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SUPPORTS OF CERTAIN INFINITELY DIVISIBLE PROBABILITY MEASURES ON LOCALLY CONVEX SPACES[1]

By Balram S. Rajput

University of Tennessee, Knoxville

Let **B** be a separable Banach space and let μ be a centered Poisson probability measure on **B** with Lévy measure M. Assume that M admits a polar decomposition in terms of a finite measure σ on the unit sphere of **B** and a Lévy measure ρ on $(0, \infty)$. The main result of this paper provides a complete description of the structure of $\mathscr{S}_\mu$, the support of μ. Specifically, it is shown that: (i) if $\int_{(0,1]} s\rho(ds) = \infty$, then $\mathscr{S}_\mu$ is a linear space and is equal to the closure of the semigroup generated by $\mathscr{S}_M$ (the support of M) and the negative of the barycenter of σ; and (ii) if $\int_{(0,1]} s\rho(ds) < \infty$ and zero is in the support of ρ, then $\mathscr{S}_\mu$ is a convex cone and is equal to the closure of the semigroup generated by $\mathscr{S}_M$. The result (i) yields an affirmative answer to the question, open for some time, of whether the support of a stable probability measure of index $1 \le \alpha < 2$ on B is a translate of a linear space. Analogs of these results, for both Poisson and stable probability measures defined on general locally convex spaces, are also provided.

1. Introduction. This paper provides a complete description of the structure of the supports of general (not necessarily symmetric) Poisson probability measures [defined on general locally convex (l.c.) spaces] whose Lévy measures admit a polar-type decomposition. This work is inspired by and completes a result of de Acosta [3] and several results of Tortrat [10], [11].

The core result is Theorem 3.1(a). It shows that if the Lévy measure M of a centered Poisson probability measure μ on a separable Banach space **B** admits a polar decomposition in terms of a finite measure σ on the unit sphere of **B** and a (Lévy) measure ρ on $(0, \infty)$ which satisfies $\int_{(0,1]} s\rho(ds) = \infty$, then $\mathscr{S}_\mu$, the support of μ, is a (closed) linear space and it is equal to the closure of the semigroup generated by $\mathscr{S}_M$, the support of M, and the negative of the barycenter of σ. This result completes and encompasses several results of Tortrat ([10], Theorems 2(i), 3, and 4 and Corollary 1; [11], Proposition 1 and Theorem 3′). It also settles two of his conjectures, which state that, under the hypotheses of Theorem 3.1(a), $\mathscr{S}_\mu$ may fail to be a linear space in the infinite-dimensional case even when **B** is a Hilbert space ([10], page 41) and that $\mathscr{S}_\mu$ "presque sûr" appears to be a linear space in the finite-dimensional case ([11], page 294). We also prove a companion result to Theorem 3.1(a) [Theorem 3.1(b)]. It shows that if $\int_{(0,1]} s\rho(ds) < \infty$ and if 0 belongs to the

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support of ρ, then $\mathscr{S}_\mu$ is a translate of a (closed) convex cone and that this cone is equal to the closure of the semigroup generated by $\mathscr{S}_M$. In this result we also provide a necessary and sufficient condition in order for this cone to be a linear space. In addition, we provide, in Theorems 4.1(a) and 4.1(b), appropriate analogs of Theorems 3.1(a) and 3.1(b), respectively, for probability measures defined on general l.c. spaces.

Let μ be an α-stable, $1 \le \alpha < 2$, probability measure on $\mathbf{B}$; the question of whether $\mathscr{S}_\mu$ is a translate of a linear space (without any restrictive hypotheses on the space or on the measure) has been open for some time (see [3], page 874, Theorem 5.2; [10], pages 38–39, Theorems 2 and 3; and [11], pages 294–295, Corollary and Remark). The core result and its analog in l.c. spaces yield two corollaries which answer this question in the affirmative for measures defined not only on $\mathbf{B}$ but also on general l.c. spaces. These complete the results of de Acosta and Tortrat just noted as well as pertinent results obtained in [2], [5] and [6].

The methods of proof used are refinements of those exploited earlier in [4], [7] and [10–12]. The organization of the rest of the paper is as follows: Section 2 contains preliminaries. Section 3 contains the core result, its companion result and a few of their corollaries. Section 4 contains the analogs of results of Section 3 for measures defined on general l.c. spaces.

2. Preliminaries. Throughout, $\mathbf{B}$ and $\mathbf{E}$ will denote, respectively, a separable Banach space and a l.c. space; further, $\mathbf{B}^*$ and $\mathbf{E}^*$ will denote, respectively, the topological duals of $\mathbf{B}$ and $\mathbf{E}$. For a subset of A of $\mathbf{B}$, the closure of the linear space (resp., the convex cone) generated by A will be denoted by $\mathbf{L}(A)$ [resp., by $\mathbf{C}(A)$]. Unless stated otherwise, all measures on a topological space X are assumed to be defined on the Borel σ-algebra of X; for a set A in X, $\bar{A}$ will denote the closure of A.

Let ν be a finite or infinite measure on a separable metric space X. Then the *support* of ν, denoted throughout by $\mathscr{S}_\nu$, is, by definition, the intersection of all closed sets F with $\nu(F^c) = 0$, where, throughout, for a set A, A^c denotes the complement of A. Clearly, $\mathscr{S}_\nu = \{x \in X\colon \nu(V) > 0$, for every open neighborhood V of $x\}$. If $X = \mathbf{B}$, then the *linear support* of ν, denoted throughout by $\mathscr{L}\mathscr{S}(\nu)$, is the intersection of all closed linear subspaces G of $\mathbf{B}$ with $\nu(G^c) = 0$. It is easy to see that

$$(2.1) \qquad \mathscr{L}\mathscr{S}(\nu) = \mathbf{L}(\mathscr{S}_\nu); \qquad \mathscr{L}\mathscr{S}(\nu) = \mathscr{S}_\nu, \quad \text{if } \mathscr{S}_\nu \text{ is linear}.$$

Let M be a finite or infinite measure on $\mathbf{B}$. If M admits a polar decomposition, that is, $M = (\sigma \times \rho) \circ \Psi^{-1}$, where Ψ is the topological isomorphism from $\partial\Delta \times (0, \infty)$ onto $\mathbf{B} \setminus \{\theta\}$ defined by $\Psi(u, t) = tu$, and where σ and ρ are, respectively, a finite measure on $\partial\Delta$ (the boundary of $\Delta \equiv \{x \in \mathbf{B}\colon \|x\| \le 1\}$ and a measure on $(0, \infty)$ (θ being the zero element of $\mathbf{B}$), then we shall write, throughout, $M = \Psi(\sigma \times \rho)$.

Now let $M = \Psi(\sigma \times \rho)$ be a Lévy measure [therefore, ρ is also Lévy on $R^+ \equiv (0, \infty)$]. Following [1], the *τ-centered Poisson probability measure* on $\mathbf{B}$ with Lévy measure M will be denoted, throughout, by c_τ-Pois(M) or by

c_τ-Pois(σ, ρ), $0 \le \tau \le \infty$. If $\tau = 0$ (resp., $\tau = \infty$), it is assumed here and elsewhere that $\int_\Delta \|x\|\, dM < \infty$ (resp., $\int_{\Delta^c} \|x\|\, dM < \infty$).

