, \tag{1.7}$$

where $\nabla_s$ denotes the gradient in the angular variables.

PROBLEM 1.8. Solve

$$\inf J(u),$$
$$u \in W^{1,\phi}_{\mathrm{loc}}(D).$$

REMARK. Note that unlike in Problem 1.6, we do not restrict our attention to the case of bounded $u$.

THEOREM 1.9. *There exists a unique solution $u_*^{\phi}$ to Problem* 1.8 *for each $\phi \in C(\partial D) \cap W^1(D)$. Moreover,*

(a) *$u_*^{\phi}$ is an admissible solution.*
(b) *$u_*^{\phi}$ satisfies* (1.3) *for each $\phi \in C^{\infty}(\partial D)$ and some $u_0$ depending on $\phi$ iff*

$$\int^{\infty} V(\rho)\left(\exp\left[2\int^{\rho} b(z)\,dz\right]\right) d\rho = \infty.$$

Theorem 1.9 identifies one specific admissible solution $u_*^{\phi}$ to $(\mathrm{EP})^{\phi}$ for each function $\phi \in C(\partial D) \cap W^1(D)$ via a variational problem. Utilizing a result in [12], we have another way to identify a specific admissible solution $\bar{u}^{\phi}$ to $(\mathrm{EP})_L$ for all $\phi \in C(\partial D)$. Namely, let $h$ and $\mu_x^h$ be as defined above with respect to the operator $L$ given by (1.1) rather than $1/2\Delta$. Then, by results of [12],

$$\mu^h_{\infty,\theta}(dz) \overset{\mathrm{def}}{=} w - \lim_{\rho\to\infty} \mu^h_{\rho,\theta}(dz) \text{ exists } \forall\, \theta \in S^{d-1}.$$

LEMMA 1.10. *There exists exactly one admissible solution $\bar{u}^{\phi}$ to $(\mathrm{EP})_L$, which satisfies* (1.3). *For this solution,*

$$u_0 = c_D(\phi) \equiv \int \phi(z) \int_{S^{d-1}} B(\theta)\mu^h_{\infty,\theta}(dz)\, l(d\theta), \tag{1.11}$$

*where*

$$B(\cdot) = \lim_{\rho \to \infty} h(\rho, \cdot) \Big/ \int_{S^{d-1}} h(\rho, \theta) l(d\theta)$$

*in* $C(S^{d-1})$ *and* $l$ *is a normalized Lebesgue measure on* $S^{d-1}$.

As we will see below, $u_*^\phi$ and $\bar{u}^\phi$ are in a sense critical cases of admissible solutions—they are respectively the least and the most ergodized ones. Note, by the way, that $u_*^\phi = \bar{u}^\phi \; \forall \; \phi \in C(\partial D) \cap W^1(D)$ iff $(\mathrm{EP})_L$ is stable.

The best we can do in answering Q2 is to give a complete description of the set of all radially admissible solutions, that is, solutions which correspond to radially symmetric perturbations of the following form:

$$(1.12)_n \qquad L_n = \frac{1}{2}\frac{\partial^2}{\partial \rho^2} + b_n(\rho)\frac{\partial}{\partial \rho} + \frac{V(\rho)}{2}\Delta_s,$$

where $b_n \to b$ uniformly on the compacts of $\mathbb{R}$. Assume that $B(0;\delta) \subset D^c$. To state the results, we have to switch to the natural scale given by the following change of radial variable:

$$(1.13) \qquad \begin{aligned} r = T(\rho) &= 1 + \int_\delta^\rho \exp\left\{-2\int_\delta^z b(s)\,ds\right\} dz, \\ R &= T(\infty) < \infty \quad \text{by transience}, \\ \gamma(r) &= V(T^{-1}(r)) \exp\left\{4\int_\delta^{T^{-1}_{(r)}} b(s)\,ds\right\}. \end{aligned}$$

Then $T$ maps $D$ into the annulus $(1, R) \times S^{d-1}$. Let $\Gamma$ denote the external boundary of this annulus: $\Gamma = \{R\} \times S^{d-1}$.

THEOREM 1.14. *Let* $u^\phi$ *be a radially admissible solution to* $(\mathrm{EP})_L$ *and set* $v^\phi$ *to be its counterpart under the transformation* $T$, $v^\phi \circ T = u^\phi$. *Then* $v^\phi$ *satisfies the following boundary condition on* $\Gamma$:

$$\frac{\partial}{\partial r} v^\phi(R, \sigma) = v.p. \int_{S^{d-1}} \left[v^\phi(R, \xi) - v^\phi(R, \sigma)\right] k(\sigma, \xi) l(d\xi),$$

*with some kernel* $k$ *which depends only on the geodesic distance* $d(\sigma, \xi)$.

The rest of the paper is organized as follows: Everywhere except for Section 6 we assume that $\overline{D}^c = B(0;\delta)$, the ball of radius $\delta$ centered at 0. Section 2 is devoted to the study of the radial stability of $(\mathrm{EP})_L$, that is, stability under radially symmetric perturbations $(1.12)_n$. In Section 3 we make use of the transformation of drift formula and John–Nirenberg-type estimates to show that in fact $(\mathrm{EP})_L$ is stable if and only if it is radially stable, which gives us the claim of Theorem 1.5. The variational connection which becomes apparent in the statement of Theorem 1.9 is discussed in Section 4. In Section 5 we investigate the properties of admissible solutions and prove Theorem 1.14. Section 6 is devoted to the generalization of the above results to the case of general smooth exterior domains. Finally, in Section 7 we show how one can extend the results to the case of general radially symmetric diffusions on the plane.

**2. Stability under radially symmetric perturbations.** In this section we assume that $D$ is the exterior of a ball, that is, $D^c = B(0:\delta)$, and consider only perturbations $\{L_n\}$, given by $(1.12)_n$. First, we rewrite everything in the natural scale given by (1.13). In the new coordinates,

$$L = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\gamma(r)}{2}\Delta_s.$$

In the same fashion we transform $L_n$ according to the scale

$$(2.1)_n \qquad \begin{aligned} r = T_n(\rho) &= 1 + \int_\delta^\rho \exp\left\{-2\int_\delta^z b(s)\,ds\right\}dz,\\ T_n(\infty) &= \infty \quad \text{by recurrence},\\ \gamma_n(r) &= V(T_n^{-1}(r))\exp\left\{4\int_\delta^{T_n^{-1}(r)} b_n(s)\,ds\right\},\end{aligned}$$

and obtain

$$L_n = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \gamma_n\frac{(r)}{2}\Delta_s$$

respectively.

Recall that $T(D) = (1, R) \times S^{d-1}$ and $T(\partial D) = \{\|x\| = 1\}$. We preserve the notation $D$ and $\partial D$ for $T(D)$ and $T(\partial D)$ respectively; it will cause no confusion as long as we remember what is the current scale. Then instead of considering admissible solutions to $(\mathrm{EP})_L$, one can investigate the class of bounded solutions to the equation in the natural scale,

$$(\mathrm{EP})_L \qquad \begin{aligned} Lu^\phi &= \left(\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\gamma(r)}{2}\Delta_s\right)u^\phi = 0 \quad \text{in } D,\\ u^\phi|_{\partial D} &= \phi,\end{aligned}$$

which can be obtained as a limit of solutions to the recurrent problems

$$(\mathrm{EP})'_n \qquad \begin{aligned} L_n u_n^\phi &= \left(\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\gamma_n(r)}{2}\Delta_s\right)u_n^\phi = 0 \quad \text{in } (1,\infty)\times S^{d-1},\\ u_n^\phi|_{\partial D} &= \phi,\end{aligned}$$

where $\{L_n\}$ is a perturbation of $L$, that is, $\gamma_n \to \gamma$ uniformly on the compacts of $[1, R)$. It will be convenient to relax slightly the definition of a radially symmetric perturbation $\{L_n\}$ (or equivalently $\{\gamma_n\}$) by demanding only that

$$(2.2) \qquad \gamma_n \to \gamma \quad \text{in } L^1_{\mathrm{loc}}([1, R)),$$

and that there are mutual bounds on the ellipticity constant; that is, one can find two continuous curves $\lambda, \Lambda$: $[1, R) \to \mathbb{R}$ such that for any $n \in \mathbb{N}$,

$$0 < \lambda(r) \le \gamma_n(r) \le \Lambda(r). \tag{2.3}$$

Let us say that $u^\phi$ is a *radially admissible* solution to $(\mathrm{EP})'_L$ if $u^\phi = \lim_{n\to\infty} u_n^\phi$, where $u_n^\phi$ solves $(\mathrm{EP})'_n$ with $\{L_n\}$ being some $\phi$-admissible radially symmetric perturbation. Similarly we will say that $(\mathrm{EP})'_L$ is *radially stable* if there is a unique radially admissible solution for each $\phi \in C(\partial D)$.

