The Annals of Probability
1993, Vol. 21, No. 4, 2157–2168

ON 1-DEPENDENT PROCESSES AND k-BLOCK FACTORS

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A stationary process $\{X_n\}_{n \in \mathbb{Z}}$ is said to be k-dependent if $\{X_n\}_{n<0}$ is independent of $\{X_n\}_{n>k-1}$. It is said to be a k-block factor of a process $\{Y_n\}$ if it can be represented as

$$X_n = f(Y_n, \dots, Y_{n+k-1}),$$

where f is a measurable function of k variables. Any $(k+1)$-block factor of an i.i.d. process is k-dependent. We answer an old question by showing that there exists a one-dependent process which is not a k-block factor of any i.i.d. process for any k. Our method also leads to generalizations of this result and to a simple construction of an eight-state one-dependent Markov chain which is not a two-block factor of an i.i.d. process.

1. Introduction. A stationary stochastic process $\{X_n\}_{n \in \mathbb{Z}}$ is said to be *k-dependent* if $\{X_n\}_{n<0}$ is independent of $\{X_n\}_{n>k-1}$. Obviously, a one-dependent process is k-dependent for any $k \geq 1$. A stationary stochastic process $\{X_n\}$ is said to be a *k-block factor* of a stochastic process $\{Y_n\}$ if it can be represented as

$$X_n = f(Y_n, Y_{n+1}, \dots, Y_{n+k-1}),$$

where f is a measurable function of k variables.

It is clear that any $(k+1)$-block factor of an i.i.d. process is also k-dependent. It was conjectured by Ibragimov and Linnik (1971) that the converse of this statement is not true: They conjectured that there are k-dependent processes which are not $(k+1)$-block factors of any i.i.d. process. Progress on this conjecture was made by Aaronson, Gilat, Keane and de Valk (1989) who showed that there exists a two-parameter family of two-state one-dependent processes which are not two-block factors of an i.i.d. process. The construction of their processes is algebraic and a more probabilistic mechanism was desired. In this paper, we construct a four-state one-dependent process which is not a k-block factor of an i.i.d. process for *any* $k \geq 2$, thereby settling the conjecture of Ibragimov and Linnik in the affirmative.

In Aaronson, Gilat and Keane (1990) it is shown that any one-dependent four-state Markov chain is necessarily a two-block factor of an i.i.d. process. This result is sharp in the sense that they constructed a five-state one-dependent Markov chain which is not. Their proof is quite complicated, and in this paper we give a very simple proof that a certain eight-state one-dependent

Received March 1992; revised October 1992.
[1]Research supported in part by AFOSR Grant 91-0215 and NSF Grant DMS-91-03738.
AMS 1991 *subject classifications*. 60G10, 54H20, 28D05.
Key words and phrases. Stationary processes, one-dependence, block factors, Markov chains.

Markov chain is not a two-block factor. For more details on one-dependent processes, we refer to Goulet (1992) or de Valk (1991) for an example.

The organization of the paper is as follows. The processes of interest are constructed in Section 2, and our main result is proved in Section 3. Section 4 deals with some generalizations of this result, and the final section is devoted to our Markov chain example.

2. Construction of the process. We construct a stochastic process $\{X_n\}$ with state space $\{0, 1, \binom{1}{2}, \binom{0}{2}\}$ as follows. First we choose an arbitrary set of indices $S \subset \mathbb{N}$. We then start off with an i.i.d. sequence $\{Z_n\}_{n \in \mathbb{Z}}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ such that $P(Z_1 = 0) = P(Z_1 = 1) = P(Z_1 = 2) = 1/3$. Let, for all $n \in \mathbb{Z}$, $\tau(n)$ be defined as

$$\tau(n) = \max_{m<n}\{m | Z_m = 2\},$$

and let $d(n) = n - \tau(n) - 1$. Thus $d(n)$ is the number of elements strictly between Z_n and the previous 2. Now the process $\{X_n\}$ corresponding to S is defined on $(\Omega, \mathscr{F}, P)$ as follows:

$$X_n = \begin{cases} Z_n, & \text{if } Z_n = 0 \text{ or } Z_n = 1, \\ \binom{i_n}{2}, & \text{if } Z_n = 2, \end{cases}$$

where

$$i_n = \begin{cases} \displaystyle\sum_{j=\tau(n)+1}^{n-1} Z_j \pmod 2, & \text{if } d(n) \notin S, \\ 1 - \displaystyle\sum_{j=\tau(n)+1}^{n-1} Z_j \pmod 2, & \text{if } d(n) \in S. \end{cases}$$

