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EXISTENCE AND CONTINUITY OF OCCUPATION DENSITIES OF STOCHASTIC INTEGRAL PROCESSES

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Let f be a square-integrable function on the unit square. Assume that the singular numbers $(a_i)_{i \in \mathbb{N}}$ of the Hilbert–Schmidt operator associated with f admit some $0 < \alpha < \frac{1}{3}$ such that $\sum_{i=1}^{\infty} |a_i|^{\alpha} < \infty$. We present a purely stochastic method to investigate the occupation densities of the Skorohod integral process U induced by f. It allows us to show that U possesses continuous square-integrable occupation densities and obviously generalizes beyond the second Wiener chaos.

1. Introduction. Integral processes related to Skorohod's [Skorohod (1975)] extension of the Itô integral of the Wiener process arise in a number of situations, for example, in the study of stochastic integral equations with boundary conditions which destroy the adaptedness of the solutions or in the disguise of Stratonovitch integrals occurring in the context of flows of diffeomorphisms on manifolds [see Ocone and Pardoux (1989), Nualart and Pardoux (1991, 1992), Donati-Martin (1991) and Buckdahn (1989)]. Although designed for stochastic purposes, Skorohod's integral owes its analytic accessibility mainly to the fact that it can be seen as the adjoint of the Malliavin derivative. Its stochastic calculus, developed in Nualart and Pardoux (1988), displays a number of stochastically interesting results. For example, for a fairly large set of integrands, Skorohod integral processes possess quadratic variations which are given in the same way as for their Itô counterparts and consequently are nontrivial. Although the oscillation strength of the samples, hidden behind this statement, calls for the investigation of their occupation densities, the stochastic calculus seems to be unable to present an equally obvious and natural access thereto. Based on the observation that Gaussian processes like Wiener's possess simple spectral properties and consequently offer an easy approach to their occupation densities via Fourier analysis, we tried an alternative method in Imkeller (1991, 1992). The Fourier transforms of occupation measures were translated into purely analytical language and an integral criterion for the existence of densities in terms of Fredholm's theory was derived. However, seen from the point of view of stochastics, this approach suffered from the same disease as other investigations in the area: The access via Malliavin's derivative eventually puts more weight on purely analytical notions and lacks stochastic intuition and flavor.

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In this article we present an approach to occupation densities of Skorohod integral processes which is purely stochastic. For the sake of clarity and brevity of the exposition of the main methods, we chose to confine our attention to the second Wiener chaos again. It will become clear along the way that it generalizes to larger parts of the Wiener space in a rather straightforward way.

To sketch the main ideas, we suppose that a Skorohod integral process U in the second chaos is given. If U is induced by, say, a symmetric function $f \in L^2([0,1]^2)$, there is an ONB $(h_i)_{i\in\mathbb{N}}$ of $L^2([0,1])$ such that we may describe f and U in the following way:

$$f = \sum_{i=1}^{\infty} a_i h_i \otimes h_i,$$

$$U_t = \sum_{i=1}^{\infty} a_i \left[\int_0^t h_i \, dW \int_0^1 h_i \, dW - \int_0^t h_i^2(s) \, ds \right],$$

$t \in [0,1]$, where W is the Wiener process. Now the body of the approach consists in using the power of semimartingale theory and its description of local times in Tanaka's formula. The obvious problem one has to face from the very beginning is the lack of adaptedness of U with respect to the natural filtration $(\mathbf{F}_t)_{t\in[0,1]}$ of W. Following an idea of Itô which was elaborated in a series of papers [see Jeulin and Yor (1985)], one could, however, think of making U a semimartingale by enlarging the Wiener filtration. Considering the preceding description of U, the obvious idea is to augment each $\mathbf{F}_t$ by the whole information present in the random vector $X = (a_i \int_0^1 h_i \, dW)_{i\in\mathbb{N}}$. But this surplus of information destroys the property "bounded variation" of the nonmartingale part in the decomposition of U with respect to the enlarged filtration: It simply is too big to make U a semimartingale. On second thought, U should be at least a semimartingale in its own natural filtration, but this is barely accessible to analysis. Thus there is no appropriate filtration at hand in which to study stochastic integrals and local times of U simultaneously.

The key observation of the method presented here starts with the following question: Is it at all necessary to have stochastic integrals with respect to U available in order to study its occupation densities? What if, just as long as we compute occupation densities, we forget about the stochastic nature of X altogether and replace it by a vector $x = (x_i)_{i\in\mathbb{N}}$ of simple real numbers running in the range of its possible values? Instead of U we could consider the processes

$$S(x) = \sum_{i=1}^{\infty} \left(x_i \int_0^{\cdot} h_i \, dW - a_i \int_0^{\cdot} h_i^2(s) \, ds \right),$$

where x is an infinite-dimensional parameter running, say, in l_2. These are not only Gaussian processes (with a deterministic drift), but also perfect semimartingales, if only we assume that $\sum_{i=1}^{\infty} |a_i| < \infty$, that is, that f is of trace class. At this point it becomes obvious that trace conditions form a natural barrier to the range of application of the method presented. In a first step, we can use Tanaka's formula to compute occupation densities $L(x)$ of $S(x)$ rather easily. With these at hand, in the second step we would like to substitute $X(\omega)$ for x again, at least for almost all ω. However, since for any x the function $L(x)(\omega)$ is an occupation density of $S(x)(\omega)$ only for almost all ω, we run into the problem of having to remove uncountably many 0-sets. But l_2 is a separable metric space. So if we knew that $L(x)$ is continuous in x, we might be allowed to remove only countably many 0-sets instead. Therefore our problem boils down to proving a.s. sample continuity of $L(x)$ in x. This would be rather hopeless on all of l_2, but, if $(a_i)_{i\in\mathbb{N}}$ decreases "rapidly enough," we can show that x may actually be assumed to be in a rather small totally bounded subspace of l_2, for which an infinite-dimensional generalization of Kolmogorov's continuity criterion [see Kono (1980) and Pisier (1980a, b)] for stochastic processes is applicable. By "rapidly enough" we mean that there exists a number $0 < \alpha < \frac{1}{3}$ such that $\sum_{i=1}^{\infty}|a_i|^{\alpha} < \infty$, but we do not attempt here to tighten our arguments for obtaining optimal results.

Given this flow of ideas, the organization of this article is straightforward.

In Section 3 the beautiful theory of local times of semimartingales, as presented in papers like Barlow and Yor (1981, 1982) and Azema and Yor (1978), is applied to derive moment estimates for $L(x)$ and $S(x)$. These are used to prove a.s. continuity on appropriate totally bounded subsets of l_2.

In Section 4, the results of Section 3 are related to Skorohod integral processes mainly by finding a way to resubstitute $X(\omega)$ for x a.s. In this manner we obtain an existence result for occupation densities of Skorohod integral processes subject to a trace condition (Theorem 3). From this point of view it is not hard to figure out the a.s. continuity of the occupation densities (Theorem 4).

It is most likely that the method of this article produces more general results about existence and continuity of occupation densities in the following three directions: for larger classes of integral processes in the second chaos (fewer restrictions on the trace), for general Wiener space beyond the second chaos and, especially, for solutions of stochastic integral equations with boundary conditions.

