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LOCAL TIMES FOR SUPERDIFFUSIONS

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In this work we study local times for a class of measure-valued Markov processes known as superprocesses. We begin by deriving analogues of well-known properties of ordinary local times. Then, restricting our attention to a class of superprocesses (which includes the important case of super-Brownian motion), we prove more detailed properties of the local times, such as joint continuity and a global Hölder condition. These are then used to obtain path properties of the superprocesses themselves. For example, we compute the Hausdorff dimension of the "level sets" of super-Brownian motion.

1. Introduction. In this paper we study local times for a class of measure-valued Markov processes known as superprocesses. Let ξ_t be an $\mathbb{R}^d$-valued Feller Markov process with infinitesimal generator A and write $M_F(\mathbb{R}^d)$ for the space of (positive) finite Borel measures on $\mathfrak{B}(\mathbb{R}^d)$, equipped with the weak topology. A continuous $M_F(\mathbb{R}^d)$-valued Markov process X_t on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is called a (Dawson–Watanabe) *superprocess* over ξ (or a *super-ξ process*) provided its Laplace functional has the representation

$$E_\mu[e^{-\langle X_t, \phi\rangle}] = e^{-\langle \mu, u(t)\rangle},$$

for all $\phi \in C_b(\mathbb{R}^d)_+$, $\mu \in M_F(\mathbb{R}^d)$ and $t \geq 0$, where $u(t) \equiv u(t,x)$ is the unique mild solution [cf. Iscoe (1986a)] of the nonlinear initial value problem

$$\frac{\partial u}{\partial t} = Au - \frac{1}{2}u^2, \qquad u(0,x) = \phi(x).$$

[Here the expectation is with respect to the probability measure P_μ, which gives the law of the process starting with initial measure μ. We write $\langle \mu, \phi\rangle \equiv \int_{\mathbb{R}^d}\phi(x)\mu(dx)$, and $C_b(\mathbb{R}^d)$ denotes the space of bounded continuous functions on $\mathbb{R}^d$.] These processes arise as solutions of measure-valued martingale problems and as high-density limits of systems of critical branching $\mathbb{R}^d$-valued Markov processes [cf. Dawson (1978), Ethier and Kurtz (1986) and Roelly-Coppoletta (1986)]. Adler and Lewin (1990) give a nice summary of the above.

Much of the work on these processes has focused on *super-Brownian motion* (i.e., ξ is Brownian motion). Here it is known that, for $d \geq 2$, the random measure X_t is singular for all $t > 0$, almost surely [cf. Perkins (1988)]. Moreover, its (Lebesgue-null, random) closed support $S(X_t)$ has Hausdorff

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dimension 2 and is compact [Dawson, Iscoe and Perkins (1989)]. Perkins (1989) has shown that, for all $t > 0$, the mass of X_t is, essentially, uniformly distributed over $S(X_t)$ according to a deterministic Hausdorff measure. This allows one to get path properties of the measure-valued process by studying the set-valued process $S(X_t)$. See Dawson, Iscoe and Perkins (1989) for numerous deep and beautiful results of this type. The special (and qualitatively different) case of $d = 1$ is handled, for example, in Konno and Shiga (1988) and Reimers (1989), where they show that X has a jointly continuous density $x(t, y)$ for all $t > 0$, a.s. So, in particular, $S(X_t)$ has positive Lebesgue measure when $d = 1$.

To define local times (i.e., occupation densities) for measure-valued processes, we must first give the appropriate notion of occupation time. This role is filled by the *weighted occupation time* [cf. Iscoe (1986a)], a measure-valued process Y_t defined by

$$Y_t(\Gamma) \equiv \int_0^t X_s(\Gamma)\, ds, \qquad \Gamma \in \mathfrak{B}(\mathbb{R}^d) \tag{1.1}$$

or, more generally, for any $B \in \mathfrak{B}(\mathbb{R}_+)$,

$$Y_B(\Gamma) \equiv \int_B X_s(\Gamma)\, ds. \tag{1.2}$$

[$\mathfrak{B}(S)$ denotes the Borel sets in S.]

For a fixed Borel set B in $\mathbb{R}_+$, the *local time*, $\alpha(x, B) = \alpha(x, \omega, B)$, of X over B is the density, when it exists, of $Y_B(dx)$ with respect to Lebesgue measure on $\mathbb{R}^d$:

$$\int_B X_s(\Gamma)\, ds = \int_\Gamma \alpha(x, B)\, dx, \qquad \forall\, \Gamma \in \mathfrak{B}(\mathbb{R}^d). \tag{1.3}$$

It is easy to see that the local time also satisfies

$$\int_B \langle X_s, f\rangle\, ds = \int_{\mathbb{R}^d} f(x)\alpha(x, B)\, dx, \tag{1.4}$$

for all bounded Borel-measurable functions f on $\mathbb{R}^d$. We will often write $\alpha_t(x)$ for $\alpha(x, [0, t])$.

Iscoe (1986b) and Sugitani (1989) showed that local times exist for super-Brownian motion when $d = 1, 2, 3$, and Dynkin (1988) proved their existence for a large class of superprocesses, which includes super-Brownian motion when $d \le 3$. Sugitani also showed that $\alpha_t(x)$ is jointly continuous in the case of super-Brownian motion. In this paper, we will prove joint continuity of the local times for a class of superprocesses which, for the moment, we shall simply call *superdiffusions*. These arise when the underlying motion is a diffusion process; super-Brownian motion is, of course, a special case. We also derive other properties of these local times, the most important of which is a global Hölder condition in the time variable. This will be used to study path properties of the superdiffusions themselves.

Before getting into the main results, let us derive some simple properties of superprocess local times, some of which will be useful later in the paper. These are analogues of properties that we have come to expect from ordinary local times [cf. Geman and Horowitz (1980)]. Unless further conditions are stated, we will be assuming simply that $\alpha_t(x)$ is a local time for an arbitrary weakly continuous superprocess on $\mathbb{R}^d$ with initial mass μ. We will often supress the ubiquitous P_μ-a.s. clause in the theorems and proofs.

First note that, by standard results, a version of $\alpha(x, \omega, B)$ can be chosen so that it is a *kernel* on $\mathbb{R}^d \times \Omega \times \mathfrak{B}(\mathbb{R}_+)$ and $\alpha_0(x, \omega) = 0$. Consequently, $t \mapsto \alpha_t(x, \omega)$ is increasing and right continuous. From now on, whenever we speak of the local time, we mean this nice version. Since $X_t \in M_F(\mathbb{R}^d)$ and $t \mapsto X_t(\mathbb{R}^d)$ is continuous (weak continuity of X_t), it follows from (1.3) that $\alpha(x, \omega, dt)$ may be chosen to be a *Radon* measure (finite on compact sets) on $\mathbb{R}_+$.

THEOREM 1.1. *For any interval B in* $[0, \infty)$ *and any bounded Borel function* f *on* $\mathbb{R}_+ \times \mathbb{R}^d$,

$$(1.5) \qquad \int_B \int_{\mathbb{R}^d} f(t, x) X_t(dx)\, dt = \int_{\mathbb{R}^d} \int_B f(t, x) \alpha(x, dt)\, dx, \quad a.s.$$

PROOF. This is obvious when f is an indicator function of a product set in $\mathbb{R}_+ \times \mathbb{R}^d$ because of (1.3). A monotone class argument does the rest. □

THEOREM 1.2. *For a.e.* $t > 0$,

$$X_t(\{x: \alpha(x, [t - \varepsilon, t + \varepsilon]) = 0 \text{ for some } \varepsilon > 0\}) = 0,$$

that is, for a.e. t, X_t *puts* 0 *mass on the collection of all points* x *whose local times* $\alpha_\cdot(x)$ *do not have* t *as a point of strict increase.*

PROOF. In (1.5), put

$$f(t, x) = I_{\{\alpha(x, [t - \varepsilon, t + \varepsilon]) = 0, \text{ some } \varepsilon > 0\}}$$

and then notice that $\{t: \alpha(x, [t - \varepsilon, t + \varepsilon]) = 0, \text{ some } \varepsilon > 0\}$ is the complement of the support of $\alpha(x, \cdot\,)$. □

Define the *x-level set* M_x of X by

$$M_x \equiv \{t \in [0, \infty): x \in S(X_t)\}.$$

This is the (random) set of times at which the support "hits" x. We sometimes write $M_x^{\mathbf{T}}$ for $M_x \cap \mathbf{T}$, when we want to emphasize a particular time set $\mathbf{T}$.

THEOREM 1.3. *For almost every* x, *the measure* $\alpha(x, dt)$ *lives on* M_x [*i.e.*, $t \mapsto \alpha_t(x)$ *can only increase on* M_x].

PROOF. Put $f(t, x) \equiv I_{\{t \in M_x^c\}} = I_{\{x \notin S(X_t)\}}$ in (1.5) to see that $\alpha(x, M_x^c) = 0$ for a.e. x. (The "c" denotes complement.) □

For a fixed interval $\mathbf{T}$ in $\mathbb{R}_+$, define the *zero set* of the local time $\alpha(x, \mathbf{T})$ by

$$\mathscr{Z}(\mathbf{T}) \equiv \{x \in \mathbb{R}^d \colon \alpha(x, \mathbf{T}) = 0\}.$$

Notice that $X_t(\mathscr{Z}(\mathbf{T})) = 0$ for a.e. $t \in \mathbf{T}$ [by (1.3)].

Below, we write λ_d for Lebesgue measure on $\mathbb{R}^d$.