In words, if $Z_n = 2$ and the number of elements between this and the previous 2 is not an element of S, then the top coordinate of X_n is the modulo 2 sum of these elements. If the number of elements between this 2 and the previous 2 is an element of S, then the top coordinate is one minus this modulo 2 sum. We will call the modulo 2 sum of a set of zeroes and ones the *parity* of this set. It is clear that $\{X_n\}$ is stationary for any S, and in order to state our main result, we define $\mathscr{S}$ to be the class of those $S \subset \mathbb{N}$, which have "arbitrarily big gaps" between successive elements. More precisely, if we write $S = \{s_1, s_2, \dots\}$ where $s_i < s_{i+1}$ for all i, then

$$\mathscr{S} = \left\{S \subset \mathbb{N} \,|\, \sup_i (s_{i+1} - s_i) = \infty\right\}.$$

Our main result is the following:

THEOREM 1. *For any S, the corresponding process $\{X_n\}$ is one-dependent, but it is not a k-block factor of any i.i.d. process, for any $k \geq 1$, whenever $S \in \mathscr{S}$.*

As a heuristic argument why processes like this should not be k-block factors, we remark that in order to determine the top coordinates, we may have to look back arbitrarily far. Some irregularity condition on S is needed though, because the process corresponding to $S = \varnothing$ is in fact a two-block factor of an i.i.d. process. See Goulet (1992) for details.

3. Proof of Theorem 1.

We first prove one-dependence:

LEMMA 1. *For any S, the corresponding process $\{X_n\}$ is one-dependent.*

PROOF. It will be enough to show that

$$P(B|A) = P(B),$$

for events A and B of the form

$$A = \{X_j = a_j,\, j = -1, \dots, -m\}$$

and

$$B = \{X_j = b_j,\, j = 1, \dots, n\},$$

where $m, n > 0$. Of course we may assume that both A and B have positive probability. If $b_j \in \{0, 1\}$ for $1 \le i \le n$, then B is clearly independent of A by construction. Hence we now assume that $b_j \notin \{0, 1\}$ for at least one value of i, and we define

$$\theta = \min\{1 \le j \le n | b_j \notin \{0, 1\}\}.$$

It is clear from the construction that of all coordinates in B, only the top coordinate of b_θ can possibly depend on A. In order to exploit this we define the following events:

$$D = \{X_1 + \cdots + X_{\theta-1} = b_1 + \cdots + b_{\theta-1} \pmod 2\},$$
$$B_\theta = \{X_\theta = b_\theta\}.$$

Conditioning on X_0 and writing $\{X_0 = \binom{*}{2}\}$ for $\{X_0 = \binom{0}{2}\} \cup \{X_0 = \binom{1}{2}\}$, we see that

$$\begin{aligned} P(B|A) &= P\left(B|A, X_0 = \binom{*}{2}\right)P\left(X_0 = \binom{*}{2}\Big|A\right) \\ &\quad + P(B|A, X_0 \in \{0,1\})P(X_0 \in \{0,1\}|A) \\ &= \left(\tfrac{1}{3}\right)^{n-1} P\left(B_\theta|A, D, X_0 = \binom{*}{2}\right)P\left(X_0 = \binom{*}{2}\Big|A\right) \\ &\quad + \left(\tfrac{1}{3}\right)^{n-1} P(B_\theta|A, D, X_0 \in \{0,1\})P(X_0 \in \{0,1\}|A). \end{aligned}$$

