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NONLINEAR TRANSFORMATIONS ON THE WIENER SPACE

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We study shift transformations on a general abstract Wiener space (E, H, μ), which have the form:

$$E \ni \omega \mapsto \mathscr{T}^{\phi}\omega \equiv \omega - \int_0^T \phi_t(\omega) Z(dt) \in E,$$

where $\phi_t(\omega)$ is a scalar function on $[0, T] \times E$ and Z is an orthogonal H-valued measure. Under suitable conditions for the kernel ϕ, we construct explicitly a probability measure μ^{ϕ} on E, which is equivalent to the standard Wiener measure μ and has the property: $\mu^{\phi}\{\mathscr{T}^{\phi} \in A\} = \mu(A)$, $A \in \mathscr{B}_E$. The main result presents an analog of the well-known Cameron–Martin–Girsanov theorem for the case where the shift is allowed to anticipate. This leads to an additional integral term in the Girsanov exponent. Also, the Wiener–Itô integral in this exponent is now replaced by an extended stochastic integral.

1. Introduction. Let $(\phi_t(\omega);\ 0 \le t \le T)$ be a process on $(C[0, T], \mu)$, where μ is defined as the standard Wiener measure, which is adapted to the natural filtration in $C[0, T]$ and satisfies

$$\int_0^T \phi_t^2(\omega)\, dt < \infty \quad \text{for } \mu\text{-a.e. } \omega \in C[0, T].$$

Set

$$R^{\phi}(\omega) = \exp\left[\int_0^T \phi_t(\omega)\, d\omega(t) - \tfrac{1}{2}\int_0^T |\phi_t(\omega)|^2\, dt\right].$$

The celebrated Cameron–Martin–Girsanov theorem (cf. [3] and [6]) states that if

$$\mathbb{E}_{\mu}\{R^{\phi}(\omega)\} = 1,$$

then

$$\mathscr{T}_t^{\phi}(\omega) \equiv \omega(t) - \int_0^t \phi_s(\omega)\, ds, \qquad 0 \le t \le T,$$

is a Wiener process relative to the measure $\mu^{\phi} \equiv R^{\phi}\, d\mu$. The expression for the Radon–Nikodym derivative R^{ϕ} suggests that the eventual extensions of this result would depend on how far one can go with the definition of the

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stochastic integral. Ramer's work [20] (see also [15]) shows that this is only a part of the problem. This pioneering work deals with the following type of shifts on a general abstract Wiener space (E, H, μ),

$$E \ni \omega \mapsto \mathscr{T}(\omega) \equiv \omega - \tau[\omega] \in E.$$

The mapping $\tau\colon E \mapsto H$ is assumed continuous and H-differentiable in the sense of Fréchet, the derivative $\tau'[\omega]$ being a Hilbert–Schmidt operator on H. It is shown that relative to the measure $\mu^\tau \mathrm{b} = R^\tau \, d\mu$, with

$$R^\tau(\omega) = |\delta(I - \tau'[\omega])| \exp\Big[\text{``}\langle \tau[\omega], \omega\rangle - \operatorname{tr}(\tau'[\omega])\text{''} - \tfrac{1}{2}\|\tau[\omega]\|_H^2\Big],$$

$\mathscr{T}(\omega)$ is white noise in E; that is, $\mu^\tau\{\mathscr{T} \in A\} = \mu(A)$, $A \in \mathscr{B}_E$. The expression in quotation marks is identified by Ramer as a version of the Wiener-Itô integral. (The quotation marks are due to Ramer. In general, neither of the two quantities makes sense; only the difference is shown to be defined unambiguously.) The appearance of the Carleman–Fredholm determinant $\delta(I - \tau'[\omega])$ clearly indicates that the stochastic integral is not the only object that requires special attention. This determinant is expressed by the eigenvalues of the operator $I - \tau'[\omega]$ and its computation does not seem to be obvious, especially if the Radon–Nikodym derivative R^τ is to be treated as a function of the time, as is the case in most applications.

In 1975 Kabanov and Skorohod [10] initiated a new type of calculus with Wiener functionals. The adaptedness requirement, typical for the Itô calculus and martingale theory, was replaced by a special type of smoothness. Based on this concept, they developed the so called extended stochastic integral, which is analogous to the Wiener-Itô integral, but, instead of adaptedness, requires smoothness of the integrand. In fact, this integral is analogous to the object which Ramer puts in quotation marks, but requires a less stringent type of smoothness. Independently and from an entirely different point of view, the concept of smoothness was developed by Malliavin [17] and Stroock [23]. With this technique at hand, Ramer's original result was generalized towards relaxing the smoothness requirement for the shift term τ in the works [22], [15] and [19].

The goal of the present article is to study a class of shifts, which is more restrictive but allows the Carleman–Fredholm determinant to be expressed in more convenient terms. To be more precise, we study shifts on a general abstract Wiener space (E, H, μ), which have the form

$$\mathscr{T}^\phi\omega = \omega - \int_0^T \phi_t(\omega) Z(dt), \qquad \omega \in E.$$

Here $Z(\cdot)$ is some orthogonal H-valued measure, which obeys certain conditions, and the scalar process (ϕ_t) is assumed to be smooth (but not necessarily adapted) in Malliavin's sense. We will obtain an analog of Cameron–Martin–Girsanov's theorem, in which the Radon–Nikodym derivative is described again as an exponent. It resembles the usual one but, because of the lack of adaptedness, contains an additional integral term, and the Itô integral is replaced by the extended stochastic integral of Kabanov and

Skorohod. The present study was initiated in the work [2], which deals with semigroups of shifts, defined by equations of the type

$$\mathscr{T}_t^{\phi}(\omega) = \omega(t) - \int_t^T \phi_s(\mathscr{T}_s^{\phi}\omega) Z(ds), \qquad \omega \in E, 0 \le t \le T.$$

The paper is organized as follows. Section 2 includes essential facts and notation that are needed in the sequel. Section 3 provides a special integral representation for determinants. It is the key technical tool for the method adopted here. The main result is proved in Section 4 by using a special factorization technique developed by Gohberg and Krein [7]. The use of such a technique should not come as a surprise. In a somewhat different context, the same factorization has been explored by Kallianpur and Oodaira [13] in the study of linear transformations on the Wiener space. Other related works are those of Shepp [21] and Hitsuda [9].

2. Preliminaries.

2.1. *The Wiener space.* We fix once and for all an abstract Wiener space (AWS) (E, H, μ) (cf. [8]), and, as usual, regard H as a proper dense subset of E, and E^*, the dual of E, as a proper dense subset of H. For $l \in E^*$ and $\omega \in E$ we set $\langle l, \omega \rangle \equiv l(\omega)$. Note that $\langle l, h \rangle \equiv (l, h)_H$, for $l \in E^*$, $h \in H \subset E$.

We also fix a compact interval $[0, T]$ and a vector-valued measure $Z(\cdot)$, defined on the Borel σ-field $\mathscr{B}_{[0,T]}$, which takes values in H and obeys the following conditions:

(i) $Z(\{t\}) = 0$, $t = [0, T]$;
(ii) $0 \neq Z([s, t)) \in E^*$, for $0 \le s < t \le T$;
(iii) $Z([s_1, t_1)) \perp Z([s_2, t_2))$, for $0 \le s_1 < t_1 \le s_2 < t_2 \le T$;
(iv) the family $\{X([0, t)) \colon 0 < t \le T\}$ spans E^*, in that all linear combinations of functionals from this family are dense in E^*, with respect to the uniform norm.

A typi al example of an AWS and a measure $Z(\cdot)$ which have these propertie, is the following:

$$E \equiv C_0[0, T] \equiv \{ f \colon f \text{ is a continuous function on } [0, T],\ f(0) = 0 \};$$

$$(2.1)\quad H \equiv \mathscr{C}'[0, T] \equiv \left\{ f \in C_0[0, T] \colon f(x) = \int_0^x \dot{f}(r)\, dr,\ \dot{f} \in L^2[0, T] \right\};$$

$$Z([s, t))(\cdot) = \int_0^{(\cdot)} 1_{[s,t)}(\tau)\, d\tau \in \mathscr{C}'[0, \mathscr{T}], \qquad 0 \le s < t \le T.$$

For any scalar function $f \in L^2(\nu; \mathbb{R})$, ν given by $\nu(A) = \|Z(A)\|_H^2$, $A \in \mathscr{B}_{[0,T]}$, we define the element $Z[f] \in H$ as the following integral:

$$Z[f] \equiv \int_0^T f(\tau) Z(d\tau).$$

This integral is defined first for simple functions,

$$f(\tau) = \sum_i a_i 1_{A_i}(\tau), \qquad a_i \in \mathbb{R},\ A_i \in \mathscr{B}_{[0,T]},$$

by the obvious expression $Z[f] = \sum_i a_i Z(A_i)$. For every simple f,

$$\|Z[f]\|_H^2 = \int_0^T |f(\tau)|^2 \nu(d\tau),$$

so that $Z[\cdot]$ extends to a unitary equivalence between $L^2(\nu; \mathbb{R})$ and H. If, for example, $Z(\cdot)$ is the measure given by (2.1), then $Z[f]$, as an element of $\mathscr{C}'[0,T]$, coincides with the function $\int_0^{(\cdot)} f(r)\,dr$.

