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SINGULAR DIFFUSION LIMITS OF A CLASS OF REVERSIBLE SELF-ORGANIZING PARTICLE SYSTEMS

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We establish hydrodynamic limits for a class of attractive, reversible particle systems with an infinite range of interaction. The limiting nonlinear diffusion equations have diffusion coefficients which are functions of the local density, and which have a singularity at a critical value of the density. On open driven systems, these singular diffusion limits explain the observed nontrivial scaling behavior known as self-organized criticality.

1. Introduction. A variety of interacting particle systems (probabilistic cellular automata) on an open driven lattice are known to exhibit large temporal fluctuations and extended spatial correlations in the absence of an obvious tuning parameter. This behavior is known as self-organized criticality, and the associated models have generated a lot of interest because of their possible application to a broad class of problems relating to the ubiquitous $1/f$ noise and fractal structures observed in nature. The prototypical examples of self-organizing systems are referred to as sand piles and were first introduced in Bak, Tang and Wiesenfeld (1987). Many variations of the model have since been studied [see, e.g., Kadanoff, Nagel, Wu and Zhou (1989) and Dhar (1989)]. In these models, the configuration at each site of a large but finite d-dimensional lattice is some nonnegative integer, interpreted as the number of grains of sand at the site. The system is driven by adding individual grains of sand randomly or deterministically. After each addition, if the local slope (or, in some cases, height) exceeds a uniform threshold value, sand is redistributed to neighboring sites or falls off the edge according to a prescribed set of rules. These rules are iterated until no sites are above threshold, in which case the event, or *avalanche*, is complete. Numerical simulations reveal that for each of these models the distributions of a variety of quantities, including the number of sites involved in an avalanche and the number of grains that fall off at the boundary, have power-law tails, which are reminiscent of the slow decay of correlations observed in equilibrium statistical mechanical systems at a critical point. For these models, there has been some ambiguity regarding the appropriate definition of time. We will take the point of view that sand is added to

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the system at some fixed finite rate, in which case the local transitions are instantaneous, and the range of interaction is given by the maximum spatial extent of an avalanche, which is governed by the system size.

From a mathematical point of view, the difficulty with the original sand pile models is that, while they are easy to simulate on a computer, they are devoid of the technical attributes which typically make interacting particle systems tractable, such as attractiveness and reversibility. In an attempt to calculate certain exponents, continuum descriptions of self-organizing models were postulated on the basis of symmetry arguments [Hwa and Kardar (1989), Grinstein, Lee and Sachdev (1990) and Garrido, Lebowitz, Maes and Spohn (1990)], although little was known rigorously, and the results did not agree with the exponents obtained from the automata models, which exhibit a wide variety of scaling behaviors. It was clear, however, that common to most, if not all, self-organizing systems are: (i) the existence of a conserved quantity; and (ii) some anisotropy in the model provided by a driving mechanism. The main point of this paper is that self-organizing models have a third essential feature: (iii) The length scale (mean range of interaction) on which the conserved quantity is redistributed diverges as the density of the conserved quantity increases to a critical point.

In this paper we examine a class of reversible, attractive models which have both a conserved quantity and a diverging mean range of interaction as the density of this conserved quantity increases (an anisotropy arises in open, driven versions of these models, where self-organizing behavior is in fact observed). We prove that the continuum limits of these one-dimensional reversible particle systems are diffusion equations with diffusion coefficients which not only vary with the local density, but which can in fact have *singularities* (poles) at a particular value of the local density.

The utility of the singular diffusion limits of self-organizing models is discussed in Carlson, Chayes, Grannan and Swindle (1990b), where numerical evidence for singular diffusion in a class of sand pile models was also presented. In Carlson, Chayes, Grannan and Swindle (1990b) it was shown that on an open driven system the local density converges to the singularity of the diffusion coefficient as the size of the system diverges. The nontrivial scaling behavior associated with this convergence was seen to depend upon the order of the pole in the diffusion coefficient, the driving mechanism and the dimension. In these calculations, in order to extract the stationary behavior the approach was to apply the continuum limits established here for a symmetric closed system to an open driven system with the appropriate boundary conditions. It remains to be proven that the hydrodynamic description for the open system is valid, although results of this type have been shown for the more conventional short range models such as the simple exclusion process [Eyink, Lebowitz and Spohn (1990)].

Establishing diffusion limits for symmetric particle systems with a conserved quantity is not new. Among previous results are those of Guo, Papanicolaou and Varadhan (1988) and Kipnis, Olla and Varadhan (1989), which use the spectral properties of the self-adjoint generator to establish the diffusion limits of two types of models with local interactions—this is often referred to as the entropy method. We will, in fact, rely heavily on the techniques developed in these references. However, the nonlocal nature of the interactions in these self-organizing models requires that we also use the attractiveness of these systems to prove a maximum principle for the particle system which will be used to bound the effective range of interaction with exponential tails.

The organization of this paper is as follows. In Section 2 we describe the one-dimensional systems that we consider, consisting of a class of two-state models (0's and 1's) with prescribed jump rates $c(j)$ which are nonincreasing functions of the jump distance j. The transition rates will depend both on the function $c(j)$ and on the configuration of the system. We then state our principal result: Under appropriate rescaling each of these stochastic systems converges to a deterministic limit described by a diffusion equation, in which the diffusion coefficients can depend on the local density. Moreover, depending on the rate of decay of the jump rates $c(j)$, the diffusion coefficient can in fact be singular at a critical point, with the order of the singularity depending on the asymptotic behavior of $c(j)$ for large j. The proof of this theorem is given in Section 3, where we begin by establishing several properties of the systems considered. In particular, we show that product measures with density $\rho \in [0,1]$ comprise a one-parameter family of invariant measures describing the occupation probabilities, and that the process is reversible with respect to these measures. Additionally, we show that the process is attractive [here it is important that the jump rates $c(j)$ are nonincreasing in j]. A superexponential estimate is then established, which plays a major role in the subsequent proof of the main result. For many of the supporting results used in Section 3, the proofs appear in the Appendix. Section 4 contains concluding remarks.

2. The main result. We begin by introducing the models that we consider. The state space is $X_N = \{0,1\}^{\mathscr{T}_N}$, where $\mathscr{T}_N$ is the discrete torus with sites: $\{i/N, i = 1, \dots, N\}$, that is, our system consists of sites which are either occupied (1) or vacant (0) on the N-site torus scaled into the unit torus. We denote the configuration of the system at time t by η_t, and we adopt the coordinate notation $\eta_t(i) = 0, 1$ according to whether the state of site i/N at time t is 0 or 1, respectively.

All transitions involve a 1 at site i hopping to a vacant site $i + j$ (j may be positive or negative and positions are defined modulo N), and we denote the new configuration, in which the occupied site i and the vacant site $i + j$ are switched, by

$$(1) \qquad \eta^{i,i+j}(k) = \begin{cases} \eta(k), & \text{if } k \neq i, i+j, \\ \eta(i+j), & \text{if } k = i, \\ \eta(i), & \text{if } k = i+j. \end{cases}$$

The transition rules are as follows. If $\eta_t(i) = 1$, and if the nearest vacant site to the right is $i + j$, then at rate $c(j)$ the 1 at site i jumps to site $i + j$. The same is true for jumps to the left (i.e., for $j < 0$), and we assume that the function $c(j)$ is symmetric [i.e., $c(j) = c(-j)$]. The generator for this process is given by

$$(2)\quad L_N f(\eta) \equiv N^2 Lf(\eta) = N^2 \sum_{i=1}^{N} \sum_{j=-N}^{N} \left\{ \left[f(\eta^{i,i+j}) - f(\eta) \right] c(j) J_\eta(i,j) \right\},$$

where f is any cylinder function (i.e., a function which depends only on the configurations of a finite set of sites), and

$$(3)\quad J_\eta(i,j) \equiv (1 - \eta(i+j)) \prod_{m=0}^{s(j)} \eta(i+m), \qquad \text{where } s(j) = \begin{cases} j-1, & \text{if } j \geq 1, \\ j+1, & \text{if } j \leq -1. \end{cases}$$

To simplify our notation, we will also write

$$(4)\quad I_\eta(i,j) \equiv \prod_{m=0}^{j} \eta(i+m).$$

In words, $J_\eta(i,j)$ is equal to 1 if there is a 1 in configuration η which could *Jump* from site i to site $i+j$, and $I_\eta(i,j)$ is equal to 1 if all sites in the *Interval* i to $i+j$ are occupied [recall that we are implicitly taking $(i+j)$ mod N]. The factor N^2 is the desired diffusion scaling of the rates, commensurate with the spatial rescaling in $\mathscr{T}_N$.