THEOREM 1.4. *If $\lambda_d(S(X_t)) = 0$ for a.e. $t > 0$, then $\lambda_1(M_x) = 0$ for a.e. x. On the other hand, if X has a jointly continuous density $x(t, z)$, then $\lambda_1(M_x^{\mathbf{T}}) > 0$ for all $x \in \{z\colon x(t, z) > 0$ for some $t \in \mathbf{T}\}$. [So, e.g., if X is super-Brownian motion and $\mu \in M_F(\mathbb{R}^d)$, then $\lambda_1(M_x) = 0$ for a.e. $x \in \mathbb{R}^d$, when $d \geq 2$, and $\lambda_1(M_x^{\mathbf{T}}) > 0$ for all $x \in \{z\colon x(t, z) > 0$ for some $t \in \mathbf{T}\}$, when $d = 1$.]*

PROOF. If $\lambda_d(S(X_t)) = 0$ for a.e. $t > 0$, then

$$\int_{\mathbb{R}^d} \lambda_1(M_x)\, dx = \int_{\mathbb{R}^d}\int_0^\infty I_{M_x}(t)\, dt\, dx = \int_0^\infty \int_{\mathbb{R}^d} I_{S(X_t)}(x)\, dx\, dt = 0.$$

To prove the second statement, suppose $x(t, z) > 0$ for some $(t, z) \in \mathbf{T} \times \mathbb{R}^d$. By joint continuity, the density must be positive for all points in some neighborhood of (t, z). In particular, for any ball $B_\varepsilon(z)$ of radius ε about z,

$$X_t(B_\varepsilon(z)) = \int_{B_\varepsilon(z)} x(t, y)\, dy > 0.$$

So X_t charges every neighborhood of z, which is the same as saying $z \in S(X_t)$, or $t \in M_z^{\mathbf{T}}$. But this must happen for all times in some open interval $B \subseteq \mathbf{T}$ which contains t, forcing $B \subseteq M_z^{\mathbf{T}}$ and proving the assertion. □

REMARK. In the case when a jointly-continuous density (and hence a local time) exists,

$$\{z\colon x(t, z) > 0 \text{ for some } t \in \mathbf{T}\} = \mathscr{Z}(\mathbf{T})^c.$$

To prove this, notice that Fubini's theorem implies $\alpha(z, \mathbf{T}) = \int_{\mathbf{T}} x(s, z)\, ds$, and then use the fact that $t \mapsto x(t, z)$ is continuous. So, in this case, Theorem 1.4 reads: If X has a jointly continuous density, then $\lambda_1(M_x^{\mathbf{T}}) > 0$ for all $x \in \mathscr{Z}(\mathbf{T})^c$.

This case points out a sharp difference between superprocess local times and ordinary local times. For ordinary processes, existence of a local time implies that the Lebesgue measure of a level set is 0.

THEOREM 1.5. *For fixed x, if $\alpha(x, \mathbf{T}) > 0$ and $t \mapsto \alpha_t(x)$ is continuous, then $M_x^{\mathbf{T}}$ is uncountable.*

PROOF. By continuity, $\alpha(x, dt)$ has no atoms. Since this measure is supported on $M_x^{\mathbf{T}}$, it follows that, if $M_x^{\mathbf{T}}$ is countable, then it must be contained in $\mathscr{Z}(\mathbf{T})$. □

See Section 2 for more information on the size of $M_x^{\mathbf{T}}$ when $\alpha(x, \mathbf{T}) > 0$ and X belongs to a particular class of superprocesses.

The *closed range* of X over $\mathbf{T}$ is defined as

$$\overline{R}(\mathbf{T}) \equiv \overline{\bigcup_{t \in \mathbf{T}} S(X_t)}.$$

As far as X is concerned, during the interval $\mathbf{T}$, this is where all the action takes place. Not surprisingly, it is also where the local time lives (as a function of x).

THEOREM 1.6. *P_μ-a.s., $\alpha(x, \mathbf{T}) = 0$ for a.e. $x \in \overline{R}(\mathbf{T})^c$.*

PROOF. From (1.3) we have $\int_{\overline{R}(\mathbf{T})^c} \alpha(x, \mathbf{T})\, dx = \int_{\mathbf{T}} X_t(\overline{R}(\mathbf{T})^c)\, dt = 0$. □

REMARK. When $x \mapsto \alpha(x, \mathbf{T})$ is *continuous*, we can say a bit more.

(i) First of all, note that the above theorem will hold for *all* $x \in \overline{R}(\mathbf{T})^c$ because $\overline{R}(\mathbf{T})^c$ is open. Thus, we have $\mathscr{G}(\mathbf{T})^c \subseteq \overline{R}(\mathbf{T})$. In fact, since $\mathscr{G}(\mathbf{T})^c$ is open, $\mathscr{G}(\mathbf{T})^c \subseteq \overline{R}(\mathbf{T})^o$. It would be interesting to know if the reverse inclusion holds.

(ii) Equation (1.3) implies that, for a.e. $t \in \mathbf{T}$, X_t puts zero mass on the closed set $\overline{R}(\mathbf{T}) \setminus \mathscr{G}(\mathbf{T})^c$ [i.e., $X_t(\overline{R}(\mathbf{T}) \cap \mathscr{G}(\mathbf{T})) = 0$]. In fact, $\mathscr{G}(\mathbf{T})$ is *nowhere dense* in $\overline{R}(\mathbf{T})$. To see this, we must show that there is no nonempty open ball $B_\varepsilon(z)$ contained in $\overline{R}(\mathbf{T}) \cap \mathscr{G}(\mathbf{T})$. Suppose there is, and pick a function $f \in C_b(\mathbb{R}^d)$ which has support equal to $B_\varepsilon(z)$ and takes value 1 on $B_{\varepsilon/2}(z)$. Then $0 = \int_{B_\varepsilon(z)} f(x)\alpha(x, \mathbf{T})\, dx = \int_{\mathbf{T}} \langle X_t, f \rangle\, dt$, and hence $\langle X_t, f \rangle = 0$ for all $t \in \mathbf{T}$, by weak continuity of X. From this it follows that $X_t(B_{\varepsilon/2}(z)) = 0$ for all $t \in \mathbf{T}$. This tells us, in particular, that $B_{\varepsilon/4}(z) \cap [\bigcup_{t \in \mathbf{T}} S(X_t)] = \varnothing$, since for x to be in $S(X_t)$, it must be that X_t charges every open ball about x. But this forces $z \notin \overline{R}(\mathbf{T})$, contradicting the assumption that $B_\varepsilon(z) \subseteq \overline{R}(\mathbf{T})$.

(iii) We also have that $\overline{R}(\mathbf{T})$ has *nonempty interior*, P_μ-a.s. (no matter how small the interval $\mathbf{T}$ is), as long as $X_t(\mathbb{R}^d) > 0$ for some $t \in \mathbf{T}$. To see this, recall that $X_t(\mathbb{R}^d)$ is a continuous function of t. Thus $0 < \int_{\mathbf{T}} X_s(\mathbb{R}^d)\, ds = \int_{\mathbb{R}^d} \alpha(x, \mathbf{T})\, dx$, so that $\mathscr{G}(\mathbf{T})^c$ is a *nonempty* open set in $\overline{R}(\mathbf{T})$.

In the case of super-Brownian motion starting at $\mu \in M_F(\mathbb{R}^d)$, we can say a bit more about the structure of M_x. Let us write S_t for $S(X_t)$, and $S_{t-} \equiv \{x: \exists\, x_n \in S_{t_n}\ \forall\, n$, with $x_n \to x$, for some sequence $t_n \to t-\}$. It is known [cf. Perkins (1990), pages 457–458] that, a.s., $S_t \subseteq S_{t-}$, $\{S_t\}$ is right continuous, and $\bigcup_{t>0}(S_{t-} \setminus S_t)$ is countable. Using this, it is easy to see that, P_μ-a.s.,

(1.6) $\qquad M_x$ is a closed set for all but countably many x.

Evans and Perkins (1991), Theorem 3.3] show that

$$\{x\} \cap \left(\bigcup_{t>0} (S_{t-} \setminus S_t) \right) = \varnothing, \qquad P_\mu\text{-a.s.},$$

for all $\mu \in M_F(\mathbb{R}^d)$. It follows that, for each fixed x,

$$M_x \text{ is closed}, \qquad P_\mu\text{-a.s.} \tag{1.7}$$

From (1.6), we have that $M_x^F \equiv M_x \cap F$ is closed for all but countably many x, a.s., when $F \subseteq [0, \infty)$ is closed. Finally, when $d \geq 2$, $P_\mu(x \notin S_t$ for a.e. $x) = 1$, since S_t has Lebesgue-measure 0. Hence, for any countable collection of times $\{t_i\}$, $P_\mu(x \notin S_{t_i}\ \forall\ i$, for a.e. $x) = 1$. Using this, it follows that, if $d \geq 2$,

$$P_\mu\big(M_x^B \text{ is closed for a.e. } x\big) = 1 \tag{1.8}$$

provided the Borel set B differs from a closed set by at most countably many points.

We conclude this section by mentioning that most of what was done above holds (with minor measurability conditions) for *any* weakly continuous measure-valued random process having an occupation density. In only a few places did we use any special properties of superprocesses.

2. Main results. We begin by specifying the class of superprocesses that we will be working with. Let ξ_t, $t \geq 0$, be the solution of the stochastic differential equation

$$d\xi_t = \sigma(\xi_t)\, dW_t + b(\xi_t)\, dt,$$

where the following assumptions are made on the coefficients. Assume that $\sigma = (\sigma_{ij})$ and $b = (b_i)$, $i, j = 1, \ldots, d$, are bounded (matrix-valued and vector-valued) functions. Let $a_{ij}(x) = \sum_{k=1}^d \sigma_{ik}(x)\sigma_{jk}(x)$ and suppose the differential operator A, given by

$$Af(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i\, \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x), \qquad f \in C_b^2(\mathbb{R}^d),$$

[$C_b^2(\mathbb{R}^d)$ denotes the family of bounded continuous functions on $\mathbb{R}^d$ with two bounded continuous derivatives] satisfies:

(i) The functions

$$a_{ij},\ b_i,\ \frac{\partial a_{ij}}{\partial x_i},\ \frac{\partial^2 a_{ij}}{\partial x_i\, \partial x_j},\ \frac{\partial b_i}{\partial x_i}$$

are bounded and Hölder continuous on $\mathbb{R}^d$, $i, j = 1, \ldots, d$.

(ii) There is a positive constant γ such that

$$\sum_{i,j=1}^d a_{ij}(x)\lambda_i\lambda_j \geq \gamma \sum_{i=1}^d \lambda_i^2$$

for any real $\lambda_1, \ldots, \lambda_d$ and all $x \in \mathbb{R}^d$.

Then ξ is an $\mathbb{R}^d$-valued diffusion with generator A. We call such a process a *smooth uniformly elliptic diffusion*.

Let X_t, $t \geq 0$, be the superprocess over ξ with values in $M_F(\mathbb{R}^d)$. We call this process a *smooth uniformly elliptic superdiffusion*. Of course, super-

Brownian motion belongs to this class. Dynkin (1988) proved that these superprocesses (and some others) have local times, when $d = 1, 2, 3$, as long as the initial mass distribution μ belongs to $M_{F,B}(\mathbb{R}^d)$, the set of measures in $M_F(\mathbb{R}^d)$ having bounded density with respect to Lebesgue measure, say $\mu(dx) \le K\,dx$ for some constant K.