The idea now is to argue that in each of the terms above, we may remove A from the conditioning event without changing the values of the probabilities involved. Clearly

$$P\left(X_0 = \binom{*}{2} \middle| A\right) = P\left(X_0 = \binom{*}{2}\right)$$

and

$$P(X_0 \in \{0,1\} | A) = P(X_0 \in \{0,1\}).$$

Also, from the construction of the process it follows that

$$P\left(B_\theta | A, D, X_0 = \binom{*}{2}\right) = P\left(B_\theta | D, X_0 = \binom{*}{2}\right).$$

Finally, by symmetry of even and odd parities, the probability of seeing a 1 (or a 0) as the top coordinate of a given 2, given any event which does not specify all coordinates back to the previous 2, is $1/2$, whatever the precise form of the event. The events $\{A, D, X_0 \in \{0,1\}\}$ and $\{D, X_0 \in \{0,1\}\}$ both fall in this category, and hence

$$P(B_\theta | A, D, X_0 \in \{0,1\}) = P(B_\theta | D, X_0 \in \{0,1\}),$$

which completes the proof. □

A cylinder event like $\{X_1 = a_1, X_2 = a_2, X_5 = a_5\}$ will be denoted by

$$X_1 X_2 \cdots X_5 = a_1 a_2 * * a_5.$$

Hence, in some sense, asterisks represent unspecified symbols. Note however that if we choose S such that, say, $2 \notin S$, the event $X_1 \cdots X_4 = \binom{1}{2} 1 * \binom{1}{2}$ is the same as the event $X_1 \cdots X_4 = \binom{1}{2} 10 \binom{1}{2}$; the symbol represented by the asterisk is determined by the other coordinates. Analogously, events like $\{X_1 = \binom{1}{2}, X_2 = a_2, X_4 = a_4\} \cup \{X_1 = \binom{0}{2}, X_2 = a_2, X_4 = a_4\}$ will be denoted by

$$X_1 \cdots X_4 = \binom{*}{2} a_2 * a_4.$$

To complete the proof of Theorem 1, we now prove:

LEMMA 2. *For any $S \in \mathscr{S}$, the corresponding process $\{X_n\}$ is not a k-block factor of any i.i.d. process, for any $k \geq 1$.*

PROOF. For ease of description, we will give the proof in the case where S is the set of squares, i.e., $S = \{n^2 | n \in \mathbb{N}\}$.

The proof will proceed by contradiction, so take k arbitrary but fixed and suppose that $\{X_n\}$ can be represented as

$$X_n = f(Y_n, \dots, Y_{n+k-1}),$$

where f is a measurable function of k variables and $\{Y_n\}$ any i.i.d. stochastic process. The *code* of a sequence of symbols $y = (y_1, y_2, \dots, y_m)$, $m \geq k$, denoted by $c(y)$, is defined as

$$c(y) = f(y_1, y_2, \dots, y_k) f(y_2, \dots, y_{k+1}) \cdots f(y_{m-k+1}, \dots, y_m).$$

Furthermore, the sequence $xxx \cdots x$ (n elements) will be denoted by $[x]^n$.

Now we choose a number n so large such that there exist $i, j \in \mathbb{N}$ satisfying (i)–(iv) below:

(i) $k < i < j < n^2$,
(ii) $n^2 - i + l \notin S$, for all $l = 1, \dots, k$,
(iii) $j - l \notin S$, for all $l = 1, \dots, k$,
(iv) $j - i + l \notin S$, for all $l = -k + 1, \dots, k - 1$.

