The Annals of Probability
1994, Vol. 22, No. 1, 469–493

INTEGRATION BY PARTS ON WIENER SPACE AND THE EXISTENCE OF OCCUPATION DENSITIES

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We present a general criterion for the existence of an occupation density, which is based on the integration by parts formula on Wiener space. This criterion is applied to two particular examples of anticipating processes. First we discuss the case of Brownian motion plus a nonadapted absolutely continuous drift, and second we consider the case of a Skorohod integral process. Finally a version of Tanaka's formula is proved for the Skorohod integral process.

1. Introduction. An extension of the Itô stochastic integral for processes which are not necessarily adapted to the Brownian motion was introduced by Skorohod in [12]. The Skorohod integral shares some of the probabilistic properties of the Itô integral and, on the other hand, it has a more analytic flavor because it can be regarded as the adjoint of the derivative operator. Recently, a stochastic calculus for Skorohod integral processes has been developed in [9] and a version of the change of variable formula (Itô's formula) for Skorohod integral processes has been established. A crucial point in the derivation of this formula is the fact that Skorohod integral processes possess quadratic variations which are given in the same way as for their Itô counterparts and consequently are nontrivial. That is, the trajectories of these processes are continuous, but highly irregular. This observation leads us to investigate their occupation densities or local times.

Different methods have been used so far to study the occupation densities of anticipating processes. Berman's idea to employ Fourier analysis (see [2]) has been applied by Imkeller in [6, 7] to find integral criteria for the existence of a square integrable local time for a Skorohod integral process which belongs to the second Wiener chaos. A more stochastic approach has been presented in [8], again for processes living in the second Wiener chaos (although this method can be generalized to deal with more general processes). Its main idea is to represent the Skorohod integral process we are dealing with as the composition of a Gaussian semimartingale depending on an infinite dimensional parameter with a Gaussian vector, and to use an infinite dimensional generalization of Kolmogorov's continuity criterion in order to find the local time of this composition.

Received June 1992.
[1]This work was done during a visit to the CRM of Barcelona.
[2]Partially supported by DGICYT Grant PB90-0452.
AMS 1991 *subject classifications*. Primary 60H05.
Key words and phrases. Perturbed Wiener process, anticipating processes, Skorohod integral process, Tanaka's formula.

In this paper we present a new general approach which can be used to show the existence of a square integrable occupation density for a wide class of anticipating processes. The starting point of our method is a sufficient criterion for the existence of an occupation density established by Geman and Horowitz in [4]. We combine it with the technique of integration by parts on Wiener space in order to deduce several sufficient criteria for the existence of occupation densities with respect to Lebesgue measure as a scale for measuring occupation. This is the content of Section 2. The main criteria are given by Theorems 2.2 and 2.3. These criteria are rather general and can be applied to different situations. Two applications of the local criterion presented in Theorem 2.2 are discussed in Sections 3 and 4.

In Section 3 we consider a nonadapted process of the form

$$X_t = W_t + \int_0^t u_s \, ds, \qquad 0 \le t \le 1, \tag{1.1}$$

where u belongs to the space $\mathbb{L}^{2,2}$. It is proved (see Theorem 3.1) that X possesses a square integrable occupation density.

Section 4 is devoted to showing the existence of a square integrable occupation density for a Skorohod integral process of the form

$$X_t = \int_0^t u_s \, dW_s, \qquad 0 \le t \le 1. \tag{1.2}$$

Some smoothness and integrability hypotheses on the process u are required (see Theorem 4.1) and, furthermore, we need a nondegeneracy condition of the form

$$E\left(\int_0^1 |u_t|^{-\gamma} \, dt\right) < \infty \quad \text{for some suitable } \gamma > 0, \tag{1.3}$$

which is natural in the chosen framework.

The continuity of the occupation density for the processes (1.1) and (1.2) under more restrictive assumptions will be discussed in a forthcoming paper. On the other hand, we believe that the local time for processes of the form (1.2) with respect to a scale measure like $u_s^2 \, ds$ (instead of Lebesgue measure) may exist without the hypothesis (1.3). But the criterion presented in Section 2 does not seem to be a convenient tool to deal with this problem.

In Section 5 we present a version of the Tanaka formula for the indefinite Skorohod integral, which is obtained as a consequence of the previous results.

2. Criteria for the existence of occupation densities. In this section we will present some criteria for the existence of occupation densities, all of which are based on the technique of integration by parts on Wiener space. We will only discuss occupation densities with respect to Lebesgue measure, denoted by λ, as scale for measuring occupation.

For a measurable function x: $T \to \mathbb{R}$, T a Borel subset of $[0, 1]$, we set

$$\mu_T(x)(C) = \int_T \mathbf{1}_C(x_s)\, ds, \qquad C \in \mathscr{B}(\mathbb{R}),$$

and we say that x *has an occupation density on* T if $\mu_T(x) \ll \lambda$. Moreover we call $d\mu_T/d\lambda$ *the occupation density of* x *on* T. If $T = [0, 1]$, we omit the reference to T and simply speak about occupation densities. An occupation density of x is called square integrable if it is in $L^2(\mathbb{R}, \lambda)$.

Consider a probability space $(\Omega, \mathscr{F}, P)$. If X is a continuous stochastic process indexed by $[0, 1]$ and $A \in \mathscr{F}$, we say that X *has an occupation density on* (T, A) if for almost all $\omega \in A$, $X(\omega)$ has an occupation density on T. If $T = [0, 1]$ and $A = \Omega$, we omit the specification (T, A) and simply speak about occupation densities.

In our analysis we will consider occupation densities on subintervals T of $[0, 1]$ and on sets $A \subset \Omega$, and we would like to patch them together to obtain occupation densities on $[0, 1]$ and Ω. The following proposition is a basic tool for this procedure.

PROPOSITION 2.1. (i) *Let* $\delta > 0$. *Suppose that* x: $[0, 1] \to \mathbb{R}$ *has a* (*resp., square integrable*) *occupation density on* T *for any subinterval* T *of diameter less than* δ. *Then* x *possesses a* (*resp., square integrable*) *occupation density.*

(ii) *Let* X *be a continuous stochastic process. Suppose that* $(\delta_n)_{n \in \mathbb{N}}$ *is a sequence of positive real numbers, and* $(A_n)_{n \in \mathbb{N}}$ *is a sequence of* $\mathscr{F}$-*measurable subsets of* Ω *such that* $A_n \uparrow \Omega$ *a.s. If* X *has a* (*resp., square integrable*) *occupation density on* (T, A_n) *for any subinterval* T *of* $[0, 1]$ *of diameter less than* δ_n, *for all* $n \in \mathbb{N}$, *then* X *has a* (*resp., square integrable*) *occupation density.*

This proposition is an immediate consequence of the preceding definitions. In the sequel we will make extensive use of the "localization" principle inherent in Proposition 2.1. On the other hand, the main tool in proving the existence criteria of occupation densities will be the following result by Geman and Horowitz [4], which we will now state in its "local" form.

PROPOSITION 2.2. *Let* T *be a Borel subset of* $[0, 1]$, $A \in \mathscr{F}$ *and* X *a continuous process. Then*

(i) *if* $\liminf_{\varepsilon \downarrow 0} (1/\varepsilon) \int_T P(A \cap \{|X_t - X_s| \le \varepsilon\})\, ds < \infty$, *for almost all* $t \in T$, X *possesses an occupation density on* (A, T).

(ii) *The process* X *possesses a square integrable occupation density on* (A, T) *if and only if*

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_T \int_T P\big(A \cap \{|X_t - X_s| \le \varepsilon\}\big)\, ds\, dt < \infty.$$

PROOF. We may assume $P(A) > 0$, and then it suffices to apply Theorems (21.12) and (21.15) of Geman and Horowitz [4] to the probability space $(A, \mathscr{F}_A, P(\cdot\,|A))$ where $\mathscr{F}_A$ is the restriction of $\mathscr{F}$ to A. □

In the sequel we will assume that $(\Omega, \mathscr{F}, P)$ is the canonical Wiener space. That is, $\Omega = C_0([0,1])$ is the space of continuous functions on $[0,1]$ which vanish at zero, and P is the Wiener measure. The canonical process $W_t(\omega) = \omega(t)$, $0 \le t \le 1$, will be a standard Wiener process. Let us recall some basic facts about Malliavin's calculus. We will denote by D the derivative operator, which is defined on a smooth random variable $F = f(W_{t_1}, \dots, W_{t_n})$, $f \in C_b^\infty(\mathbb{R}^n)$, $t_1, \dots, t_n \in [0,1]$, as follows:

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) \mathbf{1}_{[0, t_i]}(t). \tag{2.1}$$

For each $p \ge 1$, $r \in \mathbb{N}$, $\mathbb{D}^{p,r}$ will represent the Banach space of random variables on the Wiener space which is defined as the closure of smooth random variables for the norm

$$\|F\|_{p,r} = \|F\|_p + \sum_{i=1}^r \|D^i F\|_{L^p(\Omega; L^2([0,1]^i))},$$

where D^i denotes the ith iterated derivative. If $\mathbb{H}$ is a real separable Hilbert space, $\mathbb{D}^{p,r}(\mathbb{H})$ will denote the corresponding Banach space of $\mathbb{H}$-valued random variables. In particular we will denote by H the Hilbert space $L^2([0,1])$, and we will denote by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ its scalar product and norm. We will make use of the notation $\mathbb{D}^{p,r}(H) = \mathbb{L}^{p,r}$.