Let $B$ denote the one-dimensional Wiener process and $\Sigma$ the Brownian motion on $S^{d-1}$. Then if $X$ and $X^n$ are diffusions generated by the operators $L$ and $L_n$ respectively (in the natural scale), the following skew-product representations are valid:

$$X_t = \left[B_t, \Theta_t = \Sigma\left(\int_0^t \gamma(B_s)\,ds\right)\right]$$

and

$$X_t^n = \left[B_t, \Theta_t^n = \Sigma\left(\int_0^t \gamma_n(B_s)\,ds\right)\right].$$

Let $\tau_1$ be the first hitting time of $\{1\}$ by $B_t$. Then $u_n^\phi$, the solution of $(2.2)_n$, may be represented as

$$u_n^\phi(r,\theta) = E_{r,\theta}\phi(\Theta_{\tau_1}^n).$$

LEMMA 2.4. *Set $f_n(r) = \|u_n(r,\cdot)\|_{L_2}^2$, $g_n(r) = \|\nabla_s u_n(r,\cdot)\|_{L_2}^2$ and $g_n^\alpha = \|\partial_s^\alpha u_n(r,\cdot)\|_{L_2}^2$, where $\|\cdot\|_{L_2}$ denotes the norm in $L_2(S^{d-1})$, the subscript $s$ indicates the derivatives in the angular variables and $\alpha$ is any multi-index. Then $f_n$, $g_n$ and $g_n^\alpha$ are convex nonincreasing functions on $(1,\infty)$.*

PROOF. $u_n^\phi(B_t, \Theta_t^n)$ is a local martingale. Then for a point $r = \lambda q + (1-\lambda)p$, $0 \le \lambda \le 1$, and $\tau^{q,p}$, the first exit time for the interval $[q,p]$, one obtains

$$\left\| u_n^\phi(r,\theta)\right\|^2 \le E_{r,\theta}\left\| u_n^\phi(B_{\tau^{q,p}}, \Theta_{\tau^{q,p}}^n)\right\|^2. \tag{2.5}$$

Integration of (2.5) over $S^{d-1}$ gives $f_n(r) \le E_r f_n(B_\tau q, r) = (1-\lambda)f_n(p) + \lambda f_n(q)$, which is the desired convexity. Convexity of $g_n^\alpha$ follows by the same argument and the observation that $\partial^\alpha u_n^\phi$ solves $L_n\,\partial^\alpha u_n^\phi = 0$ inside $(1,\infty)\times S^{d-1}$.

To prove that $f_n$, $g_n$ and $g_n^\alpha$ are nonincreasing, it suffices therefore to show that they are bounded for $r$ large enough. There is nothing to prove for $f_n$ since by the submartingale property of $\{u_n^\phi(B_t,\Theta_t^n)\}^2$, one readily obtains that $f_n \le \|\phi\|_{L_2}^2$.

Let us prove the boundedness of $g_n^\alpha$. Suppose first that $\phi$ is in $C^\alpha(S^{d-1})$ and let $\tau_N$ be the first exit time from $[1, N]$. Then for $r \in (1, N)$,

$$\begin{aligned} u_n^\phi(r,\theta) &= E_{r,\theta}\left[\phi(\Theta_{\tau_N}^n);\ \tau_N = 1\right] \\ &\quad + E_{r,\theta}\left[u_n(N, \Theta_{\tau_1}^n);\ \tau_N = N\right] \overset{\text{def}}{=} u_{n,N}^{\phi,1} + u_{n,N}^{\phi,2}. \end{aligned} \tag{2.6}$$

Now by uniqueness,

$$\partial_s^\alpha u_{n,N}^{\phi,1}(r,\theta) = E_{r,\theta}\left[\partial_s^\alpha \phi\left(\Theta_{\tau_N}^n\right); \tau_N = 1\right].$$

On the other hand, from Schauder interior estimates [6] and the fact that $u_n(r,\cdot)$ is $C^\infty(S^{d-1})$ for each $r > 1$, it follows that for $r$ fixed and $N$ sufficiently large,

$$\left\|\partial_s^\alpha u_{n,N}^{\phi,2}(r,\cdot)\right\|_{\sup} \le C_\alpha \left\|u_{n,N}^{\phi,2}\right\|_{\sup} = O(1/N), \tag{2.7}$$

with $\|\cdot\|_{\sup}$ denoting the norm in $C(S^{d-1})$. By letting $N$ tend to $\infty$, we finally obtain from (2.6) and (2.7) that

$$\partial_s^\alpha u_n^\phi(r,\theta) = E_{r,\theta}\,\partial_s^\alpha \phi\left(\Theta_{\tau_1}^n\right). \tag{2.8}$$

For general $\phi \in C(S^{d-1})$, pick some $\hat{r} > 1$ and observe that with $\hat{r}$ instead of 1, (2.8) implies that $g_n^\alpha(\hat{r}) \ge g_n^\alpha(r)$ for any $r > \hat{r}$. Moreover, by condition (2.3) the family $\{g_n^\alpha(\cdot)\}$ is uniformly bounded on any of the intervals $[\hat{r},\infty)$ with $\hat{r} > 1$. □

CONSEQUENCE 2.9. *Let $u^\phi$ be a radially admissible solution to* $(2.2)^\phi$. *Set* $f(r) = \|u^\phi(r,\cdot)\|_{L_2}^2$, $g(r) = \|\nabla_s u^\phi(r,\cdot)\|_{L_2}^2$ *and* $g^\alpha(r) = \|\partial_s^\alpha u^\phi(r,\cdot)\|_{L_2}^2$. *Then $f$, $g$ and $g^\alpha$ are convex nonincreasing functions on* $(1, R)$.

PROOF. By definition, $u^\phi = \lim_{n\to\infty} u_n^\phi$ for some sequence $\{u_n^\phi\}$ of solutions to $(\mathrm{EP})_n^\gamma$. Since by Lemma 2.4 the sequence $\{u_n^\phi(r,\cdot)\}$ is bounded in $W^{\alpha,2}(S^{d-1})$ for $r \in (1, R)$ and any multi-index $\alpha$, the convergence holds in fact in $C^\infty$ with respect to the angular variables; hence the conclusion. □

We are now in a position to prove the main result of the section.

LEMMA 2.10. $(\mathrm{EP})_L^\gamma$ *is radially stable iff* $\gamma \notin L_1(1, R)$.

REMARK. Note that $\gamma \notin L_1(1, R)$ is exactly the integral condition of Theorem 1.5 written in the natural scale.

To prove the lemma above, we treat separately the cases $\gamma \notin L_1(1, R)$ and $\gamma \in L_1(1, R)$.

LEMMA 2.11. *Assume that $\gamma \notin L_1(1, R)$. Then there is a unique admissible solution $u^\phi$ for each $\phi \in C(\partial D)$. Moreover, $u^\phi$ satisfies*

$$u^\phi(R,\cdot) = \lim_{r\to R} u^\phi(r,\cdot) = \int_{S^{d-1}} \phi(\sigma)\, l(d\sigma) \overset{\text{def}}{=} \bar{\phi}. \tag{2.12}$$

PROOF. It turns out that for any radially admissible $u^\phi$,

$$\lim_{r\to R} g(r) = 0. \tag{2.13}$$

Now (2.13) and the Poincaré inequality imply that $\lim_{r\to R} u^\phi(r,\cdot) = \bar{\phi}$ in

$L_2(S^{d-1})$. Therefore, it remains to verify (2.13). To this end, multiply $L_n u_n^\phi = 0$ by $u_n^\phi$ and integrate over $S^{d-1}$ by parts to obtain

$$(2.14)\quad \gamma_n(r) g_n(r) = \frac{1}{2}\frac{d^2}{dr^2} f_n(r) - \left\| \frac{\partial}{\partial r} u_n^\phi(r,\cdot) \right\|_{L_2}^2 \le \frac{1}{2}\frac{d^2}{dr^2} f_n(r).$$

Now pick a point $1 < r < R$ and integrate (2.14) twice to obtain

$$\int_r^R dz \int_z^R \gamma_n(s) g_n(s)\, ds \le f_n(r) - f_n(R) + f_n'(R) \le f_n(r) \le \|\phi\|_{L_2}^2.$$

On the other hand, for any $p < r$,

$$\int_p^R dz \int_z^R \gamma_n(s) g_n(s)\, ds = \int_p^R (z-p)\gamma_n(z) g_n(z)\, ds$$

$$\ge (r-p)\int_r^R \gamma_n(z) g_n(z)\, dz.$$

Therefore,

$$\int_r^R \gamma_n(z) g_n(z)\, dz \le \|\phi\|_{L_2}^2/(r-p).$$

Since by (2.2), $\gamma_n \to \gamma$ in $L_{\text{loc}}^1$ and by assumption, $\gamma \notin L^1$, we conclude that

$$\lim_{r\to R} \lim_{n\to\infty} g_n(r) = 0,$$

which is equivalent to (2.13).

To prove the "only if" part of Lemma 2.10, we have to show that if $\gamma$ belongs to $L_1(1, R)$, then there are two different admissible solutions to $(\text{EP})_L$ for some $\vartheta \in C(\partial D)$.

To this end define $\vartheta$ as

$$(2.15)\quad \vartheta(\sigma) = \max\{(1 - d(\sigma,\xi)); 0\} - \max\{(1 - d(\sigma,\zeta)); 0\},$$

with $d(\xi,\zeta) = \pi$ and let $\{\gamma_n\}, \{\delta_n\}$ be two different perturbations which satisfy

$$(\text{C1})\quad \|\gamma_n - \gamma_m\|_{L_1(1,\infty)} \to 0 \quad \text{as } n, m \to \infty$$

and

$$(\text{C2})\quad \|\delta_n\|_{L_1(1,R')} \to \infty \quad \text{as } n \to \infty \text{ for any } R' > R$$

respectively. Let $u_n^\vartheta$ and $v_n^\vartheta$ denote solutions to $(\text{EP})_n$ with $\gamma_n$ and $\delta_n$ respectively. We may assume that $\{\gamma_n\}, \{\delta_n\}$ are $\vartheta$-admissible (in fact, they are admissible) and set $u^\vartheta$ and $v^\vartheta$ to be the corresponding admissible solutions. Then $u^\vartheta \ne v^\vartheta$. Indeed set $f(t) = \int_{S^{d-1}} q(t,\xi,\sigma)\vartheta(\sigma) l(d\sigma)$, where $q(\cdot,\cdot,\cdot)$ is the heat kernel on $S^{d-1}$. Then $f$ is a positive decreasing function with $\lim_{t\to\infty} f(t) = 0$. Set, further, $\eta_n = \int_0^\tau \gamma_n(B_s)\, ds$ and $\nu_n = \int_0^\tau \delta_n(B_s)\, ds$. Then

$$(2.16)\quad u_n^\vartheta(r,\xi) = E_r f(\eta_n), \qquad v_n^\vartheta(r,\xi) = E_r f(\nu_n).$$

The desired inequality follows now from the following lemma.