Let M_1 be the space of measurable functions $\rho(\theta)$ on the continuum unit torus $\mathscr{T} \equiv R/Z$ with $0 \leq \rho(\theta) \leq 1$ and with the weak topology. For each continuous function $G(\theta) \in C(\mathscr{T})$, we let

$$(5)\quad \langle \rho, G \rangle = \int_{\mathscr{T}} G(\theta) \rho(\theta)\, d\theta.$$

We next use the discrete process η_t to define an empirical density,

$$(6)\quad \mu_t^N(\theta, \eta_t) = \sum_{i=1}^{N} \eta_t(i) 1_{\{(i/N,(i+1)/N]\}}(\theta).$$

In words, μ_t^N is given by a density with value 1 on the interval $(i/N, (i+1)/N]$ at time t if $\eta_t(i) = 1$. This is effectively assigning mass i/N to each particle. We note that $\mu_\cdot^N \in D([0,T], M_1)$, the space of right continuous functions from any time interval $[0,T]$ to the measurable functions M_1 with left-hand limits. We choose the initial distribution of the process to be product measure ν_γ^N, where $\gamma(\theta) \in M_1$ and $\nu_\gamma^N(\eta(i) = 1) = \gamma(i/N)$, and we assume that the associated empirical densities $\mu_0^N(\theta, \eta_0)$ converge weakly to γ as $N \to \infty$. The system is out of equilibrium unless γ is taken to be a constant. We denote the path measure of the process by $\mathbf{P}_N^\gamma$.

THEOREM. *For the process* η_t *defined by* (2), *assume that the jump rates satisfy* $c(1) = 1$, $c(j+1) \leq c(j)$ *for all* $j \geq 1$, *and that* $c(-j) = c(j)$. *If the* *initial density profile is measurable with $\gamma(\theta) < 1$ for all $\theta \in \mathscr{T}$, then, with the initial distribution ν_γ^N selected as above, as $N \to \infty$ the empirical density μ^N given in* (6) *converges weakly to the unique weak solution of*

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \theta}\left(D(\rho) \frac{\partial \rho}{\partial \theta} \right) \tag{7}$$

with

$$\rho(0, \theta) = \gamma(\theta) \quad \text{and} \quad D(\rho) = \sum_{j=1}^{\infty} c(j) j^2 \rho^{j-1}. \tag{8}$$

SPECIAL CASE 1. If $c(j) \equiv 1$ for all j, then we have the self-organizing process introduced and studied in Carlson, Chayes, Grannan and Swindle (1990b) in which 1's hop to the nearest vacant site to the left and right at rate 1 each. In this case $D(\rho) = (1 + \rho)/(1 - \rho)^3$.

SPECIAL CASE 2. The jump rates $c(j) = 1/j$ correspond to the nearest-neighbor, symmetric long range exclusion process [Spitzer (1970) and Liggett (1980)], and in this case $D(\rho) = 1/(1 - \rho)^2$.

SPECIAL CASE 3. If $c(j) \sim 1/j^\lambda$ with $0 \le \lambda \le 3$ asymptotics reveals that the diffusion coefficient (8) is singular as $\rho \to 1$, where the singularity is a pole of order $\phi = 3 - \lambda$ for $\lambda < 3$, and is logarithmic for $\lambda = 3$.

The proof of the theorem is given in the next section.

3. Preliminary results and proof of the theorem. Before we proceed with the proof of the theorem, we will present two useful features of the systems that we are considering, reversibility and attractiveness. It is primarily these features which make the models we study tractable. We then use the reversibility of the processes to establish a fast (superexponential) convergence result, which plays a key role in the proof of the theorem which follows.

We begin with the reversibility of the systems (i.e., these processes satisfy a detailed balance condition).

LEMMA 3.1. *The process η_t described by* (2) *is reversible with respect to product measure ν_α for any constant density $\alpha \in [0, 1]$.*

REMARK ON THE PROOF. The process is reversible if and only if the generator [equation (2)] is self-adjoint, that is,

$$\int f L g \, d\nu_\alpha = \int g L f \, d\nu_\alpha \tag{9}$$

for all functions f and g on X_N. This result is easily established [see Liggett (1985)]. □

The next step is to establish attractiveness, essentially a partial ordering of the configurations. This property is used later in the paper to limit the effective range of interaction when the local density is less than 1.

LEMMA 3.2. *If the jump rates $c(j)$ are nonincreasing in j, then the process η_t defined by (2) is attractive.*

REMARK. The restriction that $c(j)$ be a nonincreasing function of distance j plays a crucial role in establishing this property. Attractiveness, and consequently our proof, breaks down when $c(j)$ increases with j, although we suspect that the hydrodynamic description may still be valid even when $c(j)$ increases algebraically with j.

PROOF. We use the basic coupling of two versions of the process starting from two different initial configurations $\eta_0^{(1)}$ and $\eta_0^{(2)}$ which are ordered in the sense that $\eta_0^{(1)}(x) = 1 \Rightarrow \eta_0^{(2)}(x) = 1$, so that this ordering is preserved in time. The coupling is straightforward: If a 1 at site x in the $\eta_t^{(2)}$ process jumps, and if $\eta_t^{(1)}(x) = 1$, then the 1 at site x also jumps (in the same direction) in the $\eta^{(1)}$ process. This coupling is possible due to the monotonicity of the $c(j)$'s . It is easily checked that the ordering is preserved, which is the desired result. □

The last of our preliminary results is a superexponential estimate, which is the heart of the proof of the hydrodynamic limit. In essence, here we focus our attention on a large block of sites, which is still small relative to the total number N of sites on the discrete torus $\mathscr{T}_N$. We show that the empirical average of a general cylinder function on the block of sites is essentially equal to the expected value of the cylinder function with respect to product measure at the local density of the block. In the proof of the theorem we will choose the cylinder function to be the local transition rates. The superexponential estimate reduces the dependence of these transition rates on details of the local configuration to a simple dependence on the local density field. This allows us to close the system of equations which describe the particle system as we take the diffusion limit.

LEMMA 3.3. *Let η_t be the process defined in the theorem, and let $\mathbf{P}_N^\gamma$ denote the associated path measure at density γ. For any $\delta > 0$ and any cylinder function ϕ,*

$$(10)\qquad \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbf{P}_N^\gamma\left[\int_0^t\frac{1}{N}V_{N,\varepsilon}(\eta_s)\,ds > \delta\right] = -\infty,$$

where $V_{N,\varepsilon}(\eta)$ is given by

$$(11)\quad V_{N,\varepsilon}(\eta) = \sum_{i=1}^{N}\left|\frac{1}{2\varepsilon N+1}\sum_{|i-j|\le N\varepsilon}\tau_j\phi(\eta) - \hat{\phi}\left(\frac{1}{2\varepsilon N+1}\sum_{|i-j|\le N\varepsilon}\eta(j)\right)\right|.$$

Here τ_j is the spatial shift operator and $\hat{\phi}(\alpha)$ is the expectation of the cylinder function with respect to product measure at the local density α, $\hat{\phi}(\alpha) \equiv E_{\nu_\alpha}(\phi)$.

PROOF. The proof of this result is outlined in the Appendix, and is based on techniques developed in Guo, Papanicolaou and Varadhan (1988) and Kipnis, Olla and Varadhan (1989).

We now proceed to the proof of the theorem. Several supporting lemmas will appear throughout, with the proofs provided in the Appendix.

