The Annals of Probability
1993, Vol. 21, No. 2, 979–995

SOME LIMIT THEOREMS FOR SUPER-BROWNIAN MOTION AND SEMILINEAR DIFFERENTIAL EQUATIONS

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The empirical measure, a generalization of occupation times, of a super-Brownian motion is studied. In our case the empirical measure tends almost surely to Lebesgue measure as time $t \to \infty$. Asymptotic probabilities of deviation from this central behavior by various orders (large, not very large and normal deviations) are estimated. Extension to similar superprocesses, that is, Dawson–Watanabe processes, is discussed. Our analytic approach also produces new results for semilinear PDE's.

Introduction. We study a class of measure-valued processes known as the Dawson–Watanabe processes or superprocesses. The example of super-Brownian motion is treated carefully; extension is then discussed in the concluding remarks.

Our super-Brownian motion is a stochastic process taking values in nonnegative Radon measures $\omega_t(dx)$ on $\mathbb{R}^d$, $d \geq 5$. Let P and E denote the probability distribution and expectation, respectively. The distribution is uniquely characterized by the Laplace transform of its transition function,

$$E\left\{\exp\left[-\int a(x)\omega_t(dx)\right]\right\} = \exp\left[-\int g(t,x)\,dx\right],$$

where $a(x)$ denotes a nonnegative continuous function with compact support and $g(t,x)$ is the unique solution of the parabolic PDE (Δ equals the Laplacian operator),

$$\frac{\partial g(t,x)}{\partial t} = \Delta g - g^2, \qquad t > 0,\, x \in \mathbb{R}^d,$$

$$g(0,x) = a(x).$$

This formula at $t = 0$ indicates that initially $\omega_0(dx)$ equals Lebesgue measure.

The super-Brownian motion P can be constructed [see, e.g., Dawson (1977)] as the (weak) limit of a system of many Brownian (generated by Δ) particles of small mass moving independently of one another and dying or duplicating, with probability $1/2$, after each small fixed interval of time. More precisely, if we have one particle of mass $\varepsilon \ll 1$ at each site of the fine lattice $\{\varepsilon^{1/d}x: x \in \mathbb{Z}^d\}$ initially and each particle dying or duplicating after a time interval of

Received April 1991; revised March 1992.
[1]Partially supported by the Office of Graduate Studies and Research (1990) and by NSF Grant DMS-91-06562.
AMS 1991 *subject classifications*. 60F10, 60J80, 35B40, 35K57.
Key words and phrases. Measure-valued processes, large deviations, semilinear PDE.

length ε, then the distribution converges to P as $\varepsilon \downarrow 0$. The process can also be constructed without passage to the limit [see, e.g., Dynkin (1989)].

Let $D_t(dx)$ denote the empirical (random) measure such that

$$D_t(B) = t^{-1} \int_0^t \omega_s(B)\, ds \quad \text{for bounded subsets } B \text{ of } \mathbb{R}^d.$$

For dimension $d \geq 5$ we estimate the asymptotic (as $T \to \infty$) probabilities that the empirical measure $D_T(dx)$ deviates from Lebesgue measure by the order of T^{-b}, $0 \leq b \leq 1/2$. Our results are: Theorem 1.1 for $b = 0$ (large deviations), Theorem 2.1 for $b \in (0, 1/2)$ (not very large deviations) and Theorem 2.2 for $b = 1/2$ (normal deviations), as follows.

THEOREM 1.1. *Let $d \geq 5$ and the functions $V_1, V_2, \dots, V_n$ be nonnegative Hölder continuous with disjoint compact supports and define*

$$\mathbf{D}_T = \left(\int V_1(x) D_T(dx), \dots, \int V_n(x) D_T(dx) \right),$$

$$C_c^2 = \{ f \in C^2(\mathbb{R}^d) \colon f \text{ has compact support} \},$$

$$I(\gamma) = \int \frac{(\Delta \gamma)(x)^2}{4\gamma(x)}\, dx,$$

$$\boldsymbol{\gamma} = \left(\int V_1(x)\gamma(x)\, dx, \dots, \int V_n(x)\gamma(x)\, dx \right), \qquad \gamma > 0,\ \gamma - 1 \in C_c^2.$$

Then a neighborhood O of $(\int V_1(x)\, dx, \dots, \int V_n(x)\, dx)$ exists such that if $U \subset O$ is open and $C \subset O$ is closed, then

$$\liminf_{T \to \infty} T^{-1} \log P\{\mathbf{D}_T \in U\} \geq -\inf_{\boldsymbol{\gamma} \in U} I(\gamma),$$

$$\limsup_{T \to \infty} T^{-1} \log P\{\mathbf{D}_T \in C\} \leq -\inf_{\boldsymbol{\gamma} \in C} I(\gamma).$$

Here $\inf_{\boldsymbol{\gamma} \in \phi} I(\gamma)$ is taken as $+\infty$.

CONJECTURE. *The open set O can be chosen as $(0, \infty)^n$.*

THEOREM 2.1. *Suppose $d \geq 5$, $b \in (0, 1/2)$. Let $V_1, V_2, \dots, V_n$ be as in Theorem 1.1 and define*

$$GV(x) = \frac{\Gamma((d/2) - 1)}{(4\pi^{d/2})} \int_{\mathbb{R}^d} |x - y|^{2-d} V(y)\, dy \quad \text{for } V\colon \mathbb{R}^d \to \mathbb{R},$$

$$q_{ij} = \int_{\mathbb{R}^d} GV_i(x) GV_j(x)\, dx, \qquad q = (q_{ij}),\ 1 \leq i, j \leq n.$$

Then (i) *and* (ii) *hold*:

$$\text{(i)}\quad \liminf_{T\to\infty} T^{2b-1}\log P\left\{\mathbf{D}_T - \left(\int V_1(x)\,dx,\dots,\int V_n(x)\,dx\right)\in T^{-b}U\right\}$$

$$\geq -\inf_{\sigma\in U}\frac{\sigma\cdot q^{-1}\sigma}{4}\quad \textit{for all open } U\subset\mathbb{R}^n,$$

where $T^{-b}U = \{T^{-b}\boldsymbol{\theta}: \boldsymbol{\theta}\in U\}$ *and* q^{-1} *is the inverse matrix,*

$$\text{(ii)}\quad \limsup_{T\to\infty} T^{2b-1}\log P\left\{\mathbf{D}_T - \left(\int V_1(x)\,dx,\dots,\int V_n(x)\,dx\right)\in T^{-b}C\right\}$$

$$\leq -\inf_{\sigma\in C}\frac{\sigma\cdot q^{-1}\sigma}{4}\quad \textit{for all closed } C\subset\mathbb{R}^n,$$

where $T^{-b}C = \{T^{-b}\boldsymbol{\theta}: \boldsymbol{\theta}\in C\}$.