LEMMA 2.17. (a) $\inf\{E_r f(\eta_n);\ r \in (1, R),\ n \in \mathbb{N}\} > 0$. *In particular,*

$$\liminf_{r \to R} u^{\vartheta}(r, \xi) > 0;$$

(b)
$$\lim_{r \to R} v^{\vartheta}(r, \xi) = 0.$$

PROOF. (a) follows from the fact that the mass of the distribution of $\eta_n$ is not swept out to $\infty$. To make this precise, set

$$w_n(r) = E_r \exp(-\eta_n).$$

Note that $w_n$ satisfies $(w_n)''_{rr} = \gamma_n(r) w_n$, $w_n(1) = 1$ and $w_n$ is a positive nonincreasing function on $[1, \infty)$. In fact, we have to show that $\liminf_{n \to \infty} w_n(R) > 0$. To this end pick some $q > R$ and set

$$w_n^q(r) = \frac{\int_r^q \exp\{-(q-1)\int_1^s \gamma_n(t)\,dt\}\,ds}{\int_1^q \exp\{-(q-1)\int_1^s \gamma_n(t)\,dt\}\,ds}.$$

Since $\gamma_n$ is positive, $w_n^q$ is subharmonic. Moreover, $w_n^q(1) = 1$ and $w_n^q(q) = 0$. So $w_n(r) \ge w_n^q(r)$ on $[1, q]$. But

$$w_n^q(R) \ge \frac{(q-R)}{(q-1)} \exp(-q\|\gamma_n\|_{L^1(1,q)}).$$

By condition (C1), the last expression is bounded away from 0 uniformly in $n$. Therefore, $\{w_n(R)\}$ is also bounded away from 0 uniformly in $n$; hence the conclusion. To prove part (b) of the lemma, note that $\lim_{n \to \infty} v_n^{\vartheta}(R', \cdot) = 0$ in $C(S^{d-1})$ will follow from

$$\lim_{n \to \infty} P_{R'}\{\nu_n < N\} = 0 \tag{2.18}$$

for any $N$ arbitrary large. Then we have

$$\begin{aligned} v^{\vartheta}(r, \cdot) &= \lim_{n \to \infty} \big\{E_{r,\cdot}\big[\vartheta(\Theta^n_{\tau^{1,R'}});\ \tau^{1,R'} = 1\big] \\ &\qquad + E_{r,\cdot}\big[v_n^{\vartheta}(R, \Theta^n_{\tau^{1,R'}});\ \tau^{1,R'} = R'\big]\big\} \\ &= \lim_{n \to \infty} E_{r,\cdot}\big[\vartheta(\Theta^n_{\tau^{1,R'}});\ \tau^{1,R'} = 1\big], \end{aligned}$$

where $\tau^{1,R'}$ is a first exit time from $[1, R']$. Therefore, $\|v^{\vartheta}(r, \cdot)\| \le P_r\{\tau^{1,R'} = 1\} = (R' - r)/(R' - 1)$. Thus it remains to verify (2.18). Let $w_n$ be as in (2.18) but with $\delta_n$ in place of $\gamma_n$. Then (2.18) is equivalent to

$$\lim_{n \to \infty} w_n(R') = 0 \quad \text{for any } R' > R. \tag{2.19}$$

But

$$w_n'(r) = w_n'(1) + \int_r^1 w_n \nu_n\,dz = -\int_{\infty}^r w_n \nu_n\,dz \le -w_n(R')\int_{R'}^r \nu_n\,dz,$$

and therefore,

$$w_n(r) \le (r-1) - w_n(R')\int_1^r\int_z^{R'} \nu_n\, dz \le (r-1)\left[1 - w_n(R')\int_r^{R'} \nu_n\, dz\right],$$

$w_n(1) = 1$ and $w_n$ is positive. Thus, since $w_n$ has to be nonnegative, (2.19) follows from (C2) and this concludes the proof of the lemma. □

**3. Perturbations of the general form.** If $\{L_n = L + \langle b_n, \nabla\rangle\}$ is an admissible perturbation, then its properties are characterized by the behavior of $\{b_n\}$ near $\infty$. We state this intuitively obvious fact as follows.

LEMMA 3.1. *Assume that* $\{L_n\}$ *is an admissible perturbation. Let* $\chi$ *be a smooth cutoff function, such that* $\chi = 0$ *on the ball* $B(0;p)$ *and* $\chi = 1$ *on* $B^c(0;q)$ *for some* $1 < p < q < \infty$. *Set*

$$M_n = L + \chi\langle b_n, \nabla\rangle = L_n - (1-\chi)\langle b_n, \nabla\rangle.$$

*Then* $\{M_n\}$ *is also admissible. Moreover,* $\{M_n\}$ *and* $\{L_n\}$ *are equivalent.*

PROOF. Suppose that $W$ is a $d$-dimensional Wiener process on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$ and let $X_n, Y_n$ denote the diffusions on $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x)$ generated by $L_n$ and $M_n$ respectively. Let $\sigma$ denote the diffusion matrix of $L$ in Euclidean coordinates and define $a_n = (1-\chi)\sigma^{-1}b_n$. Then $\operatorname{supp}(a_n) \subset B(0;q)$. Set

$$A_n(t) = \int_0^{t\wedge\tau_1} a_n(X_n(s))\, dW(s) - \frac{1}{2}\int_0^{t\wedge\tau_1} a_n^2(X_n(s))\, ds.$$

If $u_n^\phi$ and $v_n^\phi$ are the solutions to $(\mathrm{EP})_n$ for $L_n$ and $M_n$ respectively, then by the transformation of drift formula,

$$v_n(x) = E_x\phi(X_n(\tau_1))\exp\{A_n(\tau_1)\}.$$

Therefore,

$$\left|v_n^\phi(x) - u_n^\phi(x)\right| \le \|\phi\|_{\sup}\left(E_x \exp\{2A_n(\tau_1)\} - 1\right)^{1/2},$$

where $\|\cdot\|_{\sup}$ denotes the norm in $C(\partial D)$. Now for any two stopping times $T$ and $S$, $T > S$,

$$E_x|A_n(T\wedge\tau_1) - A_n(S\wedge\tau_1)| \le cP_x\{S < \tau_1\}, \tag{3.2}$$

where

$$c = \max_x\left(E_x\int_0^{\tau_1} a_n^2(X_n(s))\, ds\right)^{1/2} + \frac{1}{2}\max_x\left(E_x\int_0^{\tau_1} a_n^2(X_n(s))\, ds\right).$$

By virtue of the John–Nirenberg inequalities ([2] and [5]), (3.2) implies that

$$E_x \exp\{2A_n(\tau_1)\} \le 1/(1-dc),$$

with $d$ being a universal constant. On the other hand, it is easy to see that

$c = \mathscr{O}(\max_x |a_n|)$. Patching all this together, we obtain

$$|v_n^\phi(x) - u_n^\phi(x)| \le \|\phi\|_{\sup} \mathscr{O}(\max|\chi(x) b_n(x)|),$$

which gives us the claim of the lemma. □

We turn now to the proof of our stability result, Theorem 1.5. In view of Lemma 2.10, the only thing we need to verify is that the radial stability of $(\mathrm{EP})_L$ implies its stability. To do so, we proceed by working in the natural scale and adopt the convention to denote $D = T(D) = (1, R) \times S^{d-1}$. The first step is to rewrite an arbitrary perturbation $\{M_n\}$ as

$$(3.3)_n \qquad M_n = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\gamma_n(r)}{2}\Delta_s + \langle \beta_n(x), \nabla \rangle = L_n + \langle \beta_n, \nabla \rangle,$$

where $\beta_n \to 0$ uniformly on the compact sets of $D \cup \partial D$ and $\{\gamma_n\}$ satisfies (2.2) and (2.3). As above, let $X_n$ denote the diffusion generated by $L_n$ and set $\sigma_n$ to be the square root of the diffusion matrix of $L_n$ in Euclidean coordinates. Define $\alpha_n(x) = \sigma_n^{-1}(x)\beta_n(x)$. Next, as in the proof of Lemma 3.1, set

$$A_n(t) = \int_0^{t \wedge \tau_1} \alpha_n(X_n(s))\, dW(s) - \frac{1}{2}\int_0^{t \wedge \tau_1} \alpha_n^2(X_n(s))\, ds$$

and let $N_n(t)$ denote the exponential martingale, $N_n(t) = \exp\{A_n(t)\}$. Then, if $u_n^\phi$ and $v_n^\phi$ solve $(\mathrm{EP})_n$ for $L_n$ and $M_n$ respectively, we can use the transformation of drift formula once more to obtain

$$(3.4) \qquad v_n^\phi(x) = E_x \phi(X_n(\tau_1)) N_n(\tau_1).$$

Let $E_x^{n,\sigma}$ denote the expectation with respect to the process conditioned to hit $\partial D$ at $(1, \sigma)$, that is, $E_x^{n,\sigma}[\cdot] = E_x[\cdot / X_n(\tau_1) = (1, \sigma)]$. Then, for each $x \in D$, the following inequality holds:

$$(3.5) \qquad |u_n^\phi(x) - v_n^\phi(x)| \le \|\phi\|_{\sup} \|E_x^{n,\cdot} N_n(\tau_1) - 1\|_{\sup}.$$

Recall that we are going to prove that $(\mathrm{EP})_L$ is stable. Thus there is no loss of generality in assuming that $\{M_n\}$ is admissible. In fact, we need to show that for any $x \in D$,

$$(3.6) \qquad \lim_{n \to \infty} |E_x^{n,\sigma} N_n(\tau_1) - 1| = 0,$$

uniformly in $\sigma$. Since (3.5) is valid for any perturbation of the form $(3.3)_n$, it suffices to prove the following lemma.

