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GIRSANOV TRANSFORM FOR SYMMETRIC DIFFUSIONS WITH INFINITE DIMENSIONAL STATE SPACE

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A Cameron–Martin–Girsanov–Maruyama type formula for symmetric diffusions on infinite dimensional state space is proved. In particular, relaxations of the usual assumptions which still imply absolute continuity (but possibly no longer equivalence) of the path space measures are discussed. In addition a converse result is proved, that is, we show that absolute continuity of the path space measures enables us to identify the underlying Dirichlet form.

1. Introduction and main results.

A. Preliminaries. The purpose of this paper is to present a proof of a Cameron–Martin–Girsanov–Maruyama type formula for symmetric diffusions associated with Dirichlet forms which works for finite as well as for infinite dimensional state spaces E. The finite dimensional case was solved in Fukushima (1982), Oshima (1987) [generalizing the one-dimensional case studied in [Orey (1974)] by a different method which does not carry over to the infinite dimensional case. Our proof is based on recent results in Albeverio and Röckner (1991), Takeda (1990) and Röckner and Zhang (1992) (to which we also refer for further references). In order to state our results precisely we need some preparations.

Let E be a locally convex Hausdorff topological vector space over $\mathbb{R}$ which is Souslinean. Let E' be its dual equipped with strong topology. Suppose there exists a separable real Hilbert space $(H, \langle\ ,\ \rangle_H)$, densely and continuously embedded in E. Identifying H with its dual we obtain

$$(1.1)\qquad E' \subset H \subset E \quad \text{densely and continuously}$$

and $\langle\ ,\ \rangle_H$ restricted to $E' \times H$ coincides with the dualisation ${}_{E'}\langle\ ,\ \rangle_E$ between E' and E. H should be thought of as a tangent space to E at each point. Let, for $K \subset E'$,

$$(1.2)\qquad \mathscr{F}C_b^\infty(K) := \{f(l_1, \dots, l_m) | m \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^m),\ l_1, \dots, l_m \in K\},$$

where $C_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable (real) functions on $\mathbb{R}^m$ such that all partial derivatives are bounded. If $K = E'$, set $\mathscr{F}C_b^\infty :=$

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$\mathscr{F}C_b^\infty(E')$. For $k \in E$ and $u \in \mathscr{F}C_b^\infty$ we set

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk)\Big|_{s=0}, \qquad z \in E. \tag{1.3}$$

Observe that for $u \in \mathscr{F}C_b^\infty$ and $z \in E$ fixed, $h \mapsto (\partial u/\partial h)(z)$ is linear and continuous on H. Define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), h \rangle_H = \frac{\partial u}{\partial h}(z), \qquad h \in H. \tag{1.4}$$

Let μ be a probability measure on the Borel σ-algebra $\mathscr{B}(E)$ of E and for a set $\mathscr{A}$ of functions on E, we denote the corresponding set of μ-classes by $\tilde{\mathscr{A}}$. $k \in E$ is called *well-μ-admissible* if there exists $\beta_k \in L^2(E; \mu)$ such that

$$\int \frac{\partial u}{\partial k} v \, d\mu = -\int u \frac{\partial v}{\partial k} \, d\mu - \int u v \beta_k \, d\mu \quad \text{for all } u, v \in \mathscr{F}C_b^\infty. \tag{1.5}$$

We refer to Albeverio, Kusuoka and Röckner (1990) for a characterization of well-μ-admissibility [see also Röckner and Zhang (1992), Theorem 1.4]. Assume from now on that:

(1.6) There exists a dense linear subspace K of $E'(\subset H \subset E)$ such that each $k \in K$ is a well-μ-admissible element in E.

(1.6) implies that the densely defined quadratic form

$$\mathscr{E}_\mu(u, v) = \tfrac{1}{2} \int_E \langle \nabla u, \nabla v \rangle_H \, d\mu, \qquad u, v \in \widetilde{\mathscr{F}\mathscr{C}}_b^\infty, \tag{1.7}$$

is (well defined and) closable on $L^2(E; \mu)$ [cf. Albeverio and Röckner (1990), Albeverio, Kusuoka and Röckner (1990) and Röckner and Zhang (1992) for details]. We denote its closure by $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$, which is a *classical Dirichlet form* [in the sense of Albeverio and Röckner (1990); see also Fukushima (1980)].

Let $(\ ,\ )_\mu$ denote the usual inner product in $L^2(E; \mu)$. A negative definite self-adjoint operator L on $L^2(E; \mu)$ is called a *Dirichlet operator* if

$$(Lu, (u-1) \vee 0)_\mu \le 0 \quad \text{for each } u \in D(L) \tag{1.8}$$

or equivalently, if the corresponding semigroup $T_t := e^{tL}$, $t > 0$, on $L^2(E; \mu)$ is *(sub-) Markovian* (i.e., $0 \le T_t u \le 1$ whenever $0 \le u \le 1$ μ-a.e., $t > 0$). Recall that for any Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ there exists a unique Dirichlet operator $L(\mathscr{E})$ on $L^2(E; \mu)$, called its *generator*, such that

$$D(\mathscr{E}) = D\big(\sqrt{-L(\mathscr{E})}\big), \qquad \mathscr{E}(u, v) = \big(\sqrt{-L(\mathscr{E})}\,u, \sqrt{-L(\mathscr{E})}\,v\big)_\mu. \tag{1.9}$$

Recall also that there is a (1-)capacity associated to a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E; \mu)$ which we denote by $\mathscr{E}$-Cap; correspondingly we define the notions *$\mathscr{E}$-quasi-everywhere* (abbreviated $\mathscr{E}$-q.e.), *$\mathscr{E}$-quasicontinuous*, *$\mathscr{E}$-nest* and so on [cf. Fukushima (1980) and Albeverio and Röckner (1989) for details]. As Fukushima (1980), Theorem 3.13, one proves that each $u \in D(\mathscr{E}_\mu)$ has an $\mathscr{E}_\mu$-quasicontinuous (μ-)version $\tilde{u}$.

From now on we assume that either E is a conuclear space such that $\int |_{E'}\langle l, z\rangle_E| \mu(dz) < \infty$ for each $l \in E'$ or E is a separable Banach space. Then by Schmuland (1990) and Albeverio and Röckner (1989) there exists a diffusion process $\mathbf{M}_Q := (\Omega, \mathscr{F}, (X_t)_{t \geq 0}, (Q_z)_{z \in E})$ with state space E *associated with* $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$; that is, for $u: E \to \mathbb{R}$, $\mathscr{B}(E)$-measurable, bounded and $t > 0$,

$$(1.10) \quad E_z^Q[u(X_t)] = \int_\Omega u(X_t)\, dQ_z = (e^{tL(\mathscr{E}_\mu)}u)(z), \qquad \mu\text{-a.e. } z \in E.$$

Since $\mathbf{M}_Q$ is *conservative*, that is, $e^{tL(\mathscr{E}_\mu)}1 = 1$, $t \geq 0$, we may (and shall) assume that $\Omega := C([0,\infty[, E)$ and X_t: $\Omega \to E$ is evaluation at $t \in [0, \infty[$. Furthermore, $\mathscr{F} := \mathscr{F}_\infty$, where for $t \in [0, \infty]$, $\mathscr{F}_t := \sigma\{X_s | s \leq t\}$. $\mathbf{M}_Q$ is called *canonical* in this case. For brevity we also write $P \ll Q$ for two probability measures on $(\Omega, \mathscr{F})$ if P is absolutely continuous w.r.t. Q on each $\mathscr{F}_t$, $t \geq 0$, and set $P \sim Q$ if $P \ll Q$ and $Q \ll P$.

B. A converse result. Assume there exists another family of probability measures $(P_z)_{z \in E}$ on $(\Omega, \mathscr{F})$ such that $\mathbf{M}_P := (\Omega, \mathscr{F}, (X_t)_{t \geq 0}, (P_z)_{z \in E})$ is a (canonical) conservative diffusion on E which is symmetrizable; that is, there exists a probability measure m on $(E, \mathscr{B}(E))$ such that for all u, v: $E \to \mathbb{R}$, $\mathscr{B}(E)$-measurable, bounded,

$$(1.11) \qquad \int p_t u\, v\, dm = \int u p_t v\, dm \quad \text{for all } t > 0,$$

where $p_t(z, dy) := P_z[X_t \in dy]$. Consider the Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E; m)$ associated with $\mathbf{M}_P$, that is, the Dirichlet form whose generator $L(\mathscr{E})$ is the $L^2(E; m)$-generator of $\mathbf{M}_P$. Assume that

$$(1.12) \qquad \widetilde{\mathscr{F}\mathscr{C}}{}_b^\infty \text{ is dense in } D(\mathscr{E}) \text{ w.r.t. } \mathscr{E}_1 := \mathscr{E} + (\ ,\ )_m.$$

THEOREM 1.1. *Let $\mathbf{M}_Q, \mathbf{M}_P$ and $(\mathscr{E}, D(\mathscr{E}))$ be as before and set $Q_\mu := \int Q_z \mu(dz)$ and $P_m := \int P_z m(dz)$. Suppose $P_m \ll Q_\mu$. Then*:

(i) *$m = \varphi^2 \cdot \mu$ for some $\varphi \in L^2(E; \mu)$, $\varphi \geq 0$.*
(ii) *$(\mathscr{E}, D(\mathscr{E}))$ is the closure on $L^2(E; \varphi^2 \cdot \mu)$ of the quadratic form*

$$(1.13) \qquad \mathscr{E}(u, v) = \tfrac{1}{2}\int_E \langle \nabla u, \nabla v\rangle_H \varphi^2\, d\mu, \qquad u, v \in \widetilde{\mathscr{F}\mathscr{C}}{}_b^\infty.$$

Furthermore, any $\mathscr{E}_\mu$-nest is an $\mathscr{E}$-nest.