PROOF OF THEOREM. Our goal is to show that any limit point $\rho(t,\theta)$ of the sequence of discrete empirical densities μ_t^N [equation (6)] satisfies the weak form of the singular diffusion equation (7). In other words, for any $G(t,\theta) \in C^{1,3}([0,T] \times \mathscr{T})$, and for any subsequential limit $\rho(t,\theta)$:

$$\begin{aligned}\langle \rho(t,\theta), G(t,\theta)\rangle - \langle \rho(0,\theta), G(0,\theta)\rangle - \int_0^t \left\langle \rho(s,\theta), \frac{\partial G}{\partial s}(s,\theta)\right\rangle ds \\ - \int_0^t \left\langle F[\rho(s,\theta)], \frac{\partial^2 G}{\partial \theta^2}(s,\theta)\right\rangle ds = 0,\end{aligned} \tag{12}$$

where $F'(\rho) = D(\rho)$, the diffusion coefficient given in (8).

We begin by introducing the martingale

$$\begin{aligned}M_t^G = \frac{1}{N}\sum_{i=1}^N \Bigg\{ \eta_t(i) G\left(t, \frac{i}{N}\right) - \eta_0(i) G\left(0, \frac{i}{N}\right) \\ - \int_0^t \eta_s(i) \frac{\partial G}{\partial s}\left(s, \frac{i}{N}\right) ds - \int_0^t G\left(s, \frac{i}{N}\right) L_N[\eta_s(i)]\, ds \Bigg\},\end{aligned} \tag{13}$$

which should be viewed as a discrete form of (12). We will show that M_t^G converges in probability both to 0 and to (12), thereby identifying the weak limit in $D([0,T], M_1)$ of μ_t^N as the unique weak solution to (7). We denote the quadratic variation of M_t^G by Q_t^G, and a direct calculation yields

$$Q_t^G = \sum_{i=1}^N \sum_{j=-N}^N \int_0^t \left\{ \left[G\left(s, \frac{i+j}{N}\right) - G\left(s, \frac{i}{N}\right)\right]^2 c(j) J_{\eta_s}(i,j) \right\} ds. \tag{14}$$

The following lemma states that the quadratic variation of M_t^G vanishes as N diverges.

LEMMA 3.4. *Let η_t be the process described in the theorem, and let Q_t^G be given by* (14), *where $G(t,\theta) \in C^{1,3}([0,T] \times \mathscr{T})$. Denoting expectation with respect to the path measure $\mathbf{P}_N^\gamma$ by $\mathbf{E}_N^\gamma$, it follows that*

$$\lim_{N\to\infty} \mathbf{E}_N^\gamma\{Q_t^G\} = 0. \tag{15}$$

PROOF. See the Appendix.

To complete our calculations we will also need:

LEMMA 3.5. *The sequence of measures* $\{\mathbf{P}_N^\gamma\}$ *is relatively compact.*

PROOF. See the Appendix.

Lemma 3.5 implies the existence of subsequential limits, and we will now show that any limit point concentrates on paths which satisfy (12). The first three terms in (13) will converge to their counterparts in (12) due to the fact that only the empirical density μ_t^N appears in these terms. The real work is associated with the last term, which involves products of the occupation events $\{\eta_t(i) = 1\}$. We begin by rearranging the series into one indexed by the occupied sites $\{i/N\colon \eta_t(i) = 1\}$, and then splitting the result into two terms corresponding to jump sizes less than and greater than some integer C (this is done in the calculation below). The plan is to apply the superexponential estimate (Lemma 3.3) as N diverges to the rate corresponding to jump size less than C, and then to let C diverge. It is in the second step that we require the maximum principle provided by attractiveness (Lemma 3.2), to show that the terms associated with jumps exceeding C in size vanishes as $N \to \infty$. The third term in (13), which we denote by F_t^G, is

$$
\begin{aligned}
F_t^G &= \int_0^t \frac{1}{N} \sum_{i=1}^N G\left(\frac{i}{N}\right) L_N[\eta_s(i)]\, ds \\
&= \int_0^t N \sum_{i=1}^N \eta_s(i) \sum_{j=1}^N \left\{ \left[G\left(\frac{i+j}{N}\right) - G\left(\frac{i}{N}\right)\right] c(j) J_{\eta_s}(i,j) \right. \\
&\qquad\qquad \left. + \left[G\left(\frac{i-j}{N}\right) - G\left(\frac{i}{N}\right)\right] c(j) J_{\eta_s}(i,-j) \right\} ds \\
&= \int_0^t N \sum_{i=1}^N \sum_{j=1}^C \left\{ \left[G\left(\frac{i+j}{N}\right) - G\left(\frac{i}{N}\right)\right] c(j) J_{\eta_s}(i,j) \right. \\
&\qquad\qquad \left. + \left[G\left(\frac{i-j}{N}\right) - G\left(\frac{i}{N}\right)\right] c(j) J_{\eta_s}(i,-j) \right\} ds \qquad (16)\\
&\quad + \int_0^t N \sum_{i=1}^N \sum_{j=C+1}^N \left\{ \left[G\left(\frac{i+j}{N}\right) - G\left(\frac{i}{N}\right)\right] c(j) J_{\eta_s}(i,j) \right. \\
&\qquad\qquad \left. + \left[G\left(\frac{i-j}{N}\right) - G\left(\frac{i}{N}\right)\right] c(j) J_{\eta_s}(i,-j) \right\} ds,
\end{aligned}
$$

where we have neglected to write the temporal coordinate of G for convenience. We will write the last expression in (16) as

$$F_t^G = F_t^{G,C} + \mathscr{E}_t^{G,C}(N), \tag{17}$$

where $F_t^{G,C}$ corresponds to the contribution from jumps of size C or less, and $\mathscr{E}_t^{G,C}(N)$ corresponds to jumps greater than C. Working first on $F_t^{G,C}$, we regroup terms and integrate the sum by parts to obtain

$$\begin{aligned}
F_t^{G,C} &= \int_0^t N \sum_{i=1}^N \sum_{j=1}^C \left\{ \left[G\left(\frac{i-j}{N}\right) - G\left(\frac{i}{N}\right) \right] \right. \\
&\qquad \left. \times c(j)\left[J_{\eta_s}(i,-j) - J_{\eta_s}(i-j,j) \right] \right\} ds \\
&= \int_0^t N \sum_{i=1}^N \sum_{j=1}^C \left\{ \left[G\left(\frac{i-j}{N}\right) - G\left(\frac{i}{N}\right) \right] \right. \\
&\qquad \left. \times c(j)\left[I_{\eta_s}(i,-j+1) - I_{\eta_s}(i-1,-j+1) \right] \right\} ds \\
&= \int_0^t N \sum_{i=1}^N \sum_{j=1}^C c(j) I_{\eta_s}(i,-j+1) \left[G\left(\frac{i-j}{N}\right) - G\left(\frac{i}{N}\right) \right. \\
&\qquad \left. \left. - G\left(\frac{i+1-j}{N}\right) + G\left(\frac{i+1}{N}\right) \right] \right\} ds.
\end{aligned} \tag{18}$$

Using smoothness we find

$$F_t^{G,C} = \int_0^t \frac{1}{N} \sum_{i=1}^N \left\{ \left[\frac{\partial^2 G}{\partial \theta^2}\left(\frac{i}{N}\right) + O\left(\frac{C}{N}\right) \right] \sum_{j=1}^C \left[jc(j) I_{\eta_s}(i,-j+1) \right] \right\} ds. \tag{19}$$