Let $M = \Psi(\sigma \times \rho)$ be a Lévy measure on $\mathbf{B}$. The barycenter $\int_{\partial\Delta} u\, \sigma(du)$ and, for a fixed $\tau > 0$, the function $\lambda \mapsto \int_{(\lambda, \tau]} s \rho(ds)$, $0 < \lambda < \tau$, we shall denote, throughout, by $a_\sigma (\equiv a)$ and $\varphi_{\tau, \rho} (\equiv \varphi_\tau)$, respectively. Further, throughout, we shall use the notation $\mathbf{I}_j$, $j = 1, 2$, to denote the following integral conditions on M:

$$(2.2)\qquad \begin{aligned} &\mathbf{I}_1\colon \quad \int_\Delta \|x\|\, dM = \infty \quad \left(\Leftrightarrow \varphi_{1,\rho}(0^+) = \infty\right); \\ &\mathbf{I}_2\colon \quad \int_\Delta \|x\|\, dM < \infty \quad \left(\Leftrightarrow \varphi_{1,\rho}(0^+) < \infty\right). \end{aligned}$$

Finally, we shall use the notation $\tilde{\sigma}$ $(\equiv \tilde{\sigma}_a)$ and $\tilde{M}$ $(\equiv \tilde{M}_a)$ for the measures defined, respectively, by

$$(2.3)\qquad \begin{aligned} \tilde{\sigma} &= \begin{cases} \sigma, & \text{if } a = \theta, \\ \sigma + \|a\| \delta_{\{-a/\|a\|\}}, & \text{if } a \ne \theta, \end{cases} \\ \tilde{M} &= (\tilde{\sigma} \times \rho) \circ \Psi^{-1}, \end{aligned}$$

where $\delta_{\{\cdot\}}$ is the Dirac measure. The Hahn–Banach theorem yields

$$(2.4)\qquad a_\sigma \in \mathbf{C}(\mathscr{S}_\sigma), \quad \text{hence, by (2.3),} \quad \mathbf{L}(\mathscr{S}_\sigma) = \mathbf{L}(\mathscr{S}_{\tilde{\sigma}}).$$

Therefore, if $\mathbf{C}(\mathscr{S}_\sigma)$ is a linear space, then

$$(2.5)\qquad \mathbf{C}(\mathscr{S}_\sigma) = \dot{\mathbf{L}}(\mathscr{S}_\sigma) \qquad \left[= \mathbf{L}(\mathscr{S}_{\tilde{\sigma}}) = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})\right].$$

Now we state two preliminary results [Propositions 2.1(a) and 2.1(b)]; for a proof of (a) see [10], page 39; a proof of (b) can be provided using (2.1) and standard techniques. We will need one additional notation: For a measure γ on $\mathbf{B}$, following [10], page 34, we shall denote by $\Sigma(\mathscr{S}_\gamma)$ the closure of the semigroup generated by $\mathscr{S}_\gamma$ and $\{\theta\}$.

PROPOSITION 2.1. *Let $M = \Psi(\sigma \times \rho)$ by a Lévy measure on $\mathbf{B}$ and let $\mu_\tau \equiv c_\tau$-Pois(σ, ρ), $0 \le \tau \le \infty$.*

(a) *If $0 < \tau \le \infty$, then, for any sequence $\tau > \lambda_n \downarrow 0$,*

$$(2.6)\qquad \mathscr{S}_{\mu_\tau} = \bigcap_{m \ge 1} \left[\bigcup_{n \ge m} \left\{ \bar{A}_{\lambda_n} - a_\sigma \varphi_\tau(\lambda_n) \right\} \right] = \bigcap_{m \ge 1} \left[\bigcup_{n \ge m} \left\{ A_{\lambda_n} - a_\sigma \varphi_\tau(\lambda_n) \right\} \right];$$

if $\tau = 0$, then, for any sequence $\lambda_n \downarrow 0$,

$$(2.7)\qquad \mathscr{S}_{\mu_0} = \bigcap_{m \ge 1} \left[\bigcup_{n \ge m} \bar{A}_{\lambda_n} \right] = \bigcap_{m \ge 1} \left[\bigcup_{n \ge m} A_{\lambda_n} \right],$$

where A_{λ_n} is the semigroup generated by $\mathscr{S}_{M/\{\|x\| \ge \lambda_n\}} \cup \{\theta\}$.

(b) *For any* $0 \le \tau \le \infty$, *set* $\mathbf{B}_0(\mu_\tau) = \{x \in \mathbf{B}: y(x) = 0,$ *for all* $y \in B^*$ *with* $\mu_\tau \circ y^{-1} = \delta_{\{0\}}\}$. *Then*

$$(2.8)\qquad \begin{aligned} \mathbf{B}_0(\mu_\tau) &= \mathscr{L}\mathscr{S}(\mu_\tau)(=\mathbf{L}(\mathscr{S}_{\mu_\tau})) = \mathscr{L}\mathscr{S}(\sigma)(=\mathbf{L}(\mathscr{S}_\sigma)) \\ &= \mathscr{L}\mathscr{S}(M)(=\mathbf{L}(\mathscr{S}_M)). \end{aligned}$$

3. The supports of Poisson probability measures on B. We first state our main theorem and several of its implications; then we state several lemmas necessary to prove the theorem. Next we present our proof of the theorem; the proofs of the lemmas are given after the proof of the theorem.

THEOREM 3.1. *Let* $M = \Psi(\sigma \times \rho)$ *be a Lévy measure on* $\mathbf{B}$, *let* $0 \le \tau \le \infty$ *and let* $\mu_\tau = c_\tau$-Pois(M) [$\equiv c_\tau$-Pois(σ, ρ)]. *Then the following hold*:

(a) *If M satisfies* $\mathbf{I}_1$ [*see* (2.2)], *then* $\Sigma(\mathscr{S}_{\tilde{M}})$, $\mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ *and* $\mathscr{S}_{\mu_\tau}$, $0 < \tau \le \infty$, *are linear spaces and*

$$(3.1)\qquad \mathscr{S}_{\mu_\tau} = \Sigma(\mathscr{S}_{\tilde{M}}) = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) = \mathbf{B}_0(\mu_\tau) = \mathbf{L}(\mathscr{S}_\sigma) = \mathbf{L}(\mathscr{S}_M),$$

for every $0 < \tau \le \infty$; *in particular, all the supports* $\mathscr{S}_{\mu_\tau}$*'s are identical.*

(b) *If M satisfies* $\mathbf{I}_2$ *and* $0 \in \mathscr{S}_\rho$, *then* $\mathscr{S}_{\mu_0}$ *and* $\Sigma(\mathscr{S}_M)$ *are convex cones*; *moreover*,

$$(3.2)\qquad \mathscr{S}_{\mu_0} = \Sigma(\mathscr{S}_M) = \mathbf{C}(\mathscr{S}_\sigma) \quad \text{and} \quad \mathscr{S}_{\mu_\tau} = -a_\sigma \varphi_\tau(0^+) + \Sigma(\mathscr{S}_M),$$

for all $0 < \tau \le \infty$ (*recall* $\varphi_\tau = \varphi_{\tau,\rho}$). *Further*, $\mathscr{S}_{\mu_\tau}$, *for one* (*equivalently, for all*) $\tau \in [0, \infty]$, *is a linear space if and only if* $\mathscr{S}_\sigma$ (*equivalently*, $\mathscr{S}_M$) *is not contained in the half-space* $\{y \ge 0\}$, *for any* $y \in \mathbf{B}^*$ *with* $y \not\equiv 0$ *on* $\mathbf{L}(\mathscr{S}_\sigma)$. *If this condition is satisfied, then* $\forall\ \tau$,

$$(3.3)\qquad \mathscr{S}_{\mu_\tau} = \Sigma(\mathscr{S}_M) = \mathbf{C}(\mathscr{S}_\sigma) = \mathbf{B}_0(\mu_\tau) = \mathbf{L}(\mathscr{S}_\sigma) = \mathbf{L}(\mathscr{S}_M),$$