PROOF. See Section 2.

Theorem 1.1 extends Theorem 1 in Fukushima (1982), which follows from Proposition 1.2.

PROPOSITION 1.2. *Suppose that m does not charge $\mathscr{E}_\mu$-capacity zero sets and that $P_z \ll Q_z$ for m-a.e. $z \in E$. Then $P_m \ll Q_\mu$.*

PROOF. See Section 2.

REMARK. Theorem 1.1 and Proposition 1.2 remain true if in (1.6) we merely assume that each $k \in K$ is *μ-admissible* as defined in Albeverio and Röckner (1990) instead of *well-μ-admissible*.

C. A Girsanov theorem on infinite dimensional state space. For K as in (1.6) we define an operator $S_{\mu, K}$ on $L^2(E; \mu)$ with domain $\widetilde{\mathscr{F}\mathscr{C}}_b^\infty(K)$ as follows: For $u := f(l_1, \dots, l_m) \in \widetilde{\mathscr{F}\mathscr{C}}_b^\infty(K)$ and $K_0 \subset K$ an orthonormal basis of H having $l_1, \dots, l_m$ in its linear span, let

$$S_{\mu, K} u := \frac{1}{2} \sum_{k \in K_0} \left[\frac{\partial}{\partial k}\left(\frac{\partial u}{\partial k} \right) + \beta_k \frac{\partial u}{\partial k} \right], \tag{1.14}$$

where β_k is as in (1.5). Note that the sum in (1.14) is only a finite sum and that by (1.5) we have for the generator $L(\mathscr{E}_\mu)$ of $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ that $\widetilde{\mathscr{F}\mathscr{C}}_b^\infty(K) \subset D(L(\mathscr{E}_\mu))$ and

$$L(\mathscr{E}_\mu) u = S_{\mu, K} u \quad \text{for each } u \in \widetilde{\mathscr{F}\mathscr{C}}_b^\infty(K).$$

In particular, definition (1.14) is independent of the chosen basis K_0. In this subsection we assume that

(1.15) $L(\mathscr{E}_\mu)$ is the only Dirichlet operator on $L^2(E; \mu)$ extending $S_{\mu, K}$.

Sufficient conditions for (1.15) to hold have been proved in Röckner and Zhang (1992). It is fulfilled, for example, in many cases where μ is Gaussian [cf. Röckner and Zhang (1992), Proposition 3.2, A.1] or absolutely continuous w.r.t. a Gaussian measure [see Röckner and Zhang (1992), Section 2].

Let $\varphi \in D(\mathscr{E}_\mu)$, $\varphi \neq 0$ μ-a.e. such that $\beta_k \cdot \varphi \in L^2(E; \mu)$ for all $k \in K$. Then by Röckner and Zhang (1992), Proposition 2.1, each $k \in K$ is well-m-admissible where $m := \varphi^2 \cdot \mu$. Hence (as before) the quadratic form

$$\mathscr{E}_m(u, v) := \tfrac{1}{2} \int \langle \nabla u, \nabla v \rangle_H \, dm, \qquad u, v \in \widetilde{\mathscr{F}\mathscr{C}}_b^\infty \tag{1.16}$$

is (well defined and) closable on $L^2(E; m)$ and its closure $(\mathscr{E}_m, D(\mathscr{E}_m))$ has associated to it a canonical diffusion process $\mathbf{M}_P := (\Omega, \mathscr{F}, (X_t)_{t \geq 0}, (P_z)_{z \in E})$ with state space E.

THEOREM 1.3. *Consider the situation described in the preceding text. Assume furthermore that*

$$\varphi \in D(\mathscr{E}_m), \tag{1.17}$$

$$\varphi^{-1} \|\nabla \varphi\|_H \in L^2(E; \mu), \tag{1.18}$$

where we use ∇ also to denote the closure of the linear operator ∇: $\widetilde{\mathscr{F}\mathscr{C}}_b^\infty \to L^2(E \to H; \mu)$ on $L^2(E; \mu)$. Then $P_z \sim Q_z$ for $\mathscr{E}_\mu$-q.e. (resp. $\mathscr{E}_m$-q.e.) $z \in E$ and the corresponding densities are given by (3.5) [*see also* 3.4(*i*)]. *In particular, $P_m \sim Q_\mu$ and any $\mathscr{E}_\mu$-nest is an $\mathscr{E}_m$-nest and vice versa.*

PROOF. See Section 3.

REMARK 1.4. (i) Condition (1.17) can be relaxed [see 3.4(ii) below]. By Röckner and Zhang (1992), Theorem 2.3, (1.17) implies that we also have uniqueness for $L(\mathscr{E}_m)$. More precisely,

(1.19) $L(\mathscr{E}_m)$ is the only Dirichlet operator on $L^2(E; m)$ such that $\widetilde{\mathscr{F}\mathscr{C}}{}_b^\infty(K) \subset D(L(\mathscr{E}_m))$ and $L(\mathscr{E}_m)u = L(\mathscr{E}_\mu)u + \varphi^{-1}\langle \nabla\varphi, \nabla u\rangle_H$; $u \in \widetilde{\mathscr{F}\mathscr{C}}{}_b^\infty$.

Sufficient conditions for (1.17) to hold have been proved in Röckner and Zhang (1992), Proposition 2.6 [see also 4.6(i) below]. For corresponding examples see Röckner and Zhang (1992), Sections 4–7.

(ii) To call Theorem 1.3 a Girsanov theorem (on infinite dimensional state space) is justified since it follows by Albeverio and Röckner (1991), Section 6, that if $\int \langle k, z\rangle_E^2 \varphi^2(z)\mu(dz) < \infty$ for all $k \in K$, then for $\mathscr{E}_m$-q.e. $z \in E$,

$$X_t = z + W_t + N_t^\varphi, \qquad t \geq 0,\ P_z\text{-a.e.} \tag{1.20}$$

Here $(W_t)_{t\geq 0}$ is an $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion on E with covariance $\langle\ ,\ \rangle_H$ starting at 0 $(\in E)$ under P_z and $(N_t^\varphi)_{t\geq 0}$ is a continuous, E-valued, $(\mathscr{F}_t)_{t\geq 0}$-adapted process such that for each $k \in K$,

$$_{E'}\langle k, N_t^\varphi\rangle_E = \int_0^t \left[\tfrac{1}{2}\beta_k(X_s) + (\varphi^{-1}\langle k, \nabla\varphi\rangle_H)(X_s)\right] ds,$$
$$t \geq 0,\ P_z\text{-a.e., } \mathscr{E}_m\text{-q.e. } z \in E.$$

In the case E is a Banach space we have to assume in addition that E is big enough (compared with H) so that $(W_t)_{t\geq 0}$ exists [cf. Albeverio and Röckner (1991) for details].

Assumption (1.18) in Theorem 1.3 can be hard to check in applications. However, without (1.18) one cannot expect equivalence of P_z and Q_z, but only absolute continuity. Let $(\mathscr{E}_{\mu+m}, D(\mathscr{E}_{\mu+m}))$ on $L^2(E; \mu + m)$ be defined analogously to $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ on $L^2(E; \mu)$.

THEOREM 1.5. *Consider the situation described before Theorem* 1.3. *Assume that*

$$\varphi \in D(\mathscr{E}_{m+\mu}). \tag{1.21}$$

Then $P_z \ll Q_z$ *for* $\mathscr{E}_m$*-q.e.* $z \in E$. *In particular,* $P_m \ll Q_\mu$ *and any* $\mathscr{E}_\mu$*-nest is an* $\mathscr{E}_m$*-nest.*

PROOF. See Section 4.