2. Notation and preliminaries. We assume throughout that a basic probability space $(\Omega, \mathbf{F}, P)$ is given on which a Wiener process W, indexed by $[0, 1]$, lives. Stochastic (Itô) integrals of adapted integrands u with respect to W are denoted by $\int u\, dW$, the quadratic variation of martingales M of the Wiener process by $[M]$. $\mu \otimes \nu$ is the product of two measures μ, ν. Occasionally, we denote Lebesgue measure on the real line by λ. For random variables on $(\Omega, \mathbf{F}, P)$ as well as for functions on $[0, 1]^k$, p-norms are abbreviated by the same symbol, $\|\cdot\|_p$. The space of all real-valued smooth functions on $\mathbb{R}$ with compact support is denoted by $C_0^{\infty}(\mathbb{R})$. The tensor product $f \otimes g$ of functions $f, g \in L^2([0,1]^2)$ is the function $(s,t) \mapsto f(s)g(t)$ in $L^2([0,1]^2)$. Finally, l_2 is the usual Hilbert space of square-summable sequences of real numbers.

3. Occupation densities depending on a parameter. In this section we consider the Gaussian field given by the stochastic integrals of deterministic functions in $L^2([0,1])$ with respect to the Wiener process, drifted by some deterministic function of bounded variation. More precisely, we suppose that V is a continuous function of bounded variation $|V|_1$ on $[0,1]$. For $g \in L^2([0,1])$, we let

$$M(g) = \int_0^{\cdot} g(s)\, dW_s, \tag{1}$$

$$S(g) = M(g) + V. \tag{2}$$

We know that, for any g, $S(g)$ possesses an occupation density $L(S(g),\cdot\,)$ which can be represented by Tanaka's formula. For $y \in \mathbb{R}$, we have

$$(S(g)_1 - y)^+ - (-y)^+ = \int_0^1 1_{\{S(g)_s > y\}}\, dS(g)_s + \tfrac{1}{2} L(S(g), y) \tag{3}$$

[see, for example, Azema and Yor (1978) and Barlow and Yor (1981)]. Since $S(g)$ is a semimartingale, we may decompose its stochastic integral appearing in (3) and obtain the following formula, which is slightly more appropriate for our purposes. For any $y \in \mathbb{R}$, we have

$$\begin{aligned}(S(g)_1 - y)^+ &- (-y)^+ \\ &= \int_0^1 1_{\{S(g)_s > y\}}\, dM(g)_s + \int_0^1 1_{\{S(g)_s > y)}\, dV_s + \tfrac{1}{2} L(S(g), y).\end{aligned} \tag{4}$$

The purpose of this section is to derive continuity results for the processes

$$(g, t) \to S(g)_t,$$

and

$$(g, y) \mapsto \int_0^1 1_{\{S(g)_s > y\}}\, dV_s + \tfrac{1}{2} L(S(g), y).$$

We will always tacitly assume that both of these processes are measurable in all (three) variables and separable. This does not affect generality, since we have continuity in probability at least, as follows from the subsequent moment estimates, which are also used for establishing sample continuity a.s. This, indeed, is the main subject of this section. We will work our way to the appropriate refinement of Kolmogorov's continuity criterion. In our case, we obviously need a version for an infinite-dimensional parameter space, namely, $L^2([0,1])$.

Estimates of the following type have been studied in a number of papers [see Yor (1978) and Barlow and Yor (1981, 1982)]. Apart from a few minor changes, we merely have to collect them. We start with the semimartingales.

PROPOSITION 1. *There exists a constant c such that for all $p \geq 2$ and $g, h \in L^2([0,1])$ we have*

$$\left\| \sup_{t \in [0,1]} |S(g)_t - S(h)_t| \right\|_p^p \leq c^p p^{p/2} \|g - h\|_2^p.$$

PROOF. See Barlow and Yor [(1982), page 207] for the details. The essential ideas involved are the following. First, one applies Doob's maximal inequality. This is followed by an estimate for the pth moments of the modulus of a Gaussian random variable. This way the order of magnitude of the constant $p^{p/2}$ is produced, which seems to be optimal. □

Given an orthonormal basis $h = (h_j)_{j\in\mathbb{N}}$ of $L^2([0,1])$, l_2 is isomorphic with $L^2([0,1])$ via the isometry

$$i\colon l_2 \to L^2([0,1]),$$
$$x \mapsto \sum_{i=1}^{\infty} x_i h_i.$$

We will use this well-known fact to transcribe the result of Proposition 1 into the language of l_2. Doing this, we make no explicit reference to the orthonormal basis given, but always assume it to be fixed. Keeping this in mind, we use the abbreviating notation

$$M(x) = M(i(x)), \qquad S(x) = S(i(x)), \qquad x \in l_2.$$

PROPOSITION 2. *There exists a constant c such that for all $p \geq 2$ and $x, x' \in l_2$ we have*

$$\left\| \sup_{t\in[0,1]} |S(x)_t - S(x')_t| \right\|_p^p \leq c^p p^{p/2} \|x - x'\|_2^p.$$

PROOF. This is immediate from Proposition 1. □

We now turn to the corresponding inequalities for the parametrized occupation densities. To abbreviate, we denote

$$K\colon L^2([0,1]) \times \mathbb{R} \times \Omega \to \mathbb{R},$$
$$(g, y, \omega) \mapsto \int_0^1 1_{\{S(g)_s(\omega) > y\}}\, dV_s + \tfrac{1}{2} L(S(g), y)(\omega).$$

We first fix $g \in L^2([0,1])$ and compare K for different arguments in $\mathbb{R}$.

PROPOSITION 3. *There exists a constant c such that for all $p \geq 2$, $g \in L^2([0,1])$ and $y, z \in \mathbb{R}$ we have*

$$\|K(g,y,\cdot) - K(g,z,\cdot)\|_p^p \leq c^p p^{3p/4} \left\{ |y-z|^p + [\|g\|_2 + |V|_1]^{p/2} |y-z|^{p/2} \right\}.$$

PROOF. The stochastic integral part of K is treated in Barlow and Yor [(1982), page 212]. Indeed, if for $y \in \mathbb{R}$ and $t \in [0,1]$ we set

$$X_t^y = \int_0^t 1_{\{S(g)_s > y\}}\, dM(g)_s,$$

we obtain, with a universal constant a,

$$\|X_1^y - X_1^z\|_p^p \le a^p p^{3p/4} |y - z|^{p/2} \|[M(g)]_1^{1/2} + |V|_1\|_{p/2}^{p/2}.$$

However, obviously,

$$[M(g)]_1 = \|g\|_2^2,$$

so

$$\|X_1^y - X_1^z\|_p^p \le a^p p^{3p/4} |y - z|^{p/2} [\|g\|_2 + |V|_1]^{p/2}.$$

Moreover, trivially,

$$|(S(g)_1 - y)^+ - (-y)^+ - (S(g)_1 - z)^+ + (-z)^+| \le 2|y - z|.$$

Hence the desired inequality follows from (4). □

We next fix y and compare K for two different arguments in $L^2([0,1])$. This time Barlow and Yor [(1982), page 218] do not give an explicit order of magnitude estimate for the universal constants in the norm inequalities derived, but their paper contains enough information to track down such an estimate. For this reason we have to go into some details of the proof.