in particular, all supports $\mathscr{S}_{\mu_\tau}$ *are identical.*

Let ν be an infinitely divisible (i.d.) probability measure on $\mathbf{B}$ (without a Gaussian component) with Lévy measure $M = \Psi(\sigma \times \rho)$. Then, for any $0 \le \tau \le \infty$,

$$(3.4)\qquad \nu = \delta_{\{x_\tau\}} * c_\tau\text{-Pois}(M) = \delta_{\{x_\tau\}} * c_\tau\text{-Pois}(\sigma, \rho),$$

for some $x_\tau \in \mathbf{B}$, where, recall, it is assumed that the condition (i) $\int_{[1,\infty)} s\rho(ds) < \infty$ [resp., (ii) $\int_{(0,1]} s\rho(ds) < \infty$] is satisfied if $\tau = \infty$ (resp., $\tau = 0$). For $t > 0$, the measure $\nu^t = \delta_{\{tx_\tau\}} *$ Pois$(tM = \delta_{\{tx_\tau\}} *$ Pois$(\sigma, t\rho)$ is called the *t-th root* of ν. If ν is symmetric, then one can take M and σ symmetric and, hence, one can write $\nu = c_1$-Pois$(M) = c_1$-Pois(σ, ρ) [resp., c_0-Pois$(M) = c_0$-Pois(σ, ρ)] under the condition $\mathbf{I}_1$ (resp., $\mathbf{I}_2$).

Recall that a measure ν_α on $\mathbf{B}$ is called α-*stable*, $0 < \alpha < 2 \Leftrightarrow \nu_\alpha$ admits the representation (3.4), with $\rho_\alpha(ds) = s^{-(1+\alpha)}\,ds$. Clearly, since ρ_α satisfies the above condition (i) [resp., (ii)] if $1 < \alpha < 2$ (resp., $0 < \alpha < 1$), one can take $\tau = \infty$ (resp., $\tau = 0$), in (3.4), if $1 < \alpha < 2$ (resp., $0 < \alpha < 1$). Recall also that ν_α

is called *strictly* α*-stable* $\Leftrightarrow \nu_\alpha^t(\cdot) = \nu_\alpha(t^{-1/\alpha}\cdot)$ for all $t > 0$; in the case $1 < \alpha < 2$ (resp., $0 < \alpha < 1$), this is equivalent to taking $x_\infty = \theta$ (resp., $x_0 = \theta$) in (3.4).

In view of the above observations and the facts that $\mathscr{S}_{t\gamma} = \mathscr{S}_\gamma$ for a measure γ on $\mathbf{B}$, that $\widetilde{tM} = t\tilde{M}$ and that $a_\sigma = \theta$ (and, hence, $\tilde{\sigma} = \sigma$, $\tilde{M} = M$) when σ is symmetric, Theorem 3.1 immediately yields the following.

Corollary 3.1. *Let ν be an i.d. probability measure on $\mathbf{B}$ with representation* (3.4). *Then we have the following*:

(a) *If ρ satisfies $\mathbf{I}_1$ and $0 < \tau \le \infty$, then $\mathscr{S}_{\nu^t}$ is the translate by tx_τ of any one of the equal linear spaces appearing in* (3.1). *This applies, in particular, when ν is α-stable $1 \le \alpha < 2$; further, if ν is strictly α-stable, $1 < \alpha < 2$, then $\mathscr{S}_{\nu^t}$ is equal to any one of these linear spaces.*

(b) *If ρ satisfies $\mathbf{I}_2$ (and $0 \le \tau \le \infty$), then $\mathscr{S}_{\nu^t}$ is a translate by $t(x_\tau - a_\sigma\varphi_\tau(0^+))$ of the convex cone $\mathbf{C}(\mathscr{S}_\sigma)$ $[\equiv \mathbf{\Sigma}(\mathscr{S}_\nu)]$ $[\varphi_0(0^+) \equiv 0]$; further, $\mathscr{S}_{\nu^t}$ is the translate by tx_τ of any one of the equal linear spaces appearing in* (3.3) $\Leftrightarrow$ *the condition for the linearity for $\mathscr{S}_{\mu_\tau}$ in Theorem* 3.1 *is satisfied. These apply, in particular, if ν is α-stable, $0 < \alpha < 1$. Further, if ν is strictly α-stable, $0 < \alpha < 1$, then $\mathscr{S}_{\nu^t} = \mathbf{C}(\mathscr{S}_\sigma)$, and the obvious analogous statement holds for the linearity of $\mathscr{S}_{\nu^t}$.*

(c) *If ν is symmetric (with σ and M also taken symmetric) then, under either one of the conditions $\mathbf{I}_1$ or $\mathbf{I}_2$, $\mathbf{C}(\mathscr{S}_\sigma)$ and $\mathbf{\Sigma}(\mathscr{S}_M)$ are linear spaces and $\mathscr{S}_{\nu^t} = \mathbf{C}(\mathscr{S}_\sigma) = \mathbf{\Sigma}(\mathscr{S}_M) = \mathbf{L}(\mathscr{S}_\sigma) = \mathbf{L}(\mathscr{S}_M)$. All of these apply, in particular, to symmetric α-stable measures, $\forall\ \alpha$.*

Whenever we encounter several Lévy measures having the *same* first component σ but *varying* second component we will write, for clarity, $M(\rho_r)$ for M, if the second component is ρ_r. Further, we will write $\mu(\rho_r)$ for the probability measure c_1-Pois(σ, ρ_r). In addition, for a given Lévy measure ρ_r on R^+, we shall write ψ_{ρ_r} for the function $\psi_{\rho_r}(\lambda) = \rho_r([\lambda, \infty))$, $\lambda > 0$; finally, we shall also use the notation φ_{ρ_r} for φ_{1,ρ_r}.

Lemma 3.1. *Let $M = \Psi(\sigma \times \rho)$ be a finite (or infinite) Lévy measure on $\mathbf{B}$. Assume that $0 \in \mathscr{S}_\rho$. Then $\mathbf{\Sigma}(\mathscr{S}_M)$ is a convex cone and, $\forall\ m_0 \in N \equiv \{1, 2, \dots\}$,*

$$(3.5)\qquad \mathbf{C}(\mathscr{S}_\sigma) = \mathbf{\Sigma}(\mathscr{S}_M) = \bigcap_{n\ge1}\left[\bigcup_{k\ge n} A_{\lambda_k}\right] = \left[\bigcup_{k\ge m_0} A_{\lambda_k}\right],$$

where $\{\lambda_k\}$ is any sequence with $1 > \lambda_k \downarrow 0$ and the A_{λ_k}'s are as defined in Proposition 2.1.

From (3.5) one notes that $\mathbf{\Sigma}(\mathscr{S}_M)$ is independent of any other property of ρ except that $0 \in \mathscr{S}_\rho$; this will be used in the following without any specific reference.

Lemma 3.2. *Let $M(\rho_i) = \Psi(\sigma \times \rho_i)$, $i = 1, 2$, be two Lévy measures on $\mathbf{B}$. Assume $\mathscr{S}_{\rho_1} = \mathscr{S}_{\rho_2}$ and $\varphi_{\rho_1}(\lambda) - \varphi_{\rho_2}(\lambda) \to 0$ as $\lambda \downarrow 0$. Then $\mathscr{S}_{\mu(\rho_1)} = \mathscr{S}_{\mu(\rho_2)}$.*

LEMMA 3.3. *Let ρ_d be a discrete Lévy measure on R^+ with $\varphi_{\rho_d}(0^+) = \infty$. Then one can construct a discrete Lévy measure $\rho_{d0} = \Sigma_n p_n \delta_{\{t_n\}}$ on R^+ such that* $1 > t_1 > ,\cdots, > t_n, \ldots, t_n \downarrow 0, \varepsilon_n = t_n p_n \to 0$, *as* $n \to \infty, 0 < p_n \le \rho_d(\{t_n\})$, $\forall\ n$, *and that* $\varphi_{\rho_{d0}}(0^+) = \infty$.