Our next step is to convert this form of $F_t^{G,C}$ into a form which is vulnerable to the superexponential estimate. To do this we show that the transition rates appearing above can be replaced with rates averaged over boxes of $2\varepsilon N + 1$ sites. More specifically, for any fixed C, given any $a > 0$, if ε is small enough and N is large enough, then

$$\begin{aligned}
&\left| F_t^{G,C} - \int_0^t \frac{1}{N} \sum_{i=1}^N \frac{\partial^2 G}{\partial \theta^2}\left(\frac{i}{N}\right) \sum_{j=1}^C jc(j) \frac{1}{2\varepsilon N + 1} \right. \\
&\qquad \left. \times \sum_{|k-i| \le N\varepsilon} I_{\eta_s}(k,-j+1)\, ds \right| \le a
\end{aligned} \tag{20}$$

almost surely. Now, the superexponential estimate implies that for any C,

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log \mathbf{P}_N^{\gamma}\tag{21}$$
$$\times\Bigg(\frac{1}{N}\int_0^t\Bigg|\sum_{i=1}^{N}\frac{\partial^2 G}{\partial\theta^2}\Big(\frac{i}{N}\Big)\sum_{j=1}^{C} jc(j)\frac{1}{2\varepsilon N+1}\sum_{|k-i|\le N\varepsilon} I_{\eta_s}(k,-j+1)$$
$$-\sum_{i=1}^{N}\frac{\partial^2 G}{\partial\theta^2}\Big(\frac{i}{N}\Big)\sum_{j=1}^{C} jc(j)\big(\mu_s^N*\alpha_\varepsilon\big)^j\Bigg|\,ds\ge a\Bigg)$$
$$=-\infty,$$

where $\alpha_\varepsilon(\theta)=(1/2\varepsilon)1_{[-\varepsilon,\varepsilon]}(\theta)$ and $*$ denotes convolution. Note also that if $\rho(\cdot,\theta)$ denotes a weak subsequential limit of $\mu_\cdot^N$, then along the subsequence,

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\langle(\mu_t^N*\alpha_\varepsilon)^j,G\rangle\to\langle\rho(t,\cdot)^j,G\rangle.\tag{22}$$

Finally we will need to take $C\to\infty$, and to do this we use two results stated in the following lemmas. The first states that the term $\mathscr{E}_t^{G,C}(N)$ associated with jumps of size exceeding C vanishes uniformly in probability as $C\to\infty$.

LEMMA 3.6. *For any* $\delta>0$,

$$\limsup_{C\to\infty}\sup_{N>C}\mathbf{P}_N^{\gamma}\Big(\sup_{0\le t\le T}\big|\mathscr{E}_t^{G,C}(N)\big|>\delta\Big)=0.\tag{23}$$

PROOF. See the Appendix.

The second result is a maximum principle for any subsequential limit, which is an immediate consequence of attractiveness.

LEMMA 3.7. *Any subsequential limit* ρ *of* $\{\mu_\cdot^N\}$ *satisfies* $0\le\rho\le\hat{\gamma}$ *(a.e.).*

PROOF. See the Appendix.

Lemma 3.4 and (20) and (21) imply that

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbf{P}_N^{\gamma}\Bigg(\sup_{0\le t\le T}\Bigg|\langle\mu_t^N(\eta),G(t,\theta)\rangle-\langle\mu_0^N(\eta),G(0,\theta)\rangle\tag{24}$$
$$-\int_0^t\Big\langle\mu_s^N,\frac{\partial G}{\partial s}(s,\theta)\Big\rangle\,ds$$
$$-\int_0^t\Big\langle\sum_{j=1}^{C} jc(j)\big[\mu_s^N*\alpha_\varepsilon\big]^j,\frac{\partial^2 G}{\partial\theta^2}(s,\theta)\Big\rangle\,ds+\mathscr{E}_t^{G,C}(N)\Bigg|>\delta\Bigg)=0.$$

Recalling (22), we now take the limit $C \to \infty$. Lemma 3.6 annihilates the $\mathscr{E}_t^{G,C}(N)$ term, and Lemma 3.7 guarantees convergence of the series. Therefore, we see that for any $\delta > 0$, any weak limit of $\{\mu_\cdot^N\}$ is concentrated on paths $\rho(\cdot, \theta)$ such that for any $\delta > 0$, for all $t \in [0, T]$,

$$\begin{aligned}\Bigg| \langle \rho(t,\theta), G(t,\theta)\rangle - \langle \rho(0,\theta), G(0,\theta)\rangle - \int_0^t \left\langle \rho(s,\theta), \frac{\partial G}{\partial s}(s,\theta)\right\rangle ds \\ - \int_0^t \left\langle \sum_{j=1}^{\infty} jc(j)[\rho(s,\theta)]^j, \frac{\partial^2 G}{\partial \theta^2}(s,\theta) \right\rangle ds \Bigg| \le \delta.\end{aligned} \tag{25}$$

Since δ was arbitrary, the proof of the theorem is complete, provided that we have uniqueness of the subsequential limits.

To obtain uniqueness, note that $F(\rho)$ is an increasing, differentiable function of ρ on the interval $[0, 1)$. Additionally, Lemma 3.7 implies that we can consider the limiting diffusion equation with a modified diffusion coefficient,

$$\hat{D}(\rho) = \begin{cases} D(\rho), & \text{if } \rho \le \hat{\gamma}, \\ D(\hat{\gamma}), & \text{if } \rho > \hat{\gamma}, \end{cases} \tag{26}$$

and uniqueness follows from results in Aronson, Crandall and Peletier (1982). □

4. Conclusion. We have shown that a class of reversible, attractive interacting particle systems have diffusion limits which can exhibit a singularity at unit density. In particular, if we take the jumps rates to be $c(j) = j^{-\lambda}$, standard asymptotic analysis reveals that when the rates $c(j)$ decay rapidly enough ($\lambda > 3$), the diffusion coefficient is analytic. On the other hand, when the decay of the jump rates is slow enough ($\lambda < 3$), then the diffusion coefficient will have a pole of order $\phi = 3 - \lambda$ at unit density (the singularity is logarithmic when $\lambda = 3$). This singular behavior can be associated with critical phenomena on the torus [Carlson, Chayes, Grannan and Swindle (1990a, b)], which for these systems is particularly simple. Viewing the density of 0's as the order parameter $p_0 = 1 - \rho$, it follows immediately from the transition rules that p_0 is strictly conserved. Furthermore, as the density ρ approaches unity (the critical point), p_0 goes to 0, signaling a transition.

The concept of *self-organized* criticality, however, is associated with the appearance of critical behavior on open driven systems, without the special tuning of a parameter (in this case, the density). Indeed, numerical simulations of a broad class of particle systems of the type discussed here but on an open driven lattice, where the density is conserved only locally, are seen to exhibit self-organized scaling behavior. In this context the appearance of singular diffusion equations is more than a curiosity. In particular, the scaling behavior of open driven versions of these systems can be derived from the stationary solutions of the singular diffusion limits with appropriate boundary conditions [Carlson, Chayes, Grannan and Swindle (1990a)]. Furthermore, the scaling behavior is seen to depend exclusively on the order of the pole, the driving mechanism and dimension.

While this work establishes the hydrodynamic behavior of a rather large class of systems on a torus, some important questions remain. In particular, hydrodynamic limits for open systems, with either fixed densities at the boundaries or with the driving mechanism described in Carlson, Chayes, Grannan and Swindle (1990b), remain to be established. Additionally, our use of attractiveness required that the jump rates $c(j)$ be nonincreasing. This is probably not essential to the validity of the hydrodynamic limit, although establishing singular diffusion limits without attractiveness remains an open problem.

APPENDIX

Proof of auxiliary results. This appendix contains the proofs of Lemmas 3.3 through 3.7.

PROOF OF LEMMA 3.3. We present an outline of the proof of the superexponential estimate for the benefit of the reader. Only minor changes from the original proof which appears in Kipnis, Olla and Varadhan (1989) are required.

First we note that it is sufficient to establish (10) when $\gamma(\theta) \equiv 1/2$. To see that this is true, take any set $B \subset D([0,T], X_N)$ and note that

$$\mathbf{P}_N^{\gamma}(B) = \sum_{\eta \in X_N} \left(d\nu_{\gamma}^N / d\nu_{1/2}^N\right)(\eta) \mathbf{P}_N^{\eta}(B) \nu_N^{1/2}(\eta) \le 2^N \mathbf{P}_N^{1/2}(B) \tag{27}$$

since

$$|d\nu_{\gamma}^N / d\nu_{1/2}^N| \le 2^N. \tag{28}$$

Taking $\gamma \equiv 1/2$, we will now bound the expression in (10) in terms of the largest eigenvalue of the operator $L_N + aV_{N,\varepsilon}$ on the space $L^2(X_N, \nu_{1/2})$, where a is any real number. Notice that by Lemma 3.1 this operator is self-adjoint, so, using the Feynman–Kac formula and the spectral theorem we have

$$E^{\mathbf{P}_N^{1/2}}\{e^{a\int_0^t V_{N,\varepsilon}(\eta_s)\,ds}\} \le e^{t\lambda_{N,\varepsilon}(a)}, \tag{29}$$

where $\lambda_{N,\varepsilon}(a)$ is the largest eigenvalue of $L_N + aV_{N,\varepsilon}$.