For the remainder of this section, we shall assume that $\alpha(x, B)$ is a local time for a smooth uniformly elliptic superdiffusion X_t with initial mass $\mu \in M_{F,B}(\mathbb{R}^d)$, $d \le 3$. Recall that λ_d denotes Lebesgue measure on $\mathbb{R}^d$. We are now ready to state the main results of this paper. In the case of super-Brownian motion, the first two results are implicit in Sugitani (1989), whose methods play a major role in our proofs.

THEOREM 2.1. *The local time is jointly continuous P_μ-a.s., that is, $(t, a) \mapsto \alpha_t(a)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^d$. Furthermore, it satisfies the following local Hölder condition: On any rectangle $\mathbf{R} = [0, T] \times [-m, m]^d \subseteq \mathbb{R}_+ \times \mathbb{R}^d$, if $\nu \in (0, \min\{2 - d/2, 1\})$, then there is a random variable Δ and a constant $D = D(\mathbf{R})$, both positive, such that, with probability* 1,

$$|\alpha_t(y) - \alpha_s(x)| \le D|(t,y) - (s,x)|^\nu,$$

for all $(t, y), (s, x) \in \mathbf{R}$ satisfying $|(t, y) - (s, x)| < \Delta$. ($|\cdot|$ denotes the appropriate Euclidean norm.)

For certain applications (cf. Theorem 2.8), one needs a Hölder condition in the time variable which holds *uniformly* in the space variable. This is the purpose of the next result.

THEOREM 2.2. *The following global Hölder condition is satisfied for any fixed finite interval $\mathbf{T}$ in $\mathbb{R}_+$: If $\nu \in (0, \min\{2 - d/2, 1\})$, then there exists a positive random variable η and a positive constant C such that, with P_μ-probability* 1, *whenever $s, t \in \mathbf{T}$ satisfy $0 < t - s < \eta$, then*

$$|\alpha_t(x) - \alpha_s(x)| \le C(t-s)^\nu, \tag{2.1}$$

uniformly in $x \in \mathbb{R}^d$.

THEOREM 2.3. *For any finite interval $\mathbf{T} \subseteq \mathbb{R}_+$, the "approximate local time"*

$$\alpha_n(x, \mathbf{T}) \equiv \lambda_d\big(B_{1/n}(x)\big)^{-1} \int_T X_s\big(B_{1/n}(x)\big)\,ds$$

converges to $\alpha(x, B)$ as $n \to \infty$, both P_μ-a.s. and in $L^k(P_\mu)$ for every $x \in \mathbb{R}^d$ and any positive integer k. In fact we have, a.s., $\alpha_n(x, \mathbf{T}) \to \alpha(x, \mathbf{T})$ uniformly in x.

Note that the preceding theorem suggests the formal relation $\alpha(x, \mathbf{T}) = \int_{\mathbf{T}} \langle X_s, \delta_x \rangle\, ds$, so that $\alpha(x, \mathbf{T})$ can be thought of as the amount of mass that X puts on x during $\mathbf{T}$.

For ordinary processes, smoothness of the local time can be used to demonstrate irregularity of the sample paths of the process. [See Geman and Horowitz (1980) for some examples.] Theorems 2.4 and 2.8 below have a similar flavor. Namely, the Hölder continuity of the superdiffusion local time, as embodied in Theorem 2.2, is used to show that the sample paths of the superdiffusion (or, more precisely, of its support) are quite irregular. For the first application, recall that the level sets of X_t, defined as

$$M_x^{\mathbf{T}} \equiv \{t \in \mathbf{T}: x \in S(X_t)\},$$

have Lebesgue measure 0 for a.e. x, (at least for super-Brownian motion) when $d \geq 2$. We examine the Hausdorff dimension $\dim M_x^{\mathbf{T}}$ to get a better gauge of the size of such a set. [See Adler (1981) for the definition of Hausdorff dimension.]

THEOREM 2.4. *For any smooth uniformly elliptic superdiffusion X_t on $\mathbb{R}^d$, $d \leq 3$, with initial mass $\mu \in M_{F,B}(\mathbb{R}^d)$, we have, P_μ-a.s.,*

$$\dim M_x^{\mathbf{T}} \geq \min\{2 - d/2, 1\} \quad \textit{whenever } \alpha(x, \mathbf{T}) > 0. \tag{2.2}$$

(Notice that, for the case of one-dimensional super-Brownian motion, the existence of a continuous density gives $\dim M_x = 1$.)

Before stating our next theorem, we recall from Dawson, Iscoe and Perkins (1989) the following one-sided modulus of continuity for $S(X_t)$ in the case of super-Brownian motion. For this we define, for any $\delta > 0$, the *δ-enlargement* of a closed set A by

$$A^\delta \equiv \{x \in \mathbb{R}^d: \operatorname{dist}(x, A) \leq \delta\}.$$

THEOREM 2.5 (Dawson, Iscoe and Perkins). *Let X_t be super-Brownian motion on $\mathbb{R}^d$, $d \geq 1$, with initial mass $\mu \in M_F(\mathbb{R}^d)$. Then P_μ-a.s., for each $c > 2$, there is a positive random variable $\delta = \delta(c, \omega)$ such that if $s, t \geq 0$ satisfy $0 < t - s < \delta$, then*

$$S(X_t) \subseteq S(X_s)^{c \cdot h(t-s)}, \tag{2.3}$$

where $h(t) = (t \ln 1/t)^{1/2}$.

So, essentially, the support spreads no faster than its $h(\cdot)$-enlargement. This theorem will be used to get an upper bound on $\dim M_x$.

THEOREM 2.6. *Let X_t be super-Brownian motion with $\mu \in M_F(\mathbb{R}^d)$, $d \geq 1$. Then, for each $x \in \mathbb{R}^d$,*

$$\dim M_x \leq \min\{2 - d/2, 1\}, \qquad P_\mu\text{-a.s.} \tag{2.4}$$

REMARK. In case $d \geq 4$ [i.e., $\min\{2 - d/2, 1\} \leq 0$], (2.4) is to be interpreted as $\dim M_x = 0$. This is already contained in Dawson, Iscoe and Perkins (1989) where it is shown that, with probability 1, the support never hits a fixed point x when $\mu \in M_F(\mathbb{R}^d)$, $d \geq 4$.

COROLLARY 2.7. *Let X_t be super-Brownian motion with $\mu \in M_{F,B}(\mathbb{R}^d)$, $d \leq 3$. Then, for each $x \in \mathbb{R}^d$ we have, P_μ-a.s.,*

$$\alpha(x, [0,\infty)) > 0 \Rightarrow \dim M_x = \min\{2 - d/2, 1\}. \tag{2.5}$$

Our next theorem gives a sort of converse to Theorem 2.5 for $d = 3$ when the support is not enlarged fast enough.

THEOREM 2.8. *Let X_t be super-Brownian motion with $\mu \in M_{F,B}(\mathbb{R}^3)$. Let T, h, δ be fixed positive numbers and define $\phi(s) = s^{1/2+\delta}$. Set $\overline{\Omega} \equiv \{\omega: X_t(\omega, \mathbb{R}^3) > 0\ \forall\ t \in [0, T+h]\}$. Then, for every constant $q > 0$,*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \lambda\left\{s \in [t, t+\varepsilon]: S(X_s) \subseteq S(X_t)^{q \cdot \phi(s-t)}\right\} = 0 \tag{2.6}$$

a.s. on $\overline{\Omega}$, $\forall\ t \in [0, T]$.

So, provided the process does not die, the proportion of time in $[t, t+\varepsilon]$ that $S(X_s)$ spends completely within the $\phi(s-t)$-width enlargement of $S(X_t)$ goes to 0 a.s. as $\varepsilon \downarrow 0$. In fact, as we shall see in the proof, it even escapes the $\phi(\varepsilon)$-width enlargement. This result is analogous to the classical result which says that one-dimensional Brownian motion leaves the "cone" with square-root-shaped boundaries. It also shows (at least in the case $d = 3$) that the exponent $1/2$ on the t in Theorem 2.5 is the best possible.

3. Proofs.

3.1. *Preliminary estimates for the underlying motion.* Throughout this section, we will use the letter C to denote a generic positive constant (possibly depending on certain parameters) whose value may change from line to line. Let ξ_t, $t \geq 0$, be a smooth uniformly elliptic diffusion in $\mathbb{R}^d$. It can be shown [cf. Dynkin (1965), Theorem 0.5] that ξ has a transition density $p_t(x, y)$ with the following properties:

$$\text{(i)} \qquad p_t(x,y) \leq Mt^{-d/2} e^{-\beta|y-x|^2/t},$$

$$\text{(ii)} \qquad \left|\frac{\partial}{\partial x_i} p_t(x,y)\right| \leq Mt^{-(d+1)/2} e^{-\beta|y-x|^2/t}, \qquad i = 1, \dots, d,$$

where M and β are positive constants. We also have

$$\text{(iii)} \qquad \left|\frac{\partial}{\partial y_i} p_t(x,y)\right| \leq Mt^{-(d+1)/2} e^{-\beta|y-x|^2/t}, \qquad i = 1, \dots, d.$$

The last inequality can be obtained by using Kolmogorov's forward equation and the smoothness conditions on the a_{ij} and b_i (cf. Section 2) to see that the function $(t, y) \mapsto p_t(x, y)$ is the solution of a PDE which is similar to that satisfied by the function $(t, x) \mapsto p_t(x, y)$ in Dynkin's Theorem 0.5; hence (iii) follows from (ii).