The adjoint in $L^2(\Omega)$ of the derivative operator D is called the Skorohod stochastic integral and will be denoted by δ. It holds that for any process $u \in \mathbb{L}^{2,1}$, $u\mathbf{1}_{[0,t]}$ belongs to Dom δ, the domain of δ, for any $t \in [0,1]$, and we can define the indefinite Skorohod integral as

$$X_t = \delta(u \mathbf{1}_{[0,t]}) = \int_0^t u_s \, dW_s.$$

The duality relation between the operators D and δ is expressed by

$$E(F\delta(v)) = E(\langle DF, v \rangle) \tag{2.2}$$

for any $F \in \mathbb{D}^{2,1}$ and $v \in$ Dom δ. We will need the following extension of this duality property.

LEMMA 2.1. *Let $F \in \mathbb{D}^{1,1}$ be a bounded random variable and $h \in H$. Then*

$$E(F\delta(h)) = E(\langle DF, h \rangle). \tag{2.3}$$

PROOF. We can find a sequence of smooth random variables F_n, $n \in \mathbb{N}$, which converges to F a.s. and in $\mathbb{D}^{1,1}$, and such that $\sup_n \|F_n\|_\infty < \infty$. For each n we have

$$E(F_n \delta(h)) = E(\langle DF_n, h \rangle)$$

and the result follows by letting $n \to \infty$. □

We refer the reader to Nualart and Pardoux [9] and Watanabe [14] for a more detailed presentation of these notions. The construction of the spaces $\mathbb{D}^{p,r}$ for $p = 1$ can be found in Shigekawa [11].

The following lemma is a consequence of the duality relation, and it will be used in Section 3.

LEMMA 2.2. *Let $F \in \mathbb{D}^{2,1}$ be a random variable such that $E(|F|^{-2}) < \infty$. Then $P\{F > 0\}$ is 0 or 1.*

PROOF. Let $\varphi_\epsilon: \mathbb{R} \to \mathbb{R}$ be a function such that $\varphi_\epsilon(x) = x/|x|$, if $|x| > \epsilon$, and $\varphi_\epsilon(x) = x/\epsilon$, if $|x| \le \epsilon$. Then $\varphi_\epsilon(F)$ belongs to $\mathbb{D}^{2,1}$, and by the chain rule we have

$$D[\varphi_\epsilon(F)] = \varphi_\epsilon'(F)DF = \frac{1}{\epsilon}\mathbf{1}_{\{|F| \le \epsilon\}} DF.$$

Let u be a bounded process in Dom δ. The duality formula (2.2) implies

$$\begin{aligned} E[\varphi_\epsilon(F)\delta(u)] &= E\left[\frac{1}{\epsilon}\mathbf{1}_{\{|F| \le \epsilon\}}\langle DF, u\rangle\right] \\ &\le \|u\|_\infty\left(E(\|DF\|^2)\right)^{1/2}\epsilon^{-1}\left(P\{|F| \le \epsilon\}\right)^{1/2}, \end{aligned}$$

which converges to 0 as $\epsilon \downarrow 0$, due to the condition $E(|F|^{-2}) < \infty$. □

We will denote by $L = -\delta D$ the generator of the $L^2(\Omega)$-valued Ornstein–Uhlenbeck semigroup. The operators D and δ satisfy a local property which says that DF [resp., $\delta(v)$] vanishes a.s. on the set $\{F = 0\}$ (resp., $\{\|v\| = 0\}$) for any $F \in \mathbb{D}^{p,1}$ (resp., $v \in \mathbb{L}^{p,1}$), $p \ge 1$. This allows us to introduce the spaces $\mathbb{D}^{p,r}_{\text{loc}}$ and $\mathbb{L}^{p,r}_{\text{loc}}$ of random variables and stochastic processes which are locally in $\mathbb{D}^{p,r}$ and $\mathbb{L}^{p,r}$, respectively. In order to deduce a version of the Tanaka formula for the indefinite Skorohod integral we have to deal with Hilbert-valued processes which are locally Skorohod integrable but are not in $\mathbb{L}^{2,1}_{\text{loc}}$. A suitable class of locally Skorohod integrable processes can be introduced by means of the following local property, which has been proved in [3].

LEMMA 2.3. *Let $\mathbb{H}$ be a separable real Hilbert space. Suppose that u is an $\mathbb{H}$-valued measurable process such that $\int_0^1 \|u_t\|_{\mathbb{H}}^2\, dt < \infty$ a.s., and let $F \in \mathbb{D}^{2,1}$. Suppose that $Fu \in (\text{Dom}\ \delta)(\mathbb{H})$, and $E(\int_0^1 \|u_t\|_{\mathbb{H}} |D_t F|\, dt) < \infty$. Then $\delta(Fu) = 0$ a.s. on the set $\{F = 0\}$.*

Now we can introduce the following notion (see [3] Definition 2.4).

DEFINITION 2.1. We define $(\text{Dom}\ \delta)_{\text{loc}}(\mathbb{H})$ as the set of $\mathbb{H}$-valued measurable processes u satisfying $\int_0^1 \|u_t\|_{\mathbb{H}}^2\, dt < \infty$ a.s., such that there exists a sequence $\{F_n, n \ge 1\} \subset \mathbb{D}^{2,1}$ satisfying the following properties:

(i) $\{F_n = 1\} \uparrow \Omega$ a.s. and $|F_n| \le 1$ for all $n \ge 1$.
(ii) $E([\int_0^1 \|u_t\|_{\mathbb{H}} |D_t F|\, dt]^2) < \infty$ for all $n \ge 1$.
(iii) $\mathbf{1}_{[0,t]} F_n u \in (\text{Dom}\ \delta)(\mathbb{H})$ for all $n \ge 1$ and $t \in [0, 1]$.

Let us now establish two propositions, which are also consequences of the duality formula, and which will be basic tools in the derivation of the criteria for the existence of occupation densities.

PROPOSITION 2.3. *Let $U \in \mathbb{D}^{2,2}$, $h \in H$ and let $F \in \mathbb{D}^{1,1}$. Suppose that*

$$\langle DU, h \rangle > 0 \quad a.s. \text{ on } \{F \neq 0\}$$

and let $f \in C_b^\infty(\mathbb{R})$. Then

$$|E(f'(U)F)| \leq \|f\|_\infty E\left(\frac{|\delta(h)F| + |\langle DF, h\rangle|}{\langle DU, h\rangle} + \frac{|F\langle D^2U, h \otimes h\rangle|}{\langle DU, h\rangle^2} \right). \tag{2.4}$$

PROOF. Suppose first that the random variable F is bounded. We have

$$D[f(U)] = f'(U)\, DU$$

and hence

$$\langle D[f(U)], h\rangle = f'(U)\langle DU, h\rangle.$$

Consequently,

$$\frac{\langle D[f(U)], h\rangle + \varepsilon f'(U)}{\langle DU, h\rangle + \varepsilon} F = f'(U)F, \tag{2.5}$$

for any $\varepsilon > 0$. The hypotheses of the proposition imply that the random variable $f(U)F/(\langle DU, h\rangle + \varepsilon)$ belongs to the space $\mathbb{D}^{1,1}$. Therefore, Lemma 2.1 yields

$$E\left(\left\langle D\left(\frac{f(U)F}{\langle DU, h\rangle + \varepsilon}\right), h\right\rangle\right) = E\left(\frac{f(U)F\delta(h)}{\langle DU, h\rangle + \varepsilon}\right). \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$E[f'(U)F] = \varepsilon E\left(\frac{f'(U)F}{\langle DU, h\rangle + \varepsilon}\right) + E[f(U)G_\varepsilon],$$

where

$$G_\varepsilon = \frac{\delta(h)F - \langle DF, h\rangle}{\langle DU, h\rangle + \varepsilon} - F\langle D\big[(\langle DU, h\rangle + \varepsilon)^{-1}\big], h\rangle.$$

Consequently we have

$$\begin{aligned} &\left| E\left[f'(U)F \frac{\langle DU, h\rangle}{\langle DU, h\rangle + \varepsilon}\right]\right| \\ &\quad \leq \|f\|_\infty E(|G_\varepsilon|) \\ &\quad \leq \|f\|_\infty E\left(\frac{|\delta(h)F| + |\langle DF, h\rangle|}{\langle DU, h\rangle + \varepsilon} + \frac{|F\langle D^2U, h \otimes h\rangle|}{(\langle DU, h\rangle + \varepsilon)^2}\right). \end{aligned} \tag{2.7}$$

By a monotone convergence argument, the inequality (2.7) still holds for an arbitrary random variable F in the space $\mathbb{D}^{1,1}$, which is not necessarily bounded. Finally, in (2.7) we let ε tend to zero and we get (2.4). Notice that the right-hand side of the inequality (2.4) can be infinite if the random variable F is not bounded. □

PROPOSITION 2.4. *Let $U \in \mathbb{D}^{2,2}$ and $f \in C_b^\infty(\mathbb{R})$. Then*

$$\left| E\left[f'(U)\mathbf{1}_{\{\|DU\|>0\}}\right]\right| \le \|f\|_\infty E\left(\frac{|LU| + 2\|D^2U\|}{\|DU\|^2}\right). \tag{2.8}$$