LEMMA 3.4. *Let $\rho_{d0} = \Sigma_{n=1}^{\infty} p_n \delta_{\{t_n\}}$ be the Lévy measure on R^+ as in the conclusion of Lemma 3.3. Then one can construct Lévy measures ρ_1, ρ_2, ρ_{20} and ρ_{21} on R^+ which satisfy the following properties*:

(a) ρ_1 *is continuous on* R^+, $\varphi_{\rho_1}(0^+) = \infty$ *and*

$$\psi_{\rho_1}(\lambda) \le \psi_{\rho_{d0}}(2^{-1}\lambda), \quad \text{for all } \lambda > 0; \tag{3.6}$$

(b) $\mathscr{S}_{\rho_2} = \mathscr{S}_{\rho_1}$ *and* $\varphi_{\rho_2}(\lambda) - \varphi_{\rho_1}(\lambda) \to 0$, *as* $\lambda \to 0$ [*hence, also* $\varphi_{\rho_2}(0^+) = \infty$];

(c) $\rho_2 = \rho_{20} + \rho_{21}$, $\rho_{21}(R^+) < \infty$, $\rho_{20} \le \rho_{d0}$, $0 \in \mathscr{S}_{\rho_{20}} \cap \mathscr{S}_{\rho_{21}}$ *and* $(\rho_{d0} - \rho_{20})(R^+) < \infty$.

PROOF OF THEOREM 3.1(a). Clearly, $\mathbf{I}_1$ implies that $0 \in \mathscr{S}_\rho$; hence, from (3.5), $\mathbf{\Sigma}(\mathscr{S}_{\tilde{M}}) = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$. Thus, it is sufficient to prove that $\mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ is a linear space and that $\mathscr{S}_{c_1\text{-Pois}(\sigma,\rho)} = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$. In fact, once this is done, the proof of (3.1) follows from (2.1), (2.8) and the fact that $\mu_\tau = \delta_{\{k(\tau)a_\sigma\}} * c_1\text{-Pois}(\sigma, \rho)$, where $k(\tau) = -\int_{R^+} s I_{(\tau,1]}(s)\rho(ds)$, if $0 < \tau < 1$, and $k(\tau) = -\int_{R^+} s I_{(1,\tau]}(s)\rho(ds)$, if $1 < \tau \le \infty$; and the fact that $a_\sigma \in \mathbf{C}(\mathscr{S}_\sigma) \subseteq \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ [see (2.4)]. Set $\mu \equiv c_1\text{-Pois}(\sigma, \rho)$; now we will prove (i) $\mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ is a linear space and (ii) $\mathscr{S}_\mu = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$. The proof is divided into two parts.

Part I. In this part we prove the above two assertions under the additional assumption that ρ has no atom near 0. First consider the case when $a_\sigma = \theta$; in this case we have $\tilde{\sigma} = \sigma$ and $\tilde{M} = M$ [see (2.3)]. Hence, using (2.4) and (3.5), we get (ii), that is,

$$\mathscr{S}_\mu = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) = \mathbf{C}(\mathscr{S}_\sigma). \tag{3.7}$$

Next we show that if $y \in \mathbf{B}^*$ and $y(x_0) \ne 0$, for some $x_0 \in \mathbf{L}(\mathscr{S}_\sigma)$, then $\mathscr{S}_{\mu \circ y^{-1}} = R$. As $y \not\equiv 0$ on $\mathbf{L}(\mathscr{S}_\sigma)$, we can find an $u_0 \in \mathscr{S}_\sigma$ with $y(u_0) \ne 0$; hence there exists an open neighborhood V of u_0 such that y is nonzero on V. Thus, since $u_0 \in \mathscr{S}_\sigma$, we have $\sigma(W) > 0$ and $|y| > 0$ on W, where $W = V \cap \mathscr{S}_\sigma$. It is easy to verify that $\mu \circ y^{-1}$ is i.d. with Lévy measure $M_y \equiv M \circ y^{-1}/R \setminus \{0\} = (\sigma \times \rho)\circ(y \circ \Psi)^{-1}/R \setminus \{0\}$. Therefore, we have

$$\begin{aligned}\int_{\{0<|s|<1\}} |s|\, dM_y &= \int_{\{x:\, y(x)\ne 0\}} |y(x)| I_{\{0<|s|\le 1\}}(y(x))\, dM \\ &\ge \int_W |y(u)| \left(\int_{(0,\, |y(u)|^{-1}]} s\rho(ds) \right) \sigma(du) = \infty,\end{aligned}$$

since the integral $\int_{(0,|y(u)|^{-1}]} s\rho(ds) = \infty$, by $\mathbf{I}_1$, and $|y(u)| > 0$, for every $u \in W$ and $\sigma(W) > 0$. Therefore, by [11], page 293, $\mathscr{S}_{\mu \circ y^{-1}} = R$.

Now we prove that $\mathscr{S}_\mu = \mathbf{L}(\mathscr{S}_\sigma)$. We already know $\mathscr{S}_\mu = \mathbf{C}(\mathscr{S}_\sigma) \subseteq \mathbf{L}(\mathscr{S}_\sigma)$; if there is a strict inclusion, then we can find $x_0 \in \mathbf{L}(\mathscr{S}_\sigma) \setminus \mathbf{C}(\mathscr{S}_\sigma)$, and, by the Hahn–Banach theorem,

$$(3.8) \qquad y_0 \in \mathbf{B}^* \quad \text{with} \quad y_0(x_0) < r \equiv \inf\{y_0(x): x \in \mathbf{C}(\mathscr{S}_\sigma)\}.$$

Then, since $\theta \in \mathbf{C}(\mathscr{S}_\sigma)$, we have $r \le 0$; hence $y_0(x_0) < 0$. Therefore, from above, $\mathscr{S}_{\mu \circ y_0^{-1}} = R$. On the other hand, by (3.7) and (3.8), $\mu\{x \in B: y_0(x) \ge r\} = 1$; this is a contradiction. This completes the proof of the two assertions, under the condition $a_\sigma = \theta$.