LEMMA 3.7. *For every $x \in D$ and $\delta > 0$, it is possible to modify $\{M_n\}$ modulo the equivalence relation in a such a way that, uniformly in $\sigma$,*

$$(3.8) \qquad \lim_{n \to \infty} |E_x^{n,\sigma} N_n(\tau_1) - 1| < \delta.$$

PROOF. Let $k_n(x, \sigma)$ denote the density of the exit distribution of $X_n$ starting at $x$. Then $P_x^{n,\sigma}$ is in fact the $h$-transform of $P_x^n$ with $h(\cdot) = k_n(\cdot, \sigma)$.

That is, for any stopping time $S < \tau_1$ and $A \in \mathscr{F}_s$,

$$P_x^{n,\sigma}(A) = \frac{1}{k_n(x,\sigma)} E_x^n k_n(X^n(S))\chi_A.$$

Now pick two points $p, q$: $1 < p < q < R$ and let $T$ and $S$ denote the exit times from the intervals $[1, q]$ and $[p, \infty)$ respectively. If $|x| = p$ and $|y| = q$ are close to $R$, then $k_n(x, \sigma) \simeq 1/\omega_d$, and $N_n(t \wedge S)$ therefore behaves under $P_y^{n,\sigma}$ much like a martingale. On the other hand, by virtue of Lemma 3.1 we may assume that $\alpha_n = 0$ on $B(0; q)$ which, of course, implies that $E_x^{n,\sigma} N_n(T) = 1$. To make things rigorous, define in the standard way two sequences of stopping times $\{T_k\}$ and $\{S_k\}$ by

$$\begin{aligned}
&T_1 = T,\\
&\vdots\\
&S_k = T_k + \theta_{T_k} S \quad \text{if } T_k < \tau_1 \text{ and } +\infty \text{ otherwise},\\
&T_{k+1} = S_k + \theta_{s_k} T,\\
&\vdots
\end{aligned}$$

Then

$$E_x^{n,\sigma} N_n(\tau_1) = \sum_{k=1}^{\infty} E_x^{n,\sigma}[N_n(T_k); \tau_1 = T_k]. \tag{3.9}$$

Keeping in mind that

$$\sum_{k=1}^{\infty} P_x^{n,\sigma}[\tau_1 = T_k] = 1, \tag{3.10}$$

we pick the $k$th term of the series (3.9) and play with it until it fits into (3.10). To this end set $a_n(p) = \inf_{|x|=p} P_x^{n,\sigma}\{\tau_1 = T\}$ and $A_n(p) = \sup_{|x|=p} P_x^{n,\sigma}\{\tau_1 = T\}$. Note that there exists a constant $c$, such that

$$1/c \cdot (q-p) \le a_n(p) \le A_n(p) \le c \cdot (q-p) \tag{3.11}$$

for all $p$ close enough to $R$. The necessary adjustment of $E_x^{n,\sigma}[N_n(T_k); \tau_1 = T_k]$ rests on the following lemma.

LEMMA 3.12. *$\forall\ \delta > 0$, $\exists\ p_\delta$, $1 < p_\delta < R$, such that $\forall\ p \in (p_\delta, R)$ and $q = (p+R)/2$,*

$$\sup_{|y|=q} \left| E_y^{n,\sigma} N_n(S) - 1 \right| \le \delta A_n(p), \tag{3.13}$$

*provided that n is large enough.*

PROOF. It is readily seen from the results of the previous section that

$$\lim_{n\to\infty} k_n(x,\sigma) = 1/\omega_d,$$

uniformly in $|x| \geq R$ and $\sigma \in S^{d-1}$. Moreover, as will become apparent in Section 5 (see the proof of Lemma 5.12), we also have that uniformly in $\sigma$,

$$\lim_{|x| \to R} \lim_{n \to \infty} \frac{\partial k_n}{\partial r}(x, \sigma) = 0.$$

Thus for any $\delta > 0$ there exists a $p_\delta$ such that for $n$ sufficiently large,

$$1/\omega_d - \delta(R-p)/4 \leq \lim k_n(x, \sigma) \leq 1/\omega_d + \delta(R-p)/4,$$

uniformly on $|x| = p$, $p \in (p_\delta, R)$. Therefore,

$$\begin{aligned} |E_x^{n,\sigma} N_n(S) - 1| &= \left| \frac{1}{k_n(x,\sigma)} E_x^n k_n(X^n(S)) N_s(S) - 1 \right| \\ &\leq \sup_{|x|=p,\, |y|=q} k_n(x,\sigma)/k_n(y,\sigma) - 1 \leq \delta(R-p)/2. \end{aligned}$$

Since $R - p = 2(q-p)$, by the choice of $q$ and in view of (3.11), this implies the claim of the lemma. □

To conclude the proof of Lemma 3.7, note first that for $|x| = p$,

$$E_x^{n,\sigma}[N_n(\tau_1); \tau_1 = T_k] = E_x^{n,\sigma}\left[\tau_1 > T; E_{x(T)}^{n,\sigma} N_n(S)\left(E_{x(S)}^{n,\sigma} N_n(\tau_1); \tau_1 = T_{k-1}\right)\right].$$

Therefore, by (3.13),

$$\begin{aligned} (1 - \delta A_n(p))^{k-1} P_x^{n,\sigma}\{\tau_1 = T_k\} &\leq E_x^{n,\sigma}[N_n(\tau_1); \tau_1 = T_k] \\ &\leq (1 + \delta A_n(p))^{k-1} P_x^{n,\sigma}\{\tau_1 = T_k\} \end{aligned}$$

for sufficiently large $n$. Thus

$$\begin{aligned} &\sum_{k=1}^{\infty} (1 - \delta A_n(p))^{k-1} P_x^{n,\sigma}\{\tau_1 = T_k\} \\ &\quad \leq E_x^{n,\sigma} N_n(\tau_1) \\ &\quad \leq \sum_{k=1}^{\infty} (1 + \delta A_n(p))^{k-1} P_x^{n,\sigma}\{\tau_1 = T_k\}. \end{aligned} \tag{3.14}$$

But

$$P_x^{n,\sigma}\{\tau_1 = T_k\} \leq (1 - a_n(p))^{k-1} A_n(p). \tag{3.15}$$

Consequently, (3.14) and (3.15) yield that for any number $N \in \mathbb{N}$,

$$\begin{aligned} &(1 - \delta A_n(p))^N \left(1 - (1 - a_n(p))^N\right) \\ &\quad \leq E_x^{n,\sigma} N_n(\tau_1) \\ &\quad \leq 1 + (1 + \delta A_n(p))^N (1 - a_n(p))^N, \end{aligned} \tag{3.16}$$

provided only that $n$ is sufficiently large. Therefore, if we pick $p$ close enough to $R$ such that $A_n(p) \ll 1$ and set $N = [1/(\sqrt{\delta} A_n)]$, then

$$e^{-\sqrt{\delta}}\left(1 - e^{-1/c\sqrt{\delta}}\right) \leq E_x^{n,\sigma} N_n(\tau_1) \leq 1 + e^{\sqrt{\delta} - 1/c\sqrt{\delta}}.$$

Thus (3.8) follows and the proof of Lemma 3.7 is complete. □

**4. Variational connection.** We continue to restrict ourselves to the case of $D$ being the exterior of the ball $B(0;\delta)$. The first step is to restate Problem 1.8 in the natural scale. In the notation of the previous section, set

$$F(u) = \int_{(1,R)\times S^{d-1}} \|u'_r\|^2 + \gamma(r)\|\nabla_s u\|^2 \, dr\, d\sigma.$$

Then for any $\phi \in W^1(D) \cap C(\partial D)$, Problem 1.8 in the natural scale takes the following form.

PROBLEM 4.1. Solve

$$\inf_{u \in W^{1,\phi}(D)} F(u),$$

where we continue to preserve the notation $D$ and $\partial D$ for the images of those sets under (3.11), that is, for $S^{d-1} \times (1, R)$ and $S^{d-1} \times \{1\}$ respectively.