PROOF. We proceed as in the proof of Proposition 2.3 with $F = 1$ and with DU replacing h. That means, we first write

$$\frac{\langle D[f(U)], DU\rangle + \varepsilon f'(U)}{\|DU\|^2 + \varepsilon} = f'(U), \tag{2.9}$$

for any $\varepsilon > 0$. The random variable $f(U)/(\|DU\|^2 + \varepsilon)$ belongs to the space $\mathbb{D}^{2,1}$. Therefore, by the duality formula (2.2) we obtain

$$E\left(\left\langle D\left(\frac{f(U)}{\|DU\|^2 + \varepsilon}\right), DU\right\rangle\right) = E\left(\frac{f(U)\delta(DU)}{\|DU\|^2 + \varepsilon}\right). \tag{2.10}$$

From (2.9) and (2.10) we get as in the proof of Proposition 2.3,

$$\begin{aligned} &\left| E\left[f'(U)\frac{\|DU\|^2}{\|DU\|^2 + \varepsilon}\right]\right| \\ &\quad \le \|f\|_\infty E\left(\frac{|\delta DU|}{\|DU\|^2 + \varepsilon} + \frac{2|\langle D^2U, DU\otimes DU\rangle|}{(\|DU\|^2 + \varepsilon)^2}\right). \end{aligned} \tag{2.11}$$

Notice that here we do not need the fact that $\|DU\| > 0$. Applying the Cauchy–Schwarz inequality and letting $\varepsilon \downarrow 0$ we obtain the result. □

Observe that by the local property of the operators D and L we have $\{|LU| + 2\|D^2U\| = 0\} \supset \{\|DU\| = 0\}$, and in the formula (2.8) we set by convention $0/0 = 0$.

The preceding proposition can be used to derive criteria for the existence of occupation densities for processes on Wiener space. We will make use of the following criterion.

THEOREM 2.1. *Suppose that $\{X_t,\ t \in [0,1]\}$ is a continuous process such that $X_t \in \mathbb{D}^{2,2}$ for all $t \in [0,1]$. Let $F \in \mathbb{D}^{1,1}$ be a bounded random variable. Assume that there exist a constant $\alpha > 0$, a subinterval $T \subset [0,1]$ and a* *bounded and measurable function* $\beta\colon T \to \mathbb{R}$, *such that*:

(i) $\langle D(X_t - X_s), \beta \mathbf{1}_{(s,t]} \rangle \geq \alpha(t-s)$ *on* $\{F \neq 0\}$ *a.s., for all* $s \leq t$, $s, t \in T$;
(ii) *it holds that*
  (ii$_1$) $\int_T E(|\int_s^t \beta_r D_r F\, dr| / |t-s|)\, ds \leq \infty$, *and*
  (ii$_2$) $\int_T E(|F \int_s^t \int_s^t \beta_u \beta_r D_u D_r (X_t - X_s)\, du\, dr| / |t-s|^2)\, ds < \infty$,
  for a.e. $t \in T$.

Then X *possesses an occupation density on* $(\{F \neq 0\}, T)$. *Moreover if in* (ii) *the double integrals over* $T \times T$ *are finite, the occupation density is square integrable.*

Proof. Choose a nonnegative function $h \in C_0^\infty(\mathbb{R})$ such that

$$\tfrac{1}{2}\mathbf{1}_{[-1/2,1/2]} \leq h \leq \mathbf{1}_{[-1,1]}.$$

For $\varepsilon > 0$ set $h_\varepsilon(x) = (1/\varepsilon)h(x/\varepsilon)$, $x \in \mathbb{R}$. Then for $\varepsilon > 0$

$$\frac{1}{2\varepsilon}\mathbf{1}_{[-\varepsilon/2,\varepsilon/2]} \leq h_\varepsilon \leq \frac{1}{\varepsilon}\mathbf{1}_{[-\varepsilon,\varepsilon]}.$$

For $n \in \mathbb{N}$ let now $A_n = \{F \geq 1/n\}$. We will show that X possesses an occupation density on (A_n, T) for all $n \in \mathbb{N}$. Since $A_n \uparrow \{F \neq 0\}$, this will imply the first assertion. By the choice of h_ε, $\varepsilon > 0$, (i) of Proposition 2.2 is equivalent to

$$\liminf_{\varepsilon \downarrow 0} \int_T E\big(\mathbf{1}_{A_n} h_\varepsilon(X_t - X_s)\big)\, ds < \infty \tag{2.12}$$

for almost all $t \in T$. By definition $\mathbf{1}_{A_n} \leq nF$, hence (2.12) follows from

$$\liminf_{\varepsilon \downarrow 0} \int_T E\big(F h_\varepsilon(X_t - X_s)\big)\, ds < \infty, \tag{2.13}$$

for almost all $t \in T$. In order to verify (2.13), integration by parts is brought into play. Fix $s, t \in T$, $s < t$, $\varepsilon > 0$, and set $g_\varepsilon(x) = \int_{-\infty}^x h_\varepsilon(y)\, dy$. Then by Proposition 2.3 applied to $h = \beta \mathbf{1}_{(s,t]}$, $f = g_\varepsilon$, $U = X_t - X_s$, we deduce, taking into account conditions (i) and (ii),

$$\sup_{\varepsilon > 0} \int_T E\big(F h_\varepsilon(X_t - X_s)\big)\, ds < \infty$$

for almost all $t \in T$, which implies (2.13). Also, the extension to square integrable occupation densities is clear. This completes the proof. □

Theorem 2.1 provides a local result on the existence of the occupation density. In this context, Proposition 2.1 can be used to formulate a global existence result:

Theorem 2.2. *Suppose that there exist sequences* $\{F_n,\ n \geq 1\}$, $\alpha_n > 0$, $\delta_n > 0$, *and a measurable and bounded function* $\beta\colon [0,1] \to \mathbb{R}$, *such that for each* $n \geq 1$, *the hypotheses of Theorem* 2.1 *are satisfied for any interval* T *of* *length less than δ_n. Moreover, assume that $\bigcup_n \{F_n \neq 0\} = \Omega$, a.s. Then the process X possesses an occupation density [which is square integrable if we take double integrals in* (ii)].

We could also deduce global results like the following one, which in the Gaussian case reduces to a well-known criterion.

THEOREM 2.3. *Let $X = \{X_t, t \in [0,1]\}$ be a continuous process such that $X_t \in \mathbb{D}^{2,2}$ for all $t \in [0,1]$. Suppose that*

(i) $$\int_0^1 E\left(\frac{|L(X_t - X_s)|}{\|D(X_t - X_s)\|^2}\right) ds < \infty,$$

(ii) $$\int_0^1 E\left(\frac{\|D^2(X_t - X_s)\|}{\|D(X_t - X_s)\|^2}\right) ds < \infty, \quad \text{for a.e. } t \in [0,1],$$

(iii) $$\|D(X_t - X_s)\| > 0 \quad \text{for almost all } (s,t,\omega).$$

Then X possesses an occupation density. If we replace the integrals appearing in conditions (i) *and* (ii) *by double integrals over* $[0,1]^2$ *with respect to ds and dt, the occupation density is square integrable.*

PROOF. The proof is analogous to that of Theorem 2.1, but using Proposition 2.4 with $U = X_t - X_s$, instead of Proposition 2.3. □

COROLLARY 2.1. *Suppose X is a continuous process in the first chaos. Suppose that*

$$\int_0^1 \|D(X_t - X_s)\|^{-1} ds < \infty \quad \text{for a.e. } t \in T.$$

Then X possesses an occupation density. If the double integral over $[0,1]^2$ *is finite, the occupation density is square integrable.*

PROOF. In the underlying case the condition $X_t \in \mathbb{D}^{2,2}$, $t \in [0,1]$, is automatic. Moreover, $D(X_t - X_s)$ is deterministic, so that $D^2(X_t - X_s) = 0$. Finally, $L(X_t - X_s) = -(X_t - X_s)$ and

$$E(|X_t - X_s|) \le \langle D(X_t - X_s), D(X_t - X_s)\rangle^{1/2}.$$

These remarks reduce the stated criterion to the criterion of Theorem 2.3. □

REMARKS. The criterion of Corollary 2.1 is well known (see Geman and Horowitz [4] and Pitt [10]).

3. Occupation densities of the perturbed Wiener process. In this section we will apply the results of Section 2 to the perturbed Wiener process. More precisely, for some $u \in L^2(\Omega \times [0,1])$ we will consider the process $X_t =$

$W_t + \int_0^t u_s \, ds$, $t \in [0, 1]$. In case u is adapted, it is well known that X has an occupation density.

We start by presenting some inequalities for convex functions that will be useful in this section and in the following one.