Using (29) with Chebyshev's inequality,

$$\begin{aligned} \mathbf{P}_N^{1/2}\left(\frac{1}{N}\int_0^t V_{N,\varepsilon}(\eta_s)\,ds \ge \delta\right) &\le e^{t[\lambda_{N,\varepsilon}(a) - N\delta a]} \\ &= e^{N[(t/N)\lambda_{N,\varepsilon}(a) - \delta a]}. \end{aligned} \tag{30}$$

We now see that (10) will follow from showing that for any a,

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \lambda_{N,\varepsilon}(a) = 0, \tag{31}$$

since we can then let $a \to \infty$ in (30).

We will use the variational formula for $\lambda_{N,\varepsilon}(a)$. The following class of functions play the role of functions of unit norm:

$$(32)\qquad \mathscr{H}_N = \Big\{ f_N\colon X_N \to R \text{ s.t. } f_N(\eta) \ge 0 \text{ and } \sum_{\eta \in X_N} f_N(\eta) 2^{-N} = 1 \Big\}.$$

We have

$$(33)\qquad \lambda_{N,\varepsilon}(a) = \sup_{f_N \in \mathscr{H}_N} \Big\{ a \sum_{\eta \in X_N} V_{N,\varepsilon}(\eta) f_N(\eta) 2^{-N} - N^2 D_N(f_N) \Big\},$$

where D_N is the Dirichlet form

$$(34)\qquad D_N(f_N) = -\langle \sqrt{f_N}, L\sqrt{f_N} \rangle_N$$

with $\langle \phi, \psi \rangle_N = \Sigma_{\eta \in X_N} 2^{-N} \phi(\eta)\psi(\eta)$. The factor N^2 in (33) is a result of the diffusion scaling of the rates. The Dirichlet form is

$$(35)\qquad \begin{aligned} D_N(f_N) = \tfrac{1}{2} \sum_{\eta \in X_N} 2^{-N} \sum_{i=1}^{N} \sum_{j=-N}^{N} \Big\{ & c(j) J_\eta(i,j) \\ & \times \Big[\sqrt{f(\eta^{i,i+j})} - \sqrt{f(\eta)} \Big]^2 \Big\}. \end{aligned}$$

The next step is to use the convexity of the Dirichlet form D_N to reduce the class of functions considered to those elements of $\mathscr{H}_N$ which are translation invariant. That D_N is convex is an immediate consequence of the fact that for any nonnegative sequences a_i and b_i,

$$(36)\qquad \Big[\sqrt{\textstyle\sum a_i} - \sqrt{\textstyle\sum b_i} \Big]^2 \le \sum \big[\sqrt{a_i} - \sqrt{b_i} \big]^2$$

which follows from Hölder's inequality.

D_N and $V_{N,\varepsilon}$ are translation invariant [i.e, $D_N(f_N) = D_N(\tau_j f_N)$ for any j where τ_j is the shift operator, and likewise for $V_{N,\varepsilon}$], so (33) implies that the maximum eigenvalue satisfies

$$(37)\qquad \begin{aligned} \lambda_{N,\varepsilon}(a) &= \sup_{f_N \in \mathscr{H}_N} \Big\{ a \sum_{\eta \in X_N} 2^{-N} V_{N,\varepsilon}(\eta) \Big[\frac{1}{N} \sum_{i=1}^{N} \tau_i f_N(\eta) \Big] \\ &\qquad\qquad - N^2 \Big[\frac{1}{N} \sum_{i=1}^{N} D_N(\tau_i f_N) \Big] \Big\} \\ &\le \sup_{f_N \in \mathscr{H}_N} \Big\{ a \sum_{\eta \in X_N} 2^{-N} V_{N,\varepsilon}(\eta) \Big[\frac{1}{N} \sum_{i=1}^{N} \tau_i f_N(\eta) \Big] \\ &\qquad\qquad - N^2 D_N \Big\{ \Big[\frac{1}{N} \sum_{i=1}^{N} (\tau_i f_N) \Big] \Big\} \Big\}, \end{aligned}$$

where the second step used the convexity of D_N. Note that $(1/N)\Sigma_{i=1}^{N} \tau_i f_N$ is a translation invariant function, implying that it is sufficient to take the supremum in (33) over the translation invariant subset of $\mathscr{H}_N$:

$$f \in \mathscr{U}_N = \{ f_N \in \mathscr{H}_N : \tau_i f_N = f_N \}. \tag{38}$$

Noting that there exists a $C < \infty$ so that $V_{N,\varepsilon}(\eta) \le CN$ uniformly in η, if f_N is such that $D_N(f_N) > C/N$, then by (33), (31) holds. Therefore, (31) follows from

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \sup_{f_N \in \mathscr{B}_N} \frac{1}{N} \sum_{\eta \in X_N} 2^{-N} V_{N,\varepsilon} f_N(\eta) = 0, \tag{39}$$

where

$$\mathscr{B}_N = \left\{ f_N \in \mathscr{U}_N : D_N(f_N) \le \frac{C}{N} \right\}. \tag{40}$$

By translation invariance it is sufficient to show that

$$\begin{aligned} \limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \sup_{f_N \in \mathscr{B}_N} \sum_{\eta \in X_N} \Bigg| & \frac{1}{2\varepsilon N + 1} \sum_{|j| \le \varepsilon N} \tau_j \phi(\eta) \\ & - \hat{\phi}\Bigg(\frac{1}{2\varepsilon N + 1} \sum_{|j| \le \varepsilon N} \eta(j) \Bigg) \Bigg| f_N(\eta) 2^{-N} = 0. \end{aligned} \tag{41}$$

The first step in establishing (41) is to observe that for each configuration η,

$$\begin{aligned} & \frac{1}{2\varepsilon N + 1} \sum_{|j| \le \varepsilon N} \tau_j \phi(\eta) \\ & = \frac{1}{2\varepsilon N + 1} \sum_{|j| \le \varepsilon N} \left\{ \frac{1}{2k+1} \sum_{|j-l| \le k} \tau_l \phi(\eta) \right\} + O\left(\frac{k}{N} \right) \end{aligned} \tag{42}$$

and that

$$\begin{aligned} & \left| \frac{1}{2\varepsilon N + 1} \sum_{|j| \le \varepsilon N} \tau_j \phi(\eta) - \hat{\phi}\left(\frac{1}{2\varepsilon N + 1} \sum_{|j'| \le \varepsilon N} \eta(j') \right) \right| \\ & \le \frac{1}{2\varepsilon N + 1} \sum_{|j| \le \varepsilon N} \left| \frac{1}{2k+1} \sum_{|j-l| \le k} \tau_l \phi(\eta) - \hat{\phi}\left(\frac{1}{2k+1} \sum_{|j-l| \le k} \eta(l) \right) \right| \\ & \quad + \|\hat{\phi}'\|_\infty \left(\frac{1}{2\varepsilon N + 1} \right)^2 \sum_{|j| \le \varepsilon N} \sum_{|j'| \le \varepsilon N} \Bigg| \frac{1}{2k+1} \sum_{|j'-l| \le k} \eta(l) \\ & \qquad - \frac{1}{2k+1} \sum_{|j-l| \le k} \eta(l) \Bigg| + O\left(\frac{k}{N} \right), \end{aligned} \tag{43}$$

which follows from a sequence of applications of the triangle inequality and Hölder's inequality. Therefore, (41) will follow as a consequence of the following claims.

CLAIM.

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\sup_{f_N\in\mathscr{B}_N}\sum_{\eta\in X_N}\left|\frac{1}{2k+1}\sum_{|j|\le k}\tau_j\phi(\eta) - \hat{\phi}\left(\frac{1}{2k+1}\sum_{|j|\le k}\eta(j)\right)\right| f_N(\eta)2^{-N} = 0. \tag{44}$$

CLAIM.