For every $l \in E^*$, $E \ni \omega \mapsto \langle l, \omega \rangle$ is a zero-mean Gaussian r.v. on $(E, \mathscr{B}_E, \mu)$, and

$$\int_E \langle l, \omega \rangle \langle l', \omega \rangle \mu(d\omega) = (l, l')_H, \qquad l, l' \in E^*.$$

2.2. *Stochastic derivatives and integrals.* Following [14] and [18] we will introduce now stochastic derivatives for certain functionals on $(E, \mathscr{B}_E, \mu)$, and also extended stochastic integrals with respect to the random measure $\langle Z(dt), \omega \rangle$. These objects are defined first for the so-called "smooth functionals," and then the definition is extended to a larger class of functionals by an appropriate limiting procedure.

Every functional φ: $E \mapsto \mathbb{C}$, which has the form

$$\varphi(\omega) = f(\langle Z(\Delta_1), \omega \rangle, \dots, \langle Z(\Delta_k), \omega \rangle);$$
$$\Delta_i \equiv [s_i, t_i) \subset [0, T], \qquad 1 \le i \le k;\ f \in C_b^\infty(\mathbb{R}^k),$$

is called a *smooth functional* and its *stochastic derivative* $D_t\varphi(\omega)$ is defined as the following process on $(E, \mathscr{B}_E, \mu)$:

$$D_t\varphi(\omega) = \sum_{i=1}^k (\partial_i f)(\langle Z(\Delta_1), \omega \rangle, \dots, \langle Z(\Delta_k), \omega \rangle) 1_{\Delta_i}(t), \qquad 0 \le t \le T.$$

Here ∂_i stands for the differential operator $\partial/\partial x_i$. The vector space of all smooth functionals we denote by $\mathscr{S}$. The usual topological structure on $\mathscr{S}$ is given by the seminorm

$$\|\varphi\|_{2,1} = \left(\mathbb{E}_\mu\{|\varphi|^2\}\right)^{1/2} + \left(\int_0^T \mathbb{E}_\mu\{|D_t\varphi|^2\} \nu(dt)\right)^{1/2}.$$

The completion of $\mathscr{S}$ relative to the seminorm $\|\cdot\|_{2,1}$ is denoted by $\mathbb{D}^{2,1}$. The class $\mathbb{L}^{2,1}$ comprises all processes $(\phi_t(\omega)) \in L^2(\nu \times \mu; \mathbb{R})$ with $\phi_t \in \mathbb{D}^{2,1}$, $t \in [0, T]$, and with

$$\int_0^T \int_0^T \mathbb{E}_\mu\{|D_s\phi_t|^2\} \nu(ds) \nu(dt) < \infty.$$

Endowed with the norm

$$|||\phi|||_{2,1} = \left(\int_0^T \mathbb{E}_\mu\{|\phi_t|^2\}\nu(dt)\right)^{1/2} + \left(\int_0^T\int_0^T \mathbb{E}_\mu\{|D_s\phi_t|^2\}\nu(dt)\nu(ds)\right)^{1/2},$$

$\mathbb{L}^{2,1}$ is a Banach space.

Every process of the form

$$\phi_t = \sum_{i=1}^n \psi_i 1_{[t_{i-1},t_i)}(t), \qquad 0 = t_0 < t_1 < \cdots < t_n = T, \psi_i \in \mathbb{D}^{2,1},$$

we call a *simple process* and define its *extended stochastic integral* by

$$\begin{aligned}\delta(\phi) &\equiv \int_0^T : \phi_t(\omega)\langle Z(dt),\omega\rangle : \\ &= \sum_{i=1}^n \psi_i(\omega)\langle Z([t_{i-1},t_i)),\omega\rangle - \sum_{i=1}^n \int_{t_{i-1}}^{t_i} D_t\psi_i(\omega)\nu(dt).\end{aligned} \tag{2.2}$$

If we have above $\psi_i \in \mathscr{S}$, $1 \le i \le n$, then (ϕ_t) will be called a *smooth simple process*. The vector space of all simple processes will be denoted by $\mathbb{S}$ and that of all smooth simple processes will be denoted by $\mathfrak{S}$.

For $n \ge 1$ and for $0 \le i \le 2^n$, we introduce the following notation:

$$t_i^n = \frac{iT}{2^n},$$

$$\Delta_i^n = [t_{i-1}^n, t_i^n),$$

$$\zeta_i^n(\omega) = \langle Z(\Delta_i^n),\omega\rangle, \qquad \omega \in E.$$

By $\boldsymbol{\zeta}^n$ we denote the Gaussian vector $(\zeta_1^n,\dots,\zeta_{2^n}^n)$, and the probability distribution of this vector in $\mathbb{R}^{2^n}$ we denote by $\Gamma_{2^n}(dx)$. For every $(\phi_t) \in \mathbb{L}^{2,1}$ and every $n \ge 1$, define the following simple process:

$$\begin{aligned}[\mathscr{P}_n\phi]_t &= \sum_{i=1}^{2^n} \bar{\phi}_{i,n} 1_{\Delta_i^n}(t), \\ \bar{\phi}_{i,n} &= \frac{1}{\nu(\Delta_i^n)}\int_{\Delta_i^n} \phi_s\nu(ds).\end{aligned} \tag{2.3}$$

It is known (cf. [18]) that $[\mathscr{P}_n\phi] \in \mathbb{L}^{2,1}$ and $[\mathscr{P}_n\phi] \to \phi$ in $\mathbb{L}^{2,1}$. Thus, $\mathbb{S}$ is a dense subspace of $\mathbb{L}^{2,1}$, and, since $\mathbb{E}_\mu\{|\delta(\phi)|^2\} \le |||\phi|||_{2,1}^2$, whenever $\phi \in \mathbb{S}$, the integral $\delta(\cdot)$ can be extended by continuity for all elements of $\mathbb{L}^{2,1}$, due to the following closure property, established by Kusuoka and Stroock [14]: if the sequence $\{\phi^n: n \ge 1\} \subset \mathbb{S}$ converges to 0 in $L^2(\nu \times \mu;\mathbb{R})$ and $\{\delta(\phi^n): n \ge 1\}$ converges in $L^2(\mu;\mathbb{R})$, then $\{\delta(\phi^n)\ n \ge 1\}$ converges to 0.

Let, for $n \ge 1$, $\mathbb{E}_\mu^n\{\cdot\}$ denote the conditional expectation with respect to the family $\{\zeta_i^n: 1 \le i \le 2^n\}$. It is easy to see that, for every $\varphi \in \mathscr{S}$ and $n \ge 1$, one has

$$\int_0^T \left| D_t \mathbb{E}_\mu^n\{\varphi(\omega)\} \right|^2 \nu(dt) \le \mathbb{E}_\mu^n\left\{ \int_0^T \left| D_t\varphi(\omega) \right|^2 \nu(dt) \right\},$$

which shows that $\mathbb{E}_\mu^n\{\cdot\}$ is a projector in $\mathbb{D}^{2,1}$, and the last relation extends for all $\varphi \in \mathbb{D}^{2,1}$. Clearly, for $\psi \in \mathbb{D}^{2,1}$, $\mathbb{E}_\mu^n\{\psi\}$ has the form

$$\mathbb{E}_\mu^n\{\psi\}(\omega) = \Lambda_n^\psi(\zeta_1^n, \dots, \zeta_{2^n}^n),$$