PROPOSITION 4. *There exists a constant c such that for all $p \ge 2$, $y \in \mathbb{R}$ and $g, h \in L^2([0,1])$ we have*

$$\|K(g, y, \cdot) - K(h, y, \cdot)\|_p^p \le c^p p^p \Big\{ \|g - h\|_2^p + \Big[(\|g\|_2 + |V|_1)^{p/2} + (\|h\|_2 + |V|_1)^{p/2} \Big] \|g - h\|_2^{p/2} \Big\}.$$

PROOF. This time, for $g \in L^2([0,1])$, $t \in [0,1]$, the stochastic integral part is denoted by

$$X_t^g = \int_0^t 1_{\{S(g)_s > y\}} \, dM(g)_s.$$

The key estimate in the proof of Lemma 6.3 of Barlow and Yor [(1982), page 218] yields

$$\begin{aligned}
\|X_1^g - X_1^h\|_p^p &\le 2^p \Bigg[\bigg\| \int_0^1 1_{\{S(g)_s > y\}} \, d(M(g) - M(h))_s \bigg\|_p^p \\
&\qquad + \bigg\| \int_0^1 \big(1_{\{S(g)_s > y\}} - 1_{\{S(h)_s > y\}}\big) \, dM(h)_s \bigg\|_p^p \Bigg] \\
&\le 2^p \Bigg[c_1^p p^{p/2} \|g - h\|_2^p \\
&\qquad + \|L^*(S(h))\|_p^{p/2} \bigg\| \sup_{t \in [0,1]} |S(g)_t - S(h)_t| \bigg\|_2^{p/2} \Bigg],
\end{aligned} \tag{5}$$

where

$$L^*(S(h)) = \sup_{y \in \mathbb{R}} L(S(h), y),$$

and c_1 is universal. Now Corollary (5.2.2) of Barlow and Yor (1982) gives an order of magnitude for the p-norm of $L^*(S(h))$. More precisely, there are constants c_2, c_3, c_4 such that for $p \geq 2$ and $h \in L^2([0,1])$ we have

$$\begin{aligned}
\|L^*(S(h))\|_p^{p/2} &\leq c_2^{p/2} p^{p/2} \left\| \sup_{t \in [0,1]} |M(h)_t| + |V|_1 \right\|_p^{p/2} \\
&\leq c_2^{p/2} p^{p/2} 2^{p/2} \left[\left\| \sup_{t \in [0,1]} |M(h))_t| \right\|_p^{p/2} + (|V|_1)^{p/2} \right] \qquad (6) \\
&\leq c_2^{p/2} p^{p/2} 2^{p/2} \left[c_3^{p/2} p^{p/4} \|h\|_2^{p/2} + (|V|_1)^{p/2} \right] \\
&\leq c_3^p p^{3p/4} [\|h\|_2 + |V|_1]^{p/2}.
\end{aligned}$$

The third inequality hereby follows from Barlow and Yor [(1982), page 207]. To estimate the second factor in the last line of (5), we have to apply Proposition 1. This and (6), used in (5), yield the desired estimate for the stochastic integral part of K. The remaining part is treated by noting that

$$\left|(S(g)_1 - y)^+ - (S(h)_1 - y)^+\right| \leq |S(g)_1 - S(h)_1|$$

and by applying Proposition 1. □

We now combine the preceding two propositions.

PROPOSITION 5. *There exists a constant c such that for all $p \geq 2$, all $y, z \in \mathbb{R}$ and all $g, h \in L^2([0,1])$ we have*

$$\begin{aligned}
&\|K(g, y, \cdot) - K(h, z, \cdot)\|_p^p \\
&\quad \leq c^p p^p \Big\{ |y - z|^p + (\|g\|_2 + |V|_1)^{p/2} |y - z|^{p/2} + \|g - h\|_2^p \\
&\qquad + \left[(\|g\|_2 + |V|_1)^{p/2} + (\|h\|_2 + |V|_1)^{p/2} \right] \|g - h\|_2^{p/2} \Big\}.
\end{aligned}$$

PROOF. Combine Propositions 3 and 4. □

Fixing an orthonormal basis $(h_j)_{j \in \mathbb{N}}$ of $L^2([0,1])$ as before, we finally transfer the result of Proposition 5 to l_2. For less complex notation, we use the abbreviation

$$J: l_2 \times \mathbb{R} \times \Omega \to \mathbb{R},$$

$$(x, y, \omega) \mapsto K(i(x), y, \omega).$$

PROPOSITION 6. *There exists a constant c such that for all $p \geq 2$, $x, x' \in l_2$, and $y, z \in \mathbb{R}$ we have*

$$\|J(x,y,\cdot) - J(x',z,\cdot)\|_p^p$$
$$\leq c^p p^p \Big\{ |y-z|^p + (\|x\|_2 + |V|_1)^{p/2} |y-z|^{p/2} + \|x - x'\|_2^p$$
$$+ \Big[(\|x\|_2 + |V|_1)^{p/2} + (\|x'\|_2 + |V|_1)^{p/2} \Big] \|x - x'\|_2^{p/2} \Big\}.$$

PROOF. This is immediate from Proposition 5 and the isometry property of i. □

On bounded subsets of l_2, the inequalities of Propositions 1 and 6 combine to give the following exponential inequalities.

PROPOSITION 7. *There exist constants c_1, c_2 such that for all $x, x' \in l_2$,*

$$E\left(\exp\left[\frac{\sup_{t \in [0,1]} |S(x)_t - S(x')_t|}{c_2 \|x - x'\|_2} \right]^2 \right) \leq c_1.$$

PROOF. Let c be the constant given by Proposition 1. Choose $c_2 > c\sqrt{2e}$ and let

$$a_p(x,x') = \frac{\big\| \sup_{t \in [0,1]} |S(x)_t - S(x')_t| \big\|_{2p}^{2p}}{\left(c_2 \|x - x'\|_2 \right)^{2p}},$$

$x, x' \in l_2$, $p \geq 1$. Then according to Proposition 1 we have

$$1 + \sum_{p=1}^{\infty} \frac{a_p(x,x')}{p!} \leq \sum_{p=0}^{\infty} \frac{(c^2 p)^p}{c_2^{2p} p!},$$

$x, x' \in l_2$. However, the last series is summable by choice of c_2, as can be seen from the quotient criterion for convergence of series. This completes the proof. □

PROPOSITION 8. *For $r > 0$ let $K_r = \{x \in l_2 : \|x\|_2 \leq r\}$,*

$$d_r((x,y),(x',z)) = c_2 \{ |y-z| + |y-z|^{1/2} + \|x-x'\|_2 + \|x-x'\|_2^{1/2} \}$$

with $c_2 = [1 + 2(r + |V|_1)^{1/2}]ce + 1$, where c is the constant given by Proposition 6. There exists a constant c_1 such that for all $y, z \in \mathbb{R}$ and $x, x' \in K_r$ we have

$$E\left(\exp\left[\frac{|J(x,y,\cdot) - J(x',z,\cdot)|}{d_r((x,y),(x',z))} \right] \right) \leq c_1.$$

PROOF. By definition of d_r and elementary inequalities, Proposition 6 gives

$$\|J(x,y,\cdot) - J(x',z,\cdot)\|_p^p \le \left(\frac{c}{c_2}\right)^p p^p 2^p (r + |V|_1)^{p/2} d_r((x,y),(x',z))^p$$

for $x, x' \in K_r$ and $y, z \in \mathbb{R}$. Hence we may define

$$a_p(x,x',y,z) = \frac{\|J(x,y,\cdot) - J(x',z,\cdot)\|_p^p}{d_r((x,y),(x',z))^p},$$

$x, x' \in K_r$, $y, z \in \mathbb{R}$, $p \ge 2$, and proceed as in the preceding proof, using the definition of c_2. □

Our next aim is to draw conclusions from the preceding propositions concerning the a.s. continuity of the mappings

$$(x,t) \mapsto M(x)_t,$$
$$(x,y) \mapsto J(x,y,\cdot).$$

It would be unrealistic to hope for continuity on the whole infinite-dimensional space l_2. Not being totally bounded, even the sets K_r of Proposition 8 are still too big. Yet, on totally bounded subsets of l_2 we might be more lucky. Indeed, a generalization of Kolmogorov's continuity criterion due to Kono (1980) and Pisier (1980a, b), refinements of which are given in Talagrand (1990), claims that the existence of a continuous version follows from a growth condition concerning the minimal number $N(\varepsilon)$ of balls of radius ε needed to cover the parameter space. We will therefore have to fix small enough subsets of l_2 as the parameter space and give good enough estimates of $N(\varepsilon)$. Since we will have to work with different metrics on l_2, we have to state clearly which one we actually refer to. This leads to the following notation with, unfortunately, two indices. For a totally bounded metric space (X, d), $\varepsilon > 0$, we let $N^{X,d}(\varepsilon)$ be the minimal number of d-balls of radius ε needed to cover X. The following auxiliary results are elementary.