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_{r:|r|\le N\varepsilon}\sup_{f_N\in\mathscr{B}_N}\sum_{\eta\in X_N}\left|\frac{1}{2k+1}\sum_{|j|\le k}\eta(j) - \frac{1}{2k+1}\sum_{|j+r|\le k}\eta(j)\right| f_N(\eta)2^{-N} = 0. \tag{45}$$

PROOF OF (44) (One-block estimate). For any function g of $X_{2k+1} = \{0,1\}^{2k+1}$ (think of this as the space of configurations of a $2k+1$-site block of the N-site torus), let

$$D_k^*(g) = \tfrac{1}{2}\sum_{\eta\in X_{2k+1}} 2^{-(2k+1)}\sum_{i=-k}^{k}\sum_{j:|j+i|\le k} \times\left\{c(j)J_\eta(i,j)\left[\sqrt{g(\eta^{i,i+j})} - \sqrt{g(\eta)}\right]^2\right\} \tag{46}$$

be the Dirichlet form corresponding to the original process η_t restricted to X_{2k+1}. Next we define what are effectively the marginal distributions of functions f_N on X_{2k+1}:

$$\mathscr{H}_k = \left\{ f_N^k : f_N^k(\eta) = 2^{(-N+2k+1)}\sum_{\eta(n):|n|>k} f_N(\eta)\right\}. \tag{47}$$

For any function in $\mathscr{H}_k$, (46) yields

$$\begin{aligned} D_k^*(f_N^k) &= \tfrac{1}{2}\sum_{\eta(n):|n|\le k} 2^{-N}\sum_{i=-k}^{k}\sum_{j:|j+i|\le k}\left\{c(j)J_\eta(i,j)\left[\sqrt{\sum_{\eta(n):|n|>k} f_N(\eta^{i,i+j})} - \sqrt{\sum_{\eta(n):|n|>k} f_N(\eta)}\right]^2\right\} \\ &\le \tfrac{1}{2}\sum_{i=-k}^{k}\sum_{\eta}2^{-N}\sum_{j:|j+i|\le k}\left\{c(j)J_\eta(i,j)\left[\sqrt{f(\eta^{i,i+j})} - \sqrt{f(\eta)}\right]^2\right\} \\ &\le \frac{2k+1}{N}D_N(f_N), \end{aligned}$$

for $f \in \mathscr{B}_N$. The calculation above moves the sum over all coordinates of η outside of the square roots, and the last inequality is due to the additional nonnegative terms associated with transitions outside of the $2k+1$-site block and the fact that we are summing only over $2k+1$ sites out of the total of N sites. We used the convexity of the Dirichlet form in the second step. D_k^*

corresponds to a Markov process on the k-site block. We are dealing with functions f_N such that $D_N(f_N) < C/N$, so the above bound tells us that the Dirichlet form D_k^* converges to 0 as $N \to \infty$. In other words, the $2k+1$-site process is very close to equilibrium when N is large. To make this precise, we begin by replacing the supremum over $\mathscr{B}_N$ in (44) by a supremum over the appropriate functions on X_{2k+1} (acquiring an innocuous error due to the finite range of ϕ):

$$\begin{aligned}(49)\quad &\sup_{f_N \in \mathscr{B}_N} \sum_{\eta \in X_N} \left| \frac{1}{2k+1} \sum_{|j| \le k} \tau_j \phi(\eta) - \hat{\phi}\left(\frac{1}{2k+1} \sum_{|j| \le k} \eta(j) \right) \right| f_N(\eta) 2^{-N} \\ &\le \sup_{g_k \in \mathscr{G}_k} \sum_{\eta \in X_{2k+1}} \left| \frac{1}{2k+1} \sum_{|j| \le k} \tau_j \phi(\eta) - \hat{\phi}\left(\frac{1}{2k+1} \sum_{|j| \le k} \eta(j) \right) \right| \\ &\qquad \times g_k(\eta) 2^{-(2k+1)} + O\left(\frac{1}{k}\right),\end{aligned}$$

where

$$(50)\qquad \mathscr{G}_k = \left\{ g_k \in \mathscr{H}_k \colon D_k^*(g_k) \le \frac{(2k+1)C}{N^2} \right\}.$$

The compactness of the level sets of D_k^* and the fact that $(2k+1)C/N^2 \to 0$ as $N \to \infty$ implies that,

$$\begin{aligned}&\limsup_{N \to \infty} \sup_{g_k \in \mathscr{G}_k} \sum_{\eta \in X_{2k+1}} \left| \frac{1}{2k+1} \sum_{|j| \le k} \tau_j \phi(\eta) - \hat{\phi}\left(\frac{1}{2k+1} \sum_{|j| \le k} \eta(j) \right) \right| \\ &\qquad \times g_k(\eta) 2^{-(2k+1)} + O\left(\frac{1}{k}\right) \\ (51)\quad &= \sup_{g_k \colon D_k^*(g_k) = 0} \sum_{\eta \in X_{2k+1}} \left| \frac{1}{2k+1} \sum_{|j| \le k} \tau_j \phi(\eta) - \hat{\phi}\left(\frac{1}{2k+1} \sum_{|j| \le k} \eta(j) \right) \right| \\ &\qquad \times g_k(\eta) 2^{-(2k+1)} + O\left(\frac{1}{k}\right).\end{aligned}$$

We can take $g(\eta) \ge 0$, and recalling the definition of $\mathscr{G}_k$ we can view $g_k(\eta) 2^{-(2k+1)}$ as a probability distribution on X_{2k+1}. Therefore, the relevant set of probability distributions are those with Dirichlet form zero:

$$(52)\qquad S_k = \left\{ g_k(\eta) 2^{-(2k+1)} \colon D_k^*(g_k) = 0 \right\}.$$

Note that S_k is the set of convex combinations of measures with a fixed number of particles l with $0 \le l \le 2k+1$ uniformly distributed on X_{2k+1}. With this fact, the proof is completed upon observing that

$$\begin{aligned}(53)\quad &\limsup_{k \to \infty} \sup_{l \in \{1, \dots, 2k+1\}} \sum_{\eta \colon \Sigma_j \eta(j) = l} \binom{2k+1}{l}^{-1} \\ &\qquad \times \left\{ \frac{1}{2k+1} \sum_{j=1}^{2k+1} \tau_j \phi(\eta) - \hat{\phi}\left(\frac{l}{2k+1} \right) \right\} = 0,\end{aligned}$$

which follows from the fact that uniform distributions on the torus converge to product measure in the sense of uniform convergence of finite dimensional distributions (ϕ is a cylinder function).

PROOF OF (45) (Two-block estimate). This proof involves two finite blocks of sites in the same way the previous proof involved one block. Let

$$f_N^{r,k} = 2^{[-N+2(2k+1)]} \sum_{\substack{\eta(j):\ |j|>k \\ |r-j|>k}} f_N(\eta). \tag{54}$$

For $g(\eta_1, \eta_2)$ on $X_{2k+1} \times X_{2k+1}$ let

$$\begin{aligned} D_k^1(g) = \tfrac{1}{2} & \sum_{(\eta_1,\eta_2)\in X_{2k+1}\times X_{2k+1}} 2^{-2(2k+1)} \\ & \times \left\{ c(j) J_{\eta_1}(i,j) \left[\sqrt{g(\eta_1^{i,i+j}, \eta_2)} - \sqrt{g(\eta_1,\eta_2)} \right]^2 \right\} \end{aligned} \tag{55}$$

and

$$\begin{aligned} D_k^2(g) = \tfrac{1}{2} & \sum_{\eta_1,\eta_2} 2^{-2(2k+1)} \sum_{i=-k}^{k} \sum_{j:\ |j+i|\le k} \\ & \times \left\{ c(j) J_{\eta_2}(i,j) \left[\sqrt{g(\eta_1, \eta_2^{i,i+j})} - \sqrt{g(\eta_1,\eta_2)} \right]^2 \right\}. \end{aligned} \tag{56}$$