THEOREM 2.2. *For* $d\geq 5$ *the moment generating function of* $T^{1/2}[\mathbf{D}_T - (\int V_1(x)\,dx,\dots,\int V_n(x)\,dx)]$ *has the limit*

$$\lim_{T\to\infty} E\left\{\exp\left[\mathbf{a}\cdot T^{1/2}\left(\mathbf{D}_T - \left(\int V_1(x)\,dx,\dots,\int V_n(x)\,dx\right)\right)\right]\right\}$$

$$= \exp(\mathbf{a}\cdot q\mathbf{a}),\qquad \mathbf{a}\in\mathbb{R}^n.$$

In particular, this gives the following central limit theorem:

$$T^{1/2}\left(\mathbf{D}_T - \left(\int V_1(x)\,dx,\dots,\int V_n(x)\,dx\right)\right) \to_{\text{dist.}} \text{Gaussian with mean } \mathbf{0}$$

and covariance matrix $2q$ as $T\to\infty$.

For $d\geq 3$ the central limit theorem was established in Iscoe (1986). For the neighboring model of critical branching Brownian motions, various results were proved for the occupation time in Cox and Griffeath (1985). Our method is different from these papers. For systems of infinite (nonbranching) Brownian motions or Markov chains, the counterpart of Theorem 1.1 was given in Lee (1988, 1989). A general large deviation principle with less explicit rate function was given in Donsker and Varadhan (1987). The branching mechanism presents difficulty resulting from the nonlinearity of governing PDE's. Iscoe and Lee (1991) contains a counterpart of Theorem 1.1 for dimension $d = 3, 4$. Its proof requires different types of asymptotic evaluation for PDE's. The results for $d = 3, 4$ in Cox and Griffeath (1985) and Iscoe and Lee (1991) show fat large deviation tail probabilities.

While probability theory provides motivation and ideas to our investigation, we also rely on analytic techniques of PDE's. In fact we prove some PDE results which have independent interest and are more than sufficient to prove our probability theorems. Two such results are stated following some necessary notation.

Define

$$A \equiv \{\text{Hölder continuous functions with compact support in } \mathbb{R}^d\}.$$

For $V \in A$ we consider the semilinear Poisson equation,

$$(1) \qquad \Delta u(x,V) + |u|^p + V(x) = 0 \quad \text{in } \mathbb{R}^d,\ d > \frac{2p}{p-1} \text{ and } p > 1,$$

and its parabolic counterpart,

$$(2) \qquad \frac{\partial u(t,x,V)}{\partial t} = \Delta u + |u|^p + V(x), \quad \text{in } (0,\infty) \times \mathbb{R}^d,\ d > \frac{2p}{p-1},\ p > 1,$$

$$u(0,x,V) = 0.$$

A function $u(x)$ is called *proper* if $|u(x)| < c(1 + |x|)^{2-d}$ for some $c > 0$ and $u(t,x)$ is called proper if $\sup_{t>0}|u(t,x)|$ is proper. Throughout this paper we consider proper classical solutions to (1) and (2).

THEOREM 1.2. *Let $d > 2p/(p-1)$, $p > 1$ and define*

$$I_p \colon \{\gamma > 0, \gamma - 1 \in C_c^2\} \to [0,\infty),$$

$$I_p(\gamma) = \left(p^{-1/(p-1)} - p^{-p/(p-1)}\right)\int |\Delta\gamma(x)|^{p/(p-1)}\gamma(x)^{-1/(p-1)}\,dx.$$

Then a positive continuous function $\varphi = c(1 + |x|)^{(2-d)p}$, $c > 0$, exists such that if $V \in A$ and $|V| < \varphi$, then (2) has a unique proper solution $u(t,x)$. The limit function $u(x,V) \equiv \lim_{t\to\infty} u(t,x,V)$ exists pointwise and is a proper solution to (1). Moreover, the solutions satisfy

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\int u(t,x,V)\,dx &= \frac{4\pi^{d/2}}{\Gamma((d/2)-1)} \lim_{x\to\infty} u(x,V)|x|^{d-2} \\ &= \int\left[|u(x,V)|^p + V(x)\right] dx \\ &= \sup_{\gamma>0,\,\gamma-1\in C_c^2}\left[\int V(x)\gamma(x)\,dx - I_p(\gamma)\right],\end{aligned}$$

where all quantities are finite.

REMARK 1. Equation (1) can have more than one proper solution. For the case of $V \equiv 0$ with critical exponent $p = (d+2)/(d-2)$, that is, $d = 2(p+1)/(p-1)$; see, for example, Ding and Ni (1985) for a one-parameter family of explicit solutions.

REMARK 2. The Hölder continuity of V ensures that $u(t,x;V)$ [respectively, $u(x,V)$] are classical solutions to (2) [respectively, (1)]; see, for example, Gilbarg and Trudinger (1983). Theorem 1.2 actually holds for a broader class of functions V if nonclassical solutions are accepted; we will not strive for generality in this direction.

THEOREM 2.3. *Let G be as in Theorem 1.2, $b > 0$, $d > 2p/(p-1)$, $p > 1$ and $V \in A$. Then*

$$T^{-1}\int u(T,x,T^{-b}V)\,dx \sim T^{-b}\int V(x)\,dx + T^{-bp}\|GV\|_p^p \quad \text{as } T \to \infty,$$

that is,

$$\lim_{T\to\infty} T^{bp}\left[T^{-1}\int u(T,x,T^{-b}V)\,dx - T^{-b}\int V(x)\,dx\right] = \|GV\|_p^p,$$

where $\|\ \|_p$ denotes the L^p norm in the Lebesgue measure.

The remainder of this paper consists of two sections: Section 1 consists of the proofs of Theorems 1.1 and 1.2, and Section 2 consists of the proofs of Theorems 2.1, 2.2 and 2.3. Some concluding remarks follow Section 2.