PROPOSITION 9. *Let (X,d) and (X',d') be totally bounded metric spaces. Assume that $Y = X \times X'$ is endowed with metrics a and b such that*

$$a((x,x'),(y,y')) \le d(x,y) + d'(x',y'),$$
$$b((x,x'),(y,y')) \le d(x,y) \vee d'(x',y').$$

Then for $\varepsilon > 0$,

$$N^{Y,a}(\varepsilon) \le N^{X,d}\left(\frac{\varepsilon}{2}\right) N^{X',d'}\left(\frac{\varepsilon}{2}\right),$$
$$N^{Y,b}(\varepsilon) \le N^{X,d}(\varepsilon) N^{X',d'}(\varepsilon).$$

PROOF. We justify the first assertion, the second one being analogous. Let $K_{\varepsilon/2}(x)$ and $K_{\varepsilon/2}(x')$ be balls of radius $\varepsilon/2$ in X, X' centered at x, x' belonging to respective covers of X, X'. Then

$$K_\varepsilon((x,x')) = \{(y,y') \in Y: a((x,x'),(y,y')) < \varepsilon\}$$

is a ball of radius ε in Y centered at (x,x') which contains $K_{\varepsilon/2}(x) \times K_{\varepsilon/2}(x')$ and consequently belongs to a cover of Y. Hence Y can be covered by $N^{X,d}(\varepsilon/2)N^{X',d'}(\varepsilon/2)$ balls of radius ε with respect to the metric a. This implies the first assertion. □

PROPOSITION 10. *Let (X,d) be a totally bounded metric space and b a metric on X such that there exists a constant c satisfying*

$$b(x,y) \le c d(x,y),$$

$x, y \in X$. *Then for $\varepsilon > 0$ we have*

$$N^{X,b}(\varepsilon) \le N^{X,d}(c^{-1}\varepsilon).$$

PROOF. By hypothesis, any d-ball of radius $c^{-1}\varepsilon$ is contained in a b-ball of radius ε. Hence there exists a cover of X by $N^{X,d}(c^{-1}\varepsilon)$ b-balls of radius ε. This implies the asserted inequality. □

Equipped with only these elementary inequalities we will now deduce the key statement about the growth properties of $N(\varepsilon)$ as $\varepsilon \downarrow 0$. Whenever it is clear which metric space we are dealing with, the index X, d will be suppressed. For a sequence $b = (b_i)_{i\in\mathbb{N}}$ in l_2 of positive numbers we let, in the sequel,

$$B_b = \{x \in l_2: |x_i| \le b_i \text{ for } i \in \mathbb{N}\}.$$

PROPOSITION 11. *Let $b = (b_i)_{i\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that for some $\alpha < \frac{2}{5}$ we have*

$$\sum_{i=1}^{\infty} b_i^\alpha < \infty.$$

Assume that B_b is endowed with a metric d such that (B_b, d) is totally bounded and such that for some $c > 0$,

$$d(x,x') = c\|x - x'\|_2^{1/2}$$

for all $x, x' \in B_b$ satisfying $\|x - x'\|_2 \le 1$. Then the function

$$\varepsilon \mapsto \ln(N(\varepsilon) + 1)$$

is integrable on $[0,1]$.

PROOF. For $\beta > 0$ we let $c_\beta = \sum_{i=1}^{\infty} b_i^\beta$. This number may be infinite. Upon eventually replacing the metric d by the metric $c^{-1}d$ and looking at Proposition 10, we may and do assume $c = 1$. For $j \in \mathbb{N}$ we let

$$Z_j = \{x \in B_b \colon x_i = 0 \text{ for } i \neq j\},$$

$$X_j = \{x \in B_b \colon x_i = 0 \text{ for } i > j\},$$

$$X_j' = \{x \in B_b \colon x_i = 0 \text{ for } i \leq j\}.$$

On all of these spaces, we consider two metrics. The first is the restriction of d, the second the restriction of

$$d_\infty(x,y) = \sup_{i \in \mathbb{N}} |x_i - y_i|^{1/2},$$

$x, y \in l_2$. All these restrictions are denoted by the same symbols. Instead of $N^{B_b, d}(\varepsilon)$ we briefly write $N(\varepsilon)$, $\varepsilon > 0$. It is plain that for $i \in \mathbb{N}$ the interval $[-b_i, b_i]$ can be covered by at most

$$\frac{2b_i}{\varepsilon^2} \vee 1$$

d_∞-balls of radius ε. Hence for $j \in \mathbb{N}$ we have

$$\begin{aligned} N(\varepsilon) &\leq N^{X_j, d}\left(\frac{\varepsilon}{2}\right) N^{X_j', d}\left(\frac{\varepsilon}{2}\right) \\ &\leq N^{X_j, d_\infty}\left(j^{-1/4}\frac{\varepsilon}{2}\right) N^{X_j', d}\left(\frac{\varepsilon}{2}\right) \\ &\leq \prod_{i=1}^{j} N^{Z_i, d_\infty}\left(j^{-1/4}\frac{\varepsilon}{2}\right) N^{X_j', d}\left(\frac{\varepsilon}{2}\right) \\ &\leq \prod_{i=1}^{j}\left(\frac{8b_i j^{1/2}}{\varepsilon^2} \vee 1\right) N^{X_j', d}\left(\frac{\varepsilon}{2}\right). \end{aligned}$$

For the first and third inequalities, we have used Proposition 9, whereas the second is justified by Proposition 10 and the obvious inequality

$$d(x,x') \leq j^{1/4} d_\infty(x,x')$$

for $x, x' \in X_j$. Now for $\varepsilon > 0$ let us choose

$$j(\varepsilon) = \min\left\{i \in \mathbb{N} \colon c_{2/5} b_i^{8/5} < \frac{\varepsilon^4}{16}\right\}.$$

Since b converges to 0, $j(\varepsilon)$ exists. Now let $x \in X'_{j(\varepsilon)}$. Then

$$d(x,0) = \left(\sum_{i>j} x_i^2\right)^{1/4} \leq \left(\sum_{i>j} b_i^2\right)^{1/4} \leq \left(c_{2/5} b_{j(\varepsilon)}^{8/5}\right)^{1/4} < \frac{\varepsilon}{2},$$

since b is decreasing. Hence $X'_{j(\varepsilon)}$ is contained in a ball of radius $\varepsilon/2$ centered at 0 and consequently

$$N^{X'_{j(\varepsilon)}, d}\left(\frac{\varepsilon}{2}\right) = 1. \tag{7}$$

Let us next estimate $j(\varepsilon)$. For this purpose let $\gamma < \frac{1}{4}$ be big enough to ensure $c_{8\gamma/5} < \infty$. Then, by definition,

$$(j(\varepsilon) - 1)\left(\frac{\varepsilon}{2}\right)^{4\gamma} \leq \sum_{i=1}^{j(\varepsilon)} c_{2/5}^{\gamma} b_i^{8\gamma/5} \leq c_{2/5}^{\gamma} c_{8\gamma/5},$$

hence

$$j(\varepsilon) \leq 1 + c_{2/5}^{\gamma} c_{8\gamma/5}\left(\frac{2}{\varepsilon}\right)^{4\gamma}. \tag{8}$$