It is an easy matter to check that this choice is possible and we fix numbers n, i and j with the properties (i)–(iv). Now let $m = n^2 + k + 1$ and define, for any m-tuple $i_1 < i_2 < \cdots < i_m$ the event

$$E(i_1, i_2, \dots, i_m) = \left\{ c(Y_{i_1}, Y_{i_2}, \dots, Y_{i_m}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} [0]^{n^2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

By assumption we have that $P(E(i_1, \dots, i_m)) > 0$ for any m-tuple $i_1 < i_2 < \cdots < i_m$. Furthermore, we define for any k-tuple $j_1 < \cdots < j_k$ the event

$$F(j_1, \dots, j_k) = \left\{ c(Y_{j_1}, \dots, Y_{j_k}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

The idea of the proof now is to start with the event $E(1, 2, \dots, m)$, then "pull this event apart" and insert a 2 in two different places, and then show that this results in an impossible event having positive probability. To do this, remember our choice of i, j and n and consider the event

$$D_1 = E(1, 2, \dots, i, i + k + 1, \dots, m + k) \cap F(i + 1, i + 2, \dots, i + k).$$

In comparison with the event $E(1, 2, \dots, m)$, $k - 1$ zero's in $X_1 \cdots X_{n^2+2}$ are replaced by $2k - 1$ new symbols, the middle one of which is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ by construction. More formally, on the event D_1 we have

$$X_1 X_2 \cdots X_{n^2+k+2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} [0]^{i-k} \underbrace{* \cdots *}_{k-1 \text{ times}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \underbrace{* \cdots *}_{k-1 \text{ times}} [0]^{n^2-i+1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Now let the random variable n_0 on D_1 be defined as

$$n_0 = \max_{i+1 \le l \le i+k} \left\{ l | X_l = \begin{pmatrix} * \\ 2 \end{pmatrix} \right\},$$

that is, n_0 is the index of the last $\begin{pmatrix} * \\ 2 \end{pmatrix}$ preceding the designated sequence $[0]^{n^2-i+1}$. Because of the fact that $X_{n^2+k+2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and our choice of i (condition (ii) above), the parity of all the asterisks with coordinate larger than n_0 equals 1 on D_1. (Note that it follows from this that the rightmost asterisk cannot be $\begin{pmatrix} * \\ 2 \end{pmatrix}$, since then the parity of the elements between this $\begin{pmatrix} * \\ 2 \end{pmatrix}$ and the final $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ would be zero and we would reach a contradiction right away.)

Next, we consider the event D_2, defined as

$$D_2 = E(1,2,\dots,j,j+k+1,\dots,m+k) \cap F(j+1,\dots,j+k).$$

As above, we see that on D_2 we have

$$X_1X_2 \cdots X_{n^2+k+2} = \binom{1}{2}[0]^{j-k} \underbrace{* \cdots *}_{k-1 \text{ times}} \binom{1}{2} \underbrace{* \cdots *}_{k-1 \text{ times}} [0]^{n^2-j+1}\binom{1}{2},$$

Now let the random variable m_0 on D_2 be defined as

$$m_0 = \min_{j-k+2 \le l \le j+1} \left\{ l | X_l = \binom{*}{2} \right\},$$

that is, m_0 is the index of the first $\binom{*}{2}$ following the designated sequence $[0]^{j-k}$. The (random) top coordinate of X_{m_0} is denoted by q. By the same reasoning as above, the parity of the asterisks to the left of X_{m_0} must be q, this time using condition (iii) above. The contradiction now arises from combining D_1 and D_2 to construct the following event:

$$\begin{aligned} D_3 = E(1,2,\dots,i,i+k+1,i+k+2,\dots,j+k,j+2k+1,\\ j+2k+2,\dots,m+2k)\\ \cap F(i+1,i+2,\dots,i+k) \cap F(j+k+1,j+k+2,\dots,j+2k). \end{aligned}$$

We obviously have that $P(D_3) > 0$, and on D_3 we have

$$\begin{aligned} &X_1X_2 \cdots X_{n^2+2k+2}\\ &\quad= \binom{1}{2}[0]^{i-k} \underbrace{* \cdots *}_{k-1} \binom{1}{2} \underbrace{* \cdots *}_{k-1} [0]^{j-i-k+1}\\ &\qquad \underbrace{* \cdots *}_{k-1} \binom{1}{2} \underbrace{* \cdots *}_{k-1} [0]^{n^2-j+1}\binom{1}{2}. \end{aligned}$$