2. Large deviations. We establish a sequence of Lemmas (1.3–1.10) to prove Theorems 1.1 and 1.2. Let us define

$$H(t,x) = (4\pi t)^{-d/2}\exp\frac{-|x|^2}{4t}, \qquad t>0,\ x\in\mathbb{R}^d,$$

$$Hw(t,x) = \int_0^t\int H(t-s,x-y)w(s,y)\,dy\,ds$$

$$\text{for bounded functions } w\colon (0,\infty)\times\mathbb{R}^d\to\mathbb{R}, \tag{3}$$

$$G(x) = \int_0^\infty H(t,x)\,dt = c_d|x|^{2-d}, \quad \text{where } c_d = \frac{\Gamma((d/2)-1)}{(4\pi^{d/2})},$$

$$GV(x) = \int G(x-y)V(y)\,dy$$

$$\text{for bounded functions } V\colon \mathbb{R}^d\to\mathbb{R}.$$

LEMMA 1.3. *If $d > 2p/(p-1)$, $p>1$, then a function $\varphi = c(1+|x|)^{(2-d)p}$, $c>0$, exists such that*

$$G\varphi < \varphi^{1/p}/2.$$

PROOF. Simple computation based on the fact $(1+|x|)^{(2-d)p} \in L^1(\mathbb{R}^d)$ verifies this lemma. □

LEMMA 1.4. *Let $d > 2p/(p-1)$, $p > 1$, and φ be as in Lemma* 1.3. *If $V \in A$ and $|V| \le \varphi$, then* (2) *has a unique proper solution $u(t,x,V)$. The limit function*

$$u(x,V) \equiv \lim_{t\to\infty} u(t,x,V)$$

exists pointwise and is a proper solution to (1).

PROOF. We begin with the existence of a solution. From Lemma 1.3, $\bar{u}(t,x) \equiv 2G\varphi(x)$ satisfies the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta\right)\bar{u} = 2\varphi > (2G\varphi)^p + \varphi = |\bar{u}|^p + \varphi \ge |\bar{u}|^p + V(x).$$

It is readily checked that $\underline{u}(t,x) \equiv -(G\varphi)(x)$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)\underline{u} = -\varphi \le |\underline{u}|^p + V(x).$$

That is, $\bar{u}(t,x)$ is a super solution and $\underline{u}(t,x)$ is a subsolution to (2). A standard argument of monotone iteration [cf. Ladde, Laksmikantham and Vatsala (1985)] gives the existence. In addition,

(4) if $V \in A$ and $|V| \le \varphi$, then $2(G\varphi)(x) \ge u(x;V) \ge -(G\varphi)(x)$ for $t \ge 0$, $x \in \mathbb{R}^d$.

To prove the convergence as $t \to \infty$ we assert that

$$\begin{aligned}|V| \le \varphi \Rightarrow &|u(t_2,x,V) - u(t_1,x,V)|\\ &\le u(t_2,x,\varphi) - u(t_1,x,\varphi), \qquad t_2 \ge t_1 \ge 0,\ x \in \mathbb{R}^d.\end{aligned}$$

Since $u(t,x,\varphi)\uparrow u(x,\varphi)$ as $t\uparrow\infty$, this assertion implies that $u(x,v) \equiv \lim_{t\to\infty} u(t,x,V)$ exists. Using an integral version of (2) and the dominated convergence theorem, one can see that the limit function $u(x,V)$ satisfies an integral version of (1). From (4) and the Hölder continuity of V it can be proved [see, e.g., Gilbarg and Trudinger (1983)] that $u(x,V)$ is twice continuously differentiable and satisfies (1). The assertion is proved as follows:

$$\begin{aligned}|V| < \varphi &\Rightarrow |u(t,x,V)| \le u(t,x,\varphi), \qquad t \ge 0,\ x \in \mathbb{R}^d\\ &\Rightarrow \partial[u(t,x,\varphi) - u(t,x,V)]/\partial t \ge 0, \qquad t \ge 0,\ x \in \mathbb{R}^d\\ &\Leftrightarrow u(t,x,\varphi) - u(t,x,V) \quad \text{is increasing in } t\\ &\Leftrightarrow u(t_2,x,V) - u(t_1,x,V) \le u(t_2,x,\varphi) - u(t_1,x,\varphi),\\ &\qquad t_2 \ge t_1 \ge 0,\ x \in \mathbb{R}^d.\end{aligned}$$

By similar arguments we see that $u(t,x,\varphi) + u(t,x,V)$ is increasing in t, that is, $-[u(t_2,x,V) - u(t_1,x,V)] \le u(t_2,x,\varphi) - u(t_1,x,\varphi)$, $t_2 \ge t_1 \ge 0$, $x \in \mathbb{R}^d$. The assertion, thus the lemma, is proved. □

LEMMA 1.5. *If $V \in A$ and $|V| < \varphi$, φ as in Lemma 1.3, then the differential equation*

$$(5)\qquad \begin{gathered}\left(\Delta + p(\operatorname{sgn} u(x;V))|u(x;V)|^{p-1}\right)f = 0 \quad \text{in } \mathbb{R}^d,\\ f > 0 \quad \text{and} \quad f(x) \to 1 \quad \text{as } x \to \infty,\end{gathered}$$

has a unique solution, written as $f(x, u(\cdot, V))$.

PROOF. We shall call the operator $\Delta + W(x)$ subcritical if $\Delta + W + q(x) \leq 0$ for some continuous $q(x) \geq 0$, $q \not\equiv 0$. Let K be

$$K \equiv \left\{ W(x) \colon W \text{ is a bounded function on } \mathbb{R}^d \text{ and} \right.$$
$$\left. \lim_{A \uparrow \infty} \left[\sup_{x \in \mathbb{R}^d} \int_{|y| > A} \frac{|W(y)|}{|x - y|^{d-2}}\, dy \right] = 0 \right\}.$$

We take the following result from Zhao (1990).

PROPOSITION. *If the operator $\Delta + W$ is subcritical and $W \in K$, then there exists a strictly positive function $f(x) \to 1$ as $x \to \infty$ satisfying $(\Delta + W)f = 0$ in $\mathbb{R}^d$.*

From the estimate (4) and the assumption $d > 2p/(p-1)$ it is readily checked that $p(\operatorname{sgn} u(x, V))|u|^{p-1} \in K$ for all $|V| < \varphi$. By the proposition, we only need to prove that the operator $\Delta + p(\operatorname{sgn} u(x, V))|u(x, V)|^{p-1}$ is subcritical for all $|V| \leq \varphi$. To show this we write down the equation satisfied by $U(x) \equiv u(x, \varphi) - u(x, V)$,

$$\left(\Delta + \frac{|u(x,\varphi)|^p - |u(x,V)|^p}{u(x,\varphi) - u(x,V)} + \frac{\varphi(x) - V(x)}{u(x,\varphi) - u(x,V)} \right) U = 0.$$

The facts that $U > 0$ and

$$\frac{|u(x,\varphi)|^p - |u(x,V)|^p}{u(x,\varphi) - u(x,V)} \geq p(\operatorname{sgn} u(x,V))|u(x,V)|^{p-1}$$

prove the desired subcriticality property, ending the proof. □

LEMMA 1.6. *Suppose $d > 2p/(p-1)$, $p > 1$ and $V \in A$, $|V| < \varphi$, φ as in Lemma 1.3. Then*

$$(6)\qquad \begin{aligned}\lim_{t \to \infty} t^{-1} \int u(t, x, V)\, dx &= \int \left[|u(x,V)|^p + V(x) \right] dx\\ &= \frac{4\pi^{d/2}}{\Gamma((d/2) - 1)} \lim_{x \to \infty} u(x,V)|x|^{d-2}.\end{aligned}$$

We denote this functional by $J(V)$: $\{V \colon V \in A$ and $|V| < \varphi\} \to \mathbb{R}$.