PROPOSITION 3.1. *Let $g: \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing and convex function, and let $\alpha > -2$. Then there exists a constant c_α such that for any $0 \le \varphi \in L^1([0,1])$,*

$$\int_0^1 \int_0^1 g\left(\int_{x \wedge y}^{x \vee y} \varphi(u)\, du\right) |x - y|^\alpha \, dx\, dy \le c_\alpha g\left(\int_0^1 \varphi(u)\, du\right).$$

PROOF. Suppose first $\alpha \neq -1$. By symmetry, it is enough to show

$$\int_0^1 \int_0^1 \mathbf{1}_{[0,y]}(x) g\left(\int_x^y \varphi(v)\, dv\right)(y - x)^\alpha \, dx\, dy \le c_\alpha g\left(\int_0^1 \varphi(u)\, du\right).$$

We have for $x \le y$,

$$g\left(\int_x^y \varphi(u)\, du\right) = \int_x^y f\left(\int_x^z \varphi(u)\, du\right) \varphi(z)\, dz,$$

with a suitable increasing function f. Hence

$$g\left(\int_x^y \varphi(v)\, dv\right) \le \int_x^y f\left(\int_0^z \varphi(u)\, du\right) \varphi(z)\, dz.$$

This yields

$$\begin{aligned}
&\int_0^1 \int_0^1 \mathbf{1}_{[0,y]}(x) g\left(\int_x^y \varphi(u)\, du\right) |x - y|^\alpha \, dx\, dy \\
&\quad \le \int_0^1 \int_0^1 \mathbf{1}_{[0,y]}(x) \int_x^y f\left(\int_0^z \varphi(u)\, du\right) \varphi(z)\, dz |x - y|^\alpha \, dx\, dy \\
&\quad = \int_0^1 f\left(\int_0^z \varphi(u)\, du\right) \varphi(z) \left(\int_z^1 \int_0^z (y - x)^\alpha \, dx\, dy\right) dz \\
&\quad = \int_0^1 f\left(\int_0^z \varphi(u)\, du\right) \varphi(z) \frac{1}{1+\alpha} \frac{1}{2+\alpha} \left[(1 - z)^{\alpha+2} - 1 + z^{\alpha+2}\right] dz \\
&\quad \le 3 \frac{1}{|1+\alpha|} \frac{1}{2+\alpha} \int_0^1 f\left(\int_0^z \varphi(u)\, du\right) \varphi(z)\, dz \\
&\quad = 3 \frac{1}{|1+\alpha|} \frac{1}{2+\alpha} g\left(\int_0^1 \varphi(u)\, du\right),
\end{aligned}$$

which is the desired result.

If $\alpha = -1$, we have to replace

$$(1 - z)^{\alpha+2} + z^{\alpha+2} - 1$$

by

$$-z \ln z - (1 - z)\ln(1 - z)$$

in the above computations. But $z \mapsto -z \ln z - (1-z)\ln(1-z)$ is bounded on $[0,1]$. Therefore the result follows. □

We can improve Proposition 3.1 a little bit in the special case where φ is more than just integrable.

PROPOSITION 3.2. *Let $q > 1$, $\alpha > -2 - (q-1)/q$. There is a constant $c_{\alpha,q}$ such that for any $0 \le \varphi \in L^q([0,1])$,*

$$\int_0^1 \int_0^1 \left(\int_{x \wedge y}^{x \vee y} \varphi(u)\, du \right) |x-y|^\alpha \, dx\, dy \le c_{\alpha,q} \left(\int_0^1 \varphi(u)^q \, du \right)^{1/q}.$$

PROOF. Suppose $\alpha \neq -2$. In case $\alpha = -2$ only a slightly modified argument is needed. As in the proof of Proposition 3.1, we get

$$\begin{aligned} &\int_0^1 \int_0^1 \mathbf{1}_{[0,y]}(x) \left(\int_x^y \varphi(u)\, du \right) |x-y|^\alpha \, dx\, dy \\ &\quad \le \int_0^1 \varphi(u) \frac{1}{|1+\alpha|} \frac{1}{|2+\alpha|} \left[(1-u)^{2+\alpha} + u^{2+\alpha} - 1 \right] du. \end{aligned}$$

We now apply Hölder's inequality and we have

$$\int_0^1 \varphi(u) u^{2+\alpha}\, du \le \left(\int_0^1 \varphi(u)^q \, du \right)^{1/q} \left(\int_0^1 (u^{2+\alpha})^{q/(q-1)} \, du \right)^{1-(1/q)}.$$

The second factor will be finite if $\alpha > -2 - (q-1)/q$, and this was our assumption on α. □

We can now proceed to prove the main result of this section which states the existence of an occupation density for the perturbed Wiener process.

THEOREM 3.1. *Let $u \in \mathbb{L}^{2,2}$ and consider the process X defined by $X_t = W_t + \int_0^t u_s \, ds$. Then X possesses a square integrable occupation density.*

PROOF. We will apply Theorem 2.2. Notice first that X has continuous paths and $X_t \in \mathbb{D}^{2,2}$ for all $t \in [0,1]$. Choose $s, t \in [0,1]$ such that $s \le t$. Then for $r \in [0,1]$,

$$D_r(X_t - X_s) = \mathbf{1}_{[s,t]}(r) + \int_s^t D_r u_\theta \, d\theta,$$

hence

$$\left\langle D(X_t - X_s), \mathbf{1}_{[s,t]} \right\rangle = t - s + \int_s^t \int_s^t D_r u_\theta \, dr\, d\theta.$$

By the inequality of Cauchy–Schwarz we have

$$(3.1)\quad \left|\int_s^t\int_s^t D_r u_\theta \, dr\, d\theta\right| \le (t-s)\left(\int_s^t\int_s^t (D_r u_\theta)^2\, dr\, d\theta\right)^{1/2}, \qquad 0 \le s \le t \le 1.$$

Now define for $n \ge 1$,

$$Y_n = \sup_{1 \le i \le 2^n} \left(\int_{I_i}\int_{I_i} (D_r u_\theta)^2 \, dr\, d\theta\right),$$

where

$$I_i = \left[\frac{i-1}{2^n}, \frac{i+1}{2^n}\right] \cap [0,1], \qquad 1 \le i \le 2^n,$$

and take a function $\varphi \in C_0^\infty(\mathbb{R})$ satisfying $\varphi(x) = 1$ if $|x| \le 1/4$ and $\varphi(x) = 0$ if $|x| \ge 1/2$. Set $F_n = \varphi(Y_n)$. Next observe that if $|t-s| \le 2^{-n}$, then (3.1) yields

$$(3.2)\quad \left|\int_s^t\int_s^t D_r u_\theta \, dr\, d\theta\right| \le \tfrac{1}{2}|t-s|, \quad \text{on } \{F_n \ne 0\}.$$

Hence condition (i) of Theorem 2.1 is satisfied with $\beta = 1$, $\alpha = 1/2$, that is, from (3.2) we deduce

$$\left\langle D(X_t - X_s), \mathbf{1}_{[s,t]}\right\rangle \ge \tfrac{1}{2}(t-s) \quad \text{on } \{F_n \ne 0\}, \text{ if } |s-t| \le 2^{-n}.$$

We will next prove that $F_n \in \mathbb{D}^{2,1}$. Since the function $(x_1, \dots, x_{2^n}) \mapsto \sup_{1 \le i \le 2^n} |x_i|$ is Lipschitz, with a derivative bounded by 1, we have, for $\alpha \in [0,1]$,

$$|D_\alpha Y_n| \le 2 \sum_{i=1}^{2^n} \left|\int_{I_i}\int_{I_i} D_\alpha D_r u_s D_r u_s \, dr\, ds\right|.$$

This implies by Cauchy–Schwarz

$$\begin{aligned} \|DY_n\| &\le 2\left[\int_0^1 \left(\sum_{i=1}^{2^n}\left|\int_{I_i}\int_{I_i} D_\alpha D_r u_s D_r u_s\, dr\, ds\right|\right)^2 d\alpha\right]^{1/2} \\ &\le 2 \cdot 2^{n/2}\left[\sum_{i=1}^{2^n}\int_0^1\int_{I_i}\int_{I_i}(D_\alpha D_r u_s)^2\, dr\, ds\, d\alpha\, Y_n\right]^{1/2} \\ &\le 4 \cdot 2^{n/2}\|D^2 u\| Y_n^{1/2}. \end{aligned}$$

So we obtain

$$(3.3)\quad E\left(\left|\, \|DY_n\| \varphi_n'(Y)\,\right|^2\right) < \infty,$$

and F_n belongs to $\mathbb{D}^{2,1}$.

Finally, it suffices to check conditions (ii_1) and (ii_2) in Theorem 2.1 (with double integrals and $\beta = 1$). Note that

$$\left\langle DY_n, \mathbf{1}_{[s,t]}\right\rangle \le \|DY_n\|\, |t-s|^{1/2}.$$

Hence (ii_1) follows from (3.3). In order to show (ii_2), note that by definition

$$D^2_{\alpha,\beta}(X_t - X_s) = \int_s^t D_\beta D_\alpha u_\theta \, d\theta, \qquad \alpha, \beta \in [0,1].$$

Hence

$$\left| \left\langle D^2(X_t - X_s), \mathbf{1}_{[s,t]}^{\otimes 2} \right\rangle \right| = \int_s^t \int_s^t \int_s^t |D_\beta D_\alpha u_\theta| \, d\beta \, d\alpha \, d\theta.$$

Therefore we can apply Proposition 3.2 with $q = 2$ and $\alpha = -2$ to

$$\varphi(\alpha) = \int_0^1 \int_0^1 E\big(|D_\alpha D_\beta u_\theta|\big) \, d\theta \, d\beta,$$

and we obtain the result. □

Observe that the conclusion of Theorem 3.1 still holds if $u = \{u_t, 0 \le t \le 1\}$ belongs to the space $\mathbb{L}^{2,2}_{\text{loc}}$.

4. Occupation densities for the Skorohod integral process. In this section we will consider $u \in L^2(\Omega \times [0,1])$ satisfying some smoothness and integrability conditions, and we will study the occupation density of its Skorohod integral process

$$X_t = \int_0^t u_s \, dW_s, \qquad t \in [0,1].$$

In order to take into account small values of u we have to consider the following localizing variables.