where Λ_n^ψ is a function on $L^2(\Gamma_{2^n}; \mathbb{R})$. In general, although as an element of $\mathbb{D}^{2,1}$, $\mathbb{E}_\mu^n\{\psi\}$ is stochastically differentiable, Λ_n^ψ need not be differentiable in the usual sense. However, Λ_n^ψ turns out to be a function on the Sobolev class $\mathscr{W}^{2,1}(\Gamma_{2^n})$, which means the following: there exists a sequence $\{f_k: k \ge 1\} \subset C_b^\infty(\mathbb{R}^{2^n})$, such that $f_k \to \Lambda_n^\psi$ in $L^2(\Gamma_{2^n}; \mathbb{R})$ and, for every $i = 1, \dots, 2^n$, $\{\partial_i f_k: k \ge 1\}$ is Cauchy sequence in $L^2(\Gamma_{2^n}; \mathbb{R})$. It is then straightforward that $\Lambda_n^\psi \in \mathscr{D}'(\mathbb{R}^{2^n})$ and, for every i, the generalized derivative $\partial_i \Lambda_n^\psi$ is a function on $L^2(\Gamma_{2^n}; \mathbb{R})$, defined as the limit of the sequence $\{\partial_i f_k: k \ge 1\}$. We call $\{f_k: k \ge 1\} \subset C_b^\infty(\mathbb{R}^{2^n})$ the *Sobolev sequence for* Λ_n^ψ. The existence of such a sequence implies that the stochastic derivative of $\mathbb{E}_\mu^n\{\psi\}$ retains the usual form

$$D_t \mathbb{E}_\mu^n\{\psi\} = \sum_{i=1}^{2^n} (\partial_i \Lambda_n^\psi)(\zeta_1^n, \dots, \zeta_{2^n}^n) 1_{\Delta_i^n}(t), \qquad 0 \le t \le T,$$

even if $\partial_i \Lambda_n^\psi$'s are to be understood in the generalized sense. One obvious way to construct a Sobolev sequence for Λ_n^ψ, and thus to show that $\Lambda_n^\psi \in \mathscr{W}^{2,1}(\Gamma_{2^n})$, is the following. Since $\psi \in \mathbb{D}^{2,1}$, by definition there exists a sequence $\{\varphi_k: k \ge 1\} \subset \mathscr{S}$, which approximates ψ in $\mathbb{D}^{2,1}$. But each $\mathbb{E}_\mu^n\{\varphi_k\}$ has the form

$$\mathbb{E}_\mu^n\{\varphi_k\} = f_k(\zeta_1^n, \dots, \zeta_{2^n}^n),$$

with some $f_k \in C_b^\infty(\mathbb{R}^{2^n})$. We now have that $\mathbb{E}_\mu^n\{\varphi_k\} \to \mathbb{E}_\mu^n\{\psi\}$ in $\mathbb{D}^{2,1}$, which yields

$$\lim_{k, k' \to \infty} \sum_{i=1}^{2^n} \mathbb{E}_\mu\left\{ \left| (\partial_i f_k)(\zeta_1^n, \dots, \zeta_{2^n}^n) - (\partial_i f_{k'})(\zeta_1^n, \dots, \zeta_{2^n}^n) \right|^2 \right\} \nu(\Delta_i^n) = 0.$$

Hence, for every i, $[(\partial_i f_k) - (\partial_i f_{k'})] \to 0$ in $L^2(\Gamma_{2^n}; \mathbb{R})$, as $k, k' \to \infty$.

The natural topology in $\mathscr{W}^{2,1}(\Gamma_{2^n})$ is the one defined by the seminorm

$$f \to \left[\int_{\mathbb{R}^{2^n}} |f(x)|^2 \Gamma_{2^n}(dx) \right]^{1/2} + \sum_{i=1}^{2^n} \left[\int_{\mathbb{R}^{2^n}} |\partial_i f(x)|^2 \Gamma_{2^n}(dx) \right]^{1/2},$$

and in fact $\mathscr{W}^{2,1}(\Gamma_{2^n})$ is the completion of $C_b^\infty(\mathbb{R}^{2^n})$ with respect to this seminorm. It is then evident that, for $\psi \in \mathbb{D}^{2,1}$, $\{\psi_k: k \ge 1\} \subset \mathbb{D}^{2,1}$ and for some fixed $n \ge 1$, $\mathbb{E}_\mu^n\{\psi_k\} \to \mathbb{E}_\mu^n\{\psi\}$ in $\mathbb{D}^{2,1}$, as $k \to \infty$, if and only if $\Lambda_n^{\psi_k} \to \Lambda_n^\psi$ in $\mathscr{W}^{2,1}(\Gamma_{2^n})$.

2.3. *Factorization of matrices and operators.* In [7], Gohberg and Krein developed a technique, which allows certain operators on a general Hilbert space $\mathscr{H}$ to be factorized along a given chain of orthoprojectors. In the finite dimensional setting, this operation corresponds to representing a matrix as a product of two, respectively, upper and lower, triangular matrices. This technique plays a crucial role in our approach. Namely, we will use the following result, due to Gohberg and Krein [7]:

THEOREM 2.1. *Let $\mathscr{H}$ be a separable Hilbert space and let $\mathfrak{P}$ be a closed maximal chain of orthoprojectors in $\mathscr{H}$. [This means that $\mathfrak{P}$ is a family of orthoprojectors which is closed in the strong operator topology*; $0, I \in \mathfrak{P}$, *and, for* $P, P' \in \mathfrak{P}$, *either* $\operatorname{Range}(P) \subset \operatorname{Range}(P')$ *or* $\operatorname{Range}(P') \subset \operatorname{Range}(P)$ (*cf*. [7] *for details*).] *Assume that K is a Hilbert–Schmidt operator on $\mathscr{H}$ and that all operators $I - PKP$, $P \in \mathfrak{P}$, are invertible. Then the operator $(I-K)^{-1}$ admits the following representation*:

$$(I-K)^{-1} = (I+V^+)D(I+V^-), \tag{2.4}$$

where V^+ and V^- are Volterra operators, $D - I$ is a compact operator, and the following relations hold for every $P \in \mathfrak{P}$:

$$PV^+P = V^+P, \qquad P^{\perp}V^-P^{\perp} = V^-P^{\perp}, \qquad PD = DP.$$

The representation (2.4) is usually referred to as a *special factorization of the operator* $(I-K)^{-1}$ *along the chain* $\mathfrak{P}$. The operators V^+ and V^- in this representation are unique and can be expressed as special operator-valued integrals along the chain $\mathfrak{P}$, which now plays the role of an operator-valued measure (cf. [7]). We will use the above result in the case where $\mathscr{H} \equiv L^2(\nu;\mathbb{R})$, and by factorization of operators in $L^2(\nu;\mathbb{R})$ we will always mean special factorization along the chain $\mathfrak{P}_T \equiv \{P_t\colon 0 \le t \le T\}$, given by

$$P_t f = 1_{[t,T]} f, \qquad f \in L^2(\nu;\mathbb{R}).$$

Every Hilbert–Schmidt operator in $L^2(\nu;\mathbb{R})$ can be represented as an integral operator with some kernel $K \in L^2(\nu \times \nu;\mathbb{R})$; that is, can be written as

$$(Kf)(s) = \int_0^T K(s,r) f(r)\nu(dr), \qquad f \in L^2(\nu;\mathbb{R})$$

(with a slight abuse of the notation, for integral operators we will use the same symbol for the kernel and for the operator itself). We will deal with integral kernels having the following property: $\int_0^T\int_0^T |K(s,t)|^2 \nu(ds)\nu(dt) < 1$, in which case the assumption of Theorem 2.1 is met, and therefore $(I-K)^{-1}$ can be factorized along the chain $\mathfrak{P}_T$. Because of the choice of the chain and the Hilbert space, we have $D \equiv I$. Also, V^+ and V^- are integral operators with kernels $V^+(s,t)$ and $V^-(s,t)$, such that $V^+(s,t) = 0$, for $s < t$, and $V^-(s,t) = 0$, for $s > t$. We will refer to $V^+(s,t)$ and $V^-(s,t)$ respectively as the right and left Volterra kernel of $(I-K)^{-1}$. The right kernel admits the following expansion in the triangle $0 \le t < s \le T$:

$$V^+(s,t) = K(s,t) + \sum_{m=1}^{\infty} \int_t^T \cdots \int_t^T K(s,r_1)K(r_1,r_2)\cdots K(r_{m-1},r_m) \times K(r_m,t)\nu(dr_1)\cdots\nu(dr_m), \tag{2.5}$$

which yields the estimate

$$\|V^+\|_{\mathrm{HS}} \le \frac{\|K\|_{\mathrm{HS}}}{1-\|K\|_{\mathrm{HS}}}.$$

Here $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert–Schmidt norm. Obviously, the composition $K \circ V^+$ is again a Hilbert–Schmidt operator with kernel

$$(K \circ V^+)(s,t) = \int_t^T K(s,r)V^+(r,t)\nu(dr),$$

and we have

$$\int_0^T |(K \circ V^+)(t,t)|\nu(dt) \le \int_0^T \nu(dt)\left(\int_0^T |K(t,r)|^2\nu(dr)\right)^{1/2}\left(\int_0^T |V^+(r,t)|^2\nu(dr)\right)^{1/2} < \infty.$$

Let $\{K_n\colon n \ge 1\} \subset L^2(\nu \times \nu; \mathbb{R})$, be a sequence of integral kernels, all having the property $\|K_n\|_{\mathrm{HS}} < 1$, such that $\|K - K_n\|_{\mathrm{HS}} \to 0$. Then it is easy to show that $\|V^+ - V_n^+\|_{\mathrm{HS}} \to 0$ and

$$\int_0^T (K_n \circ V_n^+)(t,t)\nu(dt) \to \int_0^T (K \circ V^+)(t,t)\nu(dt). \tag{2.6}$$

Here V_n^+ is the right Volterra kernel of $(I - K_n)^{-1}$.