PROOF. Equations (1) and (2) have the integral formulations

$$u(x,V) = G\big(|u(\cdot,V)|^p + V\big)(x), \qquad x \in \mathbb{R}^d. \tag{7}$$

$$u(t,x,V) = H\big(|u(\cdot,\cdot,V)|^p + V\big)(t,x), \qquad t \geq 0,\ x \in \mathbb{R}^d. \tag{8}$$

From (8) we get

$$\lim_{t\to\infty} t^{-1}\int u(t,x,V)\,dx = \lim_{t\to\infty} t^{-1}\int_0^t\int\big[|u(s,y,V)|^p + V(y)\big]\,dy\,ds$$

$$= \lim_{t\to\infty}\int\big[|u(t,y,V)|^p + V(y)\big]\,dy,$$

where according to Cesaro's theorem the last equality holds if the limit in the right-hand side exists. It follows from Lemma 1.4 and the dominated convergence theorem [use (4)] that the last limit indeed exists and is equal to $\int[|u(y,V)|^p + V(y)]\,dy$. The last equality of the present lemma is readily checked from (7). □

LEMMA 1.7. *Let $V_1, V_2, \ldots, V_n$ be as in Theorem 1.1 and define*

$$\bar{a}_i = \min_{x\in\mathbb{R}^d}\frac{\varphi(x)}{V_i(x)}, \qquad 1 \leq i \leq n,$$

$$\Lambda(a_1,a_2,\ldots,a_n) = J\left(\sum_{i=1}^n a_iV_i\right), \qquad |a_i| < \bar{a}_i,\ 1 \leq i \leq n.$$

Then the functional Λ is strictly convex, continuously differentiable and $(\nabla\Lambda)(\mathbf{0}) = (\int V_1(x)\,dx, \int V_2(x)\,dx, \ldots, \int V_n(x)\,dx)$, where $\mathbf{0}$ is the origin in $\mathbb{R}^n$.

PROOF. To prove strict convexity, let $V \not\equiv W$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and calculate as follows:

$$\alpha J(V) + \beta J(W) - J(\alpha V + \beta W)$$

$$= \int \alpha|u(x,V)|^p + \beta|u(x,W)|^p - |u(x,\alpha V + \beta W)|^p\,dx$$

$$= \int\big\{\alpha|u(x,V)|^p + \beta|u(x,W)|^p - |\alpha u(x,V) + \beta u(x,W)|^p\big\}\,dx$$

$$+ \int\big\{|\alpha u(x,V) + \beta u(x,W)|^p - |u(x,\alpha V + \beta W)|^p\big\}\,dx.$$

The first term in the right-hand side is positive because the function $u \mapsto |u|^p$ is strictly convex and because $\alpha, \beta > 0$, and $V \not\equiv W$, thus $u(\cdot,V) \not\equiv u(\cdot,W)$. It is clear from a comparison argument that the functional $V \mapsto u(x,V)$ is convex for each fixed x, that is,

$$\alpha u(x,V) + \beta u(x,W) \geq u(x,\alpha V + \beta W).$$

Therefore the second term in the right-hand side of (9) is nonnegative if we assume $u(x,\alpha V + \beta W) \geq 0$ for $x \in \mathbb{R}^d$. Next, we remove this assumption. For

$|V| < \varphi$ consider the nonnegative function $w(t, x, V)$,

$$w(t,x,V) \equiv u(t,x,V) - u(t,x,-\varphi),$$

which satisfies

$$\frac{\partial w(t,x,V)}{\partial t} = \left(\Delta - p|u(t,x,-\varphi)|^{p-1}\right)w + h(w(t,x,V),t,x) + (V+\varphi),$$

$$\text{in} \quad t > 0, \qquad x \in \mathbb{R}^d,$$
$$w(0,x) = 0, \qquad x \in \mathbb{R}^d,$$

where $h(w, t, x) \equiv |w + u(t, x, -\varphi)|^p - |u(t, x, -\varphi)|^p + p|u(t, x, -\varphi)|^{p-1}w$ is strictly convex in w for each fixed t and x. Note that $h(0, t, x) = 0$ and $u(t, x, -\varphi) \le 0$ for $t \ge 0$, $x \in \mathbb{R}^d$. Recall from Lemma 1.5 the function $f(x, u(\cdot, -\varphi))$. It can be shown via Cesaro's theorem and the dominated convergence theorem that

$$\begin{aligned} J(V) &= \lim_{t\to\infty} \frac{1}{t}\int u(t,x,V)\,dx \\ &= \lim_{t\to\infty} \frac{1}{t}\int u(t,x,-\varphi)\,dx + \lim_{t\to\infty}\frac{1}{t}\int w(t,x,V)\,dx, \\ &= J(-\varphi) + \int [h(w(\infty,\cdot,V),\infty,x) + V(x) + \varphi(x)]\, f(x,u(\cdot,-\varphi))\,dx, \end{aligned}$$

where $w(\infty, x, V) = \lim_{t\to\infty} w(t, x, V)$ and $h(w, \infty, x) = \lim_{t\to\infty} h(w, t, x)$. It is sufficient to repeat the previous argument using this presentation of the functional J and the fact that $w(\infty, x, V) \ge 0$ for $x \in \mathbb{R}^d$, $|V| < \varphi$.