Examples with dim $E = +\infty$ have been discussed in detail in Albeverio and Röckner (1989, 1990, 1991) and Röckner and Zhang (1992), in particular those arising in Euclidean quantum field theory. It has been shown in Röckner and Zhang (1992), Section 7, that if μ is the *free field* in two space time dimensions in finite volume and φ is the exponential of a (even) *renormalized Wick polynomial*, then all assumptions for Theorem 1.3 are fulfilled and one thus has a Girsanov theorem in this case [which was the main tool in Jona-Lasinio and Mitter (1985)]. It follows from Röckner and Zhang (1992), Section 5, that if μ is the *time-zero free field on* $S'(\mathbb{R})$ and φ is the ground state of a Schrödinger operator of type $H_0 + V$, where H_0 is the *free Hamiltonian* and V is a (*space cutoff*) renormalized Wick polynomial, then Theorem 1.5 applies.

D. Compactification. We note that in order to make the standard theory of Dirichlet forms on locally compact space in Fukushima (1980) [and, e.g., also the results in Takeda (1990)] applicable we need to recall a compactification procedure developed in Albeverio and Röckner (1980, 1991) for our (possibly) infinite dimensional, hence nonlocally compact space E. It follows by Albeverio and Röckner (1989), Section 2, that there exists a Hausdorff compact separable metric space $\hat{E}$ such that $E \subset \hat{E}$ continuously and densely and such that for any Dirichlet form appearing above, $(\mathscr{E}, D(\mathscr{E}))$ say, the corresponding image Dirichlet form $(\hat{\mathscr{E}}, D(\hat{\mathscr{E}}))$ on $L^2(\hat{E}; \hat{\mu})$ [cf. Albeverio and Röckner (1991) Section 1] is a regular, local Dirichlet form. Here $\hat{\mu}$ is the image of μ under the embedding $E \subset \hat{E}$. If $\mathbf{M}$ is the diffusion process associated with $(\mathscr{E}, D(\mathscr{E}))$ on E, one can trivially extend it to a diffusion process $\hat{\mathbf{M}}$ on $\hat{E}$ by defining each $z \in \hat{E} \setminus E$ to be a *trap* for $\hat{M}$ [cf. Fukushima (1980), Theorem 4.1.3, for details] and such that E is an *invariant set* for $\hat{\mathbf{M}}$. It easily follows that $\hat{\mathbf{M}}$ is associated with $(\hat{\mathscr{E}}, D(\hat{\mathscr{E}}))$ in the sense of (1.10). Futhermore, it follows from Lyons and Röckner (1992) and Albeverio and Röckner (1989), Section 3c, that the (1-)capacity $\mathscr{E}$-Cap associated with $(\mathscr{E}, D(\mathscr{E}))$ is tight [i.e., $\mathscr{E}\text{-Cap}(E \setminus K_n) \to_{n\to\infty} 0$ for some compact sets $K_n \subset E$, $n \in \mathbb{N}$]. Hence the notions of capacity, quasicontinuous, q.e. w.r.t. $(\mathscr{E}, D(\mathscr{E}))$ and $(\hat{\mathscr{E}}, D(\hat{\mathscr{E}}))$ coincide [see Albeverio and Röckner (1991) for details]. This means that we can "lift" all questions about $(\mathscr{E}, D(\mathscr{E}))$ and $\mathbf{M}$ on the (possibly) nonlocally compact state space E to questions about $(\hat{\mathscr{E}}, D(\hat{\mathscr{E}}))$ and $\hat{\mathbf{M}}$ on the compact space $\hat{E}$ where the standard theory about abstract Dirichlet forms in Fukushima (1980) is applicable and then transfer the answers back to $(\mathscr{E}, D(\mathscr{E}))$ and $\mathbf{M}$ on E. We shall use this procedure without mentioning it explicitly or by simply adding the phrase "by compactification."

Finally, we would like to mention that the implementation of our method to prove the Girsanov-type theorems, Theorems 1.3 and 1.5, has been open for quite some time (as was communicated to us by M. Fukushima, who we would like to thank at this point) and has now become possible by exploiting the results and methods in Albeverio and Röckner (1991), Takeda (1990) and Röckner and Zhang (1992).

2. Proofs of Theorem 1.1 and Proposition 1.2.

PROOF OF THEOREM 1.1. (i) It follows by μ- (resp. m-) symmetry and conservativeness that for each $t > 0$, $Q_\mu \circ X_t^{-1} = \mu$ and $P_m \circ X_t^{-1} = m$. Now (i) is obvious.

(ii) By Fukushima (1980), Theorem 5.2.2, and compactification we know that for all $u \in \mathscr{F}C_b^\infty$,

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \qquad t \geq 0,\ P_z\text{-a.s.},\ \mathscr{E}\text{-q.e. } z \in E, \tag{2.1}$$

where $(M_t^{[u]})_{t\geq 0}$ is a martingale additive functional (abbreviated MAF) of finite energy and $(N_t^{[u]})_{t\geq 0}$ is a continuous additive functional (abbreviated CAF) of zero energy of $\mathbf{M}_P$. Furthermore [cf. Fukushima (1980), (5.2.26), (5.2.1)], for every $u \in \widetilde{\mathscr{FC}}_b^\infty$,

$$\mathscr{E}(u,u) = \lim_{t\to 0}\frac{1}{2t}E_m\Big[\big(M_t^{[u]}\big)^2\Big] = \lim_{t\to 0}\frac{1}{2t}E_m\big[\langle M^{[u]}\rangle_t\big], \tag{2.2}$$

where $E_m[\cdot] = \int \cdot\, dP_m$. Since the quadratic variation of $(N_t^{[u]})_{t\geq 0}$ vanishes we have that

$$\langle M^{[u]}\rangle_t = P_m - \lim_{n\to\infty}\sum_{i=1}^{N_n-1}\big(u(X_{t_{i+1}^n}) - u(X_{t_i^n})\big)^2, \qquad t \geq 0, \tag{2.3}$$

for a sequence $(\tau^n)_{n\in\mathbb{N}}$ of partitions $0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = t$ of $[0,t]$ with $\delta(\tau^n) := \max_i (t_{i+1}^n - t_i^n) \to_{n\to\infty} 0$. For the same reasons [cf. Albeverio and Röckner (1991), Theorem 4.3] we have for $u \in \mathscr{F}\mathscr{C}_b^\infty$,

$$u(X_t) - u(X_0) = \overline{M}_t^{[u]} + \overline{N}_t^{[u]}, \qquad t \geq 0,\ Q_z\text{-a.s.},\ \mathscr{E}_\mu\text{-q.e. } z \in E,$$

where $(\overline{M}_t^{[u]})_{t\geq 0}$, $(\overline{N}_t^{[u]})_{t\geq 0}$ are the corresponding quantities w.r.t. $\mathbf{M}_Q$ and

$$\langle \overline{M}^{[u]}\rangle_t = Q_\mu - \lim_{n\to\infty}\sum_{i=1}^{N_n-1}\big(u(X_{t_{i+1}^n}) - u(X_{t_i^n})\big)^2, \qquad t \geq 0. \tag{2.4}$$

But by Albeverio and Röckner (1991), Proposition 4.5 (cf. also Lemma 3.3 below),

$$\langle \overline{M}^{[u]}\rangle_t = \int_0^t \langle \nabla u(X_s), \nabla u(X_s)\rangle_H\, ds, \qquad t \geq 0,\ Q_\mu\text{-a.s.} \tag{2.5}$$

Since $P_m \ll Q_\mu$, (2.2)–(2.5) and the polarisation identity imply that for all $u, v \in \widetilde{\mathscr{FC}}_b^\infty$,

$$\begin{aligned}\mathscr{E}(u,v) &= \lim_{t\to 0}\frac{1}{2t}E_m\big[\langle M^{[u]}, M^{[v]}\rangle_t\big] = \lim_{t\to 0}\frac{1}{2t}E_m\left[\int_0^t \langle \nabla u(X_s), \nabla v(X_s)\rangle_H\, ds\right]\\ &= \int \langle \nabla u, \nabla v\rangle_H\, dm = \int \langle \nabla u, \nabla v\rangle_H \varphi^2\, d\mu,\end{aligned}$$

and (ii) is shown, since $\widetilde{\mathscr{FC}}_b^\infty$ is $\mathscr{E}_1$-dense in $D(\mathscr{E})$.