Suppose that

$$\Phi(s,t) = \sum_{i,j=1}^{2^n} \alpha_{i,j} 1_{\Delta_i^n}(t) 1_{\Delta_j^n}(s), \qquad 0 \le s,t \le T. \tag{2.7}$$

is a given simple function on $[0,1] \times [0,1]$ and define the matrix

$$A = \left(a_{i,j} \equiv \alpha_{i,j}\sqrt{\nu(\Delta_i^n)\nu(\Delta_j^n)}\right)_{i,j=1}^{2^n}. \tag{2.8}$$

Note that

$$\|\Phi\|_{\mathrm{HS}}^2 \equiv \|A\|_{\mathrm{HS}}^2 = \sum_{i,j=1}^{2^n} |\alpha_{i,j}|^2 \nu(\Delta_i^n)\nu(\Delta_j^n).$$

Thus, if $\|\Phi\|_{\mathrm{HS}}^2 < 1$, then the operator $(I - \Phi)^{-1}$, which acts in $L^2(\nu; \mathbb{R})$, can be factorized along the chain $\mathfrak{P}_T$. The following fact is an easy consequence of the Carleman–Fredholm theory of determinants (cf. [20] or, more recently, [12]).

LEMMA 2.1. *Let* Φ *and* A *be given respectively by* (2.7) *and* (2.8) *and suppose that* $(I-\Phi)^{-1}=(I+V^{+})(I+V^{-})$, *where* V^{+} *and* V^{-} *are Volterra operators of Hilbert–Schmidt type acting on* $L^2(\nu;\mathbb{R})$. *Then*

$$\det(I-A)=\exp\left[-\operatorname{tr}(A)-\int_0^T(\Phi\circ V^{+})(t,t)\nu(dt)\right].$$

3. A class of nonlinear transformations of the Wiener measure and their absolute continuity. In this section we will study the transformations $\mathscr{T}^{\phi}\colon E\mapsto E$, $\phi\in\mathbb{L}^{2,1}$, defined by

$$\mathscr{T}^{\phi}[\omega]=\omega-\int_0^T\phi_t(\omega)Z(dt),\qquad \omega\in E. \tag{3.1}$$

We will refer to the process ϕ above as the kernel of the shift $\mathscr{T}^{\phi}[\cdot]$. Since $\phi\in\mathbb{L}^{2,1}$ implies that $\mathbb{E}_{\mu}\{\int_0^T|\phi_t(\omega)|^2\nu(dt)\}<\infty$, then for μ-a.e. $\omega\in E$, $t\mapsto\phi_t(\omega)$ is a function of $L^2(\nu;\mathbb{R})$; and so, the integral in (3.1) is a correctly defined element of the Hilbert space $H\subset E$, for a.e. $\omega\in E$. For example, if $Z(\cdot)$ is the measure, given by (2.1), then $\mathscr{T}^{\phi}$ transforms every continuous function $\omega(\cdot)\in C_0[0,T]$ into the function $\omega(\cdot)-\int_0^{(\cdot)}\phi_t(\omega)\,dt$. Obviously, $\phi^n\to\phi$ in $\mathbb{L}^{2,1}$ implies that $\mathbb{E}_{\mu}\{\|\mathscr{T}^{\phi^n}-\mathscr{T}^{\phi}\|_H^2\}\to 0$ (the Hilbert norm can be applied here, because $\mathscr{T}^{\phi^n}-\mathscr{T}^{\phi}\in H$), and therefore, for some appropriate subsequence,

$$\mathscr{T}^{\phi^{n_k}}[\omega]\to\mathscr{T}^{\phi}[\omega]\quad\text{for }\mu\text{-a.e. }\omega\in E.$$

Another useful observation is that, for every $\phi\in\mathbb{L}^{2,1}$,

$$\int_0^T\int_0^T|D_s\phi_t(\omega)|^2\nu(ds)\nu(dt)<\infty\quad\text{for }\mu\text{-a.e. }\omega\in E.$$

Hence, for μ-a.e. $\omega\in E$, the function $(s,t)\mapsto D_s\phi_t(\omega)$ defines a Hilbert–Schmidt operator in $L^2(\nu;\mathbb{R})$, which we will denote by $D\phi(\omega)$. The corresponding right Volterra kernel in the factorization of $(I-D\phi(\omega))^{-1}$ we will denote by $V_s^{+}\phi_t(\omega)$, and the Volterra operator in $L^2(\nu;\mathbb{R})$, defined by this kernel, we will denote by $(V^{+}\phi)(\omega)$.

Our main objective now is to construct a measure μ^{ϕ} which is equivalent to μ and has the property $\mu^{\phi}\{\mathscr{T}^{\phi}\in A\}=\mu(A)$, $A\in\mathscr{B}_E$. In fact, we will study a subclass of the transformations (3.1), with the following additional requirement for the kernel $\phi\in\mathbb{L}^{2,1}$:

$$\|D\phi(\omega)\|_{\mathrm{HS}}^2\equiv\int_0^T\int_0^T|D_s\phi_t(\omega)|^2\nu(ds)\nu(dt)<1\quad\text{for }\mu\text{-a.e. }\omega\in E.$$

As one can expect, our plan is to study the problem first for smooth kernels ϕ, and then, using an appropriate approximation, to extend the result to general $\phi\in\mathbb{L}^{2,1}$.

Let h be a test function from the class $\mathscr{D}(\mathbb{R})$, such that $h\geq 0$, $\int_{\mathbb{R}}h(x)\,dx=1$ and $h(x)=0$, for $|x|\geq\frac{1}{2}$. Then the function

$$\Psi_N(x)=\int_0^x(h*1_{[-N+1,N-1]})(r)\,dr,\qquad x\in\mathbb{R},$$

has the following properties:

(i) $\Psi_N \in C^\infty$;
(ii) $|\Psi_N(x)| \le |x| \wedge (N+1)$, for all $x \in \mathbb{R}$;
(iii) $\Psi_N(x) = x$, for $x \in [-N+1, N-1]$;
(iv) $|\Psi_N'(x)| \le 1$, for all $x \in \mathbb{R}$;
(v) $|\Psi_N'(x)| = 1$, for $x \in [-N+1, N-1]$.

LEMMA 3.1. *Let* $\Lambda \in \mathscr{W}^{2,1}(\Gamma_{2^n})$. *Then* (a) *for every* $N \ge 1$, $\Psi_N(\Lambda) \in \mathscr{W}^{2,1}(\Gamma_{2^n})$, *and* $\partial_i(\Psi_N(\Lambda)) = \Psi_N'(\Lambda)\partial_i\Lambda$, $1 \le i \le 2^n$; (b) $\Psi_N(\Lambda) \to \Lambda$ *in* $\mathscr{W}^{2,1}(\Gamma_{2^n})$ *as* $N \to \infty$.

PROOF. Let $\int\{\cdot\}$ stand for $\int_{\mathbb{R}^{2^n}}\{\cdot\}\Gamma_{2^n}(dx)$ and let $f_k \in C_b^\infty(\mathbb{R}^{2^n})$, $k \ge 1$, be a Sobolev sequence for Λ. Obviously this sequence can be chosen such that $f_k \to \Lambda$ a.s. Then, by the dominated convergence theorem, we have

$$\lim_{k\to\infty} \int\left\{|\Psi_N(\Lambda) - \Psi_N(f_k)|^2\right\} = 0.$$

On the other hand, for every i,

$$\int\left\{|\Psi_N'(f_k)\partial_i f_k - \Psi_N'(\Lambda)\partial_i\Lambda|^2\right\}$$

$$\le 2\int\left\{|\Psi_N'(f_k)|^2|\partial_i f_k - \partial_i\Lambda|^2\right\} + 2\int\left\{|\partial_i\Lambda|^2|\Psi_N'(f_k) - \Psi_N'(\Lambda)|^2\right\}.$$

The first term above goes to 0, because $|\Psi_N'(f_k)| \le 1$. The second term also goes to 0, due to the dominated convergence theorem. This concludes the proof of (a). To show (b) it is enough to notice that

$$\lim_{N\to\infty}\int|\Psi_N(\Lambda) - \Lambda|^2 = 0,$$

$$\lim_{N\to\infty}\int|\Psi_N'(\Lambda)(\partial_i\Lambda) - (\partial_i\Lambda)|^2 = 0,$$

both due to the dominated convergence theorem. □

LEMMA 3.2. *Let* Γ *be a nondegenerate (i.e., supported by the whole space) Gaussian measure on* $\mathbb{R}^n$ *and let* $h_j(x) \equiv j^n h(jx)$, $x \in \mathbb{R}^n$, $j = 1, 2, \dots$, $h \in \mathscr{D}(\mathbb{R}^n)$, *be an approximate identity. Then, for every Borel function* Λ *on* $\mathbb{R}^n$, *with* $|\Lambda| \le C$, *(a.e.), one has* $h_j * \Lambda \to \Lambda$, *in* $L^2(\Gamma; \mathbb{R})$, *as* $j \to \infty$.