Now let $a_\sigma \ne \theta$, and set $\tilde{\mu} = c_1\text{-Pois}(\tilde{M}) = c_1\text{-Pois}(\tilde{\sigma}, \rho)$. Now, since $a_{\tilde{\sigma}} = \int_{\partial\Delta} u\,\tilde{\sigma}(du) = a_\sigma - a_\sigma = \theta$, it follows, from what we proved above, that $\mathscr{S}_{\tilde{\mu}}$ is a linear space and that

$$(3.9) \qquad \mathscr{S}_{\tilde{\mu}} = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) \; [= \mathbf{L}(\mathscr{S}_{\tilde{\sigma}})].$$

Next let $\gamma_1 = c_1\text{-Pois}(\|a_\sigma\|\delta_{\{1\}}, \rho)$ and $\nu_1 = \gamma_1 \circ f^{-1}$, where $f: R \to B$ is the map $f(s) = -s(a_\sigma/\|a_\sigma\|)$. Then, from $\mathbf{I}_1$ and [11], page 293, $\mathscr{S}_{\gamma_1} = R$; hence, using Lemma 1 of [7], $\mathscr{S}_{\nu_1} = a_\sigma R$. Now observe that $\nu_1 = c_1\text{-Pois}(\|a_\sigma\|\delta_{\{-a_\sigma/\|a_\sigma\|\}}, \rho)$ and that

$$\tilde{\mu} = c_1\text{-Pois}(\sigma, \rho) * c_1\text{-Pois}\big(\|a_\sigma\|\delta_{\{-a_\sigma/\|a_\sigma\|\}}, \rho\big) = \mu * \nu_1.$$

Therefore, by (3.9) and [5], page 307, we have

$$(3.10) \qquad \mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) = \mathscr{S}_{\tilde{\mu}} = \overline{\mathscr{S}_\mu + \mathscr{S}_{\nu_1}} = \overline{\mathscr{S}_\mu + a_\sigma R}.$$

Now observing that $a_\sigma \in \mathbf{C}(\mathscr{S}_\sigma)$ $[= \boldsymbol{\Sigma}(\mathscr{S}_M)]$ [see (2.4) and (3.5)], we have, by [12], page 352, $\mathscr{S}_\mu + ta_\sigma \subseteq \mathscr{S}_\mu$, for all $t \ge 0$, and, by [10], page 38 (where it is additionally required that ρ is continuous on R^+ and that $\mathscr{S}_\rho = [0, \infty)$, but these are not required for the proof of $\mathscr{S}_\mu - ta_\sigma \subseteq \mathscr{S}_\mu$, for $t \ge 0$), $\mathscr{S}_\mu - ta_\sigma \subseteq \mathscr{S}_\mu$, for all $t \ge 0$. Consequently, since $\mathscr{S}_\mu \subseteq \mathbf{L}(\mathscr{S}_\sigma) \subseteq \mathbf{L}(\mathscr{S}_{\tilde{\sigma}}) = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ [see (2.3), (2.8) and (3.9)], we have, from (3.10), that $\mathscr{S}_\mu = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$, completing the proof of Part I.

Part II. Now let ρ be arbitrary and let ρ_c and ρ_d denote the continuous and discrete parts of ρ; then, since $M(\rho_c)$ and $M(\rho_d)$ are dominated by M $[\equiv M(\rho)]$, both of these are Lévy measures. Since $M = M(\rho_c) + M(\rho_d)$, we have $\mu = \mu(\rho_c) * \mu(\rho_d)$. If $\varphi_{\rho_c}(0^+) = \infty$, then, from Part I, $\mathscr{S}_{\mu(\rho_c)}$ is a linear space and $\mathscr{S}_{\mu(\rho_c)} = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$. Therefore, since

$$\mathscr{S}_\mu = \overline{\mathscr{S}_{\mu(\rho_c)} + \mathscr{S}_{\mu(\rho_d)}} = \overline{\mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) + \mathscr{S}_{\mu(\rho_d)}},$$

and since $\mathscr{S}_{\mu(\rho_d)} \subseteq \mathbf{L}(\mathscr{S}_\sigma)$ [by (2.8)] $\subseteq \mathbf{L}(\mathscr{S}_{\tilde{\sigma}}) = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$, we have $\mathscr{S}_\mu = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ $(= \mathscr{S}_{\mu(\rho_c)})$. Thus, the only remaining case of interest is when $\varphi_{\rho_c}(0^+) < \infty$ and $\varphi_{\rho_d}(0^+) = \infty$. To take care of this case, let ρ_{d0} be the Lévy measure on R^+ constructed in Lemma 3.3 for the measure ρ_d. Then, observing that $\rho = \rho_{d0} + (\rho_c + (\rho_d - \rho_{d0}))$ (recall that $\rho_{d0} \le \rho_d$), we have that $M(\rho_{d0})$ and $M(\rho_{d'0})$ are both Lévy measures on $\mathbf{B}$, where $\rho_{d'0} = \rho_c + (\rho_d - \rho_{d0})$. Hence $\mathscr{S}_\mu = \overline{\mathscr{S}_{\mu(\rho_{d0})} + \mathscr{S}_{\mu(\rho_{d'0})}}$; thus, making use of similar arguments to those above, it follows that, in order to prove that $\mathscr{S}_\mu$ is a linear space and that $\mathscr{S}_\mu = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$, we need only prove these two assertions for $\mathscr{S}_{\mu(\rho_{d0})}$. We do this in the following:

Let ρ_1, ρ_2, ρ_{20} and ρ_{21} be the Lévy measures on R^+ constructed in Lemma 3.4 for the measure ρ_{d0}. Using (3.6) and the fact that $M(\rho_{d0})$ is a Lévy measure on $\mathbf{B}$, we have, from the contraction principle ([8], page 64; we thank Professor Jan Rosinski for bringing this contraction principle to our attention) that $M(\rho_1)$ is also a Lévy measure on $\mathbf{B}$. Thus, since ρ_1 is continuous on R^+ and $\varphi_{\rho_1}(0^+) = \infty$, we have, from Part I, that $\mathscr{S}_{\mu(\rho_1)}$ is a linear space and that $\mathscr{S}_{\mu(\rho_1)} = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$. Since $\rho_{21}(R^+) < \infty$ and $\rho_{20} \le \rho_{d0}$, both $M(\rho_{21})$ and $M(\rho_{20})$ [and, hence, also $M(\rho_2)$] are Lévy measures on $\mathbf{B}$. Hence, since $\mathscr{S}_{\rho_1} = \mathscr{S}_{\rho_2}$ and $\varphi_{\rho_2}(\lambda) - \varphi_{\rho_1}(\lambda) \to 0$ as $\lambda \to 0$, we have, by Lemma 3.2, that $\mathscr{S}_{\mu(\rho_2)} = \mathscr{S}_{\mu(\rho_1)}$. Thus $\mathscr{S}_{\mu(\rho_2)}$ is a linear space and is equal to $\mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$. Since $\rho_{21}(R^+) < \infty$, we have that $\mathscr{S}_{\mu(\rho_{21})} = \Sigma(\mathscr{S}_{M(\rho_{21})}) - \varphi_{\rho_{21}}(0^+)a_\sigma$ ([6], page 333); further, since $0 \in \mathscr{S}_{\rho_{21}} \cap \mathscr{S}_{\rho_{20}}$ we have, from Lemma 3.1, that $\Sigma(\mathscr{S}_{M(\rho_{21})}) = \Sigma(\mathscr{S}_{M(\rho_{20})})$. Therefore, since, by [12], page 352, $\mathscr{S}_{\mu(\rho_{20})} + \Sigma(\mathscr{S}_{M(\rho_{20})}) \subseteq \mathscr{S}_{\mu(\rho_{20})}$, and since $\mathscr{S}_{\mu(\rho_2)} = \overline{\mathscr{S}_{\mu(\rho_{20})} + \mathscr{S}_{\mu(\rho_{21})}}$ [recall $M(\rho_2) = M(\rho_{20}) + M(\rho_{21})$], we have that $\mathscr{S}_{\mu(\rho_2)} \subseteq \mathscr{S}_{\mu(\rho_{20})} - \varphi_{\rho_{21}}(0^+)a_\sigma$. Consequently, since $\mathscr{S}_{\mu(\rho_2)} = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ is a linear space and $a_\sigma \in \mathbf{C}(\mathscr{S}_\sigma) \subseteq \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ [recall (2.3) and (2.4)], we have $\mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) \subseteq \mathscr{S}_{\mu(\rho_{20})}$. But, by (2.4) and (2.8), $\mathscr{S}_{\mu(\rho_{20})} \subseteq \mathbf{L}(\mathscr{S}_\sigma) = \mathbf{L}(\mathscr{S}_{\tilde{\sigma}})$; therefore, since $\mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) = \mathbf{L}(\mathscr{S}_{\tilde{\sigma}})$, we have $\mathscr{S}_{\mu(\rho_{20})} = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}})$ [$\equiv \mathbf{L}(\mathscr{S}_\sigma)$]. Finally, since $\rho_{20} \le \rho_{d0}$, we have $\mathscr{S}_{\mu(\rho_{d0})} = \overline{\mathscr{S}_{\mu(\rho_{20})} + \mathscr{S}_{\mu(\rho_{d0}-\rho_{20})}}$. But, since $\mathscr{S}_{\mu(\rho_{20})} = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}}) = \mathbf{L}(\mathscr{S}_\sigma)$ and $\mathscr{S}_{\mu(\rho_{d0}-\rho_{20})} \subseteq \mathbf{L}(\mathscr{S}_\sigma)$, we have $\mathscr{S}_{\mu(\rho_{d0})} = \mathscr{S}_{\mu(\rho_{20})}$. This completes the proof of part (a) of the theorem. □