To prove the last part let $(F_n)_{n\in\mathbb{N}}$ be an $\mathscr{E}_\mu$-nest, that is, F_n is a closed subset of E such that $F_n \subset F_{n+1}$, $n \in \mathbb{N}$, and $\lim_{n\to\infty}\mathscr{E}_\mu\text{-Cap}(F_n^c) = 0$. Here $F_n^c := E \setminus F_n$. Since $e^{tL(\mathscr{E}_\mu)}1 = 1$, $t \geq 0$, we have by Fukushima (1980), Theorem 3.3.1 and Lemma 4.3.1, that

$$\mathscr{E}_\mu\text{-Cap}(F_n^c) = \int \exp(-\sigma_{F_n^c})\, dQ_\mu, \qquad n \in \mathbb{N}, \tag{2.6}$$

where $\sigma_{F_n^c} := \inf\{t > 0 | X_t \in F_n^c\}$, is the *first hitting time* of F_n^c. Hence for all $t > 0$,

$$Q_\mu\Big[\lim_{n\to\infty} \sigma_{F_n^c} \le t\Big] \le Q_\mu\Big[\lim_{n\to\infty} \sigma_{F_n^c} < \infty\Big] = 0.$$

Therefore, since $P_m \ll Q_\mu$ and $\{\lim_{n\to\infty} \sigma_{F_n^c} \le t\} \in \mathscr{F}_{t+1}$,

$$P_m\Big[\lim_{n\to\infty} \sigma_{F_n^c} \le t\Big] = 0,$$

hence

$$P_m\Big[\lim_{n\to\infty} \sigma_{F_n^c} < \infty\Big] = 0$$

and consequently, as before, $\lim_{n\to\infty} \mathscr{E}\text{-Cap}(F_n^c) = 0$; that is, $(F_n)_{n\in\mathbb{N}}$ is an $\mathscr{E}$-nest. □

Proof of Proposition 1.2. Fix $t > 0$. Let u be a nonnegative, bounded, $\mathscr{B}(E)$-measurable function with $\int u\, d\mu = 0$. Then

$$\int\int u(X_t)\, dQ_z \mu(dz) = \int u(X_t)\, dQ_\mu = \int u\, d\mu = 0,$$

hence by Albeverio and Röckner (1991), Corollary 3.8, $\int u(X_t)\, dQ_z = 0$ for $\mathscr{E}_\mu$-q.e. $z \in E$. Consequently, by assumption $\int u(X_t)\, dQ_z = 0$ for m-a.e. $z \in E$, hence $\int u(X_t)\, dP_z = 0$ for m-a.e. $z \in E$. Therefore,

$$\int u\, dm = \int u(X_t)\, dP_m = \int\int u(X_t)\, dP_z m(dz) = 0$$

and we have proved that $m \ll \mu$. If $N \in \mathscr{F}_t$ with $Q_\mu(N) = 0$, it follows that $Q_z(N) = 0$ for μ-a.e. $z \in E$, hence for m-a.e. $z \in E$. Consequently, by assumption $P_z(N) = 0$ for m-a.e. $z \in E$, and thus $P_m(N) = 0$; that is, $P_m \ll Q_\mu$. □

3. Proof of Theorem 1.3. We start with proving several lemmas. Let $\tilde{\nabla}$ denote the closure of $\nabla\colon \widetilde{\mathscr{F}\mathscr{C}_b^\infty} \to L^2(E \to H; m)$ as an operator on $L^2(E; m)$.

Lemma 3.1. *$\nabla u = \tilde{\nabla} u$ for all $u \in D(\mathscr{E}_\mu) \cap D(\mathscr{E}_m)$.*

Proof. For $l \in \mathbb{N}$, let $b_l \in C_0^\infty(\mathbb{R})$ such that $1_{[-l,l]} \le b_l \le 1_{[-l-2,l+2]}$ and $|b_l'| \le 1$ and define

$$\varphi_l := b_l(\ln \varphi), \qquad l \in \mathbb{N}.$$

Then by (1.17) and Röckner and Zhang (1992), (2.9), $\nabla((u \wedge n) \cdot \varphi_l) = \tilde{\nabla}((u \wedge n) \cdot \varphi_l)$ and $\nabla\varphi_l = \tilde{\nabla}\varphi_l$ hence by the product rule for $\nabla, \tilde{\nabla}$ [cf. Albeverio and Röckner (1991), (3.2), or Röckner and Zhang (1992), 1.8(ii)] we conclude that $\varphi_l \nabla(u \wedge n) = \varphi_l \tilde{\nabla}(u \wedge n)$. Letting $l \to \infty$ we obtain that $\nabla(u \wedge n) = \tilde{\nabla}(u \wedge n)$. Since $u \wedge n \to_{n\to\infty} u$ w.r.t. both $\mathscr{E}_\mu + (\ ,\ )_\mu$ and $\mathscr{E}_m + (\ ,\ )_m$, the assertion follows. □

LEMMA 3.2. (i) $\ln\varphi \in D(\mathscr{E}_m)$, *and for any* $\mathscr{E}_m$-*quasicontinuous* $(m\text{-})$*version* $\tilde{\varphi}$ *of* φ *we have that* $0 < \tilde{\varphi} < \infty$ $\mathscr{E}_m$-q.e.

(ii) $\varphi^{-1} \in D(\mathscr{E}_m)$.

PROOF. (i) Let $\varepsilon \in]0,1]$. Since $x \mapsto \ln(x+\varepsilon)$ belongs to $C_b^1([0,\infty[)$ and $\varphi \in D(\mathscr{E}_m)$, it follows that $\ln(\varphi+\varepsilon) \in D(\mathscr{E}_m)$. But $\ln(\varphi+\varepsilon) \le |\ln\varphi| + \ln 2 \in L^2(E;m)$ and $\|\tilde{\nabla}\ln(\varphi+\varepsilon)\|_H = \|\nabla\varphi\|_H/(\varphi+\varepsilon) \le \|\nabla\varphi\|_H/\varphi \in L^2(E;m)$, hence $\ln\varphi \in D(\mathscr{E}_m)$. Let $\widetilde{\ln\varphi}$ be an $\mathscr{E}_m$-quasicontinuous Borel $(m\text{-})$version of $\ln\varphi$. Since

$$\mathscr{E}_m\text{-Cap}\{|\widetilde{\ln\varphi}| > \lambda\} \le \frac{1}{\lambda^2}\mathscr{E}_m(\ln\varphi, \ln\varphi)$$

[cf. Fukushima (1980), Lemma 3.1.5], we obtain that

$$\mathscr{E}_m\text{-Cap}\{|\widetilde{\ln\varphi}| = +\infty\} \le \limsup_{\lambda\to\infty}\mathscr{E}_m\text{-Cap}\{|\widetilde{\ln\varphi}| > \lambda\} = 0.$$

Consequently, $\tilde{\varphi} := \exp(\widetilde{\ln\varphi})$ is an $\mathscr{E}_m$-quasicontinuous $(m\text{-})$version of φ with $0 < \tilde{\varphi} < \infty$ $\mathscr{E}_m$-q.e. Now the assertion follows by Fukushima (1980), Lemma 3.1.4 (see also Lemma 4.1 below).

(ii) is proved similarly to the first part of (i), using (1.18). □

Recall that by Albeverio and Röckner (1991), Theorem 4.3, the Fukushima decomposition holds for $(\mathscr{E}_m, D(\mathscr{E}_m))$; that is, if $u \in D(\mathscr{E}_m)$ and $\tilde{u}$ is an $\mathscr{E}_m$-quasicontinuous version, then

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \qquad t \ge 0, \tag{3.1}$$

where $M^{[u]} := (M^{[u]})_{t\ge 0}$ is an MAF of $\mathbf{M}_P$ of finite energy and $N^{[u]} := (N^{[u]})_{t\ge 0}$ is a CAF of $\mathbf{M}_P$ of zero energy [cf. Fukushima (1980), Chapter 5]. Note that since $\mathbf{M}_P$ has continuous sample paths, $M^{[u]}$ is also continuous [by Fukushima (1980), Theorem 4.3.2 and compactification]. If in particular, $u \in \mathscr{F}\mathscr{C}_b^\infty$, it follows by (1.19) and Albeverio and Röckner (1991), Remark 4.4 (ii), that

$$N_t^{[u]} = \int_0^t \big(Lu(X_s) + \varphi^{-1}(X_s)\langle\nabla\varphi(X_s), \nabla u(X_s)\rangle_H\big)\,ds, \qquad t \ge 0, \tag{3.2}$$

where $L := L(\mathscr{E}_\mu)$. Again by compactification we also have the correspondence between positive CAF's of $\mathbf{M}_P$ and *smooth measures* proved in Fukushima (1980), Theorem 5.1.3. As usual we denote the smooth measure corresponding to $\langle M^{[u]}\rangle$ for $u \in D(\mathscr{E}_m)$ by $\mu_{\langle u\rangle}$.

The following lemma is merely a special case of Albeverio and Röckner (1991), Proposition 4.5. We include the proof for the reader's convenience.