PROPOSITION 4.1. *For $q > 1$, $u \in \mathbb{L}^{2,1}$, let*

$$Z_q = \int_0^1 \frac{1}{|u_s|^q} \, ds.$$

Suppose there exist α, $\beta > 1$ such that $1/\alpha + 1/\beta = 1$ and

(i) $E(\int_0^1 |u_s|^{-(q+1)\alpha} \, ds) < \infty$,
(ii) $E(\int_0^1 \|Du_v\|^\beta \, dv) < \infty$.

Then $Z_q \in \mathbb{D}^{1,1}$.

PROOF. For $r \in [0,1]$ we have

$$D_r Z_q = (-q) \int_0^1 \frac{1}{|u_s|^{q+1}} \operatorname{sgn}(u_s) D_r u_s \, ds.$$

Hence

$$\langle DZ_q, DZ_q\rangle^{1/2} \le q\int_0^1 \frac{1}{|u_s|^{q+1}}\left(\int_0^1 (D_r u_s)^2\, dr\right)^{1/2} ds$$

$$\le q\left(\int_0^1 \frac{1}{|u_s|^{(q+1)\alpha}}\, ds\right)^{1/\alpha}\left(\int_0^1 \|Du_s\|^\beta\, ds\right)^{1/\beta}.$$

Hence (i) and (ii) imply the assertion. □

PROPOSITION 4.2. *For $p > 2$, $m > 0$, such that $p/2 - m - 1 > 0$, let $u \in \mathbb{L}^{2,3}$ satisfy:*

(i) $E(\int_0^1\int_0^1 |D_r u_v|^p\, dr\, dv) < \infty$,
(ii) $E(\int_0^1\int_0^1 (\int_0^1 (D_\alpha D_r u_v)^2\, d\alpha)^{p/2}\, dr\, dv) < \infty$,
(iii) $E(\int_0^1\int_0^1 (\int_0^1\int_0^1 (D_\beta D_\alpha D_r u_v)^2\, d\beta\, d\alpha)^{p/2}\, dr\, dv) < \infty$.

Let

$$Y_{m,p} = \int_0^1\int_0^1\int_0^1 \frac{\left|\int_{x\wedge y}^{x\vee y} D_r u_v\, dW_v\right|^p}{|x-y|^{m+2}}\, dx\, dy\, dr.$$

Then we have

(iv) $X_t \in \mathbb{D}^{2,2}$ *for any* $t \in [0,1]$, *and the process* X_t *has a continuous version*,

(v) $\langle DY_{m,p}, DY_{m,p}\rangle \mathbf{1}_{\{Y_{m,p} < n\}} \in L^{p/2}(\Omega, \mathcal{F}, P)$ *and* $Y_{m,p} \in L^1(\Omega, \mathcal{F}, P)$, $n \in \mathbb{N}$.

PROOF. Statement (iv) is clear from the hypotheses and from Theorem 5.2 of [9]. We will have to prove the first part of (v). To abbreviate, let $\xi_{x,y,r} = \int_{x\wedge y}^{x\vee y} D_r u_v\, dW_v$, $x, y, r \in [0,1]$. Then for $\alpha \in [0,1]$

$$D_\alpha Y_{m,p} = \int_0^1\int_0^1 \frac{1}{|x-y|^{m+2}} p \operatorname{sgn}(\xi_{x,y,r})|\xi_{x,y,r}|^{p-1} D_\alpha \xi_{x,y,r}\, dx\, dy\, dr.$$

Hence by Cauchy–Schwarz and Hölder's inequalities we have

$$\langle DY_{m,p}, DY_{m,p}\rangle$$

$$\le p^2\left\{\int_0^1\int_0^1\int_0^1 \frac{1}{|x-y|^{m+2}}|\xi_{x,y,r}|^{p-1}\left(\int_0^1 (D_\alpha \xi_{x,y,r})^2\, d\alpha\right)^{1/2} dx\, dy\, dr\right\}^2$$

$$\le p^2 Y_{m,p}^{2(1-1/p)}\left[\int_0^1\int_0^1\int_0^1 \frac{1}{|x-y|^{m+2}}\left(\int_0^1 (D_\alpha \xi_{x,y,r})^2\, d\alpha\right)^{p/2} dx\, dy\, dr\right]^{2/p}.$$

It is enough to prove that

$$(4.1)\quad \int_0^1\int_0^1 \frac{1}{|x-y|^{m+2}}\left(\int_0^1 (D_\alpha \xi_{x,y,r})^2\, d\alpha\right)^{p/2} dx\, dy\, dr \in L^1(\Omega, \mathscr{F}, P).$$

Now

$$D_\alpha \xi_{x,y,r} = \mathbf{1}_{[x\wedge y,\, x\vee y]}(\alpha) D_r u_\alpha + \eta_{x,y,r,\alpha},$$

where

$$\eta_{x,y,r,\alpha} = \int_{x\wedge y}^{x\vee y} D_\alpha D_r u_v\, dW_v.$$

Hence (4.1) will follow, once we have proved

$$(4.2)\quad \int_0^1\int_0^1 \frac{1}{|x-y|^{m+2}}\left(\int_{x\wedge y}^{x\vee y} (D_r u_\alpha)^2\, d\alpha\right)^{p/2} dx\, dy\, dr \in L^1(\Omega, \mathscr{F}, P),$$

$$(4.3)\quad \int_0^1\int_0^1 \frac{1}{|x-y|^{m+2}}\left(\int_0^1 (\eta_{x,y,r,\alpha})^2\, d\alpha\right)^{p/2} dx\, dy\, dr \in L^1(\Omega, \mathscr{F}, P).$$

To get (4.2), apply Hölder's inequality and Proposition 3.1 with $g(x) = x$, $\varphi(\alpha) = E(\int_0^1 |D_r u_\alpha|^p\, dr)$, integrable by (i), and we obtain a constant c_1 not depending on u such that

$$\begin{aligned} &E\left(\int_0^1\int_0^1 \frac{1}{|x-y|^{m+2}}\left(\int_{x\wedge y}^{x\vee y} (D_r u_\alpha)^2\, d\alpha\right)^{p/2} dx\, dy\, dr\right) \\ &\quad\le E\left(\int_0^1\int_0^1\int_0^1 |x-y|^{(p/2)-m-3}\int_{x\wedge y}^{x\vee y} |D_r u_\alpha|^p\, d\alpha\, dx\, dy\, dr\right) \\ &\quad\le c_1 E\left(\int_0^1\int_0^1 |D_r u_\alpha|^p\, dr\, d\alpha\right), \quad \text{if } \frac{p}{2} - m - 1 > 0. \end{aligned}$$

To get (4.3), we apply the $L^2([0,1])$-valued version of the L^p-estimate for the Skorohod integral (see Proposition 4.1 in [5]), followed by Hölder's inequality. For $x, y \in [0,1]$ we obtain a universal constant c_2 such that

$$\begin{aligned} &E\left(\left(\int_0^1 (\eta_{x,y,r,\alpha})^2\, d\alpha\right)^{p/2}\right) \\ &\quad\le c_2\left[\left(\int_0^1\int_{x\wedge y}^{x\vee y} [E(D_\alpha D_r u_v)]^2\, dv\, d\alpha\right)^{p/2} \right.\\ &\qquad\left. + E\left(\left(\int_0^1\int_{x\wedge y}^{x\vee y}\int_0^1 (D_\beta D_\alpha D_r u_v)^2\, d\beta\, dv\, d\alpha\right)^{p/2}\right)\right] \\ &\quad\le c_2|x-y|^{(p/2)-1}\left[\int_{x\wedge y}^{x\vee y}\left(\int_0^1 [E(D_\alpha D_r u_v)]^2\, d\alpha\right)^{p/2} dv \right.\\ &\qquad\left. + E\left(\int_{x\wedge y}^{x\vee y}\left(\int_0^1\int_0^1 (D_\beta D_\alpha D_r u_v)^2\, d\alpha\, d\beta\right)^{p/2} dv\right)\right]. \end{aligned}$$

To integrate this result in x, y, r, we apply Proposition 3.1 again, this time to the function

$$\varphi(v) = \int_0^1 \left(\int_0^1 [E(D_\alpha D_r u_v)]^2 \, d\alpha \right)^{p/2} dr + E\left(\int_0^1 \left(\int_0^1 \int_0^1 (D_\beta D_\alpha D_r u_v)^2 \, d\alpha \, d\beta \right)^{p/2} dr \right),$$

which is integrable due to (ii) and (iii). The integral in x, y is then finite due to $p/2 - m - 1 > 0$. This proves (4.3). Finally, the claim that $Y_{m,p} \in L^1(\Omega, \mathscr{F}, P)$ is proved in the same way as (4.1). We first have to replace $(\int_0^1 (D_\alpha \xi_{x,y,r})^2 \, d\alpha)^{1/2}$ by $\xi_{x,y,r}$ itself, which actually leads to simpler arguments. □

THEOREM 4.1. *Suppose that* $u \in \mathbb{L}^{2,3}$ *and consider the process*

$$X_t = \int_0^t u_s \, dW_s, \qquad 0 \le t \le 1.$$

Suppose further that for some $p > 4$ *we have*:

(i) $E(\int_0^1 \int_0^1 |D_r u_v|^p \, dr \, dv) < \infty$,
(ii) $E(\int_0^1 \int_0^1 (\int_0^1 (D_\alpha D_r u_v)^2 \, d\alpha)^{p/2} \, dr \, dv) < \infty$,
(iii) $E(\int_0^1 \int_0^1 (\int_0^1 \int_0^1 (D_\beta D_\alpha D_r u_v)^2 \, d\alpha \, d\beta)^{p/2} \, dr \, dv) < \infty$,
(iv) $E(\int_0^1 1/|u_r|^\gamma \, dr) < \infty$ *for* $\gamma = (3p-4)p/((p-4)(p-1))$.