Following Rosen (1987), we use these bounds to derive, for all $x, y, w \in \mathbb{R}^d$ and $t > 0$,

$$\begin{aligned}(3.1)\qquad &|p_t(x+w,y) - p_t(x,y)| \\ &\qquad \le C|w|^{\delta} t^{-(d+\delta)/2}\left(e^{-\beta|x-y|^2/t} + e^{-\beta|x+w-y|^2/t}\right),\end{aligned}$$

where δ can be any number in $[0, 1]$ and C is a constant. (There is a similar inequality for differences in the second coordinate.) We break up the proof into two cases. If $|w| \ge t^{1/2}$, then $t^{-d/2} \le t^{-(d+\delta)/2}|w|^{\delta}$. Now use (i) and the triangle inequality to get (3.1). On the other hand, if $|w| < t^{1/2}$, then $|w|/t^{1/2} \le |w|^{\delta}/t^{\delta/2}$, and hence

$$(3.2)\qquad t^{-(d+1)/2}|w| \le t^{-(d+\delta)/2}|w|^{\delta}.$$

Now by the mean value theorem, we have for some $s \in [0, 1]$,

$$\begin{aligned}|p_t(x+w,y) - p_t(x,y)| &= |\nabla_x p_t(x+sw,y)\cdot w| \\ &\le |\nabla_x p_t(x+sw,y)|\,|w| \\ &\le C|w| \sum_{i=1}^{d} \left|\frac{\partial}{\partial x_i} p_t(x+sw,y)\right| \\ &\le C|w| t^{-(d+1)/2} e^{-\beta|x-y+sw|^2/t} \quad [\text{by (ii)}].\end{aligned}$$

By (3.2), it is enough to prove that, for some constant C,

$$(3.3)\qquad e^{-\beta|x-y+sw|^2/t} \le C\left(e^{-\beta|x-y|^2/t} + e^{-\beta|x-y+w|^2/t}\right),$$

for any $s \in [0, 1]$. This can be shown by considering separately the cases $(x - y)\cdot w \ge 0$ and $(x - y)\cdot w < 0$, where in the second case the assumption $|w|^2 < t$ is used. Equation (3.1) is now proven.

If we write

$$\mathfrak{b}_t(x,y) \equiv \mathfrak{b}_t(y-x) \equiv \mathfrak{b}(t, y-x) \equiv (2\pi t)^{-d/2} e^{-|y-x|^2/2t}$$

for the transition density of standard d-dimensional Brownian motion, then $t^{-d/2}e^{-\beta|y-x|^2/t} = C\mathfrak{b}_{t/2\beta}(x, y)$. Thus, replacing $1/2\beta$ with α, we can summarize (i) and (3.1) as follows.

LEMMA 3.1. *If $p_t(x, y)$ is the transition density of a smooth uniformly elliptic diffusion in $\mathbb{R}^d$, then there are positive constants α, $c_1 = c_1(\alpha)$, $c_2 = c_2(\alpha, \delta)$, such that*

$$(3.4)\qquad p_t(x,y) \le c_1 \mathfrak{b}_{\alpha t}(x,y)$$

and

$$(3.5)\quad |p_t(x+w,y) - p_t(x,y)| \le c_2|w|^{\delta} t^{-\delta/2}\left[\mathfrak{b}_{\alpha t}(x,y) + \mathfrak{b}_{\alpha t}(x+w,y)\right],$$

for any $\delta \in [0,1]$. *There is a similar inequality for differences in the second coordinate.*

It follows easily from (3.4) that, if $\mu \in M_{F,B}(\mathbb{R}^d)$ [say $\mu(dx) \le K\,dx$], then there is a constant $c_3 = c_3(K, \alpha, d)$ such that

$$\int_{\mathbb{R}^d} p_t(x,y)\mu(dx) \le c_3, \qquad \text{for all } t, y. \tag{3.6}$$

In Section 3.2 we will need the following properties of the Brownian transition density:

$$\mathfrak{b}_s(x)\mathfrak{b}_t(x) \le C(s+t)^{-d/2}\mathfrak{b}(st/(s+t), x), \tag{3.7}$$

$$\begin{aligned}(\mathfrak{b}_s(x) + \mathfrak{b}_s(y))(\mathfrak{b}_t(x) + \mathfrak{b}_t(y))\\ \le C(s+t)^{-d/2}\big[\mathfrak{b}(st/(s+t), x) + \mathfrak{b}(st/(s+t), y)\big].\end{aligned} \tag{3.8}$$

Equation (3.7) is easy to derive. Equation (3.8) follows from (3.7) and

$$\mathfrak{b}_s(x)\mathfrak{b}_t(y) \le \mathfrak{b}_s(x)\mathfrak{b}_t(x) + \mathfrak{b}_s(y)\mathfrak{b}_t(y).$$

This last inequality is proved by showing

$$e^{-ac-bd} \le e^{-ac-bc} + e^{-ad-bd}$$

(where $a = 1/2t$, $b = 1/2s$, $c = x^2$, $d = y^2$), which is easily verified by considering separately the cases $c \le d$ and $d \le c$.

For later use, we record here two theorems from Dawson and Perkins [(1991), cf. Theorems 8.12 and 8.10]. These extend the modulus of continuity and compact support properties of super-Brownian motion to a large class of superprocesses whose underlying motion is a diffusion (strong Markov process with continuous sample paths). Let

$$\mathscr{H} = \{h: [0,1] \to [0,\infty): h \text{ continuous, nondecreasing and } h(0) = 0\}.$$

THEOREM 3.2 (Dawson and Perkins). *Suppose* $\xi(t)$ *is an* $\mathbb{R}^d$*-valued diffusion process which satisfies, for some* $h \in \mathscr{H}$,

$$\sum_{j=1}^{\infty} \theta^{-2j} \sup_{x \in \mathbb{R}^d} P^x(|\xi(r\theta^j) - x| \ge h(r\theta^j)) < \infty, \quad \forall\, \theta \in (0,1), r > 0$$

and, if $f(r) = \sup_{0<u\le 1, h(u)>0} h(ur)/h(u)$ *for* $0 < r \le 1$, *then*

$$\int_0^1 f(r)(\ln 1/r) r^{-1}\, dr < \infty.$$

If X *is the super-*ξ *process starting at* $\mu \in M_F(\mathbb{R}^d)$, *then* P_μ*-a.s., for each* $c > 1$, *there is a positive random variable* $\delta = \delta(c, \omega)$ *such that if* $s, t \ge 0$ *satisfy* $0 < t - s < \delta$, *then*

$$S(X_t) \subseteq S(X_s)^{c \cdot h(t-s)}.$$

THEOREM 3.3 (Dawson and Perkins). *Suppose ξ is an $\mathbb{R}^d$-valued diffusion such that, for some $h \in \mathscr{H}$,*

$$\sum_{n=1}^{\infty} 4^n \sup_{x \in \mathbb{R}^d} P^x(|\xi(2^{-n}) - x| \geq h(2^{-n})) < \infty,$$

$$\int_0^1 h(r)(\ln 1/r) r^{-1}\, dr < \infty.$$

If X is the super-ξ process starting at $\mu \in M_F(\mathbb{R}^d)$, then P_μ-a.s., $S(X_t)$ is compact for all $t > 0$.

We note that the bound in (i) can be used to verify directly that smooth uniformly elliptic diffusions satisfy the conditions of Theorems 3.2 and 3.3 with $h(u) = u^\nu$, for any $\nu < 1/2$. In short, and this is what we use later, any smooth uniformly elliptic superdiffusion has compact support and propagates with finite speed.

3.2. *Moment inequalities for superdiffusion local times.* It is easy to see that, when $\mu \in M_{F,B}(\mathbb{R}^d)$, the function $(t, x) \mapsto E_\mu \alpha_t(x)$ is Lipschitz continuous. Hence, when proving smoothness results for the local time, it is enough to work with $\bar{\alpha}_t(x) \equiv \alpha_t(x) - E_\mu \alpha_t(x)$.

The following lemma is central to the proofs of Theorems 2.1 and 2.2.

LEMMA 3.4. *Let $\alpha(x, B)$ be the local time for a smooth uniformly elliptic superdiffusion with initial mass $\mu \in M_{F,B}(\mathbb{R}^d)$, $d \leq 3$, and let $\bar{\alpha}(x, B) \equiv \alpha(x, B) - E_\mu \alpha(x, B)$. Fix a time interval $[0, T]$ and let*

$$\begin{cases} \gamma = \min\{2 - d/2, 1\}, & \text{if } d = 1, 3, \\ \gamma \in (0, \min\{2 - d/2, 1\}), & \text{if } d = 2. \end{cases}$$

Then, for every even integer $k \geq 2$, there is a constant c_k such that

$$E_\mu\left[\bar{\alpha}(x, B)^k\right] \leq c_k \lambda_1(B)^{k\gamma} \tag{3.9}$$

and, if $0 < \delta \leq \gamma$,

$$E_\mu|\bar{\alpha}(a, B) - \bar{\alpha}(b, B)|^k \leq c_k |a - b|^{k\delta} \lambda_1(B)^{k(\gamma - \delta/2)}, \tag{3.10}$$

for any interval $B \subseteq [0, T]$ satisfying $\lambda_1(B) \leq 1$, and any $x, a, b \in \mathbb{R}^d$. [In (3.9), c_k depends on α and T, while in (3.10) it also depends on δ.]

Our proof of Lemma 3.4 will be based on results of Sugitani (1989) which we summarize below. If X is a random variable, say that

$$E[\exp(\theta X)] = \exp\left(\sum_{n=1}^{\infty} a_n \theta^n\right) \tag{3.11}$$

holds *formally* provided $E|X|^n < \infty$ and

$$E(X^n) = \frac{d^n}{d\theta^n}\left(\exp\left(\sum_{k=1}^{n} a_k\theta^k\right)\right)\Bigg|_{\theta=0}$$

for every $n \geq 1$.

The following result is Lemma 3.2 in Sugitani (1989).

LEMMA 3.5. *Suppose* (3.11) *holds formally. If, for some integer N, there exist $r, b > 0$ such that*

$$|a_n| \leq br^n, \quad \text{for } 1 \leq n \leq 2N, \tag{3.12}$$

then there exists $C = C(b, N) > 0$ such that

$$E(X^{2N}) \leq Cr^{2N}. \tag{3.13}$$

Write $q_t(x, y) \equiv \int_0^t p_s(x, y)\,ds$ and let P_t denote the transition semigroup of a smooth uniformly elliptic diffusion with transition density $p_t(x, y)$. The next two lemmas were proved by Sugitani in the case of super-Brownian motion, and they extend trivially to the superdiffusion case.