LEMMA 3.3. *Let* $u \in D(\mathscr{E}_m)$. *Then* $\mu_{\langle u\rangle} = \langle\tilde{\nabla}u, \tilde{\nabla}u\rangle_H \cdot m$ *and*

$$\langle M^{[u]}\rangle_t = \int_0^t \langle\tilde{\nabla}u(X_s), \tilde{\nabla}u(X_s)\rangle_H\,ds, \qquad t \ge 0.$$

PROOF. If $u_n := (u \wedge n) \vee (-n)$, $n \in \mathbb{N}$, we know by Fukushima (1980), Theorem 5.2.3, and the product rule for $\tilde{\nabla}$ that for all $n \in \mathbb{N}$ and all $f \in D(\mathscr{E}_m) \cap L^\infty(E; m)$,

$$\int f \, d\mu_{\langle u_n \rangle} = 2\mathscr{E}_m(u_n f, u_n) - \mathscr{E}_m(u_n^2, f)$$

$$= \int f \langle \tilde{\nabla} u_n, \tilde{\nabla} u_n \rangle_H \, dm.$$

Since

$$\left(\sqrt{\int |f| d\mu_{\langle u \rangle}} - \sqrt{\int |f| d\mu_{\langle u_n \rangle}} \right)^2 \le 2\|f\|_\infty \mathscr{E}_m(u - u_n, u - u_n)$$

[cf. Fukushima (1980), proof of Lemma 5.4.6], we conclude that

$$\mu_{\langle u \rangle} = \langle \tilde{\nabla} u, \tilde{\nabla} u \rangle_H \cdot m. \tag{3.3}$$

Since $\langle \tilde{\nabla} u, \tilde{\nabla} u \rangle_H \cdot m$ is a smooth measure we have that

$$P_z\left[\int_0^t \left\langle \tilde{\nabla} u(X_s), \tilde{\nabla} u(X_s) \right\rangle_H ds < \infty, t \ge 0 \right] = 1$$

for $\mathscr{E}_m$-q.e. $z \in E$, which is an immediate consequence of Fukushima (1980), Lemma 5.1.6 and Theorem 3.2.3. Consequently,

$$N_t := \int_0^t \left\langle \tilde{\nabla} u(X_s), \tilde{\nabla} u(X_s) \right\rangle_H ds, \qquad t \ge 0,$$

is a positive CAF of $\mathbf{M}_P$ and for all $\mathscr{B}(E)$-measurable $f \colon E \to [0, \infty[$,

$$\begin{aligned} \frac{1}{t} \int_E E_z\left[\int_0^t f(X_s) \, dN_s \right] m(dz) &= \frac{1}{t} \int_0^t \int p_s\big(f \langle \tilde{\nabla} u, \tilde{\nabla} u \rangle_H \big) \, dm \, ds \\ &= \int f \langle \tilde{\nabla} u, \tilde{\nabla} u \rangle_H \, dm, \end{aligned} \tag{3.4}$$

where we used that $p_s(z, dy) := P_z[X_s \in dy]$ is m-symmetric and $p_s 1 \equiv 1$, $s \ge 0$. (3.3) and (3.4) imply that $(N_t)_{t \ge 0}$ and $\langle M^{[u]} \rangle$ have the same corresponding smooth measures and hence must be equivalent. □

In the following proof we apply results from Kunita and Watanabe (1963) and Takeda (1990) [and again Fukushima (1980)] only proved in the case of locally compact state spaces E. The easiest way to apply our compactification method (described in Section 1D) here is to replace, right at the beginning of the proof, $E, \mathbf{M}_P$ by $\hat{E}, \hat{\mathbf{M}}_P$, respectively, and to consider all subsequently appearing functions f on E as functions on $\hat{E}$ by putting $f \equiv 0$ on $\hat{E} \setminus E$. Then one easily transfers the final result back to $E, \mathbf{M}_P$. For simplicity, however, we drop the additional caret (^) in the notation.

PROOF OF THEOREM 1.3. We shall show that Q_z can be obtained from P_z by a Girsanov type transform. Let $\Psi := 1/\varphi$ [$\in D(\mathscr{E}_m)$ by Lemma 3.2] and let

$\tilde{\Psi} := 1/\tilde{\varphi}$, $\tilde{\varphi}$ as in Lemma 3.2. Then since $\ln \Psi \in D(\mathscr{E}_m)$ by Lemma 3.2, we have by (3.1) that

$$\ln \tilde{\Psi}(X_t) - \ln \tilde{\Psi}(X_0) = M_t^{[\ln \Psi]} + N_t^{[\ln \Psi]}, \qquad t \geq 0,\ P_z\text{-a.e.},$$

for $\mathscr{E}_m$-q.e. $z \in E$. By altering $\mathbf{M}_P$ on a set of $\mathscr{E}_m$-capacity zero, we may assume that $(M_t^{[\ln \Psi]}, \mathscr{F}_t, P_z)_{t \geq 0}$ is a real valued continuous martingale with $M_0^{[\ln \Psi]} = 0$ for all $z \in E$ [cf. Fukushima (1980), Chapters 4 and 5 for details]. Here $(\mathscr{F}_t)_{t \geq 0}$ is the minimal admissible family corresponding to $\mathbf{M}_P$ [i.e., $(\mathscr{F}_t)_{t \geq 0}$ is right continuous and completed]. Hence the multiplicative functional

$$(3.5) \qquad L_t^{\Psi} := \exp\left(M_t^{[\ln \Psi]} - \tfrac{1}{2}\langle M^{[\ln \Psi]}\rangle_t\right), \qquad t \geq 0,$$

is a continuous nonnegative local martingale, hence a supermartingale. In particular, for all $z \in E$,

$$E_z^P\left[L_t^{\Psi}\right] \leq 1 \quad \text{if } t > 0,$$
$$= 1 \quad \text{if } t = 0,$$

where E_z^P denotes expectation w.r.t. P. We want to apply Kunita and Watanabe (1963) [see also Dynkin (1965)] in order to obtain a transformed process $\mathbf{M}_{\overline{Q}}$ from $\mathbf{M}_P$ via L_t^{Ψ}, $t \geq 0$. To this end we need to replace Ω by the space Ω' of all continuous functions ω: $[0, \zeta(\omega)[\to E$. Obviously, we may consider $\mathbf{M}_P$ to be defined on the corresponding filtered space $(\Omega', \mathscr{F}', (\mathscr{F}_t')_{t \geq 0})$ such that $\zeta \equiv +\infty$ P_z-a.s. for all $z \in E$. Now we can apply Kunita and Watanabe (1963), Section 3, to conclude that there exists a standard process $\mathbf{M}_{\overline{Q}} := (\Omega', \overline{\mathscr{F}}', (\overline{\mathscr{F}}_t')_{t \geq 0}, (X_t)_{t \geq 0}, \zeta, (\overline{Q}_z)_{z \in E})$ on E such that for all $z \in E$,

$$(3.6) \qquad \overline{Q}_z(A \cap \{t < \zeta\}) = E_z^P\left[L_t^{\Psi}, A\right], \qquad A \in \mathscr{F}_t'.$$

By Takeda (1990), Theorem 1, it follows that $\mathbf{M}_{\overline{Q}}$ is μ-symmetric and conservative; that is, $\zeta \equiv \infty$ $\overline{Q}_\mu$-a.e. Note that indeed Takeda (1990) applies since $\tilde{\Psi} > 0$, $\mathscr{E}_m$-q.e. [hence if τ is as in Takeda (1990), then $\tau \equiv \infty$, P_z-a.e. for $\mathscr{E}_m$-q.e. $z \in E$] and

$$\int d\mu_{\langle \Psi \rangle} = \int \langle \tilde{\nabla}\Psi, \tilde{\nabla}\Psi\rangle_H \, dm = \int \varphi^{-2}\langle \nabla\varphi, \nabla\varphi\rangle_H \, d\mu < \infty$$

by Lemmas 3.1 and 3.3 and (1.18). Now we are going to show that the Dirichlet form associated with $\mathbf{M}_{\overline{Q}}$ is $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$. Let $u \in \widetilde{\mathscr{F}\mathscr{C}}_b^\infty(K)$. We want to prove that for all $t \geq 0$,

$$(3.7) \quad u(z) - E_z^{\overline{Q}}[u(X_t)] = -\int_0^t E_z^{\overline{Q}}[Lu(X_s)]\, ds \quad \text{for } \mu\text{-a.e. } z \in E.$$

First we note that

$$(3.8) \qquad E_z^P\left[\int_0^t L_s^{\Psi} |Lu|(X_s)\, ds\right] < \infty, \qquad t \geq 0, \text{ for } \mu\text{-a.e. } z \in E.$$