PROOF OF THEOREM 3.1(b). Clearly, we have $\mu_\tau = \delta_{\{-\varphi_\tau(0^+)a_\sigma\}} * \mu_0$; also, by (2.7), we have $\mathscr{S}_{\mu_0} = \bigcap_{m \ge 1}[\bigcup_{n \ge m} A_{\lambda_n}]$. Therefore, by Lemma 3.1, $\mathscr{S}_{\mu_0}$ and $\Sigma(\mathscr{S}_M)$ are convex cones and (3.2) holds. The proof of the stated equivalence now follows easily using the Hahn-Banach theorem as in the proof of Theorem 3.1(a) and is omitted. The proof of (3.3) follows trivially from (3.2), (2.5) and (2.8). □

PROOF OF LEMMA 3.1. Trivially, $\mathscr{S}_M \cup \{\theta\} \supseteq \mathscr{S}_{M_{\lambda_k}} \cup \{\theta\}$, $\forall\, k$, where $M_{\lambda_k} = M/\{\|x\| \ge \lambda_k\}$; consequently, $\Sigma(\mathscr{S}_M) \supseteq [\bigcup_{k \ge n} A_{\lambda_k}]$, for every $n \in N$. Further, since $\mathscr{S}_M \cup \{\theta\} \subseteq [\bigcup_{k \ge n} A_{\lambda_k}]$, $\forall\, n$, we also have $\Sigma(\mathscr{S}_M) \subseteq [\bigcup_{k \ge n} A_{\lambda_k}]$. Thus $\Sigma(\mathscr{S}_M) = \bigcap_{n \ge 1}[\bigcup_{k \ge n} A_{\lambda_k}] = [\bigcup_{k \ge m_0} A_{\lambda_k}]$, $\forall m_0 \in N$. Up to now no special property of M is used, and in fact, there is nothing new in this proof; it is included in [10], page 34. To see that $\Sigma(\mathscr{S}_M) = \mathbf{C}(\mathscr{S}_\sigma)$, we first observe that $\mathscr{S}_M = \{tu\colon t \in \mathscr{S}_\rho,\ u \in \mathscr{S}_\sigma\}$ (use Lemma 1 of [7], page 302). Therefore, as $0 \in \mathscr{S}_\rho$, $\Sigma(\mathscr{S}_M) = \{\bar{\Sigma} t_i u_i\colon t_i \in \mathscr{S}_\rho,\ u_i \in \mathscr{S}_\sigma\}$, where $\bar{\Sigma}$ denotes the sum over the finite elements. Thus, since, clearly, $\mathbf{C}(\mathscr{S}_\sigma) = \{\bar{\Sigma} t_i u_i\colon t_i \ge 0,\ u_i \in \mathscr{S}_\sigma\}$, we have that $\mathbf{C}(\mathscr{S}_\sigma) \supseteq \Sigma(\mathscr{S}_M)$. Now, let $t_0 > 0$ and $u_0 \in \mathscr{S}_\sigma$; then, as $0 \in \mathscr{S}_\rho$, we can choose $t_n \in \mathscr{S}_\rho$ and $k_n \in N$ such that $t_n k_n \to t_0$. Therefore, since $k_n t_n u_0 \in \Sigma(\mathscr{S}_M)$ and $k_n t_n u_0 \to t_0 u_0$, we have $t_0 u_0 \in \Sigma(\mathscr{S}_M)$. Thus, as $\Sigma(\mathscr{S}_M)$ is a closed semigroup, $\{\Sigma t_i u_i\colon t_i \geq 0,\ u_i \in \mathscr{S}_\sigma\}$ and, hence, $\mathbf{C}(\mathscr{S}_\sigma) = \{\overline{\Sigma t_i u_i}\colon t_i \geq 0,\ u_i \in \mathscr{S}_\sigma\} \subseteq \Sigma(\mathscr{S}_M)$, proving that $\mathbf{C}(\mathscr{S}_\sigma) = \Sigma(\mathscr{S}_M)$. □

Proof of Lemma 3.2. The hypothesis $\mathscr{S}_{\rho_1} = \mathscr{S}_{\rho_2}$ clearly implies that, for any sequence $\{\lambda_k\}$, with $1 > \lambda_1 > \cdots > \lambda_n > \cdots$ and $\lambda_n \downarrow 0$, $A_{\lambda_k}(M(\rho_1)) = A_{\lambda_k}(M(\rho_2))$ for all k (see Proposition 2.1 for the definition of A_{λ_k}'s). Now let $x \in \mathscr{S}_{\mu(\rho_1)}$ and $\varepsilon > 0$; then, by (2.6), there exists a subsequence $\{\lambda_{k(r)}\}$ of $\{\lambda_k\}$ and $x_r \in A_{\lambda_{k(r)}}(M(\rho_1))$ such that $\|x - (x_r - a_\sigma \varphi_{\rho_1}(\lambda_{k(r)}))\| < \varepsilon/2$, for large r. Then, observing that

$$\left\|x - \left(x_r - a_\sigma \varphi_{\rho_2}\left(\lambda_{k(r)}\right)\right)\right\| \leq \left\|x - \left(x_r - a_\sigma \varphi_{\rho_1}\left(\lambda_{k(r)}\right)\right)\right\| + \|a_\sigma\| \left|\varphi_{\rho_1}\left(\lambda_{k(r)}\right) - \varphi_{\rho_2}\left(\lambda_{k(r)}\right)\right|$$

and using the hypothesis $\varphi_{\rho_1}(\lambda) - \varphi_{\rho_2}(\lambda) \to 0$ as $\lambda \to 0$, we get that $\|x - (x_r - a_\sigma \varphi_{\rho_2}(\lambda_{k(r)}))\| < \varepsilon$, for large r; therefore, since $x_r \in A_{\lambda_{k(r)}}(M(\rho_2))$, for every r, we have $x \in \mathscr{S}_{\mu(\rho_2)}$. Thus $\mathscr{S}_{\mu(\rho_1)} \subseteq \mathscr{S}_{\mu(\rho_2)}$; similarly, $\mathscr{S}_{\mu(\rho_2)} \subseteq \mathscr{S}_{\mu(\rho_1)}$, showing that $\mathscr{S}_{\mu(\rho_1)} = \mathscr{S}_{\mu(\rho_2)}$. □