Then X possesses a square integrable occupation density.

PROOF. We will apply Theorem 2.2. In order to define the localizing random variables $\{F_n, n \ge 1\}$ we choose $m > 1$ such that $(p/2) - m - 1 > 0$ and let $q = 2p/(p-4)$, which implies $q > p/(m-1)$. Set $Y = Z_q + Y_{m,p}$, where Z_q and $Y_{m,p}$ are the random variables introduced in Propositions 4.1 and 4.2, respectively.

Let φ: $\mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that $\varphi(x) = 0$ if and only if $|x| \ge 1$, $\varphi(0) = 1$, and $0 < \varphi(x) < 1$ if $0 < x < 1$. Set $\varphi_n(x) = \varphi((|x| - n)^+)$. Define $F_n = \varphi_n(Y)$. From Propositions 4.1 and 4.2 it follows that $F_n \in \mathbb{D}^{1,1}$ and $\bigcup_n \{F_n \ne 0\} = \Omega$, a.s. In fact, the hypotheses of Proposition 4.2 are contained in (i)–(iii). On the other hand, choosing $\beta = p$, $\alpha = p/(p-1)$ the hypotheses of Proposition 4.1 are satisfied because

$$\alpha(q+1) = \frac{(q+1)p}{p-1} = \left(\frac{2p}{p-4} + 1 \right) \frac{p}{p-1} = \frac{(3p-4)p}{(p-4)(p-1)} = \gamma.$$

For $r \in [0,1]$, $s \le t$, we have

$$D_r(X_t - X_s) = \mathbf{1}_{[s,t]}(r) u_r + \int_s^t D_r u_\theta \, dW_\theta.$$

Observe that $\gamma > 3$. Hence condition (iv) implies, by Lemma 2.2, that for all $t \in [0,1]$, a.e., the probability $P\{u_t > 0\}$ is zero or one. Therefore there exists a measurable function β: $[0,1] \to \{-1, +1\}$, such that $\beta_t u_t = |u_t|$, for almost all

(t, ω). We are going to apply Theorem 2.1 with this function β. Let us show the inequality (i) of Theorem 2.1. We claim that there exist constants $\alpha_n > 0$, $\delta_n > 0$ such that

$$\left\langle D(X_t - X_s), \beta \mathbf{1}_{[s,t]} \right\rangle \geq \alpha_n |t - s| \quad \text{on } \{F_n \neq 0\} \text{ a.s. for all } s \leq t, |t - s| \leq \delta_n. \tag{4.4}$$

PROOF OF (4.4). We have

$$\left\langle D(X_t - X_s), \beta \mathbf{1}_{[s,t]} \right\rangle = \int_s^t |u_r| \, dr + \int_s^t \left(\int_s^t \beta_r D_r u_\theta \, dW_\theta \right) dr. \tag{4.5}$$

By Hölder's inequality we obtain

$$|t - s| \leq \left(\int_s^t |u_r| \, dr \right)^{q/(q+1)} \left(\int_s^t |u_r|^{-q} \, dr \right)^{1/(q+1)}.$$

Consequently, we have

$$\int_s^t |u_r| \, dr \geq |t - s|^{1 + (1/q)} \left(\int_0^1 \frac{1}{|u_r|^q} \, dr \right)^{-1/q} = |t - s|^{1 + (1/q)} Z_q^{-1/q}. \tag{4.6}$$

Let us apply Hölder's inequality twice, and the lemma of Garsia, Rodemich and Rumsey (see Barlow and Yor [1], page 203) to the second term on the right-hand side of (4.5) to get a universal constant c_1 such that

$$\begin{aligned} &\left| \int_s^t \left(\int_s^t \beta_r D_r u_\theta \, dW_\theta \right) dr \right| \\ &\quad \leq |t - s|^{(p-1)/p} \left(\int_s^t \left| \int_s^t D_r u_\theta \, dW_\theta \right|^p dr \right)^{1/p} \\ &\quad \leq c_1 |t - s|^{1 - (1/p) + (m/p)} \left(\int_0^1 \int_0^1 \int_0^1 \frac{\left| \int_{x \wedge y}^{x \vee y} D_r u_\theta \, dW_\theta \right|^p}{|x - y|^{m+2}} \, dx \, dy \, dr \right)^{1/p}. \end{aligned} \tag{4.7}$$

Now, from (4.6) and (4.7) it is clear how the required sequences α_n and δ_n are constructed because $q > p/(m - 1)$ and $m > 1$.

We now have to verify the integral criteria (ii) (with double integrals) of Theorem 2.1. This will be done in two steps.

STEP 1. We show that

$$\int_0^1 \int_0^1 \frac{E\left(\left| \left\langle DY, \beta \mathbf{1}_{[s,t]} \right\rangle \right| \mathbf{1}_{A_n} \right)}{|t - s|} \, ds \, dt < \infty,$$

for all $n \geq 1$, where $A_n = \{Y \leq n\}$.

Since $A_n \subset \{Z_q \le n\} \cap \{Y_{m,p} \le n\}$ we have

$$\left|\left\langle DY, \beta \mathbf{1}_{[s,t]} \right\rangle\right| \mathbf{1}_{A_n} \le \|DY\|(t-s)^{1/2} \mathbf{1}_{A_n}$$
$$\le \left(\|DZ_q\| + \|DY_{m,p}\| \mathbf{1}_{\{Y_{m,p} \le n\}}\right)(t-s)^{1/2}.$$

Hence the assertion follows from Proposition 4.1 and 4.2.

STEP 2. We show that

$$\int_0^1 \int_0^1 E\left(\left|\int_s^t \int_s^t \beta_\alpha \beta_r D_\alpha D_r (X_t - X_s)\, d\alpha\, dr\right| \varphi_n(Y)\right) |t-s|^{-2}\, ds\, dt < \infty.$$

For α, $r \in [0,1]$ and $s \le t$ we have

$$D_\alpha D_r (X_t - X_s) = \mathbf{1}_{[s,t]}(r) D_\alpha u_r + \mathbf{1}_{[s,t]}(\alpha) D_r u_\alpha + \int_s^t D_\alpha D_r u_\theta\, dW_\theta.$$

Consequently,

$$\int_s^t \int_s^t \beta_\alpha \beta_r D_\alpha D_r (X_t - X_s)\, d\alpha\, dr$$
$$= 2\int_s^t \int_s^t \beta_\alpha \beta_r D_\alpha u_r\, d\alpha\, dr + \int_s^t \int_s^t \left(\int_s^t \beta_\alpha \beta_r D_\alpha D_r u_\theta\, dW_\theta\right) d\alpha\, dr.$$

In order to prove the assertion we apply Proposition 3.2 with $\alpha = -2$, $q = 2$, and

$$\varphi(r) = \int_0^1 \left\{2E(|D_\alpha u_r|) + \sup_{0 \le s \le t \le 1} E\left(\left|\int_s^t D_\alpha D_r u_\theta\, dW_\theta\right|\right)\right\} d\alpha.$$

Notice that from the isometry property of the Skorohod integral we have $\varphi \in L^2([0,1])$, because $u \in \mathbb{L}^{2,3}$. This completes the proof of the theorem. □

REMARK. By choosing p large, we can make γ arbitrarily close to 3. On the other hand, making p close to 4 drives γ to infinity, according to (iv).

5. Tanaka's formula for Skorohod integrals. Suppose that $u = \{u_t,\ 0 \le t \le 1\}$ is a stochastic process in the space $\mathbb{L}^{p,2}$ with $p > 4$. We know (see [9]) that the Skorohod integral process defined by

$$X_t = \int_0^t u_s\, dW_s$$

possesses a continuous version. For any $\varepsilon > 0$ we introduce the functions

$$j_\varepsilon(x) = \frac{1}{2\varepsilon} \mathbf{1}_{[-\varepsilon,\varepsilon]}(x),$$
$$f_\varepsilon(x) = \int_{-\infty}^x \left(\int_{-\infty}^y j_\varepsilon(z)\, dz\right) dy.$$

We can apply the change of variable formula (Theorem 6.1 of [9]) to the function f_ε and the process X. Observe that the second derivative of f_ε is not continuous at the points $x = \pm\varepsilon$ but we can approximate j_ε by a continuous and bounded function and it is not difficult to see that the change of variable formula still holds for f_ε. In this way we obtain that

$$f_\varepsilon(X_t - x) - f_\varepsilon(-x) = \int_0^t f_\varepsilon'(X_s - x)u_s\,dW_s + \int_0^t f_\varepsilon''(X_s - x)u_s\left[\tfrac{1}{2}u_s + \int_0^s D_s u_r\,dW_r\right]ds. \tag{5.1}$$

Now we want to take the limit in the above expression as $\varepsilon \downarrow 0$. The left-hand side clearly converges to $[X_t - x]^+ - [-x]^+$. Define $v_s = u_s[\frac{1}{2}u_s + \int_0^s D_s u_r\,dW_r]$, $0 \le s \le 1$. Suppose that the process u satisfies the hypotheses of Theorem 4.1, and denote by $L(J, x)$ the local time of the process X on the Borel subset J of $[0, 1]$. We can find a version of the local time which is a measure in the variable J and such that

$$\int_J \int_{\mathbb{R}} L(ds, x)\varphi(x)\,dx = \int_J \varphi(X_s)\,ds, \tag{5.2}$$

for any bounded and measurable function $\varphi\colon \mathbb{R} \to \mathbb{R}$. Set

$$\hat{L}(t, x) = \int_0^t v_s L(ds, x). \tag{5.3}$$

Then $\hat{L}(t, x)$ is the local time of the process X if we take $v_s\,ds$ as a scale to measure the occupation time.