Now (7) and (8) imply that for $0 < \varepsilon \leq 1$,

$$\begin{aligned} \ln(N(\varepsilon)) &\leq j(\varepsilon)\left[\ln(8b_1) + \frac{1}{2}\ln j(\varepsilon) - 2\ln\varepsilon\right] \\ &\leq a_\gamma \varepsilon^{-4\gamma} \ln\frac{1}{\varepsilon}, \end{aligned}$$

with a real constant a_γ depending only on γ. But since $\gamma < \frac{1}{4}$, $\varepsilon \mapsto \ln N(\varepsilon)$ is integrable on $[0, 1]$. This implies the assertion, since $N(\varepsilon) \geq 1$ always. □

Whereas Proposition 11 is appropriate for dealing with the process J, the following proposition will be used for $\mathbb{N}$.

PROPOSITION 12. *Let $b = (b_i)_{i \in \mathbb{N}}$ be a decreasing sequence of positive numbers such that for some $\alpha < 1$ we have*

$$\sum_{i=1}^{\infty} b_i^{\alpha} < \infty.$$

Assume B_b is endowed with a metric d which is equivalent to a constant multiple of the metric induced by the l_2-norm. Then the function

$$\epsilon \mapsto [\ln(N(\varepsilon) + 1)]^{1/2}$$

is integrable on $[0, 1]$.

PROOF. We have to work with the metrics d^2 and d_∞^2 instead of d and d_∞ in the proof of Proposition 11. This yields the estimate

$$N(\varepsilon) \leq \prod_{i=1}^{j}\left(\frac{4b_i j^{1/2}}{\varepsilon} \vee 1\right) N^{X'_j, d}\left(\frac{\varepsilon}{2}\right).$$

This time, for $\varepsilon > 0$, we choose

$$j(\varepsilon) = \min\left\{i \in \mathbb{N}: c_1 b_i < \frac{\varepsilon^2}{4}\right\}.$$

Then, accordingly,

$$N^{X'_{j(\varepsilon)}, d}\left(\frac{\varepsilon}{2}\right) = 1$$

and

$$j(\varepsilon) \leq 1 + c_1^{\gamma} c_{\gamma}\left(\frac{2}{\varepsilon}\right)^{2\gamma},$$

where $\gamma < 1$ is chosen big enough to make $c_\gamma < \infty$. Hence

$$[\ln N(\varepsilon)]^{1/2} \leq a_\gamma \varepsilon^{-\gamma}\left(\ln \frac{1}{\varepsilon}\right)^{1/2}$$

with a constant a_γ depending only on γ. This obviously implies the asserted integrability. □

As corollaries of the preceding propositions we obtain the following continuity results.

THEOREM 1. *Let $b = (b_i)_{i \in \mathbb{N}}$ be a decreasing sequence of positive numbers such that for some $\alpha < \frac{2}{5}$ we have*

$$\sum_{i=1}^{\infty} b_i^{\alpha} < \infty.$$

Then J, restricted to $B_b \times \mathbb{R} \times \Omega$, possesses a version which is continuous.

PROOF. First of all, choose r large enough to ensure $K_r \supset B_b$. Then Proposition 8 states that there exists a constant c_1 and a metric d_r on $K_r \times \mathbb{R}$ for which $B_b \times I$ is totally bounded, if I is a bounded interval in $\mathbb{R}$, and which, in range 1, is equivalent to a constant multiple of the metric induced by $\|\cdot\|_2^{1/2} + |\cdot|^{1/2}$, such that for all $y, z \in \mathbb{R}$ and $x, x' \in B_b$ we have

$$E\left(\exp\left[\frac{|J(x,y,\cdot) - J(x',z,\cdot)|}{d_r((x,y),(x',z))}\right]\right) \leq c_1.$$

Since it is enough to prove continuity on $B_b \times I$ for compact intervals I, we may even suppose that the "space" variable is an additional variable in l_2. In this situation, according to Kono's theorem [Kono (1980)], in the version of Talagrand [(1990), page 2], continuity of a version of J, restricted to $B_b \times \Omega$, will follow if we can establish that

$$\varepsilon \mapsto \ln(N(\varepsilon) + 1)$$

is integrable on $[0, D]$, where D is the diameter of B_b. However, since B_b is totally bounded for d_r, this follows from integrability on [0, 1], which is established in Proposition 11. □

THEOREM 2. *Let $b = (b_i)_{i \in \mathbb{N}}$ be a decreasing sequence of positive numbers such that for some $\alpha < 1$ we have*

$$\sum_{i=1}^{\infty} b_i^{\alpha} < \infty.$$

Then

$$B_b \times [0,1] \times \Omega \ni (x,t,\omega) \mapsto S(x)_t(\omega) \in \mathbb{R}$$

possesses a continuous version.

PROOF. For this application we actually consider a version of Kono's theorem for vector-valued processes, namely, the process

$$B_b \times \Omega \ni (x,\omega) \mapsto S(x).(\omega) \in C([0,1]).$$

The topology on $C([0,1])$, the space of all continuous functions on the unit interval, is induced by the sup-norm. It is easy to see that Kono's theorem [see Kono (1980), pages 203–205] extends to this situation. We may now proceed exactly as in the preceding proof, using Propositions 7 and 12 instead of Propositions 8 and 11. □

The preceding two theorems provide enough information for what we are really up to: the study of occupation densities of integral processes with nonadapted integrands.

4. Occupation densities for integral processes in the second chaos. We may summarize the main result of the preceding section in the following statement: The occupation densities of the process

$$S(x) = \int_0^{\cdot} \sum_{i=1}^{\infty} x_i h_i(s)\, dW_s - V$$

are continuous in all variables, if x varies in a sufficiently small totally bounded subset of l_2. How is the main subject of interest of our investigation, namely, Skorohod integral processes, related to processes of this kind? To see this, let us suppose that our integral process is in the second Wiener chaos and is induced by some $f \in L^2([0,1]^2)$. For simplicity, just in the following exposition of ideas, assume f to be symmetric. Let $a = (a_i)_{i \in \mathbb{N}}$ be the family of eigenvalues of the Hilbert–Schmidt operator associated with f, of decreasing absolute value and counting multiplicities, and let $(h_i)_{i \in \mathbb{N}}$ be the corresponding ONB of $L^2([0,1])$. Then

$$f = \sum_{i=1}^{\infty} a_i h_i \otimes h_i$$

[see Weidmann (1980), page 166]. Its Skorohod integral process U can then be described in the following way. Let

$$u = \sum_{i=1}^{\infty} a_i h_i \int_0^1 h_i \, dW.$$

Then

$$U = \sum_{i=1}^{\infty} a_i \left(\int_0^{\cdot} h_i \, dW \int_0^1 h_i \, dW - \int_0^{\cdot} h_i^2 \, d\lambda \right)$$

is the integral process of u. Now let

$$V = \sum_{i=1}^{\infty} a_i \int_0^{\cdot} h_i^2 \, d\lambda.$$

Only if we suppose that f is of trace class, V will be a deterministic process of bounded variation such that

$$|V|_1 = \sum_{i=1}^{\infty} |a_i|.$$

In this case, if we set

$$X(\omega) = \left(a_i \int_0^1 h_i \, dW(\omega) \right)_{i \in \mathbb{N}},$$

$\omega \in \Omega$, we have

$$M(X) = \sum_{i=1}^{\infty} a_i \int_0^{\cdot} h_i \, dW \int_0^1 h_i \, dW;$$

so, in the terminology of Section 3,

$$S(X) = U. \tag{9}$$

It now becomes obvious that there is a relationship between the local times of the family $(S(x))_{x \in l_2}$ and the occupation density of U.