PROOF. Let, for $r > 0$,

$$B_r = \{x \in \mathbb{R}^n : |x| \le r\}$$

(here $|\cdot|$ is the Euclidean norm), and let $d > 0$ be such that $h(x) = 0$ for $x \notin B_d$. Let $\epsilon > 0$ be arbitrarily chosen and let $r_\epsilon > d$ be such that $\Gamma((B_{r_\epsilon - d})^c) \le \epsilon$. Using Fourier transforms, one can easily check that $h_j * f \to f$ in

$L^2(dx;\mathbb{R})$, as $j \to \infty$, for every $f \in L^2(dx;\mathbb{R}) \cap L^1(dx;\mathbb{R})$. Now, since $\Lambda \in L^2(\Gamma;\mathbb{R})$, we have that $1_{B_r}\Lambda \in L^2(dx;\mathbb{R}) \cap L^1(dx;\mathbb{R})$, for every $r > 0$. Hence, $h_j * (1_{B_r}\Lambda) \to 1_{B_r}\Lambda$ in $L^2(\mathbb{R}^n, dx)$, and therefore also in $L^2(\Gamma;\mathbb{R})$, as $j \to \infty$. Thus

$$\begin{aligned}\int |h_j * \Lambda - \Lambda|^2 &\le 2\int \left| h_j * \left(1_{B_{r_\epsilon}}\Lambda\right) - 1_{B_{r_\epsilon}}\Lambda \right|^2 + 2\int \left| h_j * \left(1_{B^c_{r_\epsilon}}\Lambda\right) - 1_{B^c_{r_\epsilon}}\Lambda \right|^2 \\ &\le 2\int \left| h_j * \left(1_{B_{r_\epsilon}}\Lambda\right) - 1_{B_{r_\epsilon}}\Lambda \right|^2 + 8C^2\epsilon.\end{aligned}$$

The integration here is with respect to Γ. Passing to the limit as $j \to \infty$, we get

$$\limsup_{j\to\infty} \int |h_j * \Lambda - \Lambda|^2 \le 8C^2\epsilon. \qquad \square$$

COROLLARY 3.1. *Let $\Lambda \in \mathscr{W}^{2,1}(\Gamma_{2^n})$ be such that (a.e.) $|\Lambda| \le C$ and $|\partial_i(\Lambda)| \le C$, $1 \le i \le 2^n$. Then, for any approximate identity $h_j(x)$, $x \in \mathbb{R}^{2^n}$, $j = 1, 2, \dots$, one has $h_j * \Lambda \to \Lambda$ in $\mathscr{W}^{2,1}(\Gamma_{2^n})$ as $j \to \infty$.*

PROOF. We know already, due to Lemma 3.2, that $h_j * \Lambda \to \Lambda$ in $L^2(\Gamma_{2^n};\mathbb{R})$. But for exactly the same reason, for every i,

$$\partial_i(h_j * \Lambda) \equiv h_j * (\partial_i \Lambda) \to \partial_i \Lambda$$

in $L^2(\Gamma_{2^n};\mathbb{R})$. $\square$

LEMMA 3.3. *Let $\phi \in \mathbb{L}^{2,1}$ and let $\|D\phi(\omega)\|^2_{\mathrm{HS}} \le C$, for μ-a.e. $\omega \in E$. Then there exists a sequence of smooth simple processes $\{\tilde{\phi}^n\colon n \ge 1\} \subset \mathfrak{S}$ such that $\tilde{\phi}^n \to \phi$ in $\mathbb{L}^{2,1}$, and $\sup_{\omega\in E}(\|D\tilde{\phi}^n(\omega)\|^2_{\mathrm{HS}}) < C$ for every $n \ge 1$.*

PROOF. The statement is equivalent to the following assertion: There exists a sequence $\{\tilde{\phi}^n\colon n \ge 1\} \subset \mathfrak{S}$ which converges to ϕ in $\mathbb{L}^{2,1}$, and has the property $\sup_{\omega\in E}(\|D\tilde{\phi}^n(\omega)\|^2_{\mathrm{HS}}) \le C$, $n \ge 1$. This is because if such a sequence exists, one can form the sequence $\{(1 - 1/n)\tilde{\phi}^n\colon n \ge 1\}$, which will be exactly what we are looking for.

Let $\phi \in \mathbb{L}^{2,1}$ be such that $\|D\phi\|^2_{\mathrm{HS}} \le C$, μ-a.e., and let $\{[\mathscr{P}_n\phi];\ n \ge 1\} \subset \mathbb{S}$ be the sequence constructed in (2.3). Note first that for every $n \ge 1$, we have μ-a.e.,

$$\begin{aligned}\|D[\mathscr{P}_n\phi]\|^2_{\mathrm{HS}} &= \int_0^T \nu(ds) \sum_{i=1}^{2^n} |D_s\bar{\phi}_{i,n}|^2 \nu(\Delta_i^n) \\ &= \int_0^T \nu(ds) \sum_{i=1}^{2^n} \left| \frac{1}{\nu(\Delta_i^n)} \int_{\Delta_i^n} D_s\phi_t \nu(dt) \right|^2 \\ &\le \int_0^T \int_0^T |D_s\phi_t|^2 \nu(ds)\nu(dt) \le C.\end{aligned}$$

Since the sequence $\{[\mathscr{P}_n\phi]: n \geq 1\} \subset \mathbb{S}$ is known to approximate ϕ in $\mathbb{L}^{2,1}$, the proof would be completed, if we show that the assertion holds for every process $\phi \in \mathbb{S}$ with $\|D\phi\|^2_{\mathrm{HS}} \leq C$, μ-a.e. So, let us take a simple process

$$\phi'_t = \sum_{i=1}^{2^n} \psi_{i,n} 1_{\Delta_i^n}(t), \qquad \psi_{i,n} \in \mathbb{D}^{2,1}$$

and let us assume that $\|D\phi'\|^2_{\mathrm{HS}} \leq C$, μ-a.e. Consider the following sequence of processes:

$$\mathbb{E}^m_\mu \phi'_t = \sum_{i=1}^{2^n} \mathbb{E}^m_\mu\{\psi_{i,n}\} 1_{\Delta_i^n}(t), \qquad m \geq 1.$$

For every $m \geq 1$, we have μ-a.e.,

$$\begin{aligned} \|D\mathbb{E}^m_\mu \phi'\|^2_{\mathrm{HS}} &\equiv \int_0^T \nu(ds) \sum_{i=1}^{2^n} \left| D_s \mathbb{E}^m_\mu\{\psi_{i,n}\}\right|^2 \nu(\Delta_i^n) \\ &\leq \mathbb{E}^m_\mu\left\{ \int_0^T \nu(ds) \sum_{i=1}^{2^n} |D_s\psi_{i,n}|^2 \nu(\Delta_i^n) \right\} \\ &\equiv \mathbb{E}^m_\mu\left\{\|D\phi'\|^2_{\mathrm{HS}}\right\} \leq C. \end{aligned}$$