Using (5.2) and (5.3) we can write (5.1) as follows:

$$f_\varepsilon(X_t - x) - f_\varepsilon(-x) = \int_0^t f_\varepsilon'(X_s - x)u_s\,dW_s + \int_{\mathbb{R}} f_\varepsilon''(y - x)\hat{L}(t, y)\,dy. \tag{5.4}$$

Observe that

$$\int_0^t f_\varepsilon''(y - x)\hat{L}(t, y)\,dy = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} \hat{L}(t, y)\,dy.$$

Therefore, by the Lebesgue differentiation theorem, for almost all (x, ω) we have that

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} \hat{L}(t, y)\,dy \xrightarrow{\varepsilon \downarrow 0} \hat{L}(t, x). \tag{5.5}$$

This also implies the almost sure convergence of the first summand in the right-hand side of (5.4).

The following lemma will be a basic ingredient in the proof of Tanaka's formula.

LEMMA 5.1. *Let $u \in \mathbb{L}^{2,3}$ be a stochastic process satisfying the following assumptions*:

(i) $\int_0^1 E(|u_r|^{2p})\,dr + \sum_{i=1}^3 \int_{[0,1]^{i+1}} E(|D_z^i u_r|^{2p})\,dr\,dz < \infty$, *for some* $p > 4$.
(ii) $E(\int_0^1 (1/|u_r|^{\delta})\,dr) < \infty$, *for* $\delta = 4(3p-4)/(p-4)$.

Then there exists a sequence of bounded random variables $\{F_n, n \geq 1\} \subset \mathbb{D}^{2,1}$ *such that* $\bigcup_n \{F_n \neq 0\} = \Omega$ *a.s. and*

$$\sup_{\epsilon} \int_{\mathbb{R}} E\left[F_n^2 \left|\int_0^1 f_\epsilon''(X_r - x) v_r\,dr\right|^2\right] dx < \infty, \tag{5.6}$$

for all $n \geq 1$.

PROOF. Notice first that the hypotheses (i) and (ii) imply conditions (i) to (iv) of Theorem 4.1. Let F_n be the random variable introduced in the proof of Theorem 4.1. Then F_n is bounded and we have to show that it belongs to $\mathbb{D}^{2,1}$. We have

$$DF_n = \varphi_n'(Z_q + Y_{m,p})\left[DZ_p + D(Y_{m,p})\right],$$

and we claim that this derivative is square integrable. Indeed, taking into account that $p > 4$, Proposition 4.2(v) implies that $\|DY_{m,p}\| \mathbf{1}_{\{Y_{m,p} \leq n+1\}}$ is in $L^2(\Omega, \mathscr{F}, P)$. On the other hand, for the term $\|DZ_q\|$ we have, using the estimations appearing in the proof of Proposition 4.1 with $\beta = p$,

$$\begin{aligned} &E\left(\|DZ_q\|^2\right) \\ &\quad \leq q^2 \left[E\left(\int_0^1 |u_s|^{p(q+1)/(1-p)}\,ds\right)^{4(p-1)/p} E\left(\int_0^1 \|Du_s\|^p\,ds\right)^{4/p}\right]^{1/2}. \end{aligned} \tag{5.7}$$

The second factor in (5.7) is finite by condition (i), and for the first one we use the fact that

$$E\left(\int_0^1 |u_s|^{-4(q+1)}\,ds\right) < \infty,$$

which follows from (ii) because $4(q+1) = 4(3p-4)/(p-4) = \delta$.

We know that there exist constants α_n and δ_n such that (4.4) holds with $\beta_t = \operatorname{sgn}(u_t)$. In order to show (5.6), it suffices to prove that

$$\sup_{\epsilon} \int_{\mathbb{R}} E\left[F_n^2 \left|\int_s^t f_\epsilon''(X_r - x) v_r\,dr\right|^2\right] dx < \infty, \tag{5.8}$$

for all $0 < s < t < 1$ such that $|t - s| \leq \delta_n$. Using the inequality

$$\int_{\mathbb{R}} f_\epsilon''(a - x) f_\epsilon''(b - x)\,dx \leq \frac{1}{2\epsilon} \mathbf{1}_{\{|a-b| \leq 2\epsilon\}}, \qquad a, b \in \mathbb{R},$$

we obtain

$$\int_{\mathbb{R}} E\left[F_n^2\left|\int_s^t f_\epsilon''(X_r - x)v_r\,dr\right|^2\right]dx$$

$$= \int_s^t\int_s^t E\left[F_n^2\left(\int_{\mathbb{R}} f_\epsilon''(X_r - x)\,f_\epsilon''(X_\theta - x)\,dx\right)v_r v_\theta\right]dr\,d\theta$$

$$\leq \int_s^t\int_s^t E\left[F_n^2|v_r v_\theta|\frac{1}{2\epsilon}\mathbf{1}_{\{|X_r - X_\theta| < 2\epsilon\}}\right]dr\,d\theta$$

$$\leq 4\int_s^t\int_s^t E\left[F_n^2|v_r v_\theta|h_{4\epsilon}(X_r - X_\theta)\right]dr\,d\theta,$$

where $h_{4\epsilon}$ is the function introduced in the proof of Theorem 2.1. Now we are going to use Proposition 2.3 with $U = X_r - X_\theta$, $\theta < r$, $h = \beta\mathbf{1}_{[\theta, r]}$ and $F = F_n^2|v_r v_\theta|$. Note that for almost all r, $\theta \in [s, t]$, the random variable F belongs to $\mathbb{D}^{1,1}$ due to our hypotheses. Taking into account the inequality (4.4) we obtain

$$\int_s^t\int_s^r E\left[F_n^2|v_r v_\theta|h_{4\epsilon}(X_r - X_\theta)\right]d\theta\,dr$$

$$\leq \int_s^t\int_s^r \frac{1}{\alpha_n|r-\theta|}E\left[\left|F_n^2 v_r v_\theta\int_\theta^r \beta_\sigma\,dW_\sigma\right|\right]d\theta\,dr$$

$$+ \int_s^t\int_s^r \frac{1}{\alpha_n|r-\theta|}E\left[\left|\int_\theta^r \beta_\sigma D_\sigma\left[F_n^2|v_r v_\theta|\right]d\sigma\right|\right]d\theta\,dr$$

$$+ \int_s^t\int_s^r \frac{1}{\alpha_n^2|r-\theta|^2}E\left[F_n^2\left|v_r v_\theta\int_\theta^r\int_\theta^r \beta_\sigma\beta_{\sigma'}D_\sigma D_{\sigma'}(X_r - X_\theta)\,d\sigma\,d\sigma'\right|\right]d\theta\,dr$$

$$= a_1 + a_2 + a_3.$$

We have to show that the terms a_1, a_2 and a_3 are finite.

Estimation of a_1. Using the Cauchy–Schwarz and Hölder inequalities, we obtain

$$\int_s^t\int_s^r \frac{1}{|r-\theta|}E\left[\left|v_r v_\theta\int_\theta^r \beta_\sigma\,dW_\sigma\right|\right]d\theta\,dr$$

$$\leq \int_s^t\int_s^r |r-\theta|^{-1/2}\left(E\left[|v_r v_\theta|^2\right]\right)^{1/2}d\theta\,dr$$

$$\leq \left(\int_s^t\int_s^r\left(E\left[|v_r v_\theta|^2\right]\right)^{\alpha/2}d\theta\,dr\right)^{1/\alpha}\left(\int_s^t\int_s^r (r-\theta)^{-\beta/2}\,d\theta\,dr\right)^{1/\beta},$$

where $1/\alpha + 1/\beta = 1$. The second factor in this last expression is finite provided $\beta < 2$. The first factor is also finite if $\alpha > 2$ is close enough to 2. In fact, using the assumptions on the process u, and the definition of v, we can write

$$E\left[\int_0^1 |v_r|^{2\alpha}\,dr\right] \le c_\alpha\left\{E\left[\int_0^1 |u_r|^{4\alpha}\,dr\right] + E\left[\int_0^1 |u_s|^{2\alpha}\left|\int_0^s D_s u_r\,dW_r\right|^{2\alpha} ds\right]\right\},$$

which is finite due to condition (i) if $2\alpha < p$.