Combining the conclusions obtained individually for D_1 and D_2 above, we see that the parity of *all* elements between the last $\binom{*}{2}$ preceding the designated sequence $[0]^{j-i-k+1}$ and the first $\binom{q}{2}$ following it is $1-q$. But from condition (iv) above, it follows that the number of elements between this $\binom{*}{2}$ and $\binom{q}{2}$ is not a square, and hence $\binom{q}{2}$ has the wrong top coordinate and we conclude that $P(D_3) = 0$, which is the desired contradiction. □

4. Generalizations. There are two ways to extend the example above. One may weaken the assumptions on the process $\{Y_n\}$ and no longer require that this process is i.i.d. On the other hand, one may strengthen the requirements for the process $\{X_n\}$. The process $\{X_n\}$ discussed above has the property that certain finite dimensional events have probability zero. For example, for any $s \in S$, the event

$$X_1 X_2 \cdots X_{s+2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} [0]^s \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

cannot occur. We will see in the proof of Theorem 3 below how we can construct an example of a one-dependent process which is not a k-block factor for any k, such that all finite dimensional events have positive probability. In order to state our results, we define the following concepts. A stationary process $\{Y_n\}$ is said to be *positive on cylinder events* if for any finite sequence $n_1 < n_2 < \cdots < n_l$ and any $B \subset \mathbb{R}^l$ with positive l-dimensional Lebesgue measure we have

$$P\big((Y_{n_1}, Y_{n_2}, \dots, Y_{n_l}) \in B\big) > 0.$$

A stationary process $X = \{X_n\}$ with finite state space T satisfies the *finite energy* condition of Newman and Schulman (1981) if the following is true. For any $t \in T$ and for any event B measurable with respect to the σ-field generated by $\{X_n | n \neq 0\}$ and with positive probability, we have $P(X_0 = t | B) > 0$. We remark that having finite energy is strictly weaker than being positive on cylinder events.

Each of the following theorems generalizes the result of the previous sections.

THEOREM 2. *For any $S \in \mathscr{S}$, the corresponding process $\{X_n\}$ is not a k-block factor (for any $k \geq 1$) of any process $\{Y_n\}$ such that $\{Y_n\}$ is positive on cylinder events.*

PROOF. It is an easy matter to check that the proof of Lemma 2 goes through without change if we only assume $\{Y_n\}$ to be positive on cylinder events. □

THEOREM 3. *There exists a one-dependent process with finite energy which is not a k-block factor of any i.i.d. process, for any $k \geq 1$.*

PROOF. This requires more work, though the idea is the same as in the proof of Theorem 1. Let Z_n, i_n and $d(n)$ be as defined in Section 2, where we again assume for simplicity that S is the set of squares. Let $\{\alpha_n\}$, be a sequence of positive real numbers in $(0, 1)$, decreasing to zero. The process $\{X_n\}$, which we will define now, is a perturbed version of the one constructed in Section 2. As before, if $Z_n = 0$ or $Z_n = 1$, then $X_n = Z_n$. But if $Z_n = 2$, we perturb the top coordinate of X_n independently of anything else as follows:

$$X_n = \begin{cases} \begin{pmatrix} i_n \\ 2 \end{pmatrix}, & \text{with probability } 1 - \alpha_{d(n)}, \\ \begin{pmatrix} 1 - i_n \\ 2 \end{pmatrix}, & \text{with probability } \alpha_{d(n)}. \end{cases}$$

We will say that the top coordinate of a 2 is *right* if it is equal to the parity of all elements back to the previous 2, otherwise it is *wrong*.