LEMMA 4.2. *Problem 4.1 is uniquely solvable for any $\phi \in C(\partial D) \cap W^1(D)$. The appropriate solution $u_*^\phi$ possesses the following properties*:

(a) *$u_*^\phi$ solves* $(\mathrm{EP})_L$ *inside $D$.*
(b) *$u_*^\phi$ satisfies the volume-preserving property* (2.12).
(c) *Set $\hat{X}_t = [\hat{B}_t, \hat{\Theta}_t]$, where $\hat{B}_t$ is a Brownian motion on the line, reflected in $R$ and $\hat{\Theta}_t = \Sigma(\int_0^t \gamma(\hat{B}_s)$. Then*

$$u_*^\phi(x) = E_x \phi(\hat{X}_\tau). \tag{4.3}$$

PROOF. Let $\tilde{u} \in \mathrm{Dom}(F) \cap W^{1,\phi}_{\mathrm{loc}}$. Define $\tilde{F}(v) = F(v + \tilde{u})$. Then $\tilde{F}$ is a proper strictly convex lower-semicontinuous coercive functional on the Hilbert space $W^1_{0,\gamma}(D) = \{u | F(u) < \infty \text{ and } u|_{\partial D} = 0\}$. Therefore, it possesses a unique minimizer $v_*$. Thus, $u_*^\phi = \tilde{u} + v_*$ is what we need.

Now condition (a) is just an Euler equation for the minimizer. Likewise, (b) is a very weak form of the transversality condition. Just choose a smooth function $v_p(r)$, $p \in (1, R)$, such that $v_p(1) = 0$ and $v_p(r) = 1$ for $r \geq p$. Then

$$0 = \frac{d}{dt} F\big(u_*^\phi + t v_p\big)\Big|_{t=0} = \int_{S^{d-1}} \frac{\partial}{\partial r} u_*^\phi(r,\sigma)\, dl(\sigma)$$
$$= \frac{d}{dr}\left[\int_{S^{d-1}} u^*(r,\sigma)\, dl(\sigma)\right]\Big|_{r=p}.$$

We turn now to the proof of (c). Note first that if $\gamma$ is continuously extended to the interval $[1, R]$, the claim is nothing but the usual Neuman condition for the minimizer. Otherwise, let $r_n \to R$ from the left and set

$$\gamma^n(r) = \begin{cases} \gamma(r), & \text{if } r \leq r_n, \\ 0, & \text{otherwise}. \end{cases}$$

Define $F^n(u) = \int_D [(u'_r)^2 + \gamma^n(r)\|\nabla_s u\|^2]\, dr\, d\sigma$ and let $u_*^{\phi,n}$ be the (unique) minimizer of $F^n$ on $W^{1,\phi}_{\mathrm{loc}}$. Then $u_*^{\phi,n} = E_x\phi(\hat{X}^n_\tau)$, where $\hat{X}^n = [\hat{B}_t, \hat{\Theta}^n_t =$

$\Sigma(\int_0^t \gamma^n(\hat{B}_s))]$. Using the technique of Section 2, it is not difficult to show that $u_*^\phi$, given by (4.3), is in fact the uniform limit of the sequence $\{u_*^{\phi,n}\}$. Thus for any $u$ in $W_{\mathrm{loc}}^{1,\phi}$,

$$F(u_*^\phi) = \lim_{n\to\infty} F^n(u_*^\phi) \le \lim_{n\to\infty}\lim_{m\to\infty} F^n(u_*^{\phi,m})$$
$$\le \lim_{m\to\infty} F^m(u_*^{\phi,m}) \le \lim_{m\to\infty} F^m(u) = F(u)$$

and (c) follows. □

PROOF OF THEOREM 1.9. Assume that $\gamma \notin L_1(1, R)$. Then, for any $\phi \in C(\partial D) \cap W^1(D)$, conditions (a)–(c) of the above lemma uniquely determine the only radially admissible solution $\bar{u}^\phi$ to $(\mathrm{EP})_L$. Indeed, since $F(u_*^\phi) < \infty$, one readily obtains that $\lim_{r\to R} \|\nabla_s u_*^\phi(r,\cdot)\|_{L_2} = 0$. Proceeding as in the proof of Lemma 2.11, we conclude that $\lim_{r\to R} u_*^\phi(r,\cdot) = \bar{\phi}$, which means that $u_*^\phi$ is radially admissible.

To prove the theorem in the case of $\gamma \in L_1(1, R\}$, pick an admissible perturbation $\{\gamma_n\}$, which satisfies condition (C1) and the following additional condition:

$$\text{(C3)} \qquad \lim_{n\to\infty} \|\gamma_n\|_{L^1(R,\infty)} = 0.$$

Let $\bar{u}^\phi$ denote the appropriate radially admissible solution to $(\mathrm{EP})_L$. We claim that $\bar{u}^\phi$ satisfies (4.3) and thereby solves Problem 4.1 for $\phi \in C(\partial D)$. Indeed, for any $x \in D$, the mapping $\phi \to \bar{u}^\phi(x)$ defines a continuous linear functional on the space $C(\partial D)$ or, equivalently, it defines some probability measure $\bar{\mu}_x$. We want to prove that

$$\bar{\mu}_x(dz) = P_x(\hat{X}_\tau \in dz) \stackrel{\mathrm{def}}{=} \mu_x^*(dz).$$

By the radial symmetry we can confine ourselves to some distinguished point (pole) $\xi \in S^{d-1}$ and prove that $\forall\ r \in (1, R)$,

$$\text{(4.4)} \qquad \begin{aligned} P_{r,\xi}(\hat{\Theta}_\tau \in d\sigma) &\stackrel{\mathrm{def}}{=} \mu_r^*(d\sigma) = \bar{\mu}_r(d\sigma) \\ &= \lim_{n\to\infty} \mu_r^n(d\sigma) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} P_{r,\xi}(\Theta_\tau^n \in d\sigma). \end{aligned}$$

Let $\lambda_r^*$ and $\lambda_r^n$ be the distributions of $\eta^* = \int_0^\tau \gamma(\hat{B}_s)\,ds$ and $\eta_n = \int_0^\tau \gamma_n(B_s)\,ds$ respectively, with $\hat{B}$ and $B$ both starting from $r$. Then

$$\mu_r^*(d\sigma) = \int_{\mathbb{R}^+} \lambda_r^*(dt) q(t,\xi,\sigma) l(d\sigma)$$

and

$$\mu_r^n(d\sigma) = \int_{\mathbb{R}^+} \lambda_r^n(dt) q(t,\xi,\sigma) l(d\sigma).$$

Since by condition (C2), $E_r|\eta_n - \eta_m| \to 0$ as $m, n \to \infty$ for any $r > 1$ fixed, the limit $\bar{\eta} = \lim \eta_n$ exists in $L_1$. Therefore, the sequence $\{\lambda_r^n\}$ converges weakly to some limiting measure $\bar{\lambda}_r$ and

$$\bar{\mu}_r(d\sigma) = \int_{\mathbb{R}+} \bar{\lambda}_r(dt) q(t, \xi, \sigma) l(d\sigma).$$

Thus it remains to check that $\bar{\lambda}_r = \lambda_r^*$. To this end, for any $q \in \mathbb{R}_+$ set $w_q^*(r) = E_r \exp\{-q\eta^*\}$, $w_q^n(r) = E_r \exp\{-q\eta_n\}$ and $\bar{w}_q(r) = E_r \exp\{-q\bar{\eta}\}$. Note that $\bar{w}_q = \lim w_q^n$. On the other hand, $w_q^*$ solves the equation

$$\frac{d^2}{dr^2} w(r) = q\gamma(r) w(r), \tag{4.5}$$

with boundary conditions

$$w(1) = 1 \quad \text{and} \quad \frac{d}{dr} w(R) = 0. \tag{4.6}$$

Similarly, $w_q^n$ is the positive nonincreasing solution to the equation

$$\frac{d^2}{dr^2} w(r) = q\gamma_n(r) w(r), \tag{4.5$_n$}$$

with the boundary condition $w(1) = 1$. But

$$\left\| \frac{d}{dr} w_q^n(R) \right\| = \left\| q \int_r^\infty \gamma_n(z) w_q^n(z)\, dz \right\| \le q \|\gamma_n\|_{L^1(r,\infty)}. \tag{4.7}$$

Now by the standard compactness argument, the local solvability of (4.5) and the strong Markov property, it follows that $\bar{w}_q$ satisfies (4.5), $\bar{w}_q(1) = 1$ and

$$\frac{d}{dr} \bar{w}_q(r) = \lim_{n \to \infty} \frac{d}{dr} w_q^n(R).$$

Thus (4.7) and condition (C3) imply that $\bar{w}_q$ satisfies boundary conditions (4.6) and therefore by uniqueness $\bar{w}_q(r) = w_q^*(r)\ \forall\, q \ge 0,\ r \in (1, R)$. Consequently, $\bar{\lambda}_r = \lambda_r^*$ on $(1, R)$ and (4.4) follows. □

**5. Properties of radially admissible solutions.** Our main objective in this section is to prove Theorem 1.14 which asserts that any radially admissible solution $u^\phi$ satisfies the following condition on the boundary $\Gamma = \{R\} \times S^{d-1}$ (in the natural scale):

$$\left. \frac{\partial u^\phi}{\partial r} \right|_\Gamma = v.p. \int_{S^{d-1}} \left[ u^\phi(R, \xi) - u^\phi(R, \sigma) \right] k(\sigma, \xi) l(d\xi). \tag{5.1}$$

Moreover, we will show how the kernel $k$ above is related to a particular perturbation $\{\gamma_n\}$ leading to $u^\phi$. First of all, certain regularity properties of radially admissible solutions are reflected in the following Fatou-type lemma which will also be useful later on.