To prove regularity it suffices to show that $\partial\Lambda/\partial a_i$, $1 \le i \le n$ exist and are continuous functions. We shall work on $\partial\Lambda/\partial a_1$; the same method applies to $\partial\Lambda/\partial a_i$, $2 \le i \le n$. Let us define

$$\mathbf{a} = (a_1,\dots,a_n), \qquad \bar{\mathbf{a}} = (\bar{a}_1,\dots,\bar{a}_n), \qquad \mathbf{e}_1 = (1,0,\dots,0),$$

$$u(x,\mathbf{a}) = u\left(x, \sum_{i=1}^{n} a_i V_i\right),$$

$$f(\varepsilon,x) = f(\varepsilon,x,\mathbf{a}) = \varepsilon^{-1}[u(x,\mathbf{a}+\varepsilon\mathbf{e}_1) - u(x,\mathbf{a})],$$

$$\beta(\varepsilon,x) = \beta(\varepsilon,x,\mathbf{a}) = \begin{cases} \dfrac{|u(x,\mathbf{a}+\varepsilon\mathbf{e}_1)|^p - |u(x,\mathbf{a})|^p}{u(x,\mathbf{a}+\varepsilon\mathbf{e}_1) - u(x,\mathbf{a})}, & \text{if } \varepsilon \ne 0 \\ p(\operatorname{sgn} u(x,\mathbf{a}))|u(x,\mathbf{a})|^{p-1}, & \text{if } \varepsilon = 0. \end{cases}$$

The function $f(\varepsilon, x) \equiv f(\varepsilon, x, \mathbf{a})$ satisfies the linear elliptic PDE,

$$[\Delta + \beta(\varepsilon,x)]\,f(\varepsilon,x) + V_1(x) = 0.$$

We claim that the Feynman–Kac representation holds:

$$f(\varepsilon,x) = M_x\left\{\int_0^\infty V_1(X_t)\exp\int_0^t \beta(\varepsilon,X_s)\,ds\,dt\right\},$$

where M_x, X_t denote the expectation and trajectory respectively of the Brownian motion (generated by Δ) starting from x. By this representation, the convexity of the function $u \mapsto |u|^p$, and the monotonicity of $f(\varepsilon, x, \mathbf{a})$ in $\mathbf{a}$ we notice that

$$0 \le f(\varepsilon, x, \mathbf{a}) \le f(0, x, \mathbf{a} + \varepsilon \mathbf{e}_1) \le f(0, x, \bar{\mathbf{a}}) < \infty.$$

The dominated convergence theorem then implies that

$$\frac{\partial u}{\partial a_1}(x, \mathbf{a}) = \lim_{\varepsilon \to 0} f(\varepsilon, x) = M_x\left\{\int_0^\infty V_1(X_t) \exp \int_0^t p(\operatorname{sgn} u)|u(X_s, \mathbf{a})|^{p-1}\, ds\, dt\right\}$$

and that $(\partial u/\partial a_1)(x, \mathbf{a})$ is continuous in $\mathbf{a}$. The Feynman–Kac representation can be verified by letting t tend to ∞ in the corresponding representation of the function

$$f(\varepsilon, t, x) \equiv \varepsilon^{-1}[u(t, x, \mathbf{a} + \varepsilon \mathbf{e}_1) - u(t, x, \mathbf{a})].$$

The proof also uses simple comparisons and the dominated convergence theorem.

Next we define

$$g(\varepsilon) \equiv g(\varepsilon, \mathbf{a}) \equiv \varepsilon^{-1}[\Lambda(\mathbf{a} + \varepsilon \mathbf{e}_1) - \Lambda(\mathbf{a})].$$

The function $g(\varepsilon)$ is increasing in ε and

$$\begin{aligned} g(\varepsilon) &= \int \varepsilon^{-1}\left[|u(x, \mathbf{a} + \varepsilon \mathbf{e}_1)|^p - |u(x, \mathbf{a})|^p\right] dx + \int V_1(x)\, dx \\ &= \int \beta(\varepsilon, x) f(\varepsilon, x)\, dx + \int V_1(x)\, dx. \end{aligned}$$

Again we invoke the dominated convergence theorem to get

$$\frac{\partial \Lambda}{\partial a_1}(\mathbf{a}) = \lim_{\varepsilon \to 0} g(\varepsilon) = \int p(\operatorname{sgn} u)|u(x, \mathbf{a})|^{p-1} \frac{\partial u}{\partial a_1}(x, \mathbf{a})\, dx + \int V_1(x)\, dx.$$

The continuity of this partial derivative can be seen from the following estimates ($\mathbf{b} = (b_1, \dots, b_n)$):

$$\begin{aligned} &\left|\frac{\partial \Lambda}{\partial a_1}(\mathbf{b}) - \frac{\partial \Lambda}{\partial a_1}(\mathbf{a})\right| \\ &\quad \le p \int \left|(\operatorname{sgn} u(x, \mathbf{b}))|u(x, \mathbf{b})|^{p-1} \frac{\partial u}{\partial a_1}(x, \mathbf{b}) \right. \\ &\qquad\qquad \left. - \operatorname{sgn} u(x, \mathbf{a})|u(x, \mathbf{a})|^{p-1} \frac{\partial u}{\partial a_1}(x, \mathbf{a})\right| dx \\ &\quad \le p \int \left|\operatorname{sgn} u(x, \mathbf{b})|u(x, \mathbf{b})|^{p-1} \right. \\ &\qquad\qquad \left. - \operatorname{sgn} u(x, \mathbf{a})|u(x, \mathbf{a})|^{p-1}\right| \frac{\partial u}{\partial a_1}(x, \mathbf{b})\, dx \\ &\qquad + p \int |u(x, \mathbf{a})|^{p-1} \left|\frac{\partial u}{\partial a_1}(x, \mathbf{b}) - \frac{\partial u}{\partial a_1}(x, \mathbf{a})\right| dx. \end{aligned}$$

It is a consequence of the continuity of $u(x, \mathbf{a})$ and $(\partial u/\partial a_1)(x, \mathbf{a})$ in $\mathbf{a}$ and the dominated convergence theorem that

$$\lim_{\mathbf{b} \to \mathbf{a}} \left| \frac{\partial \Lambda}{\partial a_1}(\mathbf{b}) - \frac{\partial \Lambda}{\partial a_1}(\mathbf{a}) \right| = 0,$$

that is, $\partial \Lambda/\partial a_1$ is continuous. We omit the simple computation of $(\nabla \Lambda)(\mathbf{0})$. □

LEMMA 1.8. *Let $d > 2p/(p-1)$ and $p > 1$. Then* (10) *and* (11) *hold*:

$$\sup_{V \in A, |V| < \varphi} \left[\int \gamma(x) V(x)\, dx - J(V) \right] \le I_p(\gamma) \tag{10}$$

for all $\gamma > 0$, $\gamma - 1 \in C_c^2$,

$$\sup_{\gamma > 0, \gamma - 1 \in C_c^2} \left[\int \gamma(x) V(x)\, dx - I_p(\gamma) \right] = J(V) \tag{11}$$

for all $V \in A$ *and* $|V| < \varphi$.