It is easy to see that $\{X_n\}$ has finite energy, and the proof that it is one-dependent is the same as in the proof of Lemma 1. So we need only to show that $\{X_n\}$ is not a k-block factor of any i.i.d. process, for any $k \geq 1$. Take k arbitrary but fixed and suppose that $\{X_n\}$ can be represented as

$$X_n = f(Y_n, \dots, Y_{n+k-1}),$$

where $\{Y_n\}$ is an i.i.d. process. The *code* c is defined as before. Take any n, i and j as in the proof of Lemma 2, with the extra requirement that $j - i - k + 1 > n/2$. Later we will need to choose n and j sufficiently large and i sufficiently small. Define the following events, again setting $m = n^2 + k + 1$:

$$E(i_1, \dots, i_m) = \left\{ c(Y_{i_1}, \dots, Y_{i_m}) = \begin{pmatrix} * \\ 2 \end{pmatrix} [0]^{n^2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\},$$

$$F(i_1, \dots, i_k) = \left\{ c(Y_{i_1}, \dots, Y_{i_k}) = \begin{pmatrix} * \\ 2 \end{pmatrix} \right\}.$$

By the independence of $\{Y_n\}$ we have that

$$P(E(i_1, \dots, i_m)) = \left(\tfrac{1}{3}\right)^{n^2+2} (1 - \alpha_{n^2})$$

and $P(F(i_1, \dots, i_k)) = 1/3$. As in the proof of Lemma 2, let D_1 be the event

$$D_1 = E(1, 2, \dots, i, i + k + 1, \dots, m + k) \cap F(i + 1, \dots, i + k).$$

Then we have that

$$P(D_1) = \left(\tfrac{1}{3}\right)^{n^2+3} (1 - \alpha_{n^2}).$$

On the event D_1 we have

$$X_1 \cdots X_{n^2+k+2} = \begin{pmatrix} * \\ 2 \end{pmatrix} [0]^{i-k} \underbrace{* \cdots *}_{k-1} \begin{pmatrix} * \\ 2 \end{pmatrix} \underbrace{* \cdots *}_{k-1} [0]^{n^2-i+1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We define n_0 as in the proof of Lemma 2. Let $D_1(1) \subset D_1$ be the event that the parity of the stars with coordinate larger than n_0 is 1, and let $D_1(0)$ be the event that this parity is 0. Now $D_1(0)$ is contained in the event

$$\left\{ X_1 \cdots X_{n^2+k+2} = \begin{pmatrix} * \\ 2 \end{pmatrix} [0]^{i-k} \underbrace{* \cdots *}_{k-1} \begin{pmatrix} * \\ 2 \end{pmatrix} \underbrace{* \cdots *}_{k-1} [0]^{n^2-i+1} \begin{pmatrix} * \\ 2 \end{pmatrix} \right\}$$

$$\cap \{\text{the top coordinate of } X_{n^2+k+2} \text{ is wrong}\}$$

and hence we see that

$$P(D_1(0)) \leq \left(\tfrac{1}{3}\right)^{n^2-k+4} \alpha_{n^2-i+1}.$$

This implies that we can choose n so large and i so small (depending on k)

such that

$$P(D_1(1)) \geq \tfrac{3}{4}P(D_1).$$

Next, we define the event D_2 as

$$D_2 = E(1,2,\dots,j,j+k+1,\dots,m+k) \cap F(j+1,\dots,j+k)$$

and m_0 is defined as in the proof of Lemma 2. We denote by $D_2(r) \subset D_2$ the event that the top coordinate of X_{m_0} is right. Reasoning in the same way as above, we find that

$$P(D_2(r)) \geq \tfrac{3}{4}P(D_2),$$

for j and n large enough.

Now we define D_3 by combining D_1 and D_2 as in the proof of Lemma 2. Then we have that

$$P(D_3) = \left(\tfrac{1}{3}\right)^{n^2+4}(1-\alpha_{n^2}),$$

and on D_3 we have

$$\begin{aligned} X_1X_2 \cdots X_{n^2+2k+2} &= \binom{*}{2}[0]^{i-k}\underbrace{*\cdots*}_{k-1}\binom{*}{2}\underbrace{*\cdots*}_{k-1}[0]^{j-i-k+1} \\ &\quad \underbrace{*\cdots*}_{k-1}\binom{*}{2}\underbrace{*\cdots*}_{k-1}[0]^{n^2-j+1}\binom{1}{2}. \end{aligned}$$