Indeed, by Fubini's theorem and the μ symmetry of $\mathbf{M}_{\overline{Q}}$,

$$\int E_z^P\left[\int_0^t L_s^\Psi |Lu|(X_s)\, ds\right]\mu(dz) = \int_0^t \int E_z^{\overline{Q}}[1]|Lu|(z)\mu(dz)\, ds < \infty \tag{3.9}$$

because it is dominated by $t\int|Lu|(z)\mu(dz) < \infty$. We define $(\overline{\mathscr{F}}_t)_{t\geq 0}$-stopping times

$$\tau_n := \inf\left\{t \geq 0 \,|\, L_t^\Psi \vee \int_0^t |Lu|(X_s)\, ds \vee \int_0^t \varphi^{-1}(X_s)\langle \nabla\varphi(X_s), \nabla u(X_s)\rangle_H\, ds > n\right\} \wedge n.$$

Then $\tau_n \uparrow \infty$ as $n \to \infty$ P_z-a.s. for μ-a.e. $z \in E$. By the dominated convergence theorem, Doob's optional stopping theorem and (3.1) and (3.2) we conclude that for μ-a.e. $z \in E$ and $t \geq 0$,

$$\begin{aligned}
&u(z) - E_z^{\overline{Q}}[u(X_t)] \\
&\quad = -E_z^P[L_t^\Psi(u(X_t) - u(X_0))] \\
&\quad = -\lim_{n\to\infty} E_z^P\left[L_t^\Psi\left(u(X_{\tau_n\wedge t}) - u(X_0)\right)\right] \\
&\quad = -\lim_{n\to\infty} E_z^P\left[L_{\tau_n\wedge t}^\Psi\left(u(X_{\tau_n\wedge t}) - u(X_0)\right)\right] \\
&\quad = -\lim_{n\to\infty} E_z^P\left[L_{\tau_n\wedge t}^\Psi\left(M_{\tau_n\wedge t}^{[u]} + \int_0^{\tau_n\wedge t}\left(Lu(X_s) + \frac{\langle\nabla\varphi, \nabla u\rangle_H}{\varphi}(X_s)\right) ds\right)\right] \\
&\quad = -\lim_{n\to\infty}\left(E_z^P\left[L_{\tau_n\wedge t}^\Psi M_{\tau_n\wedge t}^{[u]}\right]\right. \\
&\qquad\qquad \left.+E_z^P\left[\int_0^{\tau_n\wedge t} L_s^\Psi\left(Lu(X_s) + \frac{\langle\nabla\varphi, \nabla u\rangle_H}{\varphi}(X_s)\right) ds\right]\right),
\end{aligned}$$

where we integrated by parts in the last step. Since by Itô's formula

$$L_t^\Psi = 1 + \int_0^t L_s^\Psi\, dM_s^{[\ln\Psi]}, \qquad t \geq 0,$$

we have that

$$\begin{aligned}
E_z^P\left[L_{\tau_n\wedge t}^\Psi M_{\tau_n\wedge t}^{[u]}\right] &= E_z^P\left[\int_0^{\tau_n\wedge t} L_s^\Psi d\langle M^{[\ln\Psi]}, M^{[u]}\rangle_s\right] \\
&= -E_z^P\left[\int_0^{\tau_n\wedge t} L_s^\Psi \frac{\langle\nabla\varphi, \nabla u\rangle_H}{\varphi}(X_s)\, ds\right]
\end{aligned}$$

by Lemmas 3.1–3.3. Now (3.7) easily follows by (3.8) and the dominated convergence theorem. Let $(\mathscr{E}^{\overline{Q}}, D(\mathscr{E}^{\overline{Q}}))$ denote the Dirichlet form on $L^2(E; \mu)$

associated with $\mathbf{M}_{\overline{Q}}$. Then by (3.7) for all $v \in D(\mathscr{E}^{\overline{Q}}) \cap L^\infty(E;\mu)$,

$$\lim_{t \downarrow 0} \frac{1}{t}\left(u - E^{\overline{Q}}_{\cdot}[u(X_t)], v\right)_\mu = \lim_{t \downarrow 0}\left(-\frac{1}{t}\int_0^t \int E_z^{\overline{Q}}[Lu(X_s)]v(z)\mu(dz)\,ds\right)$$
$$= -\int Lu(z)v(z)\mu(dz),$$

where we used (3.9). Setting first $u = v$, this implies that $u \in D(\mathscr{E}^{\overline{Q}})$ [cf. Fukushima (1980), Lemma 1.3.4] and subsequently, we can conclude that the generator of $(\mathscr{E}^{\overline{Q}}, D(\mathscr{E}^{\overline{Q}}))$ coincides with L on $\widetilde{\mathscr{F}\mathscr{C}_b^\infty}$. Hence by our uniqueness assumption (1.15),

$$\left(\mathscr{E}^{\overline{Q}}, D(\mathscr{E}^{\overline{Q}})\right) = \left(\mathscr{E}_\mu, D(\mathscr{E}_\mu)\right).$$

Now the same proof as that of Fukushima (1980), Theorem 4.3.3, yields that $\mathbf{M}_{\overline{Q}}$ is properly associated with $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$; that is, $z \to E_z^{\overline{Q}}[u(X_t)]$ is $\mathscr{E}_\mu$-quasi-continuous for all $u \in L^2(E;\mu)$, $u \geq 0$, and all $t \geq 0$. Note that for the proof of Fukushima (1980), Theorem 4.3.3, one does not need $\mathbf{M}_{\overline{Q}}$ to be a Hunt process, but only that $\overline{Q}_\mu[\zeta = \infty] = 1$ and that the sample paths of $\mathbf{M}_{\overline{Q}}$ are continuous up to ζ. Since also $\mathbf{M}_Q$ is properly associated with $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$, it follows by monotone class theorems that for $\mathscr{E}_\mu$-q.e. $z \in E$,

$$\overline{Q}_z(A) = Q_z(A \cap \Omega) \quad \text{for all } A \in \mathscr{F}'. \tag{3.10}$$

Now (3.6) and (3.10) imply that $P_z \sim Q_z$ for $\mathscr{E}_\mu$-q.e. $z \in E$. Since m does not charge $\mathscr{E}_\mu$-capacity zero sets and vice versa, the second part of the assertion now follows by Proposition 1.2 and the last part of Theorem 1.1. □

REMARK 3.4. (i) We emphasize that the exponent of the Radon–Nikodym derivative L_t^Ψ in (3.5) is of the familiar form since it can be shown (by approximation) that

$$M_t^{[\ln \Psi]} = -\int_0^t \frac{\nabla\varphi}{\varphi}(X_s)\,dW_s,$$

where the stochastic integral is in the sense of Kuo (1975) and $(W_t)_{t \geq 0}$ is as in Remark 1.4(ii).

(ii) In the proof of Theorem 1.3 we have in fact only used (1.17) and (1.18) to show that $\ln\varphi, \varphi^{-1} \in D(\mathscr{E}_m)$. Hence we can weaken the hypotheses of Theorem 1.3 accordingly. We have considered the more restrictive situation since (1.17) and (1.18) are easier to check in applications.

(iii) A special case of Theorem 1.3 is also discussed in Fan (1990). However, the method of proof is different from ours and the proof does not seem to be complete to us [cf. Fan (1990), Theorem 4.1].

4. Proof of Theorem 1.5. For $\varepsilon \in [0,1]$ set

$$\varphi_\varepsilon := \varphi \vee \varepsilon \quad \text{and} \quad m_\varepsilon := \varphi_\varepsilon^2 \cdot \mu. \tag{4.1}$$

Since $\varphi_\varepsilon \in D(\mathscr{E}_\mu)$ and $\beta_k \cdot \varphi_\varepsilon \in L^2(E;\mu)$, it follows as in the case where

$\varepsilon = 0$ that each $k \in K$ is well-m_ε-admissible. Let $(\mathscr{E}_{m_\varepsilon}, D(\mathscr{E}_{m_\varepsilon}))$ denote the corresponding Dirichlet forms on $L^2(E; m_\varepsilon)$ and $\mathbf{M}_{P^\varepsilon} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z^\varepsilon)_{z\in E})$ the associated diffusion processes (cf. Section 1C). For consistency with our previous notation we set $\varphi_0 := \varphi$, $m_0 := m$ and $P^0 := P$.