The configurations η_1 and η_2 represent the configuration in two disjoint blocks of sites in η of size $2k + 1$ each; this is essentially what we have in (45). The goal is to extract a product measure of the same density in both blocks in the limit, and to do this we need to have a coupling between the two blocks. This is accomplished via

$$\Delta_k(g) = \tfrac{1}{2} \sum_{\eta_1,\eta_2} 2^{-2(2k+1)} \left[\sqrt{g((\eta_1,\eta_2)^0)} - \sqrt{g(\eta_1,\eta_2)} \right]^2, \tag{57}$$

where $(\eta_1, \eta_2)^0$ is the configuration with $\eta(0)$ exchanged with $\eta(r)$. Now, as before,

$$D_k^1(f_N^{r,k}) \le \frac{2k+1}{N} D_N(f_N), \qquad D_k^2(f_N^{r,k}) \le \frac{2k+1}{N} D_N(f_N), \tag{58}$$

so we only need to ponder Δ_k:

$$\begin{aligned} \Delta_k(f_N^{r,k}) &= \tfrac{1}{2} \sum_{\substack{\eta(j):\ |j|\le k \\ |r-j|\le k}} 2^{-2(2k+1)} \Bigg[\sqrt{\sum_{\substack{\eta(j):\ |j|>k \\ |r-j|>k}} f_N(\eta^{0,r})} \\ &\qquad\qquad - \sqrt{\sum_{\substack{\eta(j):\ |j|>k \\ |r-j|>k}} f_N(\eta)} \Bigg]^2 2^{-N+2(2k+1)} \\ &\le \tfrac{1}{2} \sum_{\eta\in X_N} 2^{-N} \left[\sqrt{f_N(\eta^{0,r})} - \sqrt{f_N(\eta)} \right]^2, \end{aligned} \tag{59}$$

where we have again used convexity. The main idea here is that if D_k^1 and D_k^2 are small, then each one separately should be close to a uniform (i.e., asymptotically product) measure. Furthermore, if Δ_k is small, then the two densities must be the same. We therefore need a bound on $\Delta_k(f_N^{r,k})$. Note that we can consider $c(1)$ to be the rate at which each nearest-neighbor pair of spins switch location, independent of configuration. Observe that we can write the switching of the spins at sites 0 and r in $\eta^{0,r}$ as a sequence of nearest neighbor switches:

$$\eta^{0,r} = \left(\cdots\left(\left(\left(\cdots\left(\left(\eta^{0,1}\right)^{1,2}\right)\cdots\right)^{r-1,r}\right)^{r-1,r-2}\right)\cdots\right)^{1,0}. \tag{60}$$

Then, writing $\sqrt{f_N(\eta^{0,r})} - \sqrt{f_N(\eta)}$ as a telescoping series, using Hölder's inequality ($[\Sigma fg]^2 \le \Sigma f^2 \Sigma g^2$, taking $g = 1$), and noting that all transitions associated with longer jumps $\{c(j)\}_{j=2}^{\infty}$ correspond to nonnegative terms in $D_N(f_N)$ yield

$$\Delta_k(f_N^{r,k}) \le \frac{(2r-1)^2}{N} D_N(f_N). \tag{61}$$

We now have three conditions on functions of the configurations on two $2k+1$-site boxes separated by at most εN sites, and we let

$$A_{N,\varepsilon}^k = \left\{ g(\eta_1, \eta_2) \colon D_k^1(g) \le \frac{2k+1}{N^2} C; \right. \\ \left. D_k^2(g) \le \frac{2k+1}{N^2} C; \Delta_k(g) \le \varepsilon^2 C \right\}. \tag{62}$$

Then

$$\begin{aligned} &\sup_{|r| \le \varepsilon N} \sup_{f_N \in \mathscr{B}_N} \sum_{\eta \in X_N} \left| \frac{1}{2k+1} \sum_{|j| \le k} \eta(j) - \frac{1}{2k+1} \sum_{|r-j| \le k} \eta(j) \right| f_N(\eta) 2^{-N} \\ &\quad \le \sup_{g \in A_{N,\varepsilon}^k} \sum_{\eta_1, \eta_2} \left| \frac{1}{2k+1} \sum_{|j| \le k} \eta_1(j) - \frac{1}{2k+1} \sum_{|j| \le k} \eta_2(j) \right| \\ &\quad\quad \times g(\eta_1, \eta_2) 2^{-2(2k+1)} \end{aligned} \tag{63}$$

and the compactness of the level sets implies that it is sufficient to show that

$$\begin{aligned} &\limsup_{k \to \infty} \sup_{\substack{D_k^1(g) = 0 \\ D_k^2(g) = 0 \\ \Delta_k(g) = 0}} \sum_{\eta_1, \eta_2} \left| \frac{1}{2k+1} \sum_{|j| \le k} \eta_1(j) - \frac{1}{2k+1} \sum_{|j| \le k} \eta_2(j) \right| \\ &\quad \times g(\eta_1, \eta_2) 2^{-2(2k+1)} = 0. \end{aligned} \tag{64}$$

Now, $D_k^1(g) = 0$, $D_k^2(g) = 0$ and $\Delta_k(g) = 0$ imply that the uniform distributions are the relevant extremal measures, and as before, the uniform distributions converge to product measure, and the claim is established. □

PROOF OF LEMMA 3.4. The goal is to use the attractiveness of the process and the exponential decay of the probability that an interval of n sites are all occupied at product measure with density less than 1, to show that the quadratic variation (14) vanishes in the limit $N \to \infty$. Let $\hat{\gamma} = \sup_{\theta \in \mathscr{T}} \gamma(\theta)$; by assumption $\hat{\gamma} < 1$. Note that all terms in (14) are nonnegative, and we may put an upper bound on Q_t^G by replacing $J_{\eta_s}(i,j)$ with $I_{\eta_s}(i, j-1)$. Then, by attractiveness and the fact that product measure $\nu_{\hat{\gamma}}$ is invariant, since $I_\eta(i,j)$ is an increasing function of η we have that (14) is bounded by

$$
\begin{aligned}
\mathbf{E}_N^{\hat{\gamma}} \int_0^t \sum_{i=1}^N \sum_{j=-N}^N c(j) I_{\eta_s}(i, j-1)\left[G\left(\frac{i+j}{N}\right) - G\left(\frac{i}{N}\right)\right]^2 ds \\
\le tE_{\nu_{\hat{\gamma}}}\left\{\sum_{i=1}^N \sum_{j=-N}^N c(j) I_\eta(i, j-1)\left[G\left(\frac{i+j}{N}\right) - G\left(\frac{i}{N}\right)\right]^2\right\} \\
\le tE_{\nu_{\hat{\gamma}}} \sum_{i=1}^N \left\{\sum_{|j| \le \sqrt{N}} c(j) I_\eta(i, j-1)\left[G\left(\frac{i+j}{N}\right) - G\left(\frac{i}{N}\right)\right]^2\right. \\
\left. + \sum_{|j| > \sqrt{N}} c(j) I_\eta(i, j-1)\left[G\left(\frac{i+j}{N}\right) - G\left(\frac{i}{N}\right)\right]^2\right\} \\
\le t \sum_{i=1}^N \left\{\left\|\frac{\partial G}{\partial \theta}\right\|_\infty^2 \sum_{|j| \le \sqrt{N}} c(j) \frac{j^2}{N^2} \hat{\gamma}^j + 4\|G\|_\infty^2 \sum_{|j| > \sqrt{N}} c(j) \hat{\gamma}^j\right\} \\
\le t\left\{\left\|\frac{\partial G}{\partial \theta}\right\|_\infty^2 \frac{1}{N} \sum_{|j| \le \sqrt{N}} j^2 c(j) \hat{\gamma}^j + 4\|G\|_\infty^2 N \sum_{|j| > \sqrt{N}} c(j) \hat{\gamma}^j\right\},
\end{aligned} \tag{65}
$$

which vanishes as $N \to \infty$, due to the boundedness of the $c(j)$'s and the exponential tails of the product measure $\nu_{\hat{\gamma}}$. Higher order terms omitted above are easily seen to vanish. □

PROOF OF LEMMA 3.5. We need to establish that, for each $G \in C(\mathscr{T})$,

$$
\lim_{a \to \infty} \limsup_{N \to \infty} \mathbf{P}_N^\gamma\left(\sup_{0 \le t \le T} |\langle \mu_t^N, G\rangle| > a\right) = 0 \tag{66}
$$

and, for any $\varepsilon > 0$,

$$
\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbf{P}_N^\gamma\left(\sup_{\substack{0 \le s < t \le T \\ t-s<\delta}} |\langle \mu_t^N, G\rangle - \langle \mu_s^N G\rangle| > \varepsilon\right) = 0. \tag{67}
$$