LEMMA 5.2. *Assume that $u^\phi$ is radially admissible. Then $\lim_{\rho \to R} u^\phi(\rho, \cdot)$ exists in $C^\infty(S^{d-1})$.*

PROOF. Note that if $\gamma \notin L_1[1, R)$, the claim of the lemma follows from Lemma 2.11. Thus it suffices to consider the remaining case $\gamma \in L^1[1, R)$. Then by virtue of Consequence 2.9, it is enough to establish the convergence only in $L_2(S^{d-1})$. So let $1 < r_0 < R$. Then the set $\{u^{\phi}(r, \cdot)| r \in (r_0, R)\}$ is precompact in $L_2(S^{d-1})$ by Consequence 2.9. Therefore, it is always possible to find a sequence of points $r_k \to k$, such that $\{u^{\phi}(r_k, \cdot)\}$ converges. Let $\{r_k\}$ be such a subsequence and denote by $\tau_k$ the exit time of $B_t$ from the interval $\Delta_k = [r_k, r_{k+1}]$. Then for $r \in \Delta_k$,

$$\| u(r, \cdot) - u^{\phi}(r_k, \cdot) \|_{L_2}^2 = f(r) + f(r_k) - 2\left\langle u^{\phi}(r_k, \cdot), E_{r,\cdot} u^{\phi}(B_{\tau_k}, \Theta_{\tau_k}) \right\rangle_{L_2},$$

where $\langle \cdot, \cdot \rangle_{L_2}$ denotes the scalar product in $L_2(S^{d-1})$. By Consequence 2.9,

$$\sup\left\{ \| \nabla u_s^{\phi}(r, \cdot) \|_{\sup} \middle| r_0 < r < R \right\} < \infty.$$

Consequently,

$$\text{(5.3)} \quad \begin{aligned} \left\langle u^{\phi}(r_k, \cdot), E_{r,\cdot} u^{\phi}(B_{\tau_k}, \Theta_{\tau_k}) \right\rangle_{L_2} &= f(r_k) + O\left( \left\| E_{r,\cdot} d(\cdot, \Theta_{\tau_k}) \right\| \right) \\ &\quad + O(f(r_k) - f(r_{k+1})), \end{aligned}$$

where $d(\cdot, \cdot)$ denotes the Riemann distance on the sphere $S^{d-1}$. The third term on the right-hand side of (5.3) is $o(1)$, both by the choice of the sequence $\{r_k\}$ and by Consequence 2.9. To estimate the second term, set $\phi(t) = E_{r,\sigma} d(\sigma, \Theta_t)$ ($\phi$, of course, does not depend on $\sigma$); then $\phi$ is a bounded continuous function on $[0, \infty)$ and $\phi(0) = 0$. Let $\lambda_k(dt)$ be the distribution of $\int_0^{\tau_k} \gamma(B_s)\, ds$. In this notation,

$$E_{r,\cdot} d(\cdot, \Theta_{\tau_k}) = \int_{\mathbb{R}+} \phi(t) \lambda_k(dt).$$

We will prove that the second term on the right-hand side of (5.3) is $o(1)$ by showing that

$$\text{(5.4)} \qquad w - \lim \lambda_k = \delta_0.$$

To prove this assertion, set

$$w_k(r) = E_r \int_0^{\tau_k} \gamma(B_s)\, ds.$$

It is not hard to see that $w_k$ satisfies $w_k'' = -\gamma(r)$ and $w_k(r_k) = w_k(r_{k+1}) = 0$. So $w_k$ can be explicitly calculated and, involving the initial assumption $\gamma \in L^1[1, R)$, we obtain

$$\lim_{k \to \infty} \max\{w_k(r) | r \in \Delta_k\} = 0,$$

and (5.4) follows. Therefore, $\|u^{\phi}(r, \cdot) - u^{\phi}(r_k, \cdot)\|_{L_2}^2 = o(1)$ and the lemma is completely proved. □

REMARK. Note that the above result combined with Lemma 4.2 suggests that as far as the radially admissible solutions are concerned, the Kuramochi compactification of $D$ is the proper one to consider.

Now let $\{\gamma_n\}$ be a radially admissible perturbation, let $\{u_n^\phi\}$ be the corresponding sequence of solutions to $(\mathrm{EP})'_n$ and let $u^\phi$ denote the limiting admissible solution

$$u^\phi(x) = \lim_{n\to\infty} u_n^\phi, \qquad x \in D.$$

Then all the information about $u^\phi$ is contained in the sequence $\{u_n^{\phi, D}\}$, where $u_n^{\phi, D}$ is the restriction of $u_n^\phi$ to $D$. Thus, to derive the desired boundary condition for $u^\phi$, we should consider not the process $X_n$ itself but rather its trace $X_n^D$ on $D$. Set $i_n(t) = \int_0^t \mathbb{I}_D(X_n(s))\,ds$, where $\mathbb{I}_D$ stands for the indicator of the set $D$, and let $I_n(t)$ denote the right-continuous inverse of $i_n$,

$$I_n(t) = \inf\{u : i_n(u) > t\}.$$

Then $X_n^D(t) = X_n(I_n(t))$ and $(X_n^D, \mathscr{F}_{I_n(t)}, P_x)$ is a family of SMP on $\overline{D}$. Furthermore,

$$u_n^\phi(x) = E_x \phi(X_n^D(\tau_1)). \tag{5.5}$$

Since $u_n^\phi$ belongs to the domain of the generator of $X_n^D$, we are in fact looking for the boundary conditions $X_n^D$ satisfies on $\Gamma$. This problem was solved in [10] in a much more general situation. Our task, however, is to exploit radial symmetry to obtain more precise information which enables us to verify Theorem 1.14. To this end set $M$ to be a strictly elliptic operator on $[1, \infty) \times S^{d-1}$ given by

$$M = \partial^2/\partial r^2 + \alpha(r)\Delta_s,$$

with $\alpha \in L_1^{\mathrm{loc}}[1, \infty)$. Let $Y(t)$ denote the diffusion generated by $M$ on $[1, \infty) \times S^{d-1}$ (with absorption on the unit sphere) and set $Y^D$ to be its trace on $\overline{D}$. Furthermore, let $A(M, D)$ denote the domain of the generator of $Y^D$. Let $Bs^a$ denote a squared Bessel process of order 0 with initial condition $a$, that is,

$$Bs^a(x) = a + 2\int_0^x \sqrt{Bs^a(y)}\,d\beta(y),$$

with $\beta$ being a one-dimensional Brownian motion. Let $T$ be a $1/2$-exponential random variable ($T \in \exp(1/2)$) independent of $\beta$. Set $Bs = Bs^T$ and define

$$\chi = \int_{\mathbb{R}+} \alpha(x + R) Bs(x)\,dx.$$

Then $\chi$ is a positive, infinitely divisible random variable; let $\mu$ denote its Lévy measure.

LEMMA 5.6. *Assume that $v \in A(M, D) \cap C^2(\overline{D})$ and set $g = v|_\Gamma$. Then*

$$\left.\frac{\partial v}{\partial r}\right|_\Gamma = \int_{\mathbb{R}+} (Q_t g - g)/t\mu(dt),$$

*where $Q_t$ is the transition operator of BM on $S^{d-1}$.*

PROOF. If $X(t) = [B(t), \Theta(t)]$ is the skew-product representation of $X$, then $X_D = [B(I(t)), \Theta(I(t))]$, where, as before, $I$ is the right-continuous inverse of the time spent by $X$ in $D$. Note that $B(I(t))$ is just a one-dimensional BM with reflection at $R$ and absorption at 1. Since $v \in C^2(\overline{D})$, an easy calculation (cf. [7]) reveals that

$$\frac{\partial v}{\partial r}\bigg|_\Gamma = \lim_{\delta\to 0}\frac{1}{\delta}\left(E_{R,\cdot}g(\Theta(\tau_\delta)) - g\right), \tag{5.7}$$

where $\tau_\delta = \inf\{t\colon B(t) = R - \delta\}$. Recall now that $\Theta(t) = \Sigma(\int_0^t \alpha(B(s)\,ds)$. Set $\chi_\delta = \int_0^\tau \delta\alpha(B(s))\,ds$. Let us rewrite (5.7) as follows:

$$\begin{aligned}\frac{\partial v}{\partial r}\bigg|_\Gamma &= \lim_{\delta\to 0}\int_{\mathbb{R}+}\left[(Q_t g - g)/t\right]\cdot\left(t/\delta P\{\chi_\delta \in dt\}\right)\\ &= \lim_{\delta\to 0}\int_{\mathbb{R}+}(Q_t g - g)/t\mu_\delta(dt).\end{aligned} \tag{5.8}$$

where $\mu_\delta(dt) = t/\delta P\{\chi_\delta \in dt\}$. We want to show that $\{\mu_\delta\}$ converges to $\mu$ in a way that enables us to pass to the limit in (5.8). We can obviously ignore the values of $\alpha$ to the left of $R$. Then by the occupation density formula and the Ray–Knight theorem ([13] and [14]), the following holds:

$$\chi_\delta \overset{d}{=} \int_{\mathbb{R}+}\alpha(R+x)Bs^T\delta(x)\,dx, \tag{5.9}$$

where $T_\delta \in \exp(1/2\delta)$ and is independent of the BM which underlies the corresponding squared Bessel process. By the scaling properties of Bessel processes [13], $\chi$ is distributed as a sum of $1/\delta$ independent copies of $\chi_\delta$. Therefore, if $f_\delta$, $f$ are the Laplace transforms of $\mu_\delta$ and $\mu$ respectively, then

$$\begin{aligned}f_\delta(\lambda) &= \int_{\mathbb{R}+}e^{-\lambda t}\mu_\delta(dt) = \frac{1}{\delta}E\chi_\delta e^{-\lambda\chi_\delta} = -\frac{1}{\delta}\frac{d}{d\lambda}\left(Ee^{-\lambda\chi_\delta}\right)\\ &= -\frac{1}{\delta}\frac{d}{d\lambda}\left(Ee^{-\lambda\chi}\right)^\delta = \frac{E\chi e^{-\lambda\chi}}{\left(Ee^{-\lambda\chi}\right)^{1-\delta}}.\end{aligned}$$