Proof of Lemma 3.3. Since $\varphi_{\rho_d}(0^+) = \infty$, we can choose distinct $0 < u_j^{(1)} < 1$, $j = 1, \dots, m_1$, such that $\sum_{j=1}^{m_1} u_j^{(1)} \rho_d(\{u_j^{(1)}\}) \geq 1$. Let $u_{j(1)}^{(1)} > \cdots > u_{j(m_1)}^{(1)}$ be the natural order of $u_j^{(1)}$'s and let $v_r = u_{j(r)}^{(1)}$, $r = 1, 2, \dots, m_1$. Now noting that $\int_{(0, v_{m_1})} s\rho_d(ds) = \infty$, with a similar argument, we get $v_{m_1+1} > \cdots > v_{m_2}$ with $\sum_{j=1}^{m_2} v_{m_1+j} \rho_d(\{v_{m_1+j}\}) \geq 2$. Continuing this process inductively, we get a sequence $\{v_r\}$ such that $1 > v_1 > \cdots > v_r > \cdots$, $v_r \downarrow 0$ and $\sum_{r=1}^{\infty} v_r \rho_d(\{v_r\}) = \infty$. Now if $v_r \rho_d(\{v_r\}) \to 0$ as $r \to \infty$, then we are done by taking $s_r = v_r$ and $p_r = \rho_d(\{v_r\})$. If not, then there must exist a subsequence $\{r_n\}$ of $\{r\}$ and $\varepsilon > 0$ such that $v_{r_n} \rho_d(\{v_{r_n}\}) \geq \varepsilon$, $\forall\ n$. In this case, it suffices to take $s_n = v_{r_n}$ and $p_n = \varepsilon(v_{r_n} n)^{-1}$, $\forall\ n$. □

Proof of Lemma 3.4. (The reader is advised to draw rough sketches of the functions φ's and ψ's to understand the proof.) Let

$$0 < \alpha_n < \min\left\{\frac{t_n}{3}, \frac{t_{n-1} - t_n}{2}, \frac{t_n - t_{n+1}}{2}\right\}, \quad \text{with } t_0 = 1;$$

set $f(s) = \sum_{n=1}^{\infty} (2\alpha_n)^{-1} p_n I_{T_n}(s)$ and define $\rho_1(ds) = f(s)\,ds$, $0 < s < \infty$, where $T_n = [t_n - \alpha_n,\ t_n + \alpha_n]$, $n \in N$. Next let $0 < \beta_n < \min\{1, p_n/4\}$, $q_n = p_n - 2\alpha_n \beta_n$, $\forall\ n \in N$ (note that $0 < q_n < p_n$). Let ρ_{20} and ρ_{21} denote, respectively, the measures on R^+ defined by $\sum_{n=1}^{\infty} q_n \delta_{\{t_n\}}$ and $g(s)\,ds$, where $g(s) = \sum_{n=1}^{\infty} \beta_n I_{T_n}(s)$, $s \in R^+$; finally, define $\rho_2 = \rho_{20} + \rho_{21}$.

Clearly, ρ_1 is a continuous measure on R^+, $\mathscr{S}_{\rho_2} = \mathscr{S}_{\rho_1}$ $(= \bigcup_n T_n \cup \{0\})$, $0 \in \mathscr{S}_{\rho_{21}}$. Trivially, $\rho_i(C) = 0$, $i = 1, 2$, where $C = R^+ \setminus \bigcup_n V_n$ and $V_n = (t_n - \alpha_n,\ t_n + \alpha_n)$. Next, observing that $\int_{T_n} \rho_{21}(ds) = 2\alpha_n \beta_n$, we have

$\rho_{21}(R^+) = 2\sum_{n=1}^{\infty}\alpha_n\beta_n < \infty$ (recall that $0 < \beta_n < 1$ and $\sum_{n=1}^{\infty}\alpha_n < \infty$). This, along with $0 < q_n < p_n$, shows that $\rho_{20} \le \rho_{d0}$, that ρ_{20} is a Lévy measure on R^+ and that $(\rho_{d0} - \rho_{20})(R^+) < \infty$. Now note that

$$\int_{T_n} s\rho_1(ds) = t_n p_n \quad \text{and}$$
$$\int_{T_n} s^2\rho_1(ds) = (2\alpha_n)^{-1}p_n\left[2\alpha_n^3 + 6t_n^2\alpha_n\right] \le 4t_n^2 p_n; \tag{3.11}$$

therefore, since $\rho_{d0} = \sum_{n=1}^{\infty}p_n\delta_{\{t_n\}}$ is a Lévy measure on R^+ and $\sum_{n=1}^{\infty}t_n p_n = \infty$, we have that ρ_2 is a Lévy measure on R^+ [recalling that $\rho_2(C) = 0$] and that $\varphi_{\rho_1}(0^+) = \infty$. Further, the first equation in (3.11), the facts that $\int_{T_n} s\rho_2(ds) = q_n t_n + \int_{T_n} sg(s)\,ds = q_n t_n + 2\alpha_n\beta_n t_n = t_n p_n$ and $\int_{T_n} s\rho_1(ds) = p_n$ and that $\rho_i(C) = 0$ together imply

$$\varphi_{\rho_{d0}}(\lambda) = \varphi_{\rho_1}(\lambda) = \varphi_{\rho_2}(\lambda) \quad \text{and} \quad \psi_{\rho_{d0}}(\lambda) = \psi_{\rho_1}(\lambda), \qquad \forall\, \lambda \in C \tag{3.12}$$

and

$$\varphi_{\rho_{d0}}(t_n + \alpha_n) < \varphi_{\rho_1}(\lambda), \qquad \varphi_{\rho_2}(\lambda) < \varphi_{\rho_{d0}}(t_n + \alpha_n) + t_n p_n, \qquad \forall\, \lambda \in V_n. \tag{3.13}$$

Therefore, since $\varepsilon_n = t_n p_n \to 0$, it follows, from (3.13) and the first equation in (3.12), that $\lim_{\lambda\to 0}(\varphi_{\rho_1}(\lambda) - \varphi_{\rho_2}(\lambda)) = 0$. The proof is thus completed except for the last part in (a), which we prove now.

From the second equation in (3.12) and the nonincreasing property of $\psi_{\rho_{d0}}$, we have that, if $\lambda \in C$, then $\psi_{\rho_1}(\lambda) \le \psi_{\rho_{d0}}(2^{-1}\lambda)$. If $\lambda \in (t_n - \alpha_n, t_n]$, then $\psi_{\rho_1}(\lambda) \le \psi_{\rho_1}(t_n - \alpha_n) = \psi_{\rho_{d0}}(t_n - \alpha_n) = \psi_{\rho_{d0}}(\lambda) \le \psi_{\rho_{d0}}(2^{-1}\lambda)$, by the nonincreasing property of ψ_{ρ_1} and $\psi_{\rho_{d0}}$, (3.12) and the facts that $t_{n-1} < (t_n - \alpha_n)$ and that ρ_{d0} has jumps only at t_n's. Next, if $t_n < \lambda < (t_n + \alpha_n)$, then, since $2^{-1}(t_n + \alpha_n) < (t_n - \alpha_n)$ (recall $\alpha_n < t_n/3$), we have $2^{-1}\lambda < (t_n - \alpha_n)$. Therefore, using the nonincreasing property of $\psi_{\rho_{d0}}$ and ψ_{ρ_1} and (3.12) again, $\psi_{\rho_{d0}}(2^{-1}\lambda) \ge \psi_{\rho_{d0}}(t_n - \alpha_n) = \psi_{\rho_1}(t_n - \alpha_n) \ge \psi_{\rho_1}(\lambda)$. □