Estimation of a_2. We decompose the term a_2 into two summands, according to the formula

$$D_\sigma\left[F_n^2|v_r v_\theta|\right] = D_\sigma\left[F_n^2\right]|v_r v_\theta| + F_n^2\left[|v_\theta|\operatorname{sgn}(v_r)D_\sigma v_r + |v_r|\operatorname{sgn}(v_\theta)D_\sigma v_\theta\right].$$

Using the definition of F_n and the Cauchy–Schwarz inequality we obtain

$$\int_s^t\int_s^r \frac{1}{|r-\theta|}E\left[\left(\int_\theta^r\left|D_\sigma\left[F_n^2\right]\right|d\sigma\right)|v_r v_\theta|\right]d\theta\,dr \le 2\int_s^t\int_s^r |r-\theta|^{-1/2}E\left[\|DF_n\|\,|v_r v_\theta|\right]d\theta\,dr.$$

By Hölder's inequality the above expression can be bounded by

$$\left(\int_s^t\int_s^r\left(E\left[\|DF_n\|\,|v_r v_\theta|\right]\right)^\alpha d\theta\,dr\right)^{1/\alpha}\left(\int_s^t\int_s^r |r-\theta|^{-\beta/2}\,d\theta\,dr\right)^{1/\beta},$$

where $1/\alpha + 1/\beta = 1$. The second factor in the above expression is finite provided $\beta < 2$, which leads us to choose $\alpha > 2$. We have

$$\left(E\left[|v_r v_\theta|\,\|DF_n\|\right]\right)^\alpha \le \left(E\left[|v_r v_\theta|^2\right]\right)^{\alpha/2}\left(E\left[\|DF_n\|^2\right]\right)^{\alpha/2} \le E\left[|v_r v_\theta|^\alpha\right]\left(E\left[\|DF_n\|^2\right]\right)^{\alpha/2},$$

which is finite if $2\alpha < p$ because $F_n \in \mathbb{D}^{2,1}$.

The second component of the term a_2 can be estimated as follows:

$$\int_s^t\int_s^r \frac{1}{|r-\theta|}E\left(\int_\theta^r |v_\theta D_\sigma v_r|\,d\sigma\right)d\theta\,dr \le \int_s^t\int_s^r |r-\theta|^{-1/2}\left(\int_\theta^r\left(E\left[|v_\theta D_\sigma v_r|\right]\right)^2 d\sigma\right)^{1/2} d\theta\,dr \le \int_s^t\int_s^r |r-\theta|^{-1/2}E\left(\|Dv_r\|\,|v_\theta|\right)d\theta\,dr.$$

By the same arguments as above, this last expression is finite provided for some $\alpha > 2$ we have

$$\int_0^1\int_0^1 E\left(\|Dv_r\|^\alpha |v_\theta|^\alpha\right)d\theta\,dr < \infty.$$

This follows from property (i) if $2\alpha < p$.

Estimation of a_3. Applying Hölder's inequality with three factors we obtain

$$\int_s^t \int_s^r |r-\theta|^{-2} \left(\int_\theta^r \int_\theta^r E\left[|v_r v_\theta D_\sigma D_{\sigma'}(X_r - X_\theta)| \right] d\sigma\, d\sigma' \right) d\theta\, dr$$

$$\leq \int_s^t \int_s^r E\left[|v_r v_\theta|\, \|D^2(X_r - X_\theta)\|_{L^8([r,\theta]^2)} \right] |r-\theta|^{-1/4}\, d\theta\, dr$$

$$\leq c_\alpha \left(\int_s^t \int_s^r E\left[|v_r v_\theta|^\alpha \right] d\theta\, dr \right)^{1/\alpha} \left(\int_s^t \int_s^r E\left[\|D^2(X_r - X_\theta)\|^8_{L^8([r,\theta]^2)} \right] d\theta\, dr \right)^{1/8},$$

with a constant c_α depending on α, provided $\alpha > 2$. We have already checked that the first factor in the above expression is finite if $2\alpha < p$. Hypothesis (i) implies that the second factor is also finite, and this completes the proof of the lemma. □

Now we can establish Tanaka's formula for an indefinite Skorohod integral, provided the integrand satisfies the assumptions of the previous lemma.

THEOREM 5.1. *Let* u *be a process which satisfies the hypotheses of Lemma* 5.1. *Suppose that* μ *is a finite measure on the real line which is absolutely continuous with respect to Lebesgue measure and has a bounded density. Then, the stochastic process* $\{\mathbf{1}_{\{X_s > x\}} u_s,\ s \in [0,1],\ x \in \mathbb{R}\}$, *regarded as a* $L^2(\mathbb{R}, \mu)$*-valued random process is locally Skorohod integrable in the sense of Definition* 2.1, *and we have*

$$(5.9)\qquad (X_t - x)^+ - (-x)^+ = \int_0^t \mathbf{1}_{\{X_s > x\}} u_s\, dW_s + \hat{L}(t,x),$$

for all $t \in [0,1]$, *where* $\hat{L}(t,x)$ *is the local time defined in* (5.3).

PROOF. First we have to introduce a sequence of localizing random variables. For each $n \geq 1$, define $G_n = \varphi_n(Y + \int_0^1 u_s^2\, ds)$, where φ_n and $Y = Z_q + Y_{m,p}$ have been introduced in Theorem 4.1 and in Lemma 5.1. As for the random variables F_n appearing in Lemma 5.1, we have $G_n \in \mathbb{D}^{2,1}$. On the other hand, the inequality (5.6) of Lemma 5.1 still holds if we replace F_n by G_n.

We claim that the random variables G_n and the $L^2(\mathbb{R}, \mu)$-valued stochastic process $\{\mathbf{1}_{\{X_s > x\}} u_s,\ s \in [0,1]\}$ satisfy the assumptions of Definition 2.1. In fact, it is clear that $\{G_n = 1\} \uparrow \Omega$, a.s., and $|G_n| \leq 1$. Using the fact that the measure μ is finite and that $\int_0^1 u_s^2\, ds$ is bounded by $n+1$ on the set $\{G_n \neq 0\}$, we have

$$(5.10)\qquad E\left(\int_{\mathbb{R}} \left| \int_0^1 u_t \mathbf{1}_{\{X_t > x\}} D_t G_n\, dt \right|^2 \mu(dx) \right) < \infty.$$

Finally we have to check that the $L^2(\mathbb{R}, \mu)$-valued process

$$\left\{ G_n u_s \mathbf{1}_{\{X_s > x\}} \mathbf{1}_{[0,t]}(s) \right\}$$

belongs to the domain of δ for all $t \in [0,1]$, and for all $n \geq 1$. The family of processes $G_n u f'_\epsilon(X_\cdot - x)\mathbf{1}_{[0,t]}$ converges to $G_n u \mathbf{1}_{\{X_\cdot > x\}}\mathbf{1}_{[0,t]}$ in $L^2(\Omega \times \mathbb{R}, P \times \mu)$, as ϵ tends to zero. The Skorohod integrals of those processes satisfy

$$\begin{aligned}\int_0^t G_n u_s f'_\epsilon(X_s - x)\,dW_s &= G_n \int_0^t u_s f'_\epsilon(X_s - x)\,dW_s - \int_0^t u_s f'_\epsilon(X_s - x) D_s G_n\,ds \\ &= G_n\left(f_\epsilon(X_t - x) - f_\epsilon(-x) - \int_0^t f''_\epsilon(X_s - x) v_s\,ds\right) \\ &\quad - \int_0^t u_s f'_\epsilon(X_s - x) D_s G_n\,ds,\end{aligned}$$

and this is bounded in $L^2(\Omega \times [0,1] \times \mathbb{R}, P \times dt \times \mu)$, uniformly in ϵ, as follows from (5.10) and (5.6). This implies that $G_n u \mathbf{1}_{\{X_\cdot > x\}}\mathbf{1}_{[0,t]}$ belongs to (Dom δ)($L^2(\mathbb{R}, \mu)$), and its Skorohod integral is the limit, in the weak topology of $L^2(\Omega \times \mathbb{R}, P \times \mu)$, of the right-hand side of the above expression.

We have seen before that $G_n(f_\epsilon(X_t - x) - f_\epsilon(-x) - \int_0^t f''_\epsilon(X_s - x)v_s\,ds)$ converges for almost all (x, ω) to $G_n \hat{L}(t, x)$ as ϵ tends to zero. This family of random variables being bounded in $L^2(\Omega \times \mathbb{R}, P \times \mu)$, we have that the convergence is in $L^q(\Omega \times \mathbb{R}, P \times \mu)$, for any $q < 2$. Therefore, we obtain

$$\begin{aligned}\int_0^t G_n u_s \mathbf{1}_{\{X_s > x\}}\,dW_s &= G_n\big((X_t - x)^+ - (-x)^+ - \hat{L}(t,x)\big) \\ &\quad - \int_0^t u_s \mathbf{1}_{\{X_s > x\}} D_s G_n\,ds,\end{aligned}$$

for all $n \geq 1$. This implies the desired formula because the sets $\{G_n = 1\}$ increase to Ω, a.s., as n tends to infinity. □

Notice that for all $x \in \mathbb{R}$ the Skorohod integral $\int_0^t \mathbf{1}_{\{X_s > x\}} u_s\,dW_s$ exists as an element of $\mathbb{D}^{2,-1}$, that is, as a distribution on the Wiener space. Using this remark one can write a version of Tanaka's formula in the distribution sense (see Ustunel [13]). However, we do not know if the hypotheses of Theorem 5.1 imply that for all $x \in \mathbb{R}$ (almost everywhere) the process $\int_0^t \mathbf{1}_{\{X_s > x\}} u_s\,dW_s$ belongs locally to Dom δ. For this reason, the Skorohod integral appearing in Tanaka's formula (5.9) has to be understood in a global sense, that is, as the integral of an $L^2(\mathbb{R}, \mu)$-valued process.

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