REMARK 4.0. Let $\varepsilon, \varepsilon' \in [0,1]$, $\varepsilon \leq \varepsilon'$. Then

$$D(\mathscr{E}_{m_{\varepsilon'}}) \subset D(\mathscr{E}_{m_\varepsilon}) \quad \text{and} \quad \mathscr{E}_{m_{\varepsilon'}}(u,u) \geq \mathscr{E}_{m_\varepsilon}(u,u) \quad \text{for all } u \in D(\mathscr{E}_{m_{\varepsilon'}}). \tag{4.2}$$

In particular, $\mathscr{E}_{m_\varepsilon}$-Cap $\leq \mathscr{E}_{m_{\varepsilon'}}$-Cap and any $\mathscr{E}_{m_{\varepsilon'}}$-quasicontinuous function is $\mathscr{E}_{m_\varepsilon}$-quasicontinuous. Additionally, if $\tilde{\nabla}^\varepsilon$ denotes the closure of $\nabla\colon \widetilde{\mathscr{F}\mathscr{C}_b^\infty} \to L^2(E \to H; m_\varepsilon)$ on $L^2(E; m_\varepsilon)$, then $\tilde{\nabla}^\varepsilon = \tilde{\nabla}^{\varepsilon'}$ on $D(\mathscr{E}_{m_{\varepsilon'}})$ (= domain of $\tilde{\nabla}^{\varepsilon'}$).

Since $(\varphi \vee 1)^2 \leq \varphi^2 + 1$, assumption (1.21) implies that

$$\varphi \in D(\mathscr{E}_{m_1}) \qquad \left[\subset D(\mathscr{E}_{m_\varepsilon}) \text{ for all } \varepsilon \in [0,1]\right]. \tag{4.3}$$

From now on we fix an $\mathscr{E}_{m_1}$-quasicontinuous Borel version $\tilde{\varphi}$ of φ. By Remark 4.0, $\tilde{\varphi}$ is $\mathscr{E}_{m_\varepsilon}$-quasicontinuous for all $\varepsilon \in [0,1]$. Since Lemma 3.2(i) does not use (1.18) we know that $0 < \tilde{\varphi} < \infty$ $\mathscr{E}_m$-q.e.

From now on we fix $\varepsilon \in [0,1]$ and set for $\delta > 0$,

$$F_\delta := \{\tilde{\varphi} \geq \delta\} \tag{4.4}$$

and

$$D(\mathscr{E}_{m_\varepsilon})_{|F_\delta} := \left\{u \in D(\mathscr{E}_{m_\varepsilon}) | \tilde{u}^\varepsilon = 0 \ \mathscr{E}_{m_\varepsilon}\text{-q.e. on } E \setminus F_\delta\right\}, \tag{4.5}$$

where in (4.5) (as below) $\tilde{u}^\varepsilon$ denotes an $\mathscr{E}_{m_\varepsilon}$-quasicontinuous version.

LEMMA 4.1. *Let $U \subset E$, U open, and let $\delta > 0$. If $u \in D(\mathscr{E}_{m_\varepsilon})$ with $u \geq 0$ m_ε-a.e. on $U \cap \{\tilde{\varphi} < \delta\}$, then $\tilde{u}^\varepsilon \geq 0$ $\mathscr{E}_{m_\varepsilon}$-q.e. on $U \cap \{\tilde{\varphi} < \delta\}$.*

PROOF. Because of the $\mathscr{E}_{m_\varepsilon}$-quasicontinuity of $\tilde{\varphi}$, the proof is completely analogous to that of Fukushima (1980), Lemma 3.1.4. □

LEMMA 4.2. $(\mathscr{E}_{m_\varepsilon}, D(\mathscr{E}_{m_\varepsilon})_{|F_{2\varepsilon}}) = (\mathscr{E}_m, D(\mathscr{E}_m)_{|F_{2\varepsilon}})$.

PROOF. By Lemma 4.1 we may replace $\mathscr{E}_{m_\varepsilon}$-q.e. in (4.5) by m_ε-a.e. Because of (4.2), it remains to show that if $u \in D(\mathscr{E}_m)$ with $u = 0$ m-a.e. on $E \setminus F_{2\varepsilon}$, then $u \in D(\mathscr{E}_{m_\varepsilon})$. But for such u and $f \in C_b^\infty(\mathbb{R})$ with $1_{[2\varepsilon,\infty[} \leq f \leq 1_{[\varepsilon,\infty[}$ we have that $u = uf(\varphi)$ m-a.e. We may assume that u is bounded, hence we can find $u_n \in \widetilde{\mathscr{F}\mathscr{C}_b^\infty}$, $n \in \mathbb{N}$, such that $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < \infty$ and $u_n \to_{n\to\infty} u$ w.r.t. $\mathscr{E}_m + (\ ,\ )_m$. Then $f(\varphi)u_n \to_{n\to\infty} f(\varphi)u = u$ in $L^2(E; m)$ and $f(\varphi)u_n \in D(\mathscr{E}_{m_1}) \subset D(\mathscr{E}_{m_\varepsilon})$ [by Fukushima (1980), Theorem 1.4.2, since $f(\varphi) \in D(\mathscr{E}_{m_1})$]. By the product rule for $\tilde{\nabla}^{\varepsilon}$ and since $\tilde{\nabla}^{\varepsilon} = \tilde{\nabla}$ on $D(\mathscr{E}_{m_\varepsilon})$ we see that for all $n, m \in \mathbb{N}$,

$$\begin{aligned}
&\mathscr{E}_{m_\varepsilon}(f(\varphi)(u_n - u_m), f(\varphi)(u_n - u_m)) \\
&\qquad + (f(\varphi)(u_n - u_m), f(\varphi)(u_n - u_m))_{m_\varepsilon} \\
&\le 2\int_{\{\varphi \ge \varepsilon\}} \Big[\|\tilde{\nabla}(u_n - u_m)\|_H^2 + \big(1 + |f'(\varphi)|^2 \|\tilde{\nabla}\varphi\|_H^2\big)(u_n - u_m)^2 \Big] \\
&\qquad \times (\varphi \vee \varepsilon)^2 \, d\mu,
\end{aligned}$$

which becomes arbitrarily small for n, m large by the assumption on $(u_n)_{n \in \mathbb{N}}$. Hence $u \in D(\mathscr{E}_{m_\varepsilon})$, since $(\mathscr{E}_{m_\varepsilon}, D(\mathscr{E}_{m_\varepsilon}))$ is closed. Since $u = u \cdot f(\varphi)$ for $u \in D(\mathscr{E}_{m_\varepsilon})|_{F_{2\varepsilon}}$, it follows by the product rule that $\tilde{\nabla}^{\varepsilon} u = \tilde{\nabla} u = 0$ μ-a.e. on $\{\varphi < \varepsilon\}$; hence $\mathscr{E}_{m_\varepsilon}(u, u) = \mathscr{E}_m(u, u)$. □

Denoting the expectation w.r.t. P_z^{ε} by E_z^{ε}, $z \in E$, we define for f: $E \to \mathbb{R}$, $\mathscr{B}(E)$-measurable, bounded and $z \in E$,

$$(4.6) \qquad {}^{\varepsilon}p_s f(z) := E_z^{\varepsilon}[f(X_s), s < \sigma_{2\varepsilon}]$$

and

$$(4.7) \qquad {}^{0}p_s f(z) := E_z[f(X_s), s < \sigma_{2\varepsilon}],$$

where $\sigma_{2\varepsilon} := \inf\{t > 0 | \tilde{\varphi}(X_t) < 2\varepsilon\}$ $(= \sigma_{E \setminus F_{2\varepsilon}})$ and E_z denotes expectation w.r.t. P_z, $z \in E$. Furthermore, for $\alpha > 0$ let

$$(4.8) \qquad {}^{\varepsilon}R_\alpha f(z) = \int_0^\infty e^{-\alpha s}\, {}^{\varepsilon}p_s f(z)\, ds, \qquad z \in E.$$

LEMMA 4.3. *Let $s > 0$. Then there exists $N \in \mathscr{B}(E)$ with $\mathscr{E}_m$-Cap$(N) = 0$ such that for all $z \in E \setminus N$ and all f: $E \to \mathbb{R}$, bounded $\mathscr{B}(E)$-measurable*:

(i) ${}^{0}p_s f(z) = {}^{\varepsilon}p_s f(z)$, $s \ge 0$.
(ii) ${}^{0}p_s f(X_t) = {}^{\varepsilon}p_s f(X_t)$, $s, t \ge 0$, P_z*-a.s.*