On the other hand, for every i, the sequence $\{\mathbb{E}^m_\mu\{\psi_{i,n}\}: m \geq 1\}$ approximates $\psi_{i,n}$ in $\mathbb{D}^{2,1}$, which yields that the sequence $\{\mathbb{E}^m_\mu \phi': m \geq 1\}$ approximates ϕ' in $\mathbb{L}^{2,1}$. Now our work is reduced to showing that the assertion holds for any process of the form:

$$\phi''_t(\omega) = \sum_{i=1}^{2^n} \Lambda_i(\boldsymbol{\zeta}^n(\omega)) 1_{\Delta_i^n}(t), \qquad \Lambda_i \in \mathscr{W}^{2,1}(\Gamma_{2^n}), \tag{3.2}$$

with $\|D\phi''\|^2_{\mathrm{HS}} \leq C$, μ-a.e. Note that the following sequence approximates ϕ'' in $\mathbb{L}^{2,1}$:

$$[\phi''_N]_t = \sum_{i=1}^{2^n} \Psi_N(\Lambda_i(\boldsymbol{\zeta}^n(\omega))) 1_{\Delta_i^n}(t), \qquad N \geq 1,$$

because, due to Lemma 3.2, for every i, $\Psi_N(\Lambda_i) \to \Lambda_i$ in $\mathbb{D}^{2,1}$ as $N \to \infty$. Also, for every $N \geq 1$, we have μ-a.e.:

$$\begin{aligned} \|D[\phi''_N]\|^2_{\mathrm{HS}} &= \sum_{i=1}^{2^n}\sum_{j=1}^{2^n} \left|\Psi'_N(\Lambda_i)(\partial_j\Lambda_i)\right|^2 \nu(\Delta_i^n)\nu(\Delta_j^n) \\ &\leq \sum_{i=1}^{2^n}\sum_{j=1}^{2^n} (\partial_j\Lambda_i)^2 \nu(\Delta_i^n)\nu(\Delta_j^n) = \|D\phi''\|^2_{\mathrm{HS}} \leq C. \end{aligned}$$

Therefore, if we show that the statement is true for every process of the type (3.2), with the additional requirement that all functions $\Lambda_i \in L^2(\Gamma_{2^n}; \mathbb{R})$ are bounded μ-a.e., that would complete the proof. But note that $\|D\phi''\|^2_{\mathrm{HS}} \leq C$, μ-a.e., yields that $|(\partial_i\Lambda_j)|^2 \nu(\Delta_i^n)\nu(\Delta_j^n) \leq C$ a.e. in $\mathbb{R}^{2^n}$, for all $i, j = 1, \ldots, 2^n$. Thus all derivatives $(\partial_i\Lambda_j)$ are bounded a.e. Then let us take some approximate identity $h_j(x)$, $x \in \mathbb{R}^{2^n}$, $j = 1, 2, \dots$ and define the following sequence:

$$\sum_{i=1}^{2^n} (h_j * \Lambda_i)(\zeta^n(\omega)) 1_{\Delta_i^n}(t), \qquad j \geq 1.$$

Due to Corollary 3.1, this sequence approximates ϕ'' in $\mathbb{L}^{2,1}$. Now the following observation completes the proof: For every $x \in \mathbb{R}^{2^n}$,

$$\begin{aligned}
&\sum_{k=1}^{2^n} \sum_{i=1}^{2^n} \left| h_j * (\partial_k \Lambda_i)(x) \right|^2 \nu(\Delta_i^n) \nu(\Delta_k^n) \\
&\quad \leq \sum_{k=1}^{2^n} \sum_{i=1}^{2^n} h_j * \left((\partial_k \Lambda_i)^2 \right)(x) \nu(\Delta_i^n) \nu(\Delta_k^n) \\
&\quad = h_j * \left(\sum_{k=1}^{2^n} \sum_{i=1}^{2^n} (\partial_k \Lambda_i)^2 \nu(\Delta_i^n) \nu(\Delta_k^n) \right)(x) \leq C
\end{aligned}$$

(here we use the fact that if $|F| \leq C$ almost everywhere then $|h_j * F| \leq C$ *everywhere*). □

For every $\phi \in \mathbb{L}^{2,1}$, with $\|D\phi\|_{\mathrm{HS}}^2 < 1$, μ-a.e., define the following function of $\omega \in E$:

$$\begin{aligned}
R^\phi(\omega) = \exp\biggl[&\int_0^T : \phi_t(\omega) \langle Z(dt), \omega \rangle : - \tfrac{1}{2} \int_0^T |\phi_t(\omega)|^2 \nu(dt) \\
&- \int_0^T \nu(dt) \int_t^T D_t \phi_s(\omega) V_s^+ \phi_t(\omega) \nu(ds) \biggr].
\end{aligned} \tag{3.3}$$

Note that, following our notation in subsection 2.3, the last integral can be written equivalently as

$$\int_0^T [D\phi(\omega) \circ (V^+ \phi)(\omega)](t, t) \nu(dt),$$

or as

$$\sum_{m=1}^{\infty} \int_0^T \nu(dt) \int_t^T \cdots \int_t^T D_t \phi_{r_1}(\omega) D_{r_1} \phi_{r_2}(\omega) \cdots D_{r_m} \phi_t(\omega) \nu(d_{r_1}) \cdots \nu(d_{r_m}),$$

and this expression is nothing but the trace in $L^2(\nu; \mathbb{R})$ of the composition of the Hilbert–Schmidt operators $D\phi(\omega)$ and $(V^+\phi)(\omega)$.

LEMMA 3.4. *Let*

$$\phi_t(\omega) = \sum_{i=1}^{2^n} f_i(\zeta^n(\omega)) 1_{\Delta_i^n}(t), \qquad f_i \in C_b^\infty(\mathbb{R}^{2^n}), \tag{3.4}$$

be a smooth simple process from the class $\mathfrak{S}$, *such that* $\sup_{\omega \in E} \|D\phi(\omega)\|_{\mathrm{HS}}^2 < 1$.

Then $\mathbb{E}_\mu\{R^\phi(\omega)\} = 1$, *and, for every functional on* E *which has the form* $\varphi(\omega) = F(\boldsymbol{\zeta}^m(\omega))$, $\omega \in E$, *for some* $m \geq 1$ *and some bounded Borel function* F: $\mathbb{R}^{2^m} \mapsto \mathbb{C}$,

$$\mathbb{E}_\mu\{\varphi(\mathscr{T}^\phi[\omega])R^\phi(\omega)\} = \mathbb{E}_\mu\{\varphi(\omega)\}. \tag{3.5}$$

PROOF. We remark first that in the representation (3.4) of the simple process $\phi \in \mathfrak{S}$, we can increase the number n arbitrarily by setting, for any $N > n$,

$$1_{\Delta_i^n}(t) = \sum_{(i-1)2^{N-n} < k \leq i 2^{N-n}} 1_{\Delta_k^N}(t),$$

$$\zeta_i^n = \sum_{(i-1)2^{N-n} < k \leq i 2^{N-n}} \zeta_k^N.$$

With the same argument, we can increase the number m in the expression $F(\boldsymbol{\zeta}^m(\omega))$. Therefore, with no loss of generality, we may and do assume that $m = n$.

For the stochastic derivative of ϕ_t we have

$$D_s\phi_t(\omega) = \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} (\partial_i f_j)(\boldsymbol{\zeta}^n(\omega)) 1_{\Delta_i^n}(s) 1_{\Delta_j^n}(t),$$

and for the stochastic integral $\delta(\phi)$, by (2.2), we have

$$\delta(\phi) \equiv \int_0^T : \phi_t(\omega)\langle Z(dt), \omega\rangle := \sum_{i=1}^{2^n} f_i(\boldsymbol{\zeta}^n(\omega))\zeta_i^n(\omega) - \sum_{i=1}^{2^n} (\partial_i f_i)(\boldsymbol{\zeta}^n(\omega))\nu(\Delta_i^n).$$

Due to Lemma 2.1, the left-hand side of (3.5) can be written as the following integral in $\mathbb{R}^{2^n}$:

$$\begin{aligned} I = {} & \frac{1}{(\sqrt{2\pi})^{2^n}} \int_{\mathbb{R}^{2^n}} F(\boldsymbol{\vartheta}_n(\boldsymbol{x})) \\ & \times \exp\left[\sum_{i=1}^{2^n} f_i(\boldsymbol{\chi}_n(\boldsymbol{x}))\sqrt{\nu(\Delta_i^n)}\, x_i - \frac{1}{2}\sum_{i=1}^{2^n} |f_i(\boldsymbol{\chi}_n(\boldsymbol{x}))|^2 \nu(\Delta_i^n)\right] \\ & \times \exp\left[-\frac{1}{2}\sum_{i=1}^{2^n} x_i^2\right] \\ & \times \det\left[I - \left((\partial_i f_j)(\boldsymbol{\chi}_n(\boldsymbol{x}))\sqrt{\nu(\Delta_i^n)\nu(\Delta_j^n)}\right)_{i,j=1}^{2^n}\right] dx_1 \cdots dx_{2^n}. \end{aligned}$$