As a consequence we obtain

$$\lim_{\delta\to 0}f_\delta(\lambda) = -\frac{d}{d\lambda}\ln\left(Ee^{-\lambda\chi}\right) = \int_{\mathbb{R}+}e^{-\lambda t}\mu(dt) = f(\lambda). \tag{5.10}$$

Therefore, $\{\mu_\delta\}$ converges weakly to $\mu$ on every finite interval $[0, N)$. On the other hand, note that

$$\int_N^\infty 1/t\mu_\delta(dt) = \frac{1}{\delta}P\{\chi_\delta > N\}.$$

We can make use of (5.9) and some standard arguments to derive the following easy estimate:

$$\frac{1}{\delta}P\{\chi_\delta > N\} \le \left(\|\alpha\|_{L_1[R,R']}\right)/N + 1/(R' - R) \tag{5.11}$$

for each $R' > R$. Consequently,

$$\lim_{N\to\infty}\int_N^\infty 1/t\mu_\delta(dt) = 0,$$

uniformly in $\delta$. To conclude the proof of the lemma, just note that by Itô's formula, $(Q_t g - g)/t$ is bounded on $[0,\infty)$ and goes to 0 as $t$ tends to $\infty$. □

We are ready now to give a description of the class of radially admissible solutions. As usual, let $\{\gamma_n\}$ be an admissible perturbation, $\{\mu_n^\phi\}$ the corresponding sequence of solutions to $(\mathrm{EP})_n$ and set $u^\phi(x) = \lim_{n\to\infty} u_n^\phi$, $x \in D$.

LEMMA 5.12. *There exists a $\sigma$-finite measure $\mu$ on $\mathbb{R}+$, such that*

$$\left.\frac{\partial u^\phi}{\partial r}\right|_\Gamma = \int_{\mathbb{R}+} (Q_t g - g)/t\mu(dt), \tag{5.13}$$

*where $g = u^\phi|_\Gamma$.*

PROOF. As in the proof of Lemma 2.17, we shall distinguish between two cases:

(i) $\forall R' > R, \qquad \limsup_{n\to\infty} \|\gamma_n\|_{L_1[1,R']} = \infty$

and

(ii) $\exists R' > R, \qquad \limsup_{n\to\infty} \|\gamma_n\|_{L_1[1,R']} < \infty.$

First assume (i). Then the claim is that $u^\phi$ satisfies (5.13) with $\mu \equiv 0$; that is,

$$\left.\frac{\partial u^\phi}{\partial r}\right|_\Gamma = \lim_{p\to R}\lim_{n\to\infty} \frac{\partial u_n^\phi(p,\cdot)}{\partial r} = 0. \tag{5.14}$$

Indeed,

$$\frac{\partial u_n^\phi(p,\cdot)}{\partial r} = \int_{\mathbb{R}+} (Q_t g_{p,n} - g_{p,n})/t\mu_{p,n}(dt), \tag{5.15}$$

where $g_{p,n} = u_n^\phi(p,\cdot)$ and $\mu_{p,n}$ is the Lévy measure of the random variable $\chi_{p,n}$,

$$\chi_{p,n} = \int_{\mathbb{R}+} \gamma_n(x+P) Bs(x)\, dx.$$

The absolute value of the expression on the right-hand side of (5.15) is bounded from above by

$$\mu_{p,n}([0,1])\|\Delta g_{p,n}\|_{\sup} + 2\sup_{t>1}\|Q_t g_{p,n} - \bar\phi\|_{\sup}\int_1^\infty 1/t\mu_{p,n}(dt). \tag{5.16}$$

We may assume without loss of generality that $\bar\phi = 0$. Note that in the notation of Lemma 2.17, $g_{p,n}$ can be represented as

$$g_{p,n}(\cdot) = E_{p,\cdot}\phi(\Theta_n(\tau)) = \int_{\mathbb{R}+} P_p\{\eta_n \in dt\} Q_t\phi = E_p Q_{\eta_n}\phi. \tag{5.17}$$

On the other hand, it follows by the very nature of the measure $\mu_{p,n}$ that for every $\lambda > 0$,

$$\max\left\{\mu_{p,n}([0,1]), \int_1^\infty 1/t\mu_{p,n}(dt)\right\} \le c(\lambda) + \ln E_p e^{-\lambda\chi_{p,n}}, \tag{5.18}$$

where $c = c(\lambda)$ is a constant. But $\|Q_t\phi\|_{\sup} \le ce^{-\lambda_0 t}$, where $\lambda_0$ is the smallest positive eigenvalue of $1/2\Delta$ on $S^{d-1}$ (recall that we have assumed $\overline{\phi}$ to be 0). Therefore, since $\chi_{p,n} \le \eta_n$,

$$\lim_{p\to R}\lim_{n\to\infty} \|E_p Q_{\eta_n}\phi\|_{\sup}\left(c(\lambda) + \ln E_p e^{-\lambda\chi_{p,n}}\right) = 0 \tag{5.19}$$

for any $\lambda < \lambda_0$. Since all the reasoning above remains true if we consider $\Delta_s\phi$ instead of $\phi$, estimates (5.16)–(5.19) imply (5.14) and this gives us the proof of the lemma in case (i).

Now assume (ii). Then there exists a number $\delta > 0$ such that for any $n \in \mathbb{N}$ and $p \le R$,

$$P\{\chi_{p,n} < 1\} > \delta. \tag{5.20}$$

Therefore, the set $\{Ee^{-\lambda\chi_{p,n}};\ n \in \mathbb{N},\ p \le R\}$ is bounded away from 0. Consequently, we obtain that $\{\mu_{p,n}\}$ is $\sigma$-finite uniformly in $p$ and $n$. Similarly,

$$\lim_{N\to\infty}\int_N^\infty 1/t\mu_{p,n}(dt) = 0,$$

uniformly in $n \in \mathbb{N}$ and $p \le R$. Thus $\{\mu_{p,n}\}$ is a tight family of measures in the sense of convergence which was implicitly introduced in the course of proving Lemma 5.6. As a consequence, we can choose a cluster point $\mu$ of $\{\mu_{p,n}\}$ and a subsequence $\{p_m, n_k\}$, such that the equality

$$\lim_{p_m\to R}\lim_{n_k\to\infty}\int_{\mathbb{R}+}\vartheta(t)/t\mu_{p_m,n_k}(dt) = \int_{\mathbb{R}+}\vartheta(t)/t\mu_{p,n}(dt)$$

holds for any continuous $\vartheta$ for which $\vartheta(t)$ and $\vartheta(t)/t$ are bounded on $\mathbb{R}+$. The result follows if we pick $\vartheta(t) = Q_t u^\phi(R,\cdot) - u^\phi(R,\cdot)$ and use Consequence 2.9 and Lemma 5.1 to justify the successive approximations of $u^\phi(p,\cdot)$ by $u_n^\phi(p,\cdot)$ and $u^\phi(R,\cdot)$ by $u^\phi(p,\cdot)$. □

Lemma 5.12 almost readily implies the claim of Theorem 1.14. Note first that if assumption (C2) holds true, we may simply pick $k = 0$. Otherwise, the above lemma asserts that any radially admissible solution $u^\phi$ satisfies

$$\left.\frac{\partial u^\phi}{\partial r}\right|_\Gamma = \int_{\mathbb{R}+} 1/t\mu(dt)\int_{S^{d-1}} q(t,\sigma,\xi)\left(u^\phi(R,\xi) - u^\phi(R,\sigma)\right)l(d\xi). \tag{5.21}$$

In this case the claim of the theorem holds with

$$k(\sigma,\xi)=\int_{\mathbb{R}+}q(t,\sigma,\xi)/t\mu(dt).$$

Indeed, changing the order of the integration in the integral on the right-hand side of (5.21), we obtain

$$\left.\frac{\partial u^{\phi}}{\partial r}\right|_{\Gamma}=\mu\{0\}\,\Delta_s u^{\phi}+v.p.\int_{S^{d-1}}k(\sigma,\xi)(u^{\phi}(R,\xi)-u^{\phi}(R,\sigma))l(d\xi).$$

Finally, the diffusion term can be omitted, as follows from the following proposition.

PROPOSITION 5.22. *Let $\mu$ be as in the statement of Lemma* 5.12. *Then $\mu$ has no atom at* 0.

PROOF. Note that under assumption (ii) of Lemma 5.12, for each $\varepsilon>0$ the family $\{P(\chi_{p,n}>\varepsilon)\}$ is bounded away from 0 by $\delta=\delta(\varepsilon)>0$. In particular, it follows that

$$\ln Ee^{-\lambda\chi_{p,n}}\geq\ln\delta(\varepsilon)-\varepsilon\lambda.$$

Therefore, for all $n\in\mathbb{N}$, $p\leq R$, $\lambda>0$ and $\varepsilon>0$, the following holds true:

$$\int_0^1(e^{-\lambda t}-1)/t\mu_{p,n}(dt)\geq\ln\delta(\varepsilon)-\varepsilon\lambda.$$

Consequently, no cluster point of the family $\{\mu_{p,n}\}$ can have an atom at 0. This completes the proof of Theorem 1.14. □

**6. General smooth exterior domains.** Let $D$ be an exterior domain with $C^{2,\alpha}$ boundary and compact complement $D^c$. We assume that $D^c$ contains an open neighborhood of the origin. Fix two numbers $\delta,\Delta>0$; $B(0;\delta)\subset D^c\subset B(0;\Delta)$. Set $B_1=(\delta,\infty)\times S^{d-1}$ and $B_2=(\Delta,\infty)\times S^{d-1}$.