4. The supports of Poisson probability measures on *E*. Let μ be a nondegenerate τ-regular probability measure on $\mathbf{E}$ (μ is defined on the Borel σ-algebra of the topology of $\mathbf{E}$); and let $\mathbf{E}_0 \equiv \mathbf{E}_0(\mu)$ be defined as in Proposition 2.1(b). Define $\mathscr{S}_\mu$, the *support* of μ, as in Section 2. Let $\mathscr{P}_{\mathbf{E}}$ be a fixed family of seminorms generating the (l.c.) topology of $\mathbf{E}$, and let $\mathscr{P} = \{p \in \mathscr{P}_{\mathbf{E}},\ p \not\equiv 0$ on $\mathbf{E}_0\}$. Then, $\forall\ p \in \mathscr{P}$, the normed space $(\mathbf{E}_0/(\{p = 0\} \cap \mathbf{E}_0), \|\cdot\|_p)$ and, hence, also its completion $\mathbf{B}_p$, is *separable*, where $\|\cdot\|_p$ is the natural induced norm. Furthermore

$$\mu(\mathbf{E}_0) = 1, \qquad \mathscr{S}_\mu = \bigcap_{\mathscr{P}} \pi_p^{-1}(\mathscr{S}_{\mu_p}), \tag{4.1}$$

$\mathscr{S}_{\mu_p} \equiv$ the support of $\mu_p \equiv \mu_0 \circ \pi_p^{-1}$, where $\mu_0 = \mu/\mathbf{E}_0$ and π_p is the natural projection of $\mathbf{E}_0$ into $\mathbf{B}_p$ (see [7], pages 299–300, and [10], page 29, for details on these results).

Now assume that μ is a weakly Poisson type (resp., an α-stable, a strictly α-stable) probability measure, that is, each finite-dimensional projection of μ is a translated Poisson (resp., an α-stable, a strictly α-stable) probability measure. It follows that μ_p is a translated centered Poisson (resp., an α-stable, a strictly α-stable) probability measure on $\mathbf{B}_p$ (the result in [9], page 313, is needed here). In the following, the Lévy measure of μ_p will be denoted by M_p, the statement that $M_p \equiv \Psi(\sigma_p \times \rho_p)$ will have a similar meaning as in Section 2, and the notations $\tilde{\sigma}_p$ and $\tilde{M}_p$ will denote measures defined as in (2.3) with σ and M replaced, respectively, by σ_p and M_p. Further, if μ_p is symmetric, we will take M_p and σ_p also symmetric. Finally, if μ_p is degenerate, we will take, by convention, $\sigma_p = M_p = \tilde{\sigma}_p = \tilde{M}_p \equiv 0$ and $\mathbf{\Sigma}(\mathscr{S}_{M_p}) = \mathbf{C}(\mathscr{S}_{\sigma_p}) = \mathbf{\Sigma}(\mathscr{S}_{\tilde{M}_p}) = \mathbf{C}(\mathscr{S}_{\tilde{\sigma}_p}) \equiv \{\theta\}$.

Using the above notation and conventions, we now state the main result and a corollary. The proofs of these follow immediately using the above facts, (4.1), Theorem 3.1 and Corollary 3.1, and are, therefore, omitted.

THEOREM 4.1. *Let μ be a τ-regular weakly Poisson type probability measure on $\mathbf{E}$ and assume $M_p = \Psi(\sigma_p \times \rho_p)$, $\forall\ p \in \mathscr{P}$. Then we have the following*:

(a) *If $\varphi_{\rho_p}(0^+)$ $[\equiv \int_{(0,1]} s\rho_p(s)\,ds] = \infty$, whenever $\mu_p \equiv \delta_{\{x_p\}} * c_1\text{-Pois}(\sigma_p, \rho_p)$ is nondegenerate, then $\bigcap_{\mathscr{P}} \pi_p^{-1}(\mathbf{\Sigma}(\mathscr{S}_{\tilde{M}_p}))$ and $\bigcap_{\mathscr{P}} \pi_p^{-1}(\mathbf{C}(\mathscr{S}_{\tilde{\sigma}_p}))$ are equal linear spaces (denoted $\mathscr{L}_\mu$) and*

$$\mathscr{S}_\mu = b_\mu + \mathscr{L}_\mu, \quad \text{for any } b_\mu \in \bigcap_{\mathscr{P}} \pi_p^{-1}\Big(x_p + \mathbf{\Sigma}\big(\mathscr{S}_{\tilde{M}_p}\big)\Big). \tag{4.2}$$

(b) *If $\mu_p = c_0\text{-Pois}(\sigma_p, \rho_p)$, $\forall\ p \in \mathscr{P}$, and if $\varphi_{\rho_p}(0^+) < \infty$ and $0 \in \mathscr{S}_{\rho_p}$ whenever μ_p is nondegenerate, then $\mathscr{S}_\mu$ is a convex cone and*

$$\mathscr{S}_\mu = \bigcap_{\mathscr{P}} \pi_p^{-1}\Big(\mathbf{\Sigma}\big(\mathscr{S}_{M_p}\big)\Big) = \bigcap_{\mathscr{P}} \pi_p^{-1}\Big(\mathbf{C}\big(\mathscr{S}_{\sigma_p}\big)\Big), \tag{4.3}$$

and $\mathscr{S}_\mu = \mathbf{E}_0(\mu) \Leftrightarrow \forall\ y \in \mathbf{E}^*$, which is nonzero on $\mathbf{E}_0$, $\mathscr{S}_{\mu \circ y^{-1}} = R$.

(c) *If μ is symmetric and if, $\forall\ p$ for which μ_p is nondegenerate, either $\rho_p(0^+) = \infty$ or $\rho_p(0^+) < \infty$ and $0 \in \mathscr{S}_{\rho_p}$, then the convex cones in (4.3) are linear spaces and each is equal to $\mathscr{S}_\mu$.*

COROLLARY 4.1. *Let μ be a τ-regular weakly α-stable probability measure on $\mathbf{E}$ so that $M_p = \psi(\sigma_p \times s_{ds}^{-(1+\alpha)})$. Then we have the following*:

(a) *If $1 \le \alpha < 2$, then $\mathscr{S}_\mu = b_\mu + \mathscr{L}_\mu$ [see (4.2)]; further, if $1 < \alpha < 2$ and μ is strictly stable, then $\mathscr{S}_\mu = \mathscr{L}_\mu$.*

(b) *If $0 < \alpha < 1$ and if μ is strictly stable, then $\mathscr{S}_\mu$ is equal to either of the equal cones appearing in (4.3), otherwise it is a translate (by x_0) of any of these equal cones, where x_0 is the centering element of μ. Further, the analog of the last statement in Theorem 4.1(b) holds.*

(c) *If μ is symmetric, then the convex cones in (4.3) are linear spaces and each is equal to $\mathscr{S}_\mu$.*

REMARK 4.2. Using the same methods as noted above, versions of both of these results can be proved for weakly Poisson type and weakly α-stable probability measures μ which are defined on the *cylinder σ-algebra* (rather than on the Borel σ-algebra) of $\mathbf{E}$. In this case, the hypothesis of τ-regularity of μ is replaced by "$\mathbf{E}$ is μ-reductible", and the support of μ is defined as in [9], pages 27–28. These results were included in an earlier version of the paper, but are excluded from this version for brevity. Theorem 3.1(b) and the just-noted versions of Theorem 4.1(b) and Corollary 4.1(b) complement Proposition 4 and Theorem 1 of [10]. However, since the proof of these [except perhaps for proving that $\mathscr{S}_\mu$ is a convex cone in Theorem 3.1(b)] follow directly from Proposition 4 of [10], we do not claim any credit for these.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
121 AYRES HALL
KNOXVILLE, TENNESSEE 37996-1300