PROOF. Recall from Lemma 1.6 and (1) that

$$J(V) = \int \left[|u(x,V)|^p + V(x) \right] dx = \int -\Delta u(x,V)\, dx.$$

Computation shows that

$$\begin{aligned} &\int \gamma(x) V(x)\, dx - J(V) \\ &\quad = \int \left\{ \gamma(x) \left[-|u(x,V)|^p - \Delta u \right] - (-\Delta u) \right\} dx \\ &\quad = \int \left\{ (\gamma(x) - 1)[-\Delta u(x,V)] - \gamma(x) |u(x,V)|^p \right\} dx \\ &\quad = \int \left\{ [-\Delta \gamma(x)] u(x,V) - \gamma(x) |u(x,V)|^p \right\} dx \quad \text{by integration by parts.} \end{aligned}$$

The simple fact

$$a > 0,\ b \in \mathbb{R} \Rightarrow \sup_{u \in \mathbb{R}} [bu - a|u|^p] = \left[p^{-1/(p-1)} - p^{-p/(p-1)} \right] \cdot |b|^{p/(p-1)} a^{-1/(p-1)} \tag{12}$$

completes the proof of (10) and shows that $J(V)$ is greater than or equal to the left-hand side of (11).

It remains to show that $J(V)$ is no greater than the left-hand side of (11). Note that the supremum in (12) is attained when u is such that

$$b = ap(\operatorname{sgn} u)|u|^{p-1}.$$

Thus the "maximizer" $\gamma(x)$ should satisfy

$$-\Delta\gamma(x) = p(\operatorname{sgn} u(x,V))|u(x,V)|^{p-1}\gamma(x) \quad \text{in } \mathbb{R}^d,$$

that is, $\gamma(x) = f(x, u(\cdot, V))$; see Lemma 1.5.

This $\gamma(x)$, however, does not satisfy $\gamma - 1 \in C_c^2$. Choose a radial function $\eta \in C_c^2\colon \mathbb{R}^d \to [0,1]$ such that $\eta(x) = 1$ for $|x| \le 1$ and set

$$\gamma_n(x) \equiv 1 + [f(x, u(\cdot, V)) - 1]\eta\left(\frac{x}{n}\right).$$

Note that $\gamma_n > 0$ and $\gamma_n - 1 \in C_c^2$. Elementary computation shows that

$$\lim_{n\to\infty}\left[\int \gamma_n(x)V(x)\,dx - I_p(\gamma_n)\right] = J(V).$$

The proof is complete. □

LEMMA 1.9. *Suppose $p = 2$, $d \ge 5$ and $V_1, V_2, \dots, V_n \ge 0$ belong to A, and $\sum_{i=1}^n V_i < \varphi$, where φ is as in Lemma 1.3. Then there exist analytic functions $F(t, z_1, \dots, z_n)$ in n complex variables $|z_1|, |z_2|, \dots, |z_n| < 1$ indexed by $t \ge 0$ such that*

$$F(t, a_1, a_2, \dots, a_n) = \int u\left(t, x, \sum a_i V_i\right) dx \quad \text{for } 0 \le t,\ -1 < a_1, \dots, a_n < 1.$$

PROOF. Define $F_n(t, x, z_1, \dots, z_n)$ recursively by

$$F_0(t, x, z_1, \dots, z_n) \equiv 0$$

$$F_n(t, x, z_1, \dots, z_n) = \int_0^t H(t-s, x-y)\left[F_n(s, y, z_1, \dots, z_n)^2 + \sum_{i=1}^n z_i V_i(y)\right] dy\,ds,$$

$$t \ge 0, \qquad x \in \mathbb{R}^d, \qquad |z_1|, |z_2|, \dots, |z_n| < 1,$$

where H is the heat kernel as in (3).

The following three properties are readily checked:

(i) $F_n(t, x, z_1, \dots, z_n)$ is analytic in $|z_1|, \dots, |z_n| < 1$ for each n, t, x.
(ii) $\sup_{n\ge 0}|F_n(t, x, z_1, \dots, z_n)| < u(t, x, \Sigma V_i) \in L^1(\mathbb{R}^d)$.
(iii) $\lim_{n\to\infty} F_n(t, x, a_1, \dots, a_n) = u(t, x, \Sigma a_i V_i)$.

By the first two properties the family $\{\int F_n(t, x, z_1, \dots, z_n)\,dx;\ n \ge 0\}$ of analytic functions is normal. Property (iii) then implies that the function

$$F(t, z_1, \dots, z_n) \equiv \lim_{n\to\infty}\int F_n(t, x, z_1, \dots, z_n)\,dx$$

exists and by the dominated convergence theorem,

$$F(t, a_1, \dots, a_n) = \lim_{n \to \infty} \int F_n(t, x, a_1, \dots, a_n)\, dx$$
$$= \int \lim_{n \to \infty} F_n(t, x, a_1, \dots, a_n)\, dx = \int u(t, x, \Sigma a_i V_i)\, dx. \quad \square$$

LEMMA 1.10. *Let E, $\omega_t(dx)$ be as in the introduction and $u(t, x, V)$ be the solution to* (2) *with $p = 2$, $d \geq 5$. Then for all $V \in A$ and $|V| < \varphi$, φ as in Lemma* 1.3, *we have*

$$E\left\{\exp \int_0^t \int V(x) \omega_s(dx)\, ds\right\} = \exp \int u(t, x, V)\, dx, \qquad t \geq 0.$$

PROOF. Let $V_+(x)$ and $V_-(x)$ be the positive and negative components of $V(x)$:

$$V_+(x) = \max\{V(x), 0\},$$
$$V_-(x) = -\min\{V(x), 0\}.$$

Note that $V_+, V_- \in A$ and $V(x) = V_+(x) - V_-(x)$. It is known [cf. Iscoe (1986)] that if $a, b \leq 0$, then

$$E\left\{\exp \int_0^t \int [aV_+(x) + bV_-(x)] \omega_s(dx)\, ds\right\} = \exp \int u(t, x, aV_+ + bV_-)]\, dx.$$

By Lemma 1.9, an analytic extension of this function exists in a complex domain containing $|a| \leq 1$, $|b| \leq 1$. The present lemma follows from properties of the Laplace transform of probability measures on $[0, \infty)^2$ [cf. Widder (1946) and Hörmander (1966)]. $\square$

PROOF OF THEOREM 1.2. This follows immediately from Lemmas 1.4, 1.6 and 1.8. $\square$

PROOF OF THEOREM 1.1. From Lemmas 1.10, 1.6 and 1.7 we have

$$\lim_{T \to \infty} T^{-1} \log E\{\exp T\mathbf{a} \cdot \mathbf{D}_T\} = \Lambda(\mathbf{a}) \quad \text{for } \mathbf{a} = (a_1, \dots, a_n),$$