In this section, we make this relationship precise and, in particular, show that the continuity of the local times of $S(x)$ in x implies the existence of an occupation density of U. Of course, we have to pay a price: In the preceding section we have seen that the local times of $S(x)$ vary continuously only if x belongs to some set of the type B_b. Via (9), we first show that $U(\omega)$ has an occupation density if $X(\omega) \in B_{\gamma|a|^{1-\delta}}$ for some $\gamma, \delta > 0$. This of course imposes a condition on the eigenvalues which goes beyond the pure existence of a trace. We assume that there is a positive number $\alpha < \frac{1}{3}$ such that

$$\sum_{i=1}^{\infty} |a_i|^{\alpha} < \infty.$$

Under this hypothesis we are also able to prove a.s. continuity of the occupation density.

We have not made any attempt either to optimize the growth condition on a or to generalize the results to the whole Wiener space. However, it is evident how to attempt a number of improvements of the results. For example, the orders of magnitude of the constants c_p in the martingale inequalities of Section 3, in particular that of Proposition 4, may not be optimal. Improving their values might lead to better exponential inequalities for J and consequently to larger subspaces of l_2 on which continuity of the local times of $S(x)$ holds. Also, the inequalities used in the proof of Proposition 11, as well as those of Proposition 14, seem to be susceptible to some improvements. Given the nature of the proposed method, to go beyond the second chaos, moreover, seems to be a rather straightforward procedure.

The assumption that f be symmetric was only for the sake of simplicity in the preceding arguments. From now on we will work with arbitrary $f \in L^2([0,1]^2)$. In this general situation we let $a = (a_i)_{i \in \mathbb{N}}$ be the sequence of "singular numbers" of f, that is, the eigenvalues of the absolute value of the Hilbert–Schmidt operator associated with f, taken in the order of decreasing modulus. The absolute value figures, for example, in the polar decomposition of an operator. It is well known that there exist orthonormal bases $(h_i)_{i \in \mathbb{N}}$ and $(g_i)_{i \in \mathbb{N}}$ of $L^2([0,1])$ such that

$$f = \sum_{i=1}^{\infty} a_i h_i \otimes g_i$$

[see Weidmann (1980), page 170]. As a minimal assumption, we suppose that f is of trace class, that is, that

$$\sum_{i=1}^{\infty} |a_i| < \infty.$$

The processes we are interested in are then

$$u = \sum_{i=1}^{\infty} a_i h_i \int_0^1 g_i \, dW,$$

and its integral process

$$U = \sum_{i=1}^{\infty} a_i \left(\int_0^{\cdot} h_i \, dW \int_0^1 g_i \, dW - \int_0^{\cdot} h_i g_i \, d\lambda \right).$$

Finally, we let

$$V = \sum_{i=1}^{\infty} a_i \int_0^{\cdot} h_i g_i \, d\lambda.$$

Then V is of bounded variation and

$$|V|_1 = \sum_{i=1}^{\infty} |a_i|.$$

Let us first conclude from the main results of the preceding section that $S(x)$ possesses continuous occupation densities "uniformly" on B_b.

PROPOSITION 13. *Let $b = (b_i)_{i \in \mathbb{N}}$ be a decreasing sequence of positive numbers such that for some $\alpha < \frac{2}{5}$ we have*

$$\sum_{i=1}^{\infty} b_i^{\alpha} < \infty.$$

Then for almost all $\omega \in \Omega$ the following two statements are true:

(i) *For all $x \in B_b$,*

$$2\left(J(x, \cdot, \omega) - \int_0^1 1_{\{S(x)_s(\omega) > \cdot\}}\, dV_s\right)$$

is an occupation density of $S(x)(\omega)$.

(ii) *$J(x, \cdot, \omega)$ is continuous for all $x \in B_b$.*

PROOF. Theorem 1 allows us to choose a set $\Omega_{Jc} \subset \Omega$ such that

$$P(\Omega_{Jc}) = 1$$

and such that for $\omega \in \Omega_{Jc}$,

$$J(\cdot, \cdot, \omega) \text{ restricted to } B_b \times \mathbb{R}$$

is continuous. Hence (ii) holds for $\omega \in \Omega_{Jc}$. Now B_b is a separable set in l_2 with respect to the metric induced by $\|\cdot\|_2^{1/2}$. Choose a countable dense sequence $(x^n)_{n \in \mathbb{N}}$ in B_b and for each $n \in \mathbb{N}$ choose a set $\Omega_n \subset \Omega$ such that

$$P(\Omega_n) = 1$$

and such that for $\omega \in \Omega_n$ we have that

$$2\left(J(x^n, \cdot, \omega) - \int_0^1 1_{\{S(x^n)_s(\omega) > \cdot\}}\, dV_s\right)$$

is an occupation density of $S(x^n)(\omega)$. Tanaka's formula (4) allows us to do so. Finally, choose a set $\Omega_{Nc} \subset \Omega$ such that

$$P(\Omega_{Nc}) = 1$$

and such that for all $\omega \in \Omega_{Nc}$,

$$[0,1] \times B_b \ni (t, x) \mapsto S(x)_t(\omega)$$

is continuous. This is justified by Theorem 2. Now define

$$\Omega_0 = \Omega_{Jc} \cap \bigcap_{n \in \mathbb{N}} \Omega_n \cap \Omega_{Nc}.$$

Then

$$P(\Omega_0) = 1.$$

To prove that (i) holds for all $\omega \in \Omega_0$, we have to show that for all $x \in B_b$ and

$\phi \in C_0^\infty(\mathbb{R})$ the following is true:

$$\int_{\mathbb{R}} \phi(z) J(x, z, \omega)\, dz$$
$$= \int_{\mathbb{R}} \phi(z) \int_0^1 1_{\{S(x)_s(\omega) > z\}}\, dV_s\, dz + \int_0^1 \phi(S(x)_s(\omega)) u_s(x)^2\, ds$$
$$= \int_0^1 \psi(S(x)_s(\omega))\, dV_s + \int_0^1 \phi(S(x)_s(\omega)) u_s(x)^2\, ds,$$

where

$$\psi(z) = \int_{-\infty}^{z} \phi(y)\, dy, \qquad u(y) = \sum_{i=1}^{\infty} y_i h_i,$$

$z \in \mathbb{R}$ and $y \in l_2$. Since $\omega \in \Omega_n$, for all $n \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} \phi(z) J(x^n, z, \omega)\, dz = \int_0^1 \psi(S(x^n)_s(\omega))\, dV_s + \int_0^1 \phi(S(x^n)_s(\omega)) u_s(x^n)^2\, ds,$$