LEMMA 3.6. *Suppose $\alpha(x, B)$ is the local time of a smooth uniformly elliptic superdiffusion starting at $\mu \in M_{F,B}(\mathbb{R}^d)$, $d \leq 3$, and let $\bar{\alpha}(x, B)$ be defined as above. Then*

$$E_\mu[\exp\theta(\bar{\alpha}_{t+s}(a) - \bar{\alpha}_s(a))] = \exp\left(2\sum_{n=2}^{\infty}(\theta/2)^n\langle\mu, v_n(s,t)\rangle\right) \tag{3.14}$$

holds formally, where the $v_n(s, t)$ are determined by

$$\begin{cases} v_1(s,t) = P_s v_1(t), \\ v_n(s,t) = P_s v_n(t) + \displaystyle\sum_{k=1}^{n-1}\int_0^s P_{s-r}(v_k(r,t)v_{n-k}(r,t))\,dr, \quad n \geq 2, \end{cases} \tag{3.15}$$

with

$$\begin{cases} v_1(t,x) = q_t(x,a), \\ v_n(t) = \displaystyle\sum_{k=1}^{n-1}\int_0^t P_{t-s}(v_k(s)v_{n-k}(s))\,ds, \quad n \geq 2. \end{cases} \tag{3.16}$$

LEMMA 3.7. *Let $\bar{\alpha}(x, B)$ and μ be as above. Then*

$$\begin{aligned} &E_\mu[\exp\theta(\bar{\alpha}(a,[s,s+t]) - \bar{\alpha}(b,[s,s+t]))] \\ &\qquad = \exp\left(2\sum_{n=2}^{\infty}(\theta/2)^n\langle\mu, u_n(s,t)\rangle\right) \end{aligned} \tag{3.17}$$

holds formally, where the $u_n(s,t)$ *are determined by*

$$(3.18)\qquad \begin{cases} u_1(s,t) = P_s u_1(t), \\ u_n(s,t) = P_s u_n(t) + \sum_{k=1}^{n-1} \int_0^s P_{s-r}(u_k(r,t)u_{n-k}(r,t))\,dr, \quad n \geq 2, \end{cases}$$

with

$$(3.19)\qquad \begin{cases} u_1(t,x) = q_t(x,a) - q_t(x,b), \\ u_n(t) = \sum_{k=1}^{n-1} \int_0^t P_{t-s}(u_k(s)u_{n-k}(s))\,ds, \quad n \geq 2. \end{cases}$$

PROOF OF LEMMA 3.4. We begin by proving (3.9). Set $\bar{q}_t(x,y) \equiv \int_0^t \mathfrak{b}_s(x,y)\,ds$. Using (3.4) and (3.7), it follows as in the proof of Lemma 3.4 in Sugitani (1989) that there exist positive constants $a_n \equiv a_n(\alpha)$ such that

$$(3.20)\quad v_n(t,x) \leq a_n t^{(n-1)(2-d/2)} \bar{q}_{2\alpha t}(x,a), \qquad n \geq 1, t \geq 0, x \in \mathbb{R}^d.$$

Next, we claim that for any $R > 0$ there exist positive constants $b_n = b_n(R,\alpha)$ such that

$$(3.21)\qquad v_n(s,t,x) \leq b_n t^{(n-1)\gamma}(P_s w(t))(x)$$

for every $s \leq R$, $t \leq 1$, $a, x \in \mathbb{R}^d$, and $n \geq 1$, where $w(t,x) \equiv \bar{q}_{2\alpha t}(x,a)$. The case $n = 1$ follows from (3.20). Now proceed by induction, assuming that (3.21) holds for $v_k(s,t)$, $k \in \{1,\dots,n-1\}$. From (3.20), we have

$$(3.22)\qquad P_s v_n(t) \leq a_n t^{(n-1)(2-d/2)} P_s w(t).$$

For $1 \leq k \leq n-1$,

$$(3.23)\qquad \begin{aligned} &\int_0^s P_{s-r}(v_k(r,t)v_{n-k}(r,t))\,dr \\ &\quad \leq b_k b_{n-k} t^{(n-2)\gamma} \int_0^s P_{s-r}(P_r w(t))^2\,dr \\ &\quad \leq b_k b_{n-k} t^{(n-2)\gamma} P_s w(t) \int_0^s \|P_r w(t)\|_\infty\,dr. \end{aligned}$$

But

$$\begin{aligned} |P_r w(t)(x)| &= \int p_r(x,y)\bar{q}_{2\alpha t}(y,a)\,dy \leq c_1 \int_0^{2\alpha t} du \int dy\, \mathfrak{b}_{\alpha r}(x,y)\mathfrak{b}_u(y,a) \\ &= c_1 \alpha \int_r^{r+2t} \mathfrak{b}_{\alpha u}(x,a)\,du \leq c_1 \alpha \int_r^{r+2t} (2\pi\alpha u)^{-d/2}\,du. \end{aligned}$$

So

$$\int_0^s \|P_r w(t)\|_\infty\,dr \leq C(\alpha) \int_0^{2t} du \int_0^s dr (r+u)^{-d/2}.$$

Straightforward calculations show that, for $d = 1$ and 3, this is bounded above by $C(R, \alpha)t^{(2-d/2)\wedge 1}$, for $s \le R$, where $C(R, \alpha)$ is a positive constant. In the case $d = 2$, use the fact that, for any $\eta \in (0, 1)$, there is a positive constant $C(\eta)$ such that $\ln(1 + t) \le C(\eta)t^{\eta}$, for all $t \ge 0$, to get

$$\int_0^{2t} du \int_0^s dr (r + u)^{-1} \le C(R, \eta) t^{1-\eta}.$$

Plugging the above estimates into (3.23), and using (3.15) and (3.22), finishes the proof of (3.21).

Equation (3.9) now follows from Lemma 3.5, Lemma 3.6, (3.21) and the fact that $\langle \mu, P_s w(t)\rangle \le 2\alpha K t$, when $\mu(dx) \le K dx$.

Next we go to the proof of (3.10). Similarly to the above, it suffices to show that the $u_n(s,t)$ in Lemma 3.7 satisfy

$$\langle \mu, |u_n(s,t)|\rangle \le b_n |a - b|^{n\delta} t^{n(\gamma - \delta/2)} \tag{3.24}$$

for all $n \ge 1$, for some positive constants b_n.

It follows by induction, along with (3.4), (3.5) and (3.8), that the $u_n(t)$ of Lemma 3.7 satisfy

$$|u_n(t)| \le a_n |a - b|^{n\delta} t^{(n-1)(\gamma-\delta/2)} \bar{u}(t), \tag{3.25}$$

where

$$\bar{u}(t, x) \equiv \int_0^{2t} s^{-\delta/2} [\mathfrak{b}_{\alpha s}(x, a) + \mathfrak{b}_{\alpha s}(x, b)]\, ds$$

and the $a_n \equiv a_n(\alpha, \delta)$ are positive constants. The proof is very similar to that of Sugitani's Lemma 3.2, so we omit it.

The next step is to show that

$$|u_n(s,t)| \le b_n |a - b|^{n\delta} t^{(n-1)(\gamma - \delta/2)} P_s \bar{u}(t), \qquad n \ge 1. \tag{3.26}$$

The case $n = 1$ is immediate from (3.25). Assume $u_k(s,t)$ satisfies (3.26) for $1 \le k \le n - 1$ and proceed by induction. First note that

$$|P_s u_n(t)| \le a_n |a - b|^{n\delta} t^{(n-1)(\gamma-\delta/2)} P_s \bar{u}(t) \tag{3.27}$$

by (3.25). Next [cf. (3.18)], we must bound

$$\begin{aligned}
&\sum_{k=1}^{n-1} \int_0^s P_{s-r}(|u_k(r,t)|\, |u_{n-k}(r,t)|)\, dr \\
&\qquad \le \sum_{k=1}^{n-1} b_k b_{n-k} |a - b|^{n\delta} t^{(n-2)(\gamma-\delta/2)} \int_0^s P_{s-r}(P_r \bar{u}(t))^2\, dr.
\end{aligned}$$

But $P_{s-r}(P_r\bar{u}(t))^2 \le \|P_r \bar{u}(t)\|_\infty P_s \bar{u}(t)$, and

$$\begin{aligned}
|P_r \bar{u}(t)(x)| &= \int dy\, p_r(x,y) \int_0^{2t} dq\, q^{-\delta/2} [\mathfrak{b}_{\alpha q}(y, a) + \mathfrak{b}_{\alpha q}(y, b)] \\
&\le c_1 \int_0^{2t} q^{-\delta/2} [\mathfrak{b}_{\alpha(r+q)}(x, a) + \mathfrak{b}_{\alpha(r+q)}(x, b)]\, dq \\
&\le C(\alpha) \int_0^{2t} q^{-\delta/2} (r + q)^{-d/2}\, dq.
\end{aligned}$$

Thus, for $s \le R$,

$$\int_0^s \|P_r\bar{u}(t)\|_\infty \, dr \le C(\alpha) \int_0^{2t} dq \, q^{-\delta/2} \int_0^R dr (r+q)^{-d/2}. \tag{3.28}$$

Handling these temporal integrals in much the same way as those in the proof of (3.9), we conclude that

$$\int_0^s P_{s-r}(P_r\bar{u}(t))^2 \, dr \le C(\alpha, R) t^{\gamma-\delta/2}, \qquad s \le R.$$

Hence we have

$$\begin{aligned} &\sum_{k=1}^{n-1} \int_0^s P_{s-r}(|u_k(r,t)| \, |u_{n-k}(r,t)|) \, dr \\ &\quad \le \left(\sum_{k=1}^{n-1} a_k a_{n-k} \right) C(\alpha, \delta, R) |a-b|^{n\delta} t^{(n-1)(\gamma-\delta/2)} P_s\bar{u}(t). \end{aligned} \tag{3.29}$$

Using (3.27) and (3.29) in (3.18) completes the induction on $u_n(s,t)$.

We complete the proof of (3.24) by noting that

$$\langle \mu, P_s\bar{u}(t) \rangle \le 4Kc_1 t. \qquad \square$$

PROOF OF THEOREM 2.1. From Lemma 3.4 we have, for all $s, t \in [0, T]$ with $|t-s| \le 1$, and all $x, y \in \mathbb{R}^d$,

$$E_\mu |\alpha_t(x) - \alpha_s(x)|^k \le c_k |t-s|^{k\gamma}$$

and

$$E_\mu |\alpha_t(y) - \alpha_t(x)|^k \le c_k |y-x|^{k\gamma},$$

when k is an even integer and c_k is a constant depending on T. [The second inequality follows from (3.10) with $\delta = \gamma$.] Using the fact that $(a+b)^k \le 2^{k-1}(a^k + b^k)$ (convexity), the above inequalities can be used to show

$$E_\mu |\alpha_t(y) - \alpha_s(x)|^k \le 2^k c_k |(t,y) - (s,x)|^{k\gamma},$$

which is enough to yield the conditions in the multiparameter version of Kolmogorov's lemma [cf. Karatzas and Shreve (1988), page 55], thus proving the theorem. $\square$

PROOF OF THEOREM 2.2. This proof is modeled after the proof of Theorem 3 in Geman, Horowitz and Rosen (1984). We will write $\lambda(B)$ rather than $\lambda_1(B)$.