Here $\boldsymbol{\vartheta}_n(\boldsymbol{x})$ and $\boldsymbol{\chi}_n(\boldsymbol{x})$ are vector-functions given by

$$\begin{aligned}\boldsymbol{\vartheta}_n(\boldsymbol{x}) = \Big(&\sqrt{\nu(\Delta_1^n)}\,x_1 - f_1\Big(\sqrt{\nu(\Delta_1^n)}\,x_1,\dots,\sqrt{\nu(\Delta_j^n)}\,x_j,\dots,\sqrt{\nu(\Delta_{2^n}^n)}\,x_{2^n}\Big)\,\Delta_1^n,\dots,\\ &\sqrt{\nu(\Delta_i^n)}\,x_i - f_i\Big(\sqrt{\nu(\Delta_1^n)}\,x_1,\dots,\sqrt{\nu(\Delta_j^n)}\,x_j,\dots,\sqrt{\nu(\Delta_{2^n}^n)}\,x_{2^n}\Big)\Delta_i^n,\dots,\\ &\sqrt{\nu(\Delta_{2^n}^n)}\,x_{2^n} - f_{2^n}\Big(\sqrt{\nu(\Delta_1^n)}\,x_1,\dots,\sqrt{\nu(\Delta_j^n)}\,x_j,\dots,\sqrt{\nu(\Delta_{2^n}^n)}\,x_{2^n}\Big)\,\Delta_{2^n}^n\Big),\\ \boldsymbol{\chi}^n(\boldsymbol{x}) = \Big(&\sqrt{\nu(\Delta_1^n)}\,x_1,\dots,\sqrt{\nu(\Delta_j^n)}\,x_j,\dots,\sqrt{\nu(\Delta_{2^n}^n)}\,x_{2^n}\Big).\end{aligned}$$

Next, change the variables as follows:

$$y_i = x_i - f_i\Big(\sqrt{\nu(\Delta_1^n)}\,x_1,\dots,\sqrt{\nu(\Delta_j^n)}\,x_j,\dots,\sqrt{\nu(\Delta_{2^n}^n)}\,x_{2^n}\Big)\sqrt{\nu(\Delta_i^n)}\,,\qquad 1 \le i \le 2^n.$$

An easy application of Picard's method shows that the above transformation on $\mathbb{R}^{2^n}$ is one-to-one. Jacobi's transformation formula now yields

$$\begin{aligned}I = \frac{1}{(\sqrt{2\pi})^{2^n}}\int_{\mathbb{R}^{2^n}} &F\Big(\sqrt{\nu(\Delta_1^n)}\,y_1,\dots,\sqrt{\nu(\Delta_j^n)}\,y_j,\dots,\sqrt{\nu(\Delta_{2^n}^n)}\,y_{2^n}\Big)\\ &\times \exp\left[-\frac{1}{2}\sum_{i=1}^{2^n} y_i^2\right] dy_1 \cdots dy_{2^n}.\end{aligned}$$

This last integral obviously coincides with the right-hand side of (3.5). This proves (3.5). In particular, for $F \equiv 1$, we get $\mathbb{E}_\mu\,\{R^\phi(\omega)\} = 1$. □

Corollary 3.2. *Let $\phi \in \mathfrak{S}$ be exactly as in Lemma 3.4. Then, for every $l \in E^*$,*

$$\mathbb{E}_\mu\Big\{e^{i\langle l,\,\mathscr{T}^\phi[\omega]\rangle}R^\phi(\omega)\Big\} = e^{-1/2\|l\|_H^2}.$$

Proof. Obviously

$$\mathscr{T}^\phi[\omega] = \omega - \sum_{i=1}^{2^n} f_i(\boldsymbol{\zeta}^n(\omega))Z(\Delta_i^n).$$

Also, all functionals on E which have the form

$$\langle l', \cdot\rangle = \sum_{i=1}^{2^m}\alpha_i\big\langle Z(\Delta_i^m), \cdot\big\rangle,\qquad \alpha_i \in \mathbb{R},$$

for some $m \ge 1$, form a dense linear subspace of E^* (dense with respect to the uniform norm). Then, due to the dominated convergence theorem, we only have to prove the statement for functionals of this type (we know already, from the preceding lemma, that R^ϕ is an integrable function). Consider the functional $E \ni \omega \mapsto \langle l', \mathscr{T}^\phi[\omega]\rangle$, where l' is as above. The argument in the last proof shows that, with no loss of generality, one can assume that $m = n$.

This implies

$$e^{i\langle l', \mathscr{T}^{\phi}[\omega]\rangle} = \exp\left[i \sum_{i=1}^{2^n} \alpha_i \zeta_i^n(\mathscr{T}^{\phi}[\omega])\right].$$

The statement now follows from Lemma 3.5, by taking there

$$F(x_1, \dots, x_{2^n}) = e^{i(\alpha_1 x_1 + \cdots + \alpha_{2^n} x_{2^n})}. \qquad \square$$

Here is our main result.

THEOREM 3.1. *Let $\phi \in \mathbb{L}^{2,1}$ and let $\|D\phi(\omega)\|_{\mathrm{HS}}^2 < 1$, for μ-a.e. $\omega \in E$. Assume that $\mathbb{E}_{\mu}\{R^{\phi}(\omega)\} = 1$. Then, for every $l \in E^*$,*

$$\mathbb{E}_{\mu}\left\{e^{i\langle l, \mathscr{T}^{\phi}[\omega]\rangle} R^{\phi}(\omega)\right\} = e^{-1/2\|l\|_H^2}.$$

PROOF. According to Lemma 3.3, we choose a sequence $\{\tilde{\phi}^n \colon n \geq 1\} \subset \mathfrak{S}$, such that $\tilde{\phi}^n \to \phi$ in $\mathbb{L}^{2,1}$, and $\sup_{\omega \in E} \|D\tilde{\phi}^n(\omega)\|_{\mathrm{HS}}^2 < 1$, for every $n \geq 1$. Obviously, we can replace this sequence by an appropriate subsequence, so that, for μ-a.e. $\omega \in E$,

$$\lim_{n\to\infty} \int_0^T : \tilde{\phi}_t^n(\omega) \langle Z(dt), \omega\rangle : = \int_0^T : \phi_t(\omega)\langle Z(dt), \omega\rangle :,$$

$$\lim_{n\to\infty} \int_0^T \left|\tilde{\phi}_t^n(\omega)\right|^2 \nu(dt) = \int_0^T \left|\phi_t(\omega)\right|^2 \nu(dt),$$

$$\lim_{n\to\infty} \int_0^T \int_0^T \left|D_s\tilde{\phi}_t^n(\omega) - D_s\phi_t(\omega)\right|^2 \nu(ds)\nu(dt) = 0.$$

Note that [cf. (2.6)] the last relation yields that, for μ-a.e. $\omega \in E$,

$$\lim_{n\to\infty} \int_0^T \nu(dt) \int_t^T D_t\tilde{\phi}_s^n(\omega) V_s^+ \tilde{\phi}_t^n(\omega) \nu(ds)$$
$$= \int_0^T \nu(dt) \int_t^T D_t\phi_s(\omega) V_s^+ \phi_t(\omega)\nu(ds).$$

Hence, $\lim_{n\to\infty} R^{\tilde{\phi}^n}(\omega) = R^{\phi}(\omega)$, μ-a.e.; and since $1 = \mathbb{E}_{\mu}\{R^{\tilde{\phi}^n}(\omega)\} \to 1 = \mathbb{E}_{\mu}\{R^{\phi}(\omega)\}$, as $n \to \infty$, we conclude that the family $\{R^{\tilde{\phi}^n}(\omega) \colon n \geq 1\}$ is uniformly integrable. But then, for every $l \in E^*$, the family $\{e^{i\langle l, \mathscr{T}^{\tilde{\phi}^n}[\omega]\rangle} R^{\tilde{\phi}^n}(\omega) \colon n \geq 1\}$ is also uniformly integrable; and so, since

$$\lim_{n\to\infty} e^{i\langle l, \mathscr{T}^{\tilde{\phi}^n}[\omega]\rangle} R^{\tilde{\phi}^n}(\omega) = e^{i\langle l, \mathscr{T}^{\phi}[\omega]\rangle} R^{\phi}(\omega),$$

for μ-a.e. $\omega \in E$, we conclude that

$$\mathbb{E}_{\mu}\left\{e^{i\langle l, \mathscr{T}^{\phi}[\omega]\rangle} R^{\phi}(\omega)\right\} = \lim_{n\to\infty} \mathbb{E}_{\mu}\left\{e^{i\langle l, \mathscr{T}^{\tilde{\phi}^n}[\omega]\rangle} R^{\tilde{\phi}^n}(\omega)\right\} = e^{-1/2\|l\|_H^2}. \qquad \square$$

The conclusion of the last theorem can be paraphrased as follows: The distribution of the element $\mathscr{T}^{\phi}[\omega] \in E$, with respect to the measure $d\mu^{\phi} \equiv$

$R^\phi\, d\mu$, is the same as the distribution of $\omega \in E$, with respect to the measure μ; that is, $\mathscr{T}^\phi[\omega]$ is a white noise in E, relative to the measure μ^ϕ.