PROOF OF THEOREM 1.14. Suppose that $u^{\phi}$ is a radially admissible solution of $(\mathrm{EP})_L$ in $D$. Set $\tilde{\phi}(z)=u^{\phi}(z)$ for $|z|=\Delta$. Then it is easy to see that $u^{\phi}|_{B_2}$ is a radially admissible solution to

$$Lu=0\quad\text{in } B_2,$$
$$u|_{\partial B_2}=\tilde{\phi}.$$

Thus the proof of Section 5 applies and hence the result. □

PROOF OF LEMMA 1.10. Assume that $u^{\phi}$ is admissible and satisfies (1.3) with some number $u_0$. Then it is easy to see that

$$\int_{S^{d-1}}u^{\phi}(r,\sigma)l(d\sigma)=u_0$$

for each $r>\Delta$. On the other hand, using the notation of Section 1, we may represent $u^{\phi}$ as

$$u^{\phi}(x) = h(x)\int_{\partial D}\phi(z)\mu_x^h(dz) + (1 - h(x))u_0. \tag{6.1}$$

Integrating (6.1) over $S^{d-1}$ for $\rho = |x| > \Delta$, we therefore obtain

$$u_0 = \int_{S^{d-1}} h(\rho,\theta)l(d\theta)\int_{\partial D}\phi(z)\mu_{\rho,\theta}^h(dz) + \left(1 - \int_{S^{d-1}} h(\rho,\theta)l(d\theta)\right)u_0.$$

Thus it remains to prove that

$$\lim_{\rho\to\infty} h(\rho,\cdot)\bigg/\int_{S^{d-1}} h(\rho,\theta)l(d\theta)$$

exists, and Lemma 1.10 is established. To this end, let us switch to the natural scale (1.13) and investigate

$$\lim_{r\to R} g(r,\cdot)\bigg/\int_{S^{d-1}} g(r,\theta)l(d\theta),$$

where $g(r,\cdot) = h(T^{-1}(r),\cdot)$. Since $g$ satisfies

$$\left(\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{\gamma(p)}{2}\Delta_s\right)g = 0$$

and $g(R,\cdot) \equiv 0$, it follows that

$$\int_{S^{d-1}} g(r,\theta)l(d\theta) = 1/c(R - r)$$

for some constant $c > 0$. On the other hand, set $p = T(\Delta)$ and let $\tau^{p,R}$ denote the first exit time from $(p, R) \times S^{d-1}$. Then for $r \in (p, R)$,

$$\begin{aligned} g(r,\cdot) &= P_r\{B(\tau^{p,R}) = p\}E_{r,\cdot}\left[h(p,\Theta(\tau^{p,R}))/B(\tau^{p,R}) = p\right] \\ &= \frac{(R-r)}{(R-p)}E_{r,\cdot}\left[h(p,\Theta(\tau^{p,R}))/B(\tau^{p,R}) = p\right]. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} g(r,\cdot)\bigg/\int_{S^{d-1}} g(r,\theta)l(d\theta) &= cg(r,\cdot)/(R-r) \\ &= c(R-p)E_{r,\cdot}\left[h(p,\Theta(\tau^{p,R}))/B(\tau^{p,R}) = p\right]. \end{aligned}$$

But this last quantity converges to some $B \in C(S^{d-1})$ as $r \to R$; (1.11) follows. To complete the proof of Lemma 1.10, pick a $\phi$-admissible perturbation which satisfies (C2) and note that by Lemma 2.17, $u^{\phi}|_{B_2}$ satisfies (1.3). □

PROOF OF THEOREM 1.5. If $\int^{\infty} V(\rho)(\exp[2\int^{\rho} b(z)\,dz])\,d\rho = \infty$, then one can apply Theorem 1.5 for the case of radially symmetric domains to check that $u^{\phi}|_{B_2}$ satisfies (1.3) for each admissible solution $u^{\phi}$. Hence by Lemma 1.10, there is exactly one admissible solution for each $\phi \in C(\partial D)$. Therefore, $(\mathrm{EP})_L$ is indeed stable.

To prove the "only if" part of the theorem, let us return to the natural scale and assume that $\gamma \in L_1(1, R)$. We then construct a boundary function $\phi$ and a $\phi$-admissible perturbation $\{L_n\}$ in such a fashion that $u^\phi$ does not satisfy (1.3). By virtue of Lemma 1.10, this means that there are two different admissible solutions; therefore, (EP) cannot be stable.

Recall now that $B(0;1) \subset D^c$ [in the natural scale (3.11)] and define $v$ on $\partial B(0;1)$ as in (2.15). Pick some radially admissible perturbation $\{\gamma_n\}$ and denote by $v^v$ the corresponding admissible solution in $T(B_1)$. Note that $v^v$ does not satisfy (1.3). So $u^\phi = v^v|_D$ also does not satisfy (1.3). But $u^\phi$ is admissible with $\phi = v^v|_{\partial D}$ and the claim follows. □

Proof of Theorem 1.9. Note that under the transformation (1.13) all the conclusions of Lemma 3.2 still hold true. In particular, the (unique) solution to Problem 1.8 is still given by

$$u_*^\phi(r,\theta) = E_{r,\theta}\phi\left(\hat{B}_\tau, \hat{\Theta}_\tau\right),$$

where $\tau$ is the first exit time from $T(D)$. Pick a radially admissible perturbation $\{\gamma_n\}$, which satisfies conditions (C1) and (C3). Let $u^\phi$ denote the corresponding admissible solution. We claim that $u^\phi = u_*^\phi$. Indeed, pick $p, q \in (T(\Delta), R)$; $p < q$. Furthermore, let $\tau_q$ denote the first exit time from $T(D) \cap B(0;q)$ and let $\tau_p$ denote the first hitting time of $B(0;p)$. Then

$$u^\phi(x) = E_x u^\phi\left(\hat{B}_{\tau_p}, \hat{\Theta}_{\tau_p}\right) \tag{6.2}$$

for $|x| = q$. On the other hand, for $|x| = p$,

$$u^\phi(x) = E_x\left[u^\phi\left(\hat{X}_{\tau_q}\right); |\hat{X}_{\tau_q}| = q\right] + E_x\left[\phi\left(\hat{X}_{\tau_q}\right); \hat{X}_{\tau_q} \in T(\partial D)\right]. \tag{6.3}$$

But in view of the strong Markov property, (6.2) and (6.3) are equivalent to

$$u^\phi(r,\theta) = E_{r,\theta}\phi\left(\hat{B}_\tau, \hat{\Theta}_\tau\right),$$

and the result follows. □

**7. Radially symmetric diffusions in the plane.** The results of the previous sections can be extended readily to the case of the general elliptic self-adjoint radially symmetric operators in the plane. Let $L$ be a generator of a transient diffusion, given by

$$L = \nabla(A(\rho)\nabla) + A(\rho)\nabla q\, \nabla,$$

where $\nabla = (\partial/\partial\rho, \partial/\partial\phi)^T$, $q = q(\rho)$ and $A$ is $2 \times 2$ matrix:

$$A(\rho) = \begin{pmatrix} a(\rho) & b(\rho) \\ b(\rho) & c(\rho) \end{pmatrix}.$$

Let $D$ be the exterior of the unit ball. We are going to find a factorization of $L$

and $D$ which fits the natural scale setting of Section 2. This amounts to the proper choice of new radial and angular variables, which we denote by $r$ and $\xi$ respectively. We will skip a rather obvious geometrical interpretation of this choice and set

$$r(\rho) = 1 + \int_1^\rho e^{-q(z)}/a(z)\,dz, \qquad r(\infty) = R < \infty,$$

$$\xi(\rho,\phi) = \phi - \int_1^\rho b(z)/a(z)\,dz.$$

Then an easy computation reveals that in the new coordinates, $L$ takes the form

$$L = \partial^2/\partial r^2 + e^{2q(\rho)} \det A(\rho)\Delta_s$$

and $D$ is mapped one to one onto the annulus $(1, R) \times S^{d-1}$. The following lemma is a direct consequence of the results of the previous sections and a characterization of the Martin boundary at $\infty$ given in [12]:

LEMMA 7.1. *Let $L$ be as above. Then*:

(i) *The following three statements are equivalent*:
(a) *The exterior Dirichlet problem is stable in the sense of Section* 1.
(b) *The Kuramochi boundary at $\infty$ for $L$ is a singleton.*

(c)
$$\int_1^\infty (e^{q(\rho)}/a(\rho)) \det A(\rho)\,d\rho = \infty.$$

(ii) *The Martin boundary at $\infty$ for $L$ is a singleton $(S^1)$ iff*

$$\int_1^\infty e^{-q(\rho)}a(\rho)(R - r(\rho))^2 \int_\rho^\infty e^{q(Z)} \det A(z)/(a(z))(R - r(z)))^2\,dz\,d\rho$$
$$= \infty\ (< \infty).$$

REMARK. Note that claim (ii) of the above lemma partially complements the result in [3] which asserts that the Martin boundary at $\infty$ for radially symmetric diffusions in the plane is either a singleton or $S^1$.

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DEPARTMENT OF MATHEMATICS
TECHNION–ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA 32000
ISRAEL