Since $t \mapsto E_\mu \alpha_t(x)$ is Lipschitz continuous in t (uniformly in x) when $\mu \in M_{F,B}(\mathbb{R}^d)$, it suffices to show, P_μ-a.s.,

$$|\bar{\alpha}_t(x) - \bar{\alpha}_s(x)| \le C(t-s)^\nu, \qquad 0 < t-s < \eta, \tag{3.30}$$

uniformly in $x \in \mathbb{R}^d$, where we recall that $\bar{\alpha}_t(x) \equiv \alpha_t(x) - E_\mu \alpha_t(x)$.

To simplify writing, we consider only the case $\mathbf{T} = [0, 1]$. Let $\mathscr{D}_n$ denote the collection of intervals obtained by dyadically partitioning $[0, 1]$ into 2^n intervals, each of length 2^{-n}. Let $\theta_n = 2^{-n/2}$ and set $G_n = \{x \in \mathbb{R}^d: |x| \le n, x = \theta_n p \text{ for some } p \in \mathbb{Z}^d\}$. So G_n consists of points in the lattice of step size θ_n which are contained in the ball of radius n, centered at the origin. Note that $\#G_n \le Cn^d 2^{nd/2}$.

Let $\varepsilon > 0$ be arbitrary, but fixed, and γ as in Lemma 3.4. We have, for any even k,

$$P\Big[\max_{x \in G_n} \bar{\alpha}(x, B) \ge \lambda(B)^{\gamma - \varepsilon} \text{ for some } B \in \mathscr{D}_n\Big]$$
$$\le \sum_{x \in G_n} \sum_{B \in \mathscr{D}_n} P\big[\bar{\alpha}(x, B) \ge \lambda(B)^{\gamma - \varepsilon}\big] \le Cn^d 2^{n(1 + d/2 - k\varepsilon)},$$

where the last inequality follows from Markov's inequality and (3.9). By choosing k large enough, this will be summable in n. The Borel–Cantelli lemma yields, a.s.,

$$\max_{x \in G_n} \bar{\alpha}(x, B) \le \lambda(B)^{\gamma - \varepsilon} \quad \text{for all } B \in \mathscr{D}_n, \text{ when } n \ge N_1, \tag{3.31}$$

where N_1 is some positive random variable.

Notice that this proves the theorem for $x \in G_n$ when $[s, t]$ is a dyadic interval with small enough length. Now we start "filling in" the space between the points in the lattice G_n by defining for $x \in G_n$ and $n, h \in \mathbb{Z}_+$,

$$F(n, h, x) = \left\{ y \in \mathbb{R}^d: y = x + \theta_n \sum_{j=1}^{h} 2^{-j} \varepsilon_j \text{ for } \varepsilon_j \in \{0, 1\}^d \right\},$$

that is, the set of "dyadic points up to order h in the θ_n-cube with lower left corner at x." Two points $y_1 \ne y_2 \in F(n, h, x)$ are said to be *linked* if they are adjacent, that is, if $y_1 - y_2 = \psi \theta_n 2^{-h}$ for some $\psi \in \{0, \pm 1\}^d$. There are 2^{dh} points in $F(n, h, x)$, and so there can be no more than $2^{dh} d$ linked pairs in $F(n, h, x)$.

We have

$$P\Big\{|\bar{\alpha}(y_1, B) - \bar{\alpha}(y_2, B)| > |y_1 - y_2|^{\varepsilon/2} \lambda(B)^{\gamma - \varepsilon},$$
$$\text{for some } B \in \mathscr{D}_n, x \in G_n, h \ge 1,$$
$$\text{and linked pair } y_1, y_2 \in F(n, h, x)\Big\}$$
$$\le \sum_{B \in \mathscr{D}_n} \sum_{x \in G_n} \sum_{h=1}^{\infty} \sum_{y_1, y_2} P\Big\{|\bar{\alpha}(y_1, B) - \bar{\alpha}(y_2, B)| > |y_1 - y_2|^{\varepsilon/2} \lambda(B)^{\gamma - \varepsilon}\Big\}$$
$$\le C 2^n n^d 2^{nd/2} \sum_{h=1}^{\infty} d 2^{dh} |y_1 - y_2|^{k\varepsilon/2} \lambda(B)^{k\varepsilon/2} \quad [\text{by } (3.10)]$$
$$\le Cn^d 2^{n(1 + d/2 - 3k\varepsilon/4)} \sum_{h=1}^{\infty} 2^{h(d - k\varepsilon/2)}, \tag{3.32}$$

where the innermost sum after the first inequality is over all linked pairs $y_1, y_2 \in F(n, h, x)$, and the last inequality follows from $|y_1 - y_2| \le 2^{-(n/2+h)}\sqrt{d}$.

Now the sum over h in (3.32) is finite when $k > 2d/\varepsilon$, and hence the sum over n of the quantity in (3.32) is finite when k also satisfies $3k\varepsilon/4 > 1 + d/2$. Since our bounds hold for all (positive) even k, we may invoke Borel–Cantelli to get the existence of a positive random variable N_2 such that, a.s.,

$$|\alpha(y_1, B) - \alpha(y_2, B)| \le |y_1 - y_2|^{\varepsilon/2}\lambda(B)^{\gamma-\varepsilon}, \tag{3.33}$$

for every $B \in \mathscr{D}_n$, $x \in G_n$, $h \ge 1$, and linked pair $y_1, y_2 \in F(n, h, x)$, if $n \ge N_2$.

Since X_t has compact support and propagates with finite speed, there is a random variable $N_3 > 0$ such that, a.s., $\overline{R}(\mathbf{T}) \subseteq B_n(0)$ for all $t \in \mathbf{T}$ when $n \ge N_3$. Now pick $n \ge N_1, N_2, N_3$ and let z be an arbitrary point in $\mathbb{R}^d$. If $z \notin B_n(0)$ then $\alpha(z, \mathbf{T}) = 0$ a.s. (cf. Theorem 1.6 and the remarks following it), and so (3.30) holds trivially for all such z. If $z \in B_n(0)$, then there is an $x \in G_n$ such that $z = \lim_{h\to\infty} y_h$, where $y_h = x + \theta_n \sum_{j=1}^{h} 2^{-j}\psi_j$, with $\psi_j \in \{0,1\}^d$ and $y_0 = x$. Since each pair y_h, y_{h-1} is linked in $F(n, h, x)$, (3.33) implies

$$\begin{aligned}
|\bar{\alpha}(z, B) - \bar{\alpha}(x, B)| &\le \sum_{h=1}^{\infty} |\bar{\alpha}(y_h, B) - \bar{\alpha}(y_{h-1}, B)| \\
&\le \sum_{h=1}^{\infty} |y_h - y_{h-1}|^{\varepsilon/2}\lambda(B)^{\gamma-\varepsilon} \\
&\le C\lambda(B)^{\gamma-\varepsilon+\varepsilon/4} \sum_{h=1}^{\infty} 2^{-h\varepsilon/2} \le C\lambda(B)^{\gamma-\varepsilon}, \qquad B \in \mathscr{D}_n.
\end{aligned}$$

So using this and (3.31), we have that (3.30) holds for all $x \in \mathbb{R}^d$ whenever $[s, t] \in \mathscr{D}_n$, $n \ge N_1, N_2, N_3$ (i.e., for $|t - s|$ small). To complete the proof, use the fact [cf. Billingsley (1965), page 140] that any interval $[s, t]$ of length less than η can be covered by 4 or fewer *dyadic* intervals, each of length less than η. □

3.3. *Remaining proofs.*

PROOF OF THEOREM 2.3. Set $v_k^{\mu}(z_1, \dots, z_k; \mathbf{T}) \equiv E_{\mu}[\alpha(z_1, \mathbf{T}) \cdots \alpha(z_k, \mathbf{T})]$. We need the following lemma.

LEMMA 3.8. *For any compact interval* $\mathbf{T} \subseteq \mathbb{R}_+$, $(z_1, \dots, z_k) \mapsto v_k^{\mu}(z_1, \dots, z_k; \mathbf{T})$ *is uniformly continuous on* $\mathbb{R}^{dk}$.

PROOF. Let $\bar{z} = (z_1, \dots, z_k)$, $\bar{w} = (w_1, \dots, w_k) \in \mathbb{R}^{dk}$ and set $\alpha_i = \alpha(z_i + w_i; \mathbf{T})$, $\beta_i = \alpha(z_i; \mathbf{T})$. Then

$$
\begin{aligned}
(3.34)\qquad & v_k^\mu(\bar{z} + \bar{w}; \mathbf{T}) - v_k^\mu(\bar{z}; \mathbf{T}) \\
&= E_\mu[\alpha_1 \cdots \alpha_k - \beta_1 \cdots \beta_k] \\
&= E_\mu[(\alpha_1 - \beta_1)\alpha_2 \cdots \alpha_k + (\alpha_2 - \beta_2)\beta_1\alpha_3 \cdots \alpha_k \\
&\qquad + \cdots + (\alpha_k - \beta_k)\beta_1 \cdots \beta_{k-1}].
\end{aligned}
$$

Now, for example,

$$|E_\mu[(\alpha_1 - \beta_1)\alpha_2 \cdots \alpha_k]| \le \|\alpha_1 - \beta_1\|_k \|\alpha_2\|_k \cdots \|\alpha_k\|_k$$

follows from the generalized Hölder inequality [cf. Gilbarg and Trudinger (1983), page 146], where $\|\cdot\|_k$ is the norm in $L^k(P_\mu)$. Next note that $\|\alpha_2\|_k, \dots, \|\alpha_k\|_k$ can be bounded using (3.9), and $\|\alpha_1 - \beta_1\|_k \le C|w_1|^\delta$ follows from (3.10). (Clearly it is enough to consider even k.) The other terms in (3.34) can be handled in the same way. Putting all this together yields

$$|v_k^\mu(\bar{z} + \bar{w}; \mathbf{T}) - v_k^\mu(\bar{z}; \mathbf{T})| \le C\|\bar{w}\|^\delta,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{dk}$. □

We now return to the proof of Theorem 2.3. Let $b_n = \lambda_d(B_{1/n}(x))$. Then, using (1.3), we have a.s. $\alpha_n(x, \mathbf{T}) = b_n^{-1}\int_{B_{1/n}(x)} \alpha(Z, \mathbf{T})\,dz \to \alpha(x, \mathbf{T})$ as $n \to \infty$, for all $x \in \mathbb{R}^d$, where we have used the continuity of $z \mapsto \alpha(z, B)$.