Let $D_3(1) \subset D_3$ be the event that the parity of the asterisks between the designated sequence $[0]^{j-i-k+1}$ and the previous 2 equals 1. We claim that

$$P(D_3(1)) \geq \tfrac{3}{4}P(D_3).$$

To see this note that we have $P(D_3) = P(D_1)P(F(j+k+1,\dots,j+2k))$, and also $P(D_3(1)) = P(D_1(1))P(F(j+k+1,\dots,j+2k))$. Hence

$$\frac{P(D_3(1))}{P(D_3)} = \frac{P(D_1(1))}{P(D_1)} \geq \frac{3}{4}$$

and the claim follows.

The event $D_3(r) \subset D_3$ is the event that the top coordinate of the first 2 following the designated sequence $[0]^{j-i-k+1}$ is equal to the parity of the asterisks between this 2 and the sequence. In the same way as above we see that

$$P(D_3(r)) \geq \tfrac{3}{4}P(D_3).$$

Hence

$$P(D_3(1) \cap D_3(r)) \geq \tfrac{1}{2}P(D_3) = \tfrac{1}{2}\left(\tfrac{1}{3}\right)^{n^2+4}(1-\alpha_{n^2}).$$

On the other hand, we remark that $D_3(1) \cap D_3(r)$ is contained in the event

$$\Big\{X_1 \cdots X_{n^2+2k+2} = \binom{*}{2}[0]^{i-k}\underbrace{*\cdots*}_{k-1}\binom{*}{2}\underbrace{*\cdots*}_{k-1}[0]^{j-i-k+1}$$
$$\underbrace{*\cdots*}_{k-1}\binom{*}{2}\underbrace{*\cdots*}_{k-1}[0]^{n^2-j+1}\binom{1}{2}\Big\}$$
$$\cap\,\{\text{the top coordinate of the first 2 following the designated sequence } [0]^{j-i-k+1} \text{ is wrong}\}.$$

This event has probability at most $(1/3)^{n^2-2k+6}\alpha_{n/2}$, by our choice of j and i. So it follows that

$$\left(\tfrac{1}{3}\right)^{n^2-2k+6}\alpha_{n/2} \geq \tfrac{1}{2}\left(\tfrac{1}{3}\right)^{n^2+4}(1-\alpha_{n^2}),$$

which is impossible for n sufficiently large. □

5. Markov chains and two-block factors. Consider the process $\{X_n\}$ defined in Section 2, corresponding to $S = \{1\}$, that is, the top coordinate of a 2 is wrong if and only if there is exactly one element between this 2 and the previous 2. We now define a Markov chain $\{M_n\}$ with state space $\{\binom{0}{2}, \binom{1}{2}, \binom{1}{0}, \binom{1}{1}, \binom{e}{0}, \binom{e}{1}, \binom{d}{0}, \binom{d}{1}\}$ (in this order), and transition matrix P given by

$$P = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

It is not hard to see that $\{X_n\}$ can be represented as a one-block factor of $\{M_n\}$, that is, $X_n = f(M_n)$, for all n, where $f(\binom{0}{2}) = \binom{0}{2}$, $f(\binom{1}{2}) = \binom{1}{2}$, $f(\binom{*}{0}) = 0$ and $f(\binom{*}{1}) = 1$, where $* \in \{e, d\}$. The interpretation of the Markov chain $\{M_n\}$ is that if $M_n = \binom{1}{1}$, then $X_n = 1$ and X_n is preceded by a 2. If $M_n = \binom{e}{1}$, then $X_n = 1$ and the parity of all elements (including X_n) back to the previous 2 is 0. If $M_n = \binom{d}{1}$, this parity is 1. (e stands for even parity, d for odd.) The interpretation when the lower coordinate is 0 is analogous. We now have the following result.