REMARK. The Radon–Nikodym derivative R^ϕ was described here by the process (ϕ_t) and the two integral kernels: $D_s\phi_t$ and $V_s^+\phi_t$. Since in our approach "past" and "future" play equal roles, one may ask: Why is the right Volterra kernel $V_s^+\phi_t$ in the factorization of $(I - D\phi)^{-1}$ more important than the left Volterra kernel $V_s^-\phi_t$? In fact, it is not, for it is easy to check that

$$\int_0^T \nu(dt)\int_t^T D_t\phi_s(\omega)V_s^+\phi_t(\omega)\nu(ds) = \int_0^T \nu(dt)\int_0^t D_t\phi_s(\omega)V_s^-\phi_t(\omega)\nu(ds),$$

and so, R^ϕ can be described with the left kernel as well. However, choosing the right kernel better explains why in the classical nonanticipative case, neither of the kernels $D_s\phi_t$, $V_s^+\phi_t$ or $V_s^-\phi_t$ appears in the Radon–Nikodym derivative. Indeed, if the process (ϕ_t) in Theorem 3.1 is such that each random variable ϕ_t is measurable with respect to the σ-field generated by the family $\{\langle Z([s',t')), \cdot\,\rangle\colon 0 \le s' < t' \le t\}$, then $D_s\phi_t = 0$ whenever $s \ge t$, which yields that $V_s^+\phi_t \equiv 0$. Thus, in the latter case, the third integral in (3.3) vanishes, and the extended stochastic integral becomes the usual Itô integral, and so we come to the standard form of the Radon–Nikodym derivative, known from the classical Cameron–Martin–Girsanov theorem (cf. [3] and [6]). It should be noted, however, that this theorem requires no differentiability of any kind for the shift term, and therefore cannot be viewed as a particular case of Theorem 3.1 above. After the original works of Cameron–Martin and Girsanov, many different proofs of their result were published. All these proofs essentially use martingale methods and the Itô calculus (cf. [4] for one of the most recent expositions). It turns out that an independent proof, based on the approximation technique used above, is also possible (cf. [5]). This proof involves no martingale methods or Itô calculus.

REMARK. After the present paper was completed, the author learned about the work [1] which deals with the shift transformations $\mathscr{T}^\phi$, discussed above, in the case where these transformations act on the classical Wiener space and the vector-valued measure $Z(\cdot)$ is taken to be as in (2.1). The fact that one can construct a probability measure $d\mu^\phi = R^\phi\, d\mu$ with $\mu^\phi\{\mathscr{T}^\phi \in A\} = \mu(A)$, $A \in \mathscr{B}_E$, established here in Theorem 4.1 under the assumption $\int_0^T\int_0^T |D_s\phi_t(\omega)|^2\nu(ds)\nu(dt) < 1$, for μ-a.e. $\omega \in E$, and $\mathbb{E}_\mu\{R^\phi\} = 1$, is obtained in [1] under the following assumptions:

(a) There exists a constant $C < 1$ such that, for μ-a.e. $\omega \in E$,

$$\|D\phi(\omega)\|_{\mathrm{HS}} \equiv \left(\int_0^T\int_0^T |D_s\phi_t(\omega)|^2\nu(ds)\nu(dt)\right)^{1/2} \le C < 1.$$

(b) There exists a constant $\alpha > 1$ such that,

$$\mathbb{E}_\mu\left\{\exp\left[\frac{\alpha}{2}\int_0^T |\phi_t(\omega)|^2 \nu(dt)\right]\right\} < \infty.$$

The method used in [1] is entirely different and the Radon–Nikodym derivative R^ϕ is described there by different means. Moreover, (a) and (b) imply our condition $\mathbb{E}_\mu\{R^\phi(\omega)\} = 1$. This follows easily by the standard argument of the classical theory (cf. [16] and [11]). Indeed, (a) and (b) imply that there exists a sequence of smooth simple processes $\phi^n \in \mathfrak{S}$, $n = 1, 2, \dots,$ which approximates ϕ in $\mathbb{L}^{2,1}$ and is such that $\|D\phi^n(\omega)\|_{\mathrm{HS}} \le C < 1$ and

$$\sup_n \mathbb{E}_\mu\left\{\exp\left[\frac{\alpha}{2}\int_0^T |\phi_t^n(\omega)|^2 \nu(dt)\right]\right\} < \infty$$

(cf. Lemma 4.2 in [1]; this also follows easily by the argument in the proof of our Lemma 4.3). Let $\delta > 0$ be chosen so that $(1+\delta)(1+\delta^2)(1+\delta+\delta^2) < \alpha$ and $(1+\delta)(1+\delta^2)C < 1$. Set $\epsilon = \delta^2$, $p = 1+\delta$ and $q = (1+\delta)/\delta$. Let, for $n \ge 1$, $\varphi^n = (1+\epsilon)p(\phi^n)$ and let us denote respectively by U_+^n and V_+^n the right Volterra kernels in the factorization of the Fredholm operators $D\varphi^n(\omega)$ and $D\phi^n(\omega)$. Then we have, for every $n \ge 1$,

$$\begin{aligned}
(R^{\phi^n})^{1+\epsilon} = \exp&\left[(1+\epsilon)\int_0^T :\phi_t^n\langle Z(dt),\omega\rangle: - \frac{(1+\epsilon)^2 p}{2}\int_0^T |\phi_t^n|^2 \nu(dt)\right. \\
&\qquad\left. - \frac{1}{p}\int_0^T [D\varphi^n \circ U_+^n](t,t)\nu(dt)\right] \\
&\times \exp\left[\left(\frac{(1+\epsilon)^2 p}{2} - \frac{1+\epsilon}{2}\right)\int_0^T |\phi_t^n|^2 \nu(dt)\right] \\
&\times \exp\left[\frac{1}{p}\int_0^T [D\varphi^n \circ U_+^n](t,t)\nu(dt)\right. \\
&\qquad\left. -(1+\epsilon)\int_0^T [D\phi^n \circ V_+^n](t,t)\nu(dt)\right] \\
&\equiv X_n \times Y_n \times Z_n.
\end{aligned}$$

Note that

$$\left|\int_0^T [D\varphi^n \circ U_+^n](t,t)\nu(dt)\right| < \|U_+^n\|_{\mathrm{HS}} \le \frac{\|D\varphi^n\|_{\mathrm{HS}}}{1 - \|D\varphi^n\|_{\mathrm{HS}}} \le \frac{(1+\epsilon)pC}{1-(1+\epsilon)pC},$$

$$\left|\int_0^T [D\phi^n \circ U_+^n](t,t)\nu(dt)\right| < \|V_+^n\|_{\mathrm{HS}} \le \frac{\|D\phi^n\|_{\mathrm{HS}}}{1 - \|D\phi^n\|_{\mathrm{HS}}} \le \frac{C}{1-C},$$

and so all exponents Z_n are globally bounded by some constant M_δ. Hence

$$\mathbb{E}_\mu\Big\{\big(R^\phi\big)^{1+\epsilon}\Big\} \le M_\delta\Big(\mathbb{E}_\mu\big\{(X_n)^p\big\}^{1/p}\Big)\Big(\mathbb{E}_\mu\big\{(Y_n)^q\big\}^{1/q}\Big).$$

But $(X_n)^p \equiv R^{\varphi^n}$, and therefore, due to our Lemma 4.4, we have $\mathbb{E}_\mu\{(X_n)^p\} = 1$. On the other hand, due to our choice of δ,

$$\exp\left[q\left(\frac{(1+\epsilon)^2 p}{2} - \frac{1+\epsilon}{2}\right)\int_0^T |\phi_t^n|^2 \nu(dt)\right] \le \exp\left[\frac{\alpha}{2}\int_0^T \big|\phi_t^n(\omega)\big|^2 \nu(dt)\right].$$

Thus

$$\sup_n \mathbb{E}_\mu\Big\{\big(R^{\phi^n}\big)^{1+\epsilon}\Big\} < \infty,$$

which yields that $\{R^{\phi^n},\ n = 1,2,\dots\}$ is a uniformly integrable family. Since, $\mathbb{E}_\mu\{R^{\phi^n}\} = 1$, for every $n \ge 1$, and since there is a subsequence of R^{ϕ^n}, $n = 1,2,\dots,$ which converges a.s. to R^ϕ, it follows that $\mathbb{E}_\mu\{R^\phi\} = 1$.

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