$n \in \mathbb{N}$. Now choose a subsequence $(y^n)_{n \in \mathbb{N}}$ of $(x^n)_{n \in \mathbb{N}}$ such that $y^n \to x$ for $n \to \infty$. It is then sufficient to prove the following three statements:

$$(10) \qquad \int_{\mathbb{R}} \phi(z) J(x^n, z, \omega)\, dz \to \int_{\mathbb{R}} \phi(z) J(x, z, \omega)\, dz,$$

$$(11) \qquad \int_0^1 \psi(S(y^n)_s(\omega))\, dV_s \to \int_0^1 \psi(S(x)_s(\omega))\, dV_s,$$

$$(12) \qquad \int_0^1 \phi(S(y^n)_s(\omega)) u_s(y^n)^2\, ds \to \int_0^1 \phi(S(x)_s(\omega)) u_s(x)^2\, ds,$$

as $n \to \infty$. Since $\omega \in \Omega_{Jc}$, (10) is a consequence of dominated convergence. Since $\omega \in \Omega_{Nc}$, we have

$$\psi(S(y^n)_s(\omega)) \to \psi(S(x)_s(\omega)),$$
$$\phi(S(y^n)_s(\omega)) \to \phi(S(x)_s(\omega))$$

as $n \to \infty$, boundedly. Hence (11) follows again from dominated convergence. Moreover,

$$u(y^n) \to u(x) \quad \text{in } L^2([0,1])$$

as $n \to \infty$. Hence (12) follows from convergence in $L^2([0,1])$. This completes the proof. □

We now consider the random vector

$$X = (X_i)_{i \in \mathbb{N}},$$

where

$$X_i = a_i \int_0^1 g_i\, dW,$$

$i \in \mathbb{N}$. We want to show that the possible values X takes are essentially contained in sets of the type B_b.

PROPOSITION 14. *Suppose that for some* $0 < \delta < 1$ *such that* $2\delta \leq 1 - \delta$ *we have*

$$\sum_{i=1}^{\infty} |a_i|^{2\delta} < \infty.$$

Then there exists an increasing sequence of sets $(A_n)_{n \in \mathbb{N}}$ *in* $\mathbf{F}$ *such that*

(i) $P(A_n) \to 1$ *for* $n \to \infty$,

(ii) $X(\omega) \in B_{n|a|^{1-\delta}}$ *for* $\omega \in A_n$,

$n \in \mathbb{N}$.

PROOF. For $i \in \mathbb{N}$ let

$$Y_i = |a_i|^{\delta} \int_0^1 g_i \, dW.$$

Then we have

$$E\Big(\sup_{i \in \mathbb{N}} Y_i^2\Big) \leq \sum_{i=1}^{\infty} E(Y_i^2) = \sum_{i=1}^{\infty} |a_i|^{2\delta} < \infty,$$

since, for $i \in \mathbb{N}$, $\int_0^1 g_i \, dW$ is a Gaussian unit variable. Now let

$$A_n = \Big\{\omega \in \Omega \colon \sup_{i \in \mathbb{N}} |Y_i(\omega)| \leq n\Big\}, \qquad n \in \mathbb{N}.$$

Then $(A_n)_{n \in \mathbb{N}}$ is obviously increasing in $\mathbf{F}$ and

$$P(A_n^c) \leq \frac{1}{n^2} E\Big(\sup_{i \in \mathbb{N}} Y_i^2\Big) \to 0$$

as $n \to \infty$. This implies (i). Now let $\omega \in A_n$. Then for $i \in \mathbb{N}$,

$$\begin{aligned} |X_i(\omega)| &= |a_i|^{1-\delta} |a_i|^{\delta} \left| \int_0^1 g_i \, dW(\omega) \right| \\ &= |a_i|^{1-\delta} |Y_i(\omega)| \leq |a_i|^{1-\delta} n. \end{aligned}$$

This implies (ii) and finishes the proof. □

A combination of Propositions 13 and 14 yields existence of occupation densities for U on the sets A_n, as we now show.

PROPOSITION 15. *Suppose that for some* $\alpha < \frac{1}{3}$ *we have*

$$\sum_{i=1}^{\infty} |a_i|^{\alpha} < \infty.$$

For $n \in \mathbb{N}$ *let* A_n *be according to Proposition* 14, *chosen for* $\delta = \alpha/2$. *Then for* *almost all* $\omega \in A_n$, *the following two statements are true*:

(i)
$$2\left(J(X(\omega),\cdot,\omega) - \int_0^1 1_{\{S(X(\omega))_s(\omega) > \cdot\}}\, dV_s\right)$$
is an occupation density of $U(\omega)$.

(ii) $J(X(\omega),\cdot,\omega)$ *is continuous.*

PROOF. Proposition 14 implies that for $\omega \in A_n$ we have
$$X(\omega) \in B_{n|a|^{1-\alpha/2}}.$$
Now, since $\alpha < \frac{1}{3}$, $1 - \alpha/2 > \frac{5}{6}$. Hence there exists some $\beta < \frac{2}{5}$ satisfying
$$\sum_{i=1}^{\infty} |a_i|^{(1-\alpha/2)\beta} < \infty.$$
Now we apply Proposition 13 to $b_i = n|a_i|^{1-\alpha/2}$, $i \in \mathbb{N}$. It implies that
$$2\left(J(x,\cdot,\omega) - \int_0^1 1_{\{S(x)_s(\omega) > \cdot\}}\, dV_s\right)$$
is an occupation density of $S(x)(\omega)$ for all $x \in B_{n|a|^{1-\alpha/2}}$, and that $J(x,\cdot,\omega)$ is continuous for all $x \in B_{n|a|^{1-\alpha/2}}$. For almost all $\omega \in A_n$, we may therefore substitute $X(\omega)$ for x in order to obtain the desired conclusion. □

Proposition 15 enables us to state our first main result.

THEOREM 3. *Suppose that for some* $\alpha < \frac{1}{3}$, *we have* $\sum_{i=1}^{\infty}|a_i|^{\alpha} < \infty$. *Then for almost all* $\omega \in \Omega$, $U(\omega)$ *possesses an occupation density which is given by*
$$2\left(J(X(\omega),\cdot,\omega) - \int_0^1 1_{\{U_s(\omega) > \cdot\}}\, dV_s\right),$$
and such that $J(X(\omega),\cdot,\omega)$ *is continuous.*

PROOF. According to Proposition 15, to fulfill the requirements of the assertion, we only have to remove a 0-set from each A_n. On the other hand, Proposition 14 states that $\bigcup_{n\in\mathbb{N}} A_n$ is a 1-set. This yields the result. □

Concerning the a.s. continuity of the occupation density of U, Theorem 3 only leaves one question open: Is
$$y \mapsto \int_0^1 1_{\{U_s > \cdot\}}\, dV_s$$
continuous a.s.?

PROPOSITION 16. *For almost all* $\omega \in \Omega$ *the mapping*
$$y \to \int_0^1 1_{\{U_s(\omega) > y\}}\, dV_s$$
is continuous.