The first condition is immediate, and the second is established using techniques developed in Holley and Stroock (1979), which establish (67) using moment conditions from the Censov criterion and Doob's inequality. The computations are analogous to those used to prove Lemma 3.4. □

PROOF OF LEMMA 3.6. Using the integration by parts described in the proof of the theorem [see (18)] we see that

$$\begin{aligned}
&\limsup_{C\to\infty}\limsup_{N\to\infty}\mathbf{P}_N^{\gamma}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t N\sum_{i=1}^{N}\sum_{j=C+1}^{N}c(j)\Bigg\{\Bigg[G\Bigg(\frac{i+j}{N}\Bigg)-G\Bigg(\frac{i}{N}\Bigg)\Bigg]J_{\eta_s}(i,j)\\
&\qquad\qquad+\Bigg[G\Bigg(\frac{i-j}{N}\Bigg)-G\Bigg(\frac{i}{N}\Bigg)\Bigg]J_{\eta_s}(i,-j)\Bigg\}\,ds\Bigg|>\delta\Bigg)\\
&\le\limsup_{C\to\infty}\limsup_{N\to\infty}\mathbf{P}_N^{\gamma}\Bigg(\int_0^T\Bigg|\sum_{j=C+1}^{T}c(j)N\sum_{i=1}^{N}\Bigg\{\Bigg[G\Bigg(\frac{i-j}{N}\Bigg)-G\Bigg(\frac{i}{N}\Bigg)\\
&\qquad\qquad-G\Bigg(\frac{i+1-j}{N}\Bigg)+G\Bigg(\frac{i+1}{N}\Bigg)\Bigg]I_{\eta_s}(i,-j+1)\Bigg\}\Bigg|\,ds>\delta\Bigg).
\end{aligned}\tag{68}$$

Now, by attractiveness and the fact that $I_\eta(i,j)$ is an increasing function of η, we replace $\mathbf{P}_N^{\gamma}$ with $\mathbf{P}_N^{\hat{\gamma}}$ obtaining the following bound for the right-hand side of (68):

$$\begin{aligned}
&\limsup_{C\to\infty}\limsup_{N\to\infty}\mathbf{P}_N^{\hat{\gamma}}\Bigg(\int_0^T\sum_{j=C+1}^{N}c(j)N\sum_{i=1}^{N}\Bigg\{\Bigg|G\Bigg(\frac{i-j}{N}\Bigg)-G\Bigg(\frac{i}{N}\Bigg)\\
&\qquad\qquad-G\Bigg(\frac{i+1-j}{N}\Bigg)+G\Bigg(\frac{i+1}{N}\Bigg)\Bigg|I_{\eta_s}(i,-j+1)\Bigg\}\,ds>\delta\Bigg)\\
&\le\limsup_{C\to\infty}\limsup_{N\to\infty}\frac{1}{\delta}\mathbf{E}_N^{\hat{\gamma}}\Bigg(\int_0^T\sum_{j=C+1}^{N}c(j)N\sum_{i=1}^{N}\Bigg\{\Bigg|G\Bigg(\frac{i-j}{N}\Bigg)-G\Bigg(\frac{i}{N}\Bigg)\\
&\qquad\qquad-G\Bigg(\frac{i+1-j}{N}\Bigg)+G\Bigg(\frac{i+1}{N}\Bigg)\Bigg|I_{\eta_s}(i,-j+1)\Bigg\}\,ds\Bigg),
\end{aligned}\tag{69}$$

where we have used Chebyshev's inequality. Therefore, we need only show that

$$\begin{aligned}
&\limsup_{C\to\infty}\limsup_{N\to\infty}\mathbf{E}_N^{\hat{\gamma}}\Bigg(\int_0^T\sum_{j=C+1}^{N}c(j)\Bigg\{N\sum_{i=1}^{N}\Bigg|G\Bigg(\frac{i-j}{N}\Bigg)-G\Bigg(\frac{i}{N}\Bigg)\\
&\qquad\qquad-G\Bigg(\frac{i+1-j}{N}\Bigg)+G\Bigg(\frac{i+1}{N}\Bigg)\Bigg|I_{\eta_s}(i,-j+1)\Bigg\}\,ds\Bigg)=0.
\end{aligned}\tag{70}$$

Using Fubini's theorem and the invariance of $\nu_{\hat{\gamma}}$, we have

$$\begin{aligned}
&\limsup_{C\to\infty}\limsup_{N\to\infty}\int_0^T \mathbf{E}_N^{\hat{\gamma}}\Bigg(\sum_{j=C+1}^{N} c(j)\Bigg\{N\sum_{i=1}^{N}\Bigg|G\left(\frac{i-j}{N}\right)-G\left(\frac{i}{N}\right)\\
&\qquad -G\left(\frac{i+1-j}{N}\right)+G\left(\frac{i+1}{N}\right)\Bigg|I_{\eta_s}(i,-j+1)\Bigg\}\Bigg)\,ds\\
&=\limsup_{C\to\infty}\limsup_{N\to\infty} TE_{\nu_{\hat{\gamma}}}\Bigg(\sum_{j=C+1}^{N} c(j)\Bigg\{N\sum_{i=1}^{N}\Bigg|G\left(\frac{i-j}{N}\right)-G\left(\frac{i}{N}\right)\\
&\qquad -G\left(\frac{i+1-j}{N}\right)+G\left(\frac{i+1}{N}\right)\Bigg|I_{\eta}(i,-j+1)\Bigg\}\Bigg)\\
&=\limsup_{C\to\infty}\limsup_{N\to\infty}\Bigg\{TE_{\nu_{\hat{\gamma}}}\Bigg(\sum_{j=C+1}^{\sqrt{N}} c(j)N\sum_{i=1}^{N}\Bigg|G\left(\frac{i-j}{N}\right)-G\left(\frac{i}{N}\right)\\
&\qquad -G\left(\frac{i+1-j}{N}\right)+G\left(\frac{i+1}{N}\right)\Bigg|I_{\eta}(i,-j+1)\Bigg)\\
&\qquad +TE_{\nu_{\hat{\gamma}}}\Bigg(\sum_{j=\sqrt{N}+1}^{N} c(j)N\sum_{i=1}^{N}\Bigg|G\left(\frac{i-j}{N}\right)-G\left(\frac{i}{N}\right)\\
&\qquad -G\left(\frac{i+1-j}{N}\right)+G\left(\frac{i+1}{N}\right)\Bigg|I_{\eta}(i,-j+1)\Bigg)\Bigg\}\\
&=\limsup_{C\to\infty}\limsup_{N\to\infty}\Bigg\{TE_{\nu_{\hat{\gamma}}}\sum_{i=1}^{N}\Bigg(\sum_{j=C+1}^{\sqrt{N}} c(j)\frac{j}{N}\Bigg|\frac{\partial^2 G}{\partial\theta^2}\left(\frac{i}{N}\right)+O\left(\frac{j}{N}\right)\Bigg|\\
&\qquad \times I_{\eta}(i,-j+1)+\sum_{j=\sqrt{N}+1}^{N} c(j)4N\|G\|_\infty I_{\eta}(i,-j+1)\Bigg)\Bigg\}\\
&\le\limsup_{C\to\infty}\limsup_{N\to\infty} T\Bigg\{\left\|\frac{\partial^2 G}{\partial\theta^2}\right\|_\infty\sum_{j=C+1}^{\sqrt{N}} c(j)j(\hat{\gamma})^j\\
&\qquad +4\|G\|_\infty N^2\sum_{j=\sqrt{N}+1}^{N} c(j)(\hat{\gamma})^j\Bigg\}\\
&=0,
\end{aligned}\tag{71}$$

since $\hat{\gamma} < 1$ and the $c(j)$'s are bounded. Higher order terms also vanish. □

PROOF OF LEMMA 3.7. The lower bound is trivial. The upper bound follows immediately from attractiveness as used in the proof of Lemma 3.4 in (65). □

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