PROOF. Let $f \in \mathscr{F}\mathscr{C}_b^\infty$ and $\alpha > 0$. By Fukushima (1980), Lemma 4.4.2, it follows that ${}^{\varepsilon}R_\alpha f$ is $\mathscr{E}_{m_\varepsilon}$-quasicontinuous, ${}^{\varepsilon}R_\alpha f \in D(\mathscr{E}_{m_\varepsilon})|_{F_{2\varepsilon}}$ and

$$\mathscr{E}_{m_\varepsilon}({}^{\varepsilon}R_\alpha f, v) + \alpha({}^{\varepsilon}R_\alpha f, v)_{m_\varepsilon} = (f, v)_{m_\varepsilon} \quad \text{for all } v \in D(\mathscr{E}_{m_\varepsilon})|_{F_{2\varepsilon}}$$

and a corresponding statement with ${}^{0}R_\alpha f, m$ replacing ${}^{\varepsilon}R_\alpha f, m_\varepsilon$, respectively. By Lemma 4.2 this entails that

$$\mathscr{E}_m({}^{\varepsilon}R_\alpha f - {}^{0}R_\alpha f, {}^{\varepsilon}R_\alpha f - {}^{0}R_\alpha f) + \alpha({}^{\varepsilon}R_\alpha f - {}^{0}R_\alpha f, {}^{\varepsilon}R_\alpha f - {}^{0}R_\alpha f)_m = 0,$$

hence ${}^{\varepsilon}R_\alpha f = {}^{0}R_\alpha f$ m-a.e. By Remark 4.0 and Lemma 4.1 it follows that the latter equality holds $\mathscr{E}_m$-q.e. By the uniqueness of the Laplace transform and the right continuity of $s \mapsto {}^{\varepsilon}p_s f(z)$, $z \in E$, we obtain that for all $z \in E$ outside some $\mathscr{E}_m$-capacity zero set,

$$(4.9) \qquad {}^{0}p_s f(z) = {}^{\varepsilon}p_s f(z) \quad \text{for all } s > 0.$$

Furthermore, since ${}^{\varepsilon}R_{\alpha}f$ is $\mathscr{E}_m$-quasicontinuous, using Fukushima (1980), Theorem 4.3.2, we can find an $\mathscr{E}_m$-capacity zero set $N_0 \in \mathscr{B}(E)$ such that for all $z \in E \setminus N_0$,

$$ {}^{0}R_{\alpha} f(X_t) = {}^{\varepsilon}R_{\alpha} f(X_t), \qquad \alpha > 0, t \ge 0,\ P_z\text{-a.s.} $$

We deduce as before that for all $z \in E \setminus N_0$,

$$ {}^{0}p_s f(X_t) = {}^{\varepsilon}p_s f(X_t), \qquad s, t \ge 0,\ P_z\text{-a.s.} \tag{4.10} $$

By the Hahn–Banach theorem we see that there exists a countable set $\mathscr{K} \subset \mathscr{F}C_b^{\infty}$ which is closed under multiplication, contains the constant function 1 and separates the points of E. By Schwartz (1973), Lemma 18, page 108, $\mathscr{K}$ generates $\mathscr{B}(E)$. Applying monotone class theorems with $\mathscr{K}$, we obtain (i) and (ii) from (4.9) and (4.10). □

The proof of Lemma 4.4 is now standard, but we include it for the reader's convenience.

LEMMA 4.4. *Let $N \in \mathscr{B}(E)$ be as in Lemma* 4.3 *and* $0 < t_1 < \cdots < t_n < \infty$. *Let $f_0, \dots, f_n\colon E \to \mathbb{R}$ be bounded, $\mathscr{B}(E)$-measurable and let $z \in E \setminus N$. Then*

$$ \begin{aligned} &E_z\big[f_0(X_0) f_1(X_{t_1}) \cdots f_n(X_{t_n}), t_n < \sigma_{2\varepsilon}\big] \\ &\quad = E_z^{\varepsilon}\big[f_0(X_0) f_1(X_{t_1}) \cdots f_n(X_{t_n}), t_n < \sigma_{2\varepsilon}\big]. \end{aligned} $$

PROOF. For $n = 1$ the assertion is clear by Lemma 4.3(i). Suppose the assertion holds for $n-1$. Then by induction, the Markov property of $\mathbf{M}_P$ and $\mathbf{M}_{P^{\varepsilon}}$ and Lemma 4.3(ii):

$$ \begin{aligned} &E_z\big[f_0(X_0) \cdots f_{n-1}(X_{t_{n-1}}) f_n(X_{t_n}), t_n < \sigma_{2\varepsilon}\big] \\ &\quad = E_z\big[f_0(X_0) \cdots f_{n-1}(X_{t_{n-1}}) f_n(X_{t_n - t_{n-1}} \circ \Theta_{t_{n-1}}), t_{n-1} < \sigma_{2\varepsilon}, \\ &\qquad\qquad t_n < t_{n-1} + \sigma_{2\varepsilon} \circ \Theta_{t_{n-1}}\big] \\ &\quad = E_z\big[f_0(X_0) \cdots f_{n-1}(X_{t_{n-1}})\, {}^{0}p_{t_n - t_{n-1}} f_n(X_{t_{n-1}}), t_{n-1} < \sigma_{2\varepsilon}\big] \\ &\quad = E_z\big[f_0(X_0) \cdots f_{n-1}(X_{t_{n-1}})\, {}^{\varepsilon}p_{t_n - t_{n-1}} f_n(X_{t_{n-1}}), t_{n-1} < \sigma_{2\varepsilon}\big] \cdot \\ &\quad = E_z^{\varepsilon}\big[f_0(X_0) \cdots f_{n-1}(X_{t_{n-1}})\, {}^{\varepsilon}p_{t_n - t_{n-1}} f_n(X_{t_{n-1}}), t_{n-1} < \sigma_{2\varepsilon}\big] \cdot \\ &\quad = E_z^{\varepsilon}\big[f_0(X_0) \cdots f_{n-1}(X_{t_{n-1}}) f_n(X_{t_n}), t_n < \sigma_{2\varepsilon}\big]. \end{aligned} $$

□

COROLLARY 4.5. *Let $t > 0$ and N be as in Lemma* 4.3. *Then for all $z \in E \setminus N$ and $A \in \mathscr{F}_t$,*

$$ P_z[A, t < \sigma_{2\varepsilon}] = P_z^{\varepsilon}[A, t < \sigma_{2\varepsilon}]. $$

Now we are prepared to prove Theorem 1.5.

PROOF OF THEOREM 1.5. By Corollary 4.5 and Theorem 1.3 it follows that for $\mathscr{E}_m$-q.e. $z \in E$,

$$P_z[\cdot, t < \sigma_{2\varepsilon}] \ll Q_z \quad \text{for all } t > 0. \tag{4.11}$$

Since $\tilde{\varphi}$ is $\mathscr{E}_m$-quasicontinuous, $t \mapsto \tilde{\varphi}(X_t)$ is continuous P_z-a.s. and $\sigma_{2\varepsilon} \uparrow \sigma_{\{\tilde{\varphi}=0\}}$ as $\varepsilon \downarrow 0$ P_z-a.s. for $\mathscr{E}_m$-q.e. $z \in E$. Since $\tilde{\varphi} > 0$ $\mathscr{E}_m$-q.e., we have that

$$E_m\left[\exp(-\sigma_{\{\tilde{\varphi}=0\}})\right] = 0.$$

But $z \mapsto E_z[\exp(-\sigma_{\{\tilde{\varphi}=0\}})]$ is $\mathscr{E}_m$-quasicontinuous [cf. Fukushima (1980), Theorem 4.3.5, hence by Lemma 4.1, $\sigma_{\{\tilde{\varphi}=0\}} = +\infty$ P_z-a.s. for $\mathscr{E}_m$-q.e. $z \in E$. Now the first part of the assertion follows, by letting $\varepsilon \downarrow 0$ in (4.11). The second is again a consequence of Proposition 1.2 and the last part of Theorem 1.1. □

REMARK 4.6. (i) It might not be easy to verify condition (1.21) directly in applications. Note that it is enough to check that $\varphi_N := \varphi \wedge N \in D(\mathscr{E}_{m+\mu})$ for all $N \in \mathbb{N}$. But for this to hold it suffices, for example, to check whether there exist $u_n \in \widetilde{\mathscr{F}\mathscr{C}}_b^\infty$, $n \in \mathbb{N}$, and $p, q \in [2, \infty]$ with $1/p + 1/q = \frac{1}{2}$ such that $\varphi \in L^p(E; \mu)$, $u_n \to_{n\to\infty} \varphi_N$ in $L^q(E; \mu)$ and

$$\int \|\nabla(u_n - u_m)\|_H^q \, d\mu \xrightarrow[n, m \to \infty]{} 0.$$

Since we may assume that $\sup_{n \in \mathbb{N}} \|u_n\|_\infty < \infty$, this is immediate by Hölder's inequality. In particular, if E, H and μ form an abstract Wiener space, then the preceding condition is fulfilled for all $\varphi \in D_{q,1}$ [see Sugita (1985)]. For more general examples we refer to Röckner and Zhang (1992), Sections 3 and 5.

(ii) If we consider Theorem 1.5 in the case $\dim E < \infty$, $E = \mathbb{R}^d$ say, we see that we cannot hope to get $P_z \sim Q_z$ for $\mathscr{E}_m$-q.e. $z \in E$ since we have dropped condition (1.18), which has been shown in Fukushima (1982), Theorem 2, to be (essentially) a necessary condition for this.

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