PROOF. We assume in the following proof that f is continuous. It is easy to generalize to arbitrary f, given its basis description. We first prove that the signed measure dV_s is absolutely continuous with respect to the measure $u_s^2(\omega)\,ds$ for almost all $\omega \in \Omega$. Now note that

$$dV_s = f(s,s)\,ds$$

and

$$u_s^2(\omega)\,ds = \left(\int_0^1 f(s,u)\,dW_u(\omega)\right)^2 ds.$$

Hence it is enough to show

$$\lambda \otimes P\left(\left\{(s,\omega)\colon f(s,s) \neq 0, \int_0^1 f(s,u)\,dW_u(\omega) = 0\right\}\right) = 0.$$

According to Fubini's theorem, the expression on the left-hand side equals

$$\int_0^1 P\left(\int_0^1 f(s,u)\,dW_u = 0\right) 1_{\{f(s,s)\neq 0\}}\,ds,$$

but this expression vanishes, since for $s \in [0,1]$ the random variable

$$\int_0^1 f(s,u)\,dW_u$$

is Gaussian with variance $\int_0^1 f(s,u)^2\,du$, which is nontrivial a.e. on the set $\{f(s,s) \neq 0\}$. This proves absolute continuity. Now observe that, for all $\omega \in \Omega$ for which both absolute continuity holds and $U(\omega)$ possesses an occupation density, we have

$$\int_0^1 1_{\{U_s(\omega)=y\}}\,dV_s = 0$$

for all $y \in \mathbb{R}$, since

$$\int_0^1 1_{\{U_s(\omega)=y\}} u_s^2(\omega)\,ds = 0$$

for all $y \in \mathbb{R}$. This completes the proof. □

Theorem 3 and Proposition 16 together yield the second main result of the paper on continuity of occupation densities.

THEOREM 4. *Suppose that for some $\alpha < \frac{1}{3}$, we have $\sum_{i=1}^{\infty} |a_i|^{\alpha} < \infty$. Then for almost all $\omega \in \Omega$, $U(\omega)$ possesses a continuous occupation density.*

PROOF. Combine Theorem 3 and Proposition 16. □

REMARK 1. We do not know whether the orders of magnitude of the constants in the L^p-inequalities taken from Barlow and Yor (1982) are optimal. Optimizing these estimates would lead to a better modulus of continuity for the processes and consequently to a better estimate for the number $N(\varepsilon)$ in

Proposition 7. This in turn could allow for bigger subspaces of l_2 than the B_b and consequently enlarge the class of $f \in L^2([0,1]^2)$ for whose associated integral processes occupation densities exist. We have not tried to make these conjectures precise.

REMARK 2. The proof of Proposition 13 by no means depends on the independent random variables it is applied to. This is not the only source of hope that the method developed here and exemplified in the case of the second Wiener chaos leads to much more general results. We conjecture that at least it can be transferred to any Skorohod integral process generated by finitely many chaos with additional restrictions concerning the trace of the Malliavin derivative Du of the integrand u. More precisely, suppose u is a Skorohod integrable process not necessarily in the second chaos with integral process U, and let $(h_i)_{i \in \mathbb{N}}$ be an arbitrary ONB of $L^2([0,1])$. In a first step one would describe U under assumptions concerning the trace of Du as $\sum_{i=1}^{\infty} \int_0^1 u_s\, h_i(s)\, ds \int_0^t h_i\, dW$ plus a trace term. In the next step, the l_2-valued random variable $X = (\int_0^1 u_s\, h_i(s)\, ds,\ i \in \mathbb{N})$ would be introduced and replaced by a "parameter" $x \in l_2$. Now a new problem arises, due to the random nature of the trace term. One possibility to solve it could consist in parametrizing randomness in it in the step made before. In a final step one would have to make assumptions on u to ensure that X is a.s. in a countable union of sets of the form B_b with b as in Proposition 11. This should lead to a construction of occupation densities of U.

REMARK 3. The approach of this article can also be applied to yield results on occupation densities of solutions of stochastic integral equations, for example, with boundary conditions [see Nualart and Pardoux (1991, 1992), Ocone and Pardoux (1989) and Donati-Martin (1991)]. The proof of this statement is deferred to a forthcoming paper.

REMARK 4. In the same way as Theorem 3 was deduced from Theorem 1, we could have obtained a continuity result for the process $U = S(X)$, based on the method presented, under certain trace conditions. These trace conditions, however, turn out to be unnecessarily restrictive, considering the continuity results for Skorohod integral processes figuring in Nualart and Pardoux (1988). This fact is by no means surprising and indicates that one of the main aspects of our method, namely, to forget completely about the stochastic nature of the random variables $\int_0^1 g_i\, dW$, $i \in \mathbb{N}$, and their mutual interactions, may prohibit obtaining optimal results in specific situations. We conjecture that our existence and continuity results, even for occupation densities in the second chaos, can be essentially improved.

REFERENCES

AZEMA, J. and YOR, M. (1978). En guise d'introduction. *Temps locaux. Astérisque* **52–53** 3–16.

BARLOW, M. and YOR, M. (1981). (Semi-) Martingale inequalities and local times. *Z. Wahrsch. Verw. Gebiete* **55** 237–254.

BARLOW, M. and YOR, M. (1982). Semi-martingale inequalities via the Garsia–Rodemich–Rumsey lemma, and applications to local times. *J. Funct. Anal.* **49** 198–229.

BUCKDAHN, R. (1989). Transformations on the Wiener space and Skorohod-type stochastic differential equations. Seminarbericht Nr. 105, Sektion Math., Humboldt-Univ., Berlin.

DONATI-MARTIN, C. (1991). Equations différentielles stochastiques dans $\mathbb{R}$ avec conditions aux bords. *Stochastics* **35** 143–173.

IMKELLER, P. (1991). On the existence of occupation densities of stochastic integral processes via operator theory. In *Seminar on Stochastic Processes* (E. Çinlar, ed.). *Progress in Probability* **24** 207–240. Birkhäuser, Boston.

IMKELLER, P. (1992). Occupation densities for stochastic integral processes in the second Wiener chaos. *Probab. Theory Related Fields.* To appear.

JEULIN, T. and YOR, M., (eds). (1985). Grossissement de filtrations: Exemples et applications. *Séminaire de Calcul Stochastique, Paris 1982/83. Lecture Notes in Math.* **1118**. Springer, Berlin.

KONO, N. (1980). Sample path properties of stochastic processes. *J. Math. Kyoto Univ.* **20** 295–313.

NUALART, D. and PARDOUX, E. (1988). Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields* **78** 535–581.

NUALART, D. and PARDOUX, E. (1991). Boundary value problems for stochastic differential equations. *Ann. Probab.* **19** 1118–1144.

NUALART, D. and PARDOUX, E. (1992). Second order stochastic differential equations with Dirichlet boundary conditions. *Stochastic Process. Appl.* To appear.

OCONE, D. and PARDOUX, E. (1989). Linear stochastic differential equations with boundary conditions. *Probab. Theory Related Fields* **82** 489–526.

PISIER, G. (1980a). Conditions d'entropie assurant la continuité de certains processus. In *Séminaire d'Analyse Fonctionnelle 1979–1980. Exposes VIII–XIV.* Ecole Polytechnique, Palaiseau.

PISIER, G. (1980b). Some applications of the metric entropy condition to harmonic analysis. *Proc. in Harmonic Analysis (Storrs). Lecture Notes in Math.* **995** 123–154. Springer, Berlin.

SIMON, B. (1979). Trace ideals and their applications. *London Math. Soc. Lecture Notes Series* **35**. Cambridge Univ. Press.

SKOROHOD, A. V. (1975). On a generalization of a stochastic integral. *Theory Probab. Appl.* **20** 219–233.

TALAGRAND, M. (1990). Sample boundedness of stochastic processes under increment conditions. *Ann. Probab.* **18** 1–49.

WEIDMANN, J. (1980). *Linear Operators in Hilbert Space*. Springer, Berlin.

YOR, M. (1978). Sur la continuité des temps locaux associés a certaines semi-martingales. *Temps locaux. Astérisque* **52–53** 23–35.

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