To see that this convergence is uniform in x, first note that $\alpha(x, \mathbf{T}) \to 0$ as $|x| \to 0$. This is obvious when μ has compact support, by the modulus of continuity result in Theorem 3.2. The general case of $\mu \in M_F(\mathbb{R}^d)$ can be reduced to the previous one as follows. Using the particle picture in Dawson and Perkins (1991), if we start with $\mu_M(\cdot) \equiv \mu(\cdot \cap B_M(0)^c)$, then the small amount of mass in $B_M(0)^c$ will die very quickly, say by time $\varepsilon(M)$, where $\varepsilon(M) \to 0$ as $M \to \infty$. Hence, under P_{μ_M}, we have for any t, $\alpha_t(x) = \alpha_{\varepsilon(M)}(x)$ for all $|x| > M$. This goes to 0 as $M \to \infty$, by continuity in t of the local time.

Thus we have that $z \mapsto \alpha(z, \mathbf{T})$ is *uniformly* continuous. Now an easy argument shows that $b_n^{-1}\int_{B_{1/n}(x)} \alpha(z, \mathbf{T})\,dz \to \alpha(x, \mathbf{T})$ uniformly in x.

To prove convergence in $L^k(P_\mu)$, for any even integer k,

$$
\begin{aligned}
& E_\mu[\alpha_n(x, \mathbf{T}) - \alpha(x, \mathbf{T})]^k \\
&\quad = \sum_{m=0}^k \binom{k}{m}(-1)^m E_\mu[\alpha_n^m(x, \mathbf{T})\alpha^{k-m}(x, \mathbf{T})] \\
&\quad = \sum_{m=0}^k \binom{k}{m}(-1)^m E_\mu\left[b_n^{-m}\int_{B_{1/n}(x)^m} \alpha(z_1, \mathbf{T}) \cdots \alpha(z_m, \mathbf{T})\alpha^{k-m}(x, \mathbf{T})\,d\bar{z}\right] \\
&\quad = \sum_{m=0}^k \binom{k}{m}(-1)^m b_n^{-m}\int_{B_{1/n}(x)^m} v_k^\mu(z_1, \dots, z_m, x, \dots, x; \mathbf{T})\,d\bar{z}.
\end{aligned}
$$

Thus, by the continuity of $v_k^\mu(z_1, \dots, z_k; \mathbf{T})$, we have for all x,

$$\lim_{n\to\infty} E_\mu[\alpha_n(x,\mathbf{T}) - \alpha(x,\mathbf{T})]^k$$

$$= \sum_{m=0}^{k} \binom{k}{m}(-1)^m v_k^\mu(x,\dots,x;\mathbf{T}) = v_k^\mu(x,\dots,x;\mathbf{T})(1-1)^k = 0.$$

(Note that the L^k convergence actually holds uniformly in x.) □

Proof of Theorem 2.4. Fix an ω for which (2.1) holds and suppose that $\dim M_x^{\mathbf{T}} = \nu < \min\{2 - d/2, 1\}$. It is enough to treat the case when $\mathbf{T}$ is a finite interval. We will show that $\alpha(x,\mathbf{T}) = 0$. For small intervals $B \subseteq \mathbf{T}$, it follows from (2.1) that $\alpha(x, B) \leq C\lambda_1(B)^\nu$. So, since $\alpha(x, \cdot\,)$ is a finite Borel measure on $\mathbf{T}$ and $M_x^{\mathbf{T}}$ is a Borel set, Lemma 8.7.1. of Adler (1981) implies that $\alpha(x, M_x^{\mathbf{T}}) = 0$, and hence $\alpha(x,\mathbf{T}) = 0$. □

Proof of Theorem 2.6. For $d \leq 2$, the result is trivial because $M_x \subseteq \mathbb{R}_+$. In the case of $d \geq 3$, it is enough to consider a finite time interval; we work with the interval [0, 1] for simplicity.

We begin by showing that, P_μ-a.s.,

$$\dim M_x \leq \min\{2 - d/2, 1\} \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{3.35}$$

For each n, we form a dyadic cover, $\{O_{mn}\}_{m=1}^{2^n}$, of [0, 1] by defining

$$O_{mn} = [(m-1)/2^n, m/2^n].$$

Note that $\sum_{m=1}^{2^n} \operatorname{diam} O_{mn} = 1$, for each n. Following Kahane [(1985), page 140], we set

$$O_{mn}(x) = \begin{cases} O_{mn}, & \text{if } x \in \overline{R}(O_{mn}),\\ \varnothing, & \text{if } x \notin \overline{R}(O_{mn}).\end{cases}$$

[$\overline{R}(\mathbf{T})$ was defined in Section 1.] Then, for each x and n, $\{O_{mn}(x)\}_{m=1}^{2^n}$ is a (random) cover for $M_x = M_x^{[0,1]}(\omega)$. So, roughly speaking, we use only those O_{mn} during which the support visits x.

It is easy to see that the theorem will be proved if we can show that, P_μ-a.s.,

$$\liminf_{n\to\infty} \sum_{m=1}^{2^n} (\operatorname{diam} O_{mn}(x))^\beta < \infty \tag{3.36}$$

holds for a.e. x, whenever $\beta > 2 - d/2$.

Recall the notation of Theorem 2.5. Using an idea in Tribe (1989), for $a > 0$, we let $\Omega_a \equiv \{\omega\colon \delta(3, \omega) \geq a\}$ be the set of paths for which the one-sided modulus of continuity holds (with $c = 3$) on intervals of length less than a. By Theorem 2.5, Ω_a increases to a set of full probability as $a \downarrow 0$. Clearly then, it suffices to show that, for any fixed $a > 0$ and $A \subseteq \mathbb{R}^d$ of finite Lebesgue measure, (3.36) holds for a.e. $x \in A$ whenever $\omega \in \Omega_a$. This will be accomplished by showing

$$E_\mu\left[\int_A \liminf_{n\to\infty} \sum_{m=1}^{2^n} (\operatorname{diam} O_{mn}(x))^\beta\, dx; \Omega_a\right] < \infty, \tag{3.37}$$

when $\beta > 2 - d/2$.

Now

$$\begin{aligned} E_\mu&\left[\int_A \sum_{m=1}^{2^n} (\operatorname{diam} O_{mn}(x))^\beta\, dx; \Omega_a\right] \\ &= E_\mu\left[\sum_{m=1}^{2^n} \int_{A\cap \overline{R}(O_{mn})} (\operatorname{diam} O_{mn})^\beta\, dx; \Omega_a\right] \\ &= \sum_{m=1}^{2^n} 2^{-n\beta} E_\mu\left[\lambda_d\left(A \cap \overline{R}(O_{mn})\right); \Omega_a\right] \\ &\le \sum_{m=1}^{2^n} 2^{-n\beta} E_\mu\left[\lambda_d\left(A \cap S(X_{(m-1)/2^n}\right)^{3h(2^{-n})}; \Omega_a\right] \end{aligned}$$

as long as $2^{-n} < a$, where the last line follows from Theorem 2.5 with $c = 3$.

Now from Dawson, Iscoe and Perkins [(1989), page 159], given $\varepsilon > 0$, there is a constant $c = c(d)$ such that, for all $t \ge \varepsilon^2$ and all $x \in \mathbb{R}^d$, $d \ge 3$,

$$\begin{aligned} P_\mu[X_t(B_\varepsilon(x)) > 0] &\le c\varepsilon^{d-2} \int_{\mathbb{R}^d} \mathfrak{b}_{t+\varepsilon^2}(x, y)\mu(dy) \\ &= c\varepsilon^{d-2} \int_{\mathbb{R}^d} \mathfrak{b}_{t+\varepsilon^2}(y, x)\mu(dy). \end{aligned}$$

Since $x \in S(X_t)^\varepsilon$ if and only if $X_t(B_\varepsilon(x)) > 0$, this implies that, for all $t \ge \varepsilon^2$,

$$\begin{aligned} E_\mu\left[\lambda_d\left(A \cap S(X_t)^\varepsilon\right); \Omega_a\right] &\le E_\mu\left[\lambda_d\left(A \cap S(X_t)^\varepsilon\right)\right] \\ &= \int_A P_\mu[X_t(B_\varepsilon(x)) > 0]\, dx \\ &\le C\mu(\mathbb{R}^d)\varepsilon^{d-2}. \end{aligned} \tag{3.38}$$

Apply this, with $\varepsilon = 3h(2^{-n})$, to get

$$\begin{aligned} E_\mu\left[\lambda_d\left(A \cap S(X_t)^{3h(2^{-n})}\right); \Omega_a\right] &\le C\mu(\mathbb{R}^d)3^{d-2}(n2^{-n}\ln 2)^{(d-2)/2} \\ &\equiv C(n2^{-n})^{d/2-1}, \qquad \forall\, t \ge (9\ln 2)n2^{-n}. \end{aligned}$$

Thus we have, for $2^{-n} < a$,

$$\begin{aligned} E_\mu&\left[\int_A \sum_{m=1}^{2^n} (\operatorname{diam} O_{mn}(x))^\beta\, dx; \Omega_a\right] \\ &\le \sum_{m=1}^{[9n\ln 2]+1} 2^{-n\beta}\lambda_d(A) + C \sum_{m=[9n\ln 2]+2}^{2^n} 2^{-n(\beta+d/2-1)} n^{d/2-1}, \end{aligned}$$

where $[b]$ denotes the greatest integer $\leq b$. The first sum goes to 0 as $n \to \infty$, as long as $\beta > 0$. The second sum is bounded above by $C\sum_{m=1}^{2^n} 2^{-n(\beta+d/2-1)} n^{d/2-1} = C2^n 2^{-n(\beta+d/2-1)} n^{d/2-1}$ which also tends to 0 whenever $\beta + d/2 - 1 > 1$. Thus, for any $\beta > 2 - d/2$,

$$E_\mu\left[\int_A \liminf_{n\to\infty} \sum_{m=1}^{2^n} (\operatorname{diam} O_{mn}(x))^\beta \, dx; \Omega_a\right]$$
$$\leq \liminf_{n\to\infty} E_\mu\left[\int_A \sum_{m=1}^{2^n} (\operatorname{diam} O_{mn}(x))^\beta \, dx; \Omega_a\right] = 0,$$

proving (3.37), as desired.