THEOREM 4. *The Markov chain $\{M_n\}$ is one-dependent, but it is not a two-block factor of an i.i.d. process.*

PROOF. An easy calculation shows that the stationary distribution π of $\{M_n\}$ is given by

$$\pi = \left(\tfrac{2}{9}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{9}\right).$$

Straightforward calculations now show that $P^2 = \Pi$, where Π is the matrix whose rows consist of π. It is not difficult to check that this implies that $\{M_n\}$ is one-dependent.

To show that $\{M_n\}$ is not a two-block factor, it is enough to show that $\{X_n\}$ is not a two-block factor, as a one-block factor of a two-block factor is obviously a two-block factor. So we now assume that $\{X_n\}$ can be represented as

$$X_n = f(Y_n, Y_{n+1}),$$

for some measurable function f and an i.i.d. process $\{Y_n\}$, where we can assume without loss of generality that Y_1 is uniformly distributed on $[0, 1]$. The code c is defined as before, and k-dimensional Lebesgue measure is denoted by λ_k. For $w, x, y, z \in [0, 1]$, we define the following events:

$$E = \left\{(w, x, y, z) \in [0, 1]^4 | c(w, x, y, z) = \binom{1}{2} 0 \binom{1}{2}\right\},$$

$$E_3(w, x, z) = \{y \in [0, 1] | (w, x, y, z) \in E\},$$

$$E_3'(w, x, z) = \{(y, y') \in E_3(w, x, z) \times E_3(w, x, z) | f(y, y') = 1\},$$

$$E_3''(w, x, z) = \{(y, y', y'') | (y, y'), (y', y'') \in E_3'(w, x, z)\}.$$

Of course, $\lambda_4(E) > 0$, and because of the fact that

$$\lambda_4(E) = \int_0^1 \int_0^1 \int_0^1 \lambda_1(E_3(w, x, z))\, dw\, dx\, dz,$$

we see that $\lambda_1(E_3(w, x, z))$ is positive on a set of positive measure in $[0, 1]^3$.

We next show that for almost all (w, x, z) we have (up to a set of measure zero)

$$E_3'(w, x, z) = E_3(w, x, z) \times E_3(w, x, z). \tag{1}$$

To see this, let

$$D(w, x, z) = (E_3(w, x, z) \times E_3(w, x, z)) \setminus E_3'(w, x, z),$$

and

$$D = \{(w, x, y, y', z) | (y, y') \in D(w, x, z)\}.$$

On D we have either $c(w, x, y, y', z) = \binom{1}{2} 00 \binom{1}{2}$ or $c(w, x, y, y', z) = \binom{1}{2} 0 \binom{*}{2} \binom{1}{2}$, and from the definition of $\{X_n\}$ we see that $\lambda_5(D) = 0$. However,

$$\lambda_5(D) = \int_0^1 \int_0^1 \int_0^1 \lambda_2(D(w, x, z))\, dw\, dx\, dz,$$

and hence $\lambda_2(D(w, x, z)) = 0$ for almost all (w, x, z), and (1) follows. But then we conclude that for almost all (w, x, z) we have that up to a set of measure zero,

$$E_3''(w, x, z) = E_3(w, x, z) \times E_3(w, x, z) \times E_3(w, x, z).$$

Because $\lambda_1(E_3(w, x, z))$ is positive on a set of positive measure, we conclude that

$$\lambda_3((w, x, z) \in [0, 1] | \lambda_3(E_3''(w, x, z))) > 0.$$

Now define the set G as

$$G = \{(w, x, y, y', y'', z) | (y, y', y'') \in E_3''(w, x, z)\}.$$

We then have

$$\lambda_6(G) = \int_0^1 \int_0^1 \int_0^1 \lambda_3(E_3''(w, x, z))\, dw\, dx\, dz > 0.$$

On the other hand, on G we have

$$c(w, x, y, y', y'', z) = \left(\frac{1}{2}\right) 011 \left(\frac{1}{2}\right),$$

which is impossible by the definition of the process $\{X_n\}$. □

Acknowledgments. Robert M. Burton and Marc Goulet are grateful to the Technical University of Delft for its hospitality.

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