$-\bar{a}_i \leq a_i \leq \bar{a}_i$, $1 \leq i \leq n$ and that $O \equiv \{(\nabla\Lambda)(\mathbf{a})\colon -\bar{a}_i < a_i < \bar{a}_i, 1 \leq i \leq n\}$ is an open neighborhood of $(\nabla\Lambda)(\mathbf{0})$ which is $(\int V_1(x)\, dx, \dots, \int V_n(x)\, dx)$. With this limit of a cumulant generating function, a general large deviation result [cf. Ellis (1985) and Gärter (1977)] ensures two estimates:

$$\liminf_{T \to \infty} T^{-1} \log P\{\mathbf{D}_T \in U\} \geq -\inf_{\sigma \in U} \sup_{\substack{-\bar{a}_i < a_i < \bar{a}_i \\ 1 \leq i \leq n}} [\boldsymbol{\sigma} \cdot \mathbf{a} - \Lambda(\mathbf{a})],$$

$$\limsup_{T \to \infty} T^{-1} \log P\{\mathbf{D}_T \in C\} \leq -\inf_{\sigma \in C} \sup_{\substack{-\bar{a}_i < a_i < \bar{a}_i \\ 1 \leq i \leq n}} [\boldsymbol{\sigma} \cdot \mathbf{a} - \Lambda(\mathbf{a})].$$

Comparing with the desired two estimates it suffices to prove

$$\inf_{\gamma=\sigma} I(\gamma) = \sup_{\substack{-\bar{a}_i<a_i<\bar{a}_i\\ 1\le i\le n}} [\sigma\cdot\mathbf{a} - \Lambda(\mathbf{a})] \quad \text{for } \sigma\in O, \tag{13}$$

where γ is as in Theorem 1.1. Note that $I = I_2$.

The left-hand side of (13) is no smaller than the right-hand side due to (10) and the fact that

$$\sup_{|V|<\varphi}\left[\int\gamma(x)V(x)\,dx - J(V)\right]$$
$$\ge \sup_{\substack{|a_i|<\bar{a}_i\\ 1\le i\le n}} [\sigma\cdot\mathbf{a} - J(\Sigma a_iV_i)] = \sup_{\substack{|a_i|<\bar{a}_i\\ 1\le i\le n}} [\sigma\cdot\mathbf{a} - \Lambda(\mathbf{a})].$$

To prove the other direction of this inequality, we note from (11) for $V = \Sigma a_iV_i$, $\mathbf{a}\in\mathbb{R}^n$ that

$$\sup_{\sigma_i>0,\,1\le i\le n}\left\{\sigma\cdot\mathbf{a} - \left[\inf_{\gamma:\,\gamma=\sigma} I(\gamma)\right]\right\} \ge \tilde{\Lambda}(\mathbf{a}),$$

where γ is as in Theorem 1.1 and $\tilde{\Lambda}$ denotes the smallest convex extension of Λ ($\tilde{\Lambda}(\mathbf{a}) = \Lambda(\mathbf{a})$ for $|a_i| < \bar{a}_i$, $1\le i\le n$). It is easy to check that the expression in the bracket [ ] is a lower semicontinuous convex function of $\sigma, \sigma_i > 0$, $1\le i\le n$. An elementary property of Legendre's transform now concludes that the left-hand side of (13) is no greater than the right-hand side, ending the proof. □

2. Not very large deviations, normal deviations and concluding remarks. Theorems 2.1, 2.2 and 2.3 are consequences of a number of lemmas in Section 1 and the following:

LEMMA 2.4. *Suppose $V\in A$, $b, s > 0$ and that G is as in Theorem 2.1. Then*

$$\lim_{T\to\infty}\|T^bu(Ts,\cdot,T^{-b}V) - GV\|_p = 0,$$

thus

$$\lim_{T\to\infty}\|T^bu(Ts,\cdot,T^{-b}V)\|_p = \|GV\|_p.$$

PROOF. First assume $|V| < \varphi$, φ as in Lemma 1.3. It is readily checked by a comparison argument that if $T\ge 1$, then

$$|T^bu(t,x,T^{-b}V)| \le u(t,x,\varphi) < u(x,\varphi), \quad t\ge 0,\ x\in\mathbb{R}^d. \tag{14}$$

We use the integral equation (8) of $u(t,x,T^{-b}V)$ to get

$$0\le T^bu(Ts,x,T^{-b}V) - HV(Ts,x) = T^bH\big(|u(\cdot,\cdot,T^{-b}V)|^p\big)(Ts,x)$$
$$\le T^{-b(p-1)}H\big(|u(\cdot,\varphi)|^p\big)(Ts,x) \le T^{-b(p-1)}G\big(|u(\cdot,\varphi)|^p\big)(x).$$

The desired result now follows from the simple facts that $G(|u(\cdot,\varphi)|^p) \in L^p(\mathbb{R}^d)$ and $HV(Ts,\cdot)$ converges in L^p to GV as $T \to \infty$.

For general V notice that $|\varepsilon V| < \varphi$ for sufficiently small $\varepsilon > 0$. Regard $T^{-b}V$ as $(T^{-b}\varepsilon^{-1})\varepsilon V$. It suffices to repeat the same argument. □

PROOF OF THEOREM 2.3. Integrating (8) shows

$$\begin{aligned} 0 &\le T^{-1}\int u(T,x,T^{-b}V)\,dx - T^{-b}\int V(y)\,dy \\ &= T^{-1}\int_0^T\int |u(\tau,y,T^{-b}V)|^p\,dy\,d\tau \\ &= \int_0^1 \|u(Ts,\cdot,T^{-b}V)\|_p^p\,ds \\ &= T^{-bp}\int_0^1 \|T^b u(Ts,\cdot,T^{-b}V)\|_p^p\,ds. \end{aligned}$$

This equality, together with lemma 2.4 and the uniform (in the time parameter) bound (14) proves the theorem by the dominated convergence theorem. □

PROOF OF THEOREMS 2.1 AND 2.2. By Lemma 1.10 and Theorem 2.3 with $p = 2$ we have for $V = \sum_{i=1}^n a_i V_i$ that

$$\begin{aligned} &T^{2b-1}\log E\left\{\exp \mathbf{a}\cdot T^{1-b}\left(\mathbf{D}_T - \left(\int V_1(x)\,dx,\dots,\int V_n(x)\,dx\right)\right)\right\} \\ &\quad = T^{2b}\left[T^{-1}\int u(T,x,T^{-b}V)\,dx - T^{-b}\int V(x)\,dx\right] \\ &\quad \to \|GV\|_2^2 = \mathbf{a}\cdot q\mathbf{a}, \qquad \mathbf{a}\in\mathbb{R}^n, \quad \text{as } T\to\infty. \end{aligned}$$

The case $b = 1/2$ proves Theorem 2.2. The case $b \in (0,1/2)$ proves Theorem 2.1 via the general large deviation result as used in the proof of Theorem 1.1. The rate function is

$$\sup_{\mathbf{a}\in\mathbb{R}^n}[\boldsymbol{\sigma}\cdot\mathbf{a} - \mathbf{a}\cdot q\mathbf{a}] = \frac{\boldsymbol{\sigma}\cdot q^{-1}\boldsymbol{\sigma}}{4}, \qquad \boldsymbol{\sigma}\in\mathbb{R}^n. \qquad \square$$

Concluding remarks. Our method is applicable to many other measure processes. For example consider, instead of Brownian motion, a transient Markov chain on a countable state space X (transition probability matrix $= \pi$) and make two assumptions.