To get Theorem 2.6 from (3.35), we employ a result in Evans and Perkins (1991). Remember that we only have to treat the case $d \geq 3$. We know

$$E_\mu \int_{\mathbb{R}^d} 1(\dim M_x > \beta) \, dx = 0,$$

for $\beta > 2 - d/2$. Fubini's theorem (the requisite measurability will be proved later) implies

$$P_\mu(\dim M_x > \beta) = 0, \quad \text{a.e. } x.$$

But Theorem 1.1 in Evans and Perkins (1991) implies that, for any $t > 0$ and $\mu, \nu \in M_F(\mathbb{R}^d)$, $P_\mu(X_{t+\cdot} \in dw)$ and $P_\nu(X_{t+\cdot} \in dw)$, considered as measures on paths in $M_F(\mathbb{R}^d)$, are absolutely continuous with respect to each other. If, for some $y \in \mathbb{R}^d$, $P_\mu(\dim M_y > \beta) > 0$, then translating μ by $x - y$ we must have $P_\mu(\dim M_x > \beta) > 0$ for every x. This contradicts (3.35), so the theorem is proved. □

We now prove the measurability which was needed in the above application of Fubini's theorem.

LEMMA 3.9. *If X_t is super-Brownian motion with $\mu \in M_F(\mathbb{R}^d)$, $d \geq 1$, then $(x, \omega) \mapsto \dim M_x(\omega)$ is measurable with respect to the completion of $\mathfrak{B}(\mathbb{R}^d) \times \mathcal{F}$.*

PROOF. The case $d = 1$ is trivial. We consider the composition

$$(x, \omega) \overset{M}{\mapsto} M_x(\omega) \overset{\dim}{\longmapsto} \dim M_x(\omega).$$

Cutler (1984) showed that dim: $\mathbf{F}(\mathbb{R}) \to \mathbb{R}$ is a measurable map, where $\mathbf{F}(\mathbb{R})$ is the collection of closed subsets of $\mathbb{R}$ under the compact topology (see Cutler for definitions). Thus it remains to prove that M: $\mathbb{R}^d \times \Omega \to \mathbf{F}(\mathbb{R})$ is measurable. Equation (1.6) implies that, off a set of $\lambda_d \times P_\mu$-measure 0, $M_x(\omega)$ is indeed a closed set. Similar to the proof of Corollary 4.4.1.1 in Cutler (1984), it is enough to show that, for any open set $G \subseteq \mathbb{R}$,

$$\{(x, \omega): M_x(\omega) \cap G \neq \varnothing\} \in \overline{\mathfrak{B}(\mathbb{R}^d) \times \mathcal{F}}, \tag{3.39}$$

where the bar denotes completion.

We can write G as a disjoint union of a countable collection of open intervals, so it is enough to prove (3.39) when G is of the form $[0, b)$ or (a, b), where $0 \le a < b \le \infty$. The case $b = \infty$ can be handled with a countable union, so we need only consider $0 \le a < b < \infty$. Finally, since $\lambda_d \times P_\mu(\{x, \omega): M_x(\omega) \cap \{a, b\} \neq \varnothing\}) = 0$ when $a, b > 0$ (S_t is singular for all $t > 0$, a.s., when $d \ge 2$), and $\{(x, \omega)\colon 0 \in M_x(\omega)\} = S_0 \times \Omega$ is measurable, it is enough to show that $\{(x, \omega)\colon M_x(\omega) \cap [a, b] \neq \varnothing\}$ is measurable when $0 < a, b < \infty$.

Let $A_1 \equiv \{(x, \omega)\colon x \in \bigcup_{t>0}(S_{t-}(\omega) \setminus S_t(\omega))\}$. Then $\lambda_d \times P_\mu(A_1) = 0$. We have

$$\begin{aligned}
&\{(x, \omega)\colon M_x(\omega) \cap [a, b] \neq \varnothing\}\\
&\quad= \bigcup_{t \in [a, b]} \bigcap_{n \ge 1} \{(x, \omega)\colon X_t(B_{1/n}(x)) > 0\}\\
(3.40)\qquad &\quad= \bigcap_{n \ge 1} \bigcup_{t \in [a, b]} \{(x, \omega) \in A_1^c\colon X_t(B_{1/n}(x)) > 0\} \cup A\\
&\quad= \bigcap_{n \ge 1} \bigcup_{t \in [a, b] \cap \mathbb{Q}} \{(x, \omega) \in A_1^c\colon X_t(B_{1/n}(x)) > 0\} \cup A,
\end{aligned}$$

where $A \subseteq A_1$. The last equality is a consequence of the weak continuity of X_t; the second equality is proved as follows. The inclusion "$\subseteq$" is trivial. To prove the reverse inclusion, suppose

$$(x, \omega) \in \bigcap_{n \ge 1} \bigcup_{t \in [a, b} \{(x, \omega) \in A_1^c\colon X_t(B_{1/n}(x)) > 0\}.$$

Then there is a sequence $\{t_n\} \subseteq [a, b]$ such that $X_{t_n}(B_{1/n}(x)) > 0$, $n \ge 1$. Compactness implies that $\{t_n\}$ has a subsequence which converges to some point in $[a, b]$. Without loss of generality, assume $t_n \mapsto t \in [a, b]$. Then it follows that $x \in S_{t-}$. But for $(x, \omega) \in A_1^c$, $x \in S_{t-}(\omega)$ implies $x \in S_t(\omega)$, and hence $X_t(B_{1/n}(x)) > 0\ \forall\ n \ge 1$, proving the desired inclusion.

The proof is finished by showing that $\{(x, \omega)\colon X_t(B_{1/n}(x)) > 0\} \in \mathfrak{B}(\mathbb{R}^d) \times \mathscr{F}$. A monotone class argument shows that $\{(t, x, \omega)\colon \int_{\mathbb{R}^d} f(x, y) X_t(dy) > 0\}$ is jointly measurable for all bounded measurable $f(x, y)$, and this yields the desired result. □

REMARK. The results in Chapter 1 can also be improved slightly for a large class of superprocesses (e.g., when the underlying process has a transition density) by using Theorem 1.1 of Evans and Perkins (1991). These results can be shown to hold for *each* fixed x (or t) P_μ-a.s., rather than P_μ-a.s. for a.e. x (or t). For example, in Theorem 1.2 we have

$$P_\mu(X_t(\{x\colon \alpha(x, [t - \varepsilon, t + \varepsilon]) = 0, \text{some } \varepsilon > 0\}) = 0 \text{ for a.e. } t > 0) = 1.$$

This can be improved to say, for *each* $t > 0$,

$$P_\mu(X_t(\{x\colon \alpha(x, [t - \varepsilon, t + \varepsilon]) = 0, \text{some } \varepsilon > 0\}) = 0) = 1.$$

We omit the details.

PROOF OF THEOREM 2.8. Recall the notation in Theorem 2.8. Let $m \equiv m(\omega) \equiv \min_{s \in [0, T+h]} X_s(\mathbb{R}^3) > 0$ for all $\omega \in \overline{\Omega}$. We prove the theorem only in the case of $q = 1$; the same proof works for other values of q with only slight notational changes. Begin by noticing that, for $t \in [0, T]$, $0 < \varepsilon < h$, and $\omega \in \overline{\Omega}$,

$$\int_t^{t+\varepsilon} I_{\{S(X_s) \subseteq S(X_t)^{\phi(s-t)}\}}(s)\, ds \le \frac{1}{m} \int_t^{t+\varepsilon} X_s\big(S(X_t)^{\phi(s-t)}\big)\, ds,$$

since if $S(X_s) \subseteq S(X_t)^{\phi(s-t)}$, then $X_s(S(X_t)^{\phi(s-t)})/m = X_s(\mathbb{R}^3)/m \ge 1$. So it suffices to show

$$\varepsilon^{-1} \int_t^{t+\varepsilon} X_s\big(S(X_t)^{\phi(\varepsilon)}\big)\, ds \to 0 \quad \text{a.s. on } \overline{\Omega}.$$

To this end, set $\varepsilon_n \equiv 2^{-n}$ and use the definition of local time to get

$$\begin{aligned}
\sum_{n=1}^{\infty} E_\mu&\left[\varepsilon_{n+1}^{-1} \int_t^{t+\varepsilon_n} X_s\big(S(X_t)^{\phi(\varepsilon_n)}\big)\, ds; \overline{\Omega}\right] \\
&= \sum_{n=1}^{\infty} \varepsilon_{n+1}^{-1} E_\mu\left[\int_{S(X_t)^{\phi(\varepsilon_n)}} \alpha(x, [t, t+\varepsilon_n])\, dx; \overline{\Omega}\right] \\
&\le C \sum_{n=1}^{\infty} \varepsilon_{n+1}^{-1} \varepsilon_n^{1/2-\delta} E_\mu\left[\lambda_d\big(S(X_t)^{\phi(\varepsilon_n)}\big)\right] \\
&\qquad\qquad (\text{Theorem 2.2 with } \nu = 1/2 - \delta/2).
\end{aligned}$$

Since $d = 3$, we may bound the expectation as in (3.38) to see that this sum is finite. It follows that

$$\varepsilon_{n+1}^{-1} \int_t^{t+\varepsilon_n} X_s\big(S(X_t)^{\phi(\varepsilon_n)}\big)\, ds \to 0, \quad \text{as } n \to \infty,$$

a.s. on $\overline{\Omega}$ for all $t \in [0, T]$.

Now note that if $\varepsilon_{n+1} \le \varepsilon < \varepsilon_n$, then

$$\varepsilon^{-1} \int_t^{t+\varepsilon} X_s\big(S(X_t)^{\phi(\varepsilon)}\big)\, ds \le \varepsilon_{n+1}^{-1} \int_t^{t+\varepsilon_n} X_s\big(S(X_t)^{\phi(\varepsilon_n)}\big)\, ds.$$

The result follows by letting $\varepsilon \downarrow 0$. □

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