ASSUMPTION 1. A positive constant c exists such that for all $x, y \in X$,

$$\sum_y \pi_{x,y}\left(\sum_{k=0}^{\infty}\pi_{y0}^{(k)}\right)^2 < c\sum_{k=0}^{\infty}\pi_{y0}^{(k)},$$

where 0 is a fixed element of X.

ASSUMPTION 2. A function α: $X \mapsto (0,\infty)$ exists such that $\sum_{x\in X}\alpha(x) = \infty$ and

$$\sum_{x\in X} \alpha(x)\pi_{xy} = \alpha(y).$$

Assumption 1 is a counterpart of Lemma 1.3; $\alpha(x)$ is a counterpart of Lebesgue measure. Denote by P the resulting measure process ω_t on X with $\omega_0(x) = \alpha(x)$, $x \in X$ from a construction similar to that of the introduction. Let E be the associated expectation and $B_1, B_2, \dots, B_n$ be n distinct elements of X. Then

$$E\left\{\exp\left[-\sum_{i=1}^{n} a_i\omega_t(B_i)\right]\right\} = \exp\left[-\sum_{x\in X} g(t,x)\alpha(x)\right],$$

where $n \in \mathbb{N}$, $a_i \geq 0$ and $g(t,x)$ satisfies the equation

$$\frac{\partial g(t,x)}{\partial t} = \left[\sum_{y\in X}(\pi_{xy} - \delta_{xy})g(t,y)\right] - g(t,x)^2, \qquad t > 0,\ x\in X,$$

$$g(0,B_i) = a_i, \qquad g(0,x) = 0 \quad \text{if } x \notin \{B_1, B_2, \dots, B_n\},$$

$$\text{where} \quad \delta_{xy} = 1 \quad \text{if and only if} \quad x = y.$$

Theorems 1.1, 2.1 and 2.2 are expected to hold with the new definition of the functional I and matrix (q_{ij}):

$$I\colon \{\gamma\colon \gamma - \alpha \text{ has a finite support}\} \to [0,\infty],$$

$$I(\gamma) = \sum_{x\in X}\alpha(x)\left\{\left[\sum_{y}\gamma(y)(\pi_{yx} - \delta_{yx})\right]^2 \Big/ 4\gamma(x)\right\},$$

$$q_{ij} = \sum_{x\in X}\alpha(x)\left(\sum_{k=0}^{\infty}\pi_{xB_i}^{(k)}\right)\left(\sum_{k=0}^{\infty}\pi_{xB_j}^{(k)}\right).$$

With some modifications one can extend the PDE results to systems of semilinear differential equations such as

$$\Delta u_k + |u_k|^p + V_k(x) + \sum_{j=1}^{m} d_{k,j}(x)(u_j - u_k) = 0 \quad \text{in } \mathbb{R}^d,$$

$$|u_k|^p \in L^1(\mathbb{R}^d), \qquad 1 \leq k \leq m$$

and its parabolic counterpart, where $d_{kj}(x) \geq 0$. First order terms $b_k(x)\cdot\nabla u_k$, with appropriate $b_k(x)$, can also be included in the equations.

While Theorem 1.1 can be strengthened in many ways, the conjecture that $\underline{O}$ can be $(0,\infty)^n$ remains unsolved. To prove this conjecture requires more or less that Theorem 1.2 hold without the condition $|V| < \varphi$ (allowing the four quantities to be simultaneously $+\infty$).

Acknowledgments. The author is thankful to M. I. Freidlin, M. Grillakis, W.-M. Ni, S. R. S. Varadhan and the referee for valuable comments.

REFERENCES

COX, J. T. and GRIFFEATH, D. (1985). Occupation times for critical branching Brownian motions. *Ann. Probab.* **13** 1108–1132.

DAWSON, D. (1977). The critical measure diffusion process. *Z. Wahrsch. Verw. Gebiete* **40** 125–145.

DING, W.-Y. and NI, W.-M. (1985).. On the elliptic equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ and related topics. *Duke Math. J.* **52** 485–505.

DONSKER, M. D. and VARADHAN, S. R. S. (1987). Large deviations for noninteracting infinite particle systems. *J. Statist. Phys.* **46** 1195–1232.

DYNKIN, E. B. (1989). Superprocesses and their linear additive functionals. *Trans. Amer. Math. Soc.* **314** 255–282.

ELLIS, R. S. (1985). *Entropy, Large Deviations and Statistical Mechanics*. Springer, New York.

GÄRTNER, J. (1977). On large deviations from the invariant measure. *Theory Probab. Appl.* **22** 24–39.

GILBARG, D. and TRUDINGER, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin.

HÖRMANDER, L. (1966). *An Introduction to Complex Analysis in Several Variables*. Van Nostrand, Princeton.

ISCOE, I. (1986). A weighted occupation time for a class of measure-valued branching processes. *Probab. Theory Related Fields* **71** 85–116.

ISCOE, I. and LEE, T.-Y. (1991). Large deviations for occupation times of measured-valued Brownian motions. *Stochastics Stochastic Rep.* To appear.

LADDE, G. S., LAKSHMIKANTHAM, V. and VATSALA, A. S. (1985). *Monotonic Iterative Techniques in Nonlinear Differential Equations*. Pitman, New York.

LEE, T.-Y. (1988). Large deviations for noninteracting infinite particle systems. *Ann. Probab.* **16** 1537–1558.

LEE, T.-Y. (1989). Large deviations for systems of noninteracting recurrent particles. *Ann. Probab.* **17** 46–57.

WIDDER, D. (1946). *The Laplace Transform*. Princeton Univ. Press.

ZHAO, Z. (1990). Subcriticality, positivity, and gaugeability of the Schrödinger operator. *Bull. Amer. Math. Soc.* **23** 513–517.

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