DOMINATION BETWEEN TREES AND APPLICATION TO AN EXPLOSION PROBLEM

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We define a notion of stochastic domination between trees, where one tree dominates another if, when the vertices of each are labeled with independent, identically distributed random variables, one tree is always more likely to contain a path with a specified property. Sufficient conditions for this kind of domination are (1) more symmetry and (2) earlier branching. We apply these conditions to the problem of determining how fast a tree must grow before first-passage percolation on the tree exhibits an explosion, that is to say, infinitely many vertices are reached in finite time. For a tree in which each vertex at distance $n - 1$ from the root has $f(n)$ offspring, f nondecreasing, an explosion occurs with exponentially distributed passage times if and only if $\sum f(n)^{-1} < \infty$.

1. Introduction. Let Γ be any locally finite tree with some vertex ρ chosen as the root and total height $N \leq \infty$. Label the vertices of Γ with independent, identically distributed real random variables $\{X(\sigma): \sigma \in \Gamma\}$, and let $B \subseteq \mathbb{R}^N$ be some Borel set. Let $P(B; \Gamma)$ denote the probability that $(X(\sigma_1), X(\sigma_2), \dots) \in B$ for some non-self-intersecting path $\rho, v_1, \dots, v_N$ in Γ. This probability arises in many contexts. The problem as such is studied in [10] and [8]. For first-passage percolation, the probability of reaching level N by time T is $P(B; \Gamma_{|N})$, where B is the set $\sum_{i=1}^{N} v_i \leq T$ and $\Gamma_{|N}$ is the first N levels of Γ. The same quantity arises when studying diffusion-limited aggregation on trees via the exponential representation [1, 4]. A random walk in a random environment on a tree will be transient when some path is itself transient, which probability can be reduced via the electrical representation to $P(B; \Gamma)$ for independent, identically distributed resistances, the set B being again the summable sequences; see [12] and [13]. The study of tree-indexed Markov chains [6] can be reduced to computations of $P(B; \Gamma)$ by representing the Markov chain as a function of independent, identically distributed uniform $[0, 1]$ random variables.

Let $|v|$ denote the distance from ρ to v, and write $w \leq v$ if w is on the path from ρ to v. Let $\Gamma_n = \{v: |v| = n\}$ denote the nth level of Γ, and let $\Gamma_{|n}$ denote the first n levels of Γ. Say that Γ is *spherically symmetric* with growth function f if every vertex $v \in \Gamma_n$ has $f(|v| + 1)$ children (a child of v is a neighbor w with $v \leq w$). We examine the following notion of stochastic domination between trees.

DEFINITION 1. Let Γ and Γ' be two finite or infinite trees with roots ρ and ρ', respectively. We say that Γ dominates Γ' if, whenever

$$\{ X(v) \colon v \in \Gamma \} \cup \{ X'(w) \colon w \in \Gamma' \}$$

is a collection of *i.i.d.* random variables, $n \geq 1$ is an integer and $B \subseteq \mathbb{R}^n$ is a Borel set, then

$$P(B; \Gamma_{|n}) \geq P(B; \Gamma'_{|n}).$$

Spherically symmetric trees are easier to compute with, which is the principal motivation for developing inequalities that compare $P(B; \Gamma)$ and $P(B; \Gamma')$ when one of Γ or Γ' is spherically symmetric. Our main comparison result, proved in Section 2, is the following.

THEOREM 1. *Let Γ and Γ' be finite or infinite trees with Γ spherically symmetric. Then Γ dominates Γ' if and only if for every $n \geq 1$ the nth generation of Γ,*

$$\Gamma_n \overset{\text{def}}{=} \{ v \in \Gamma \colon |v| = n \},$$

is at least as big as the nth generation of Γ'.

REMARKS.

(i) One half of Theorem 1 is easy: Take $\{X(v)\}$ to be i.i.d. Bernoulli random variables and $B = \{(x_1, x_2, \ldots) \colon x_n = 1\}$ to see that $|\Gamma_n| \geq |\Gamma'_n|$ is necessary for domination.

(ii) We understand the domination partial order completely only for spherically symmetric trees and for trees of height 2. In the latter case, it reduces to the classical notion of Hardy majorization (Proposition 6).

(iii) Comparison results for $P(B; \Gamma)$ as B varies may be found in [13].

A consequence of Theorem 1 is that among all trees Γ of height n having $|\Gamma_n| = k$, the tree $T(n, k)$ consisting of k disjoint paths of length n joined at the root is maximal in the domination order. If the common law of the $X(\sigma)$ is μ and $B \subseteq \mathbb{R}^n$, then $1 - P(B; T(n, k)) = (1 - \mu^n(B))^k$, where μ^n is the n-fold product measure; thus, for any Γ of height n,

(1) $$1 - P(B; \Gamma) \geq (1 - \mu^n(B))^k.$$

The definition of $P(B; \Gamma)$ extends naturally to any *graded graph*, this being a finite graph whose vertices are partitioned into levels $1, \ldots, n$ with oriented edges allowed only from levels i to $i + 1$, $i = 1, \ldots, n - 1$. A natural conjecture is the following.

CONJECTURE 2. *If G is a graded graph of height n, let $K(G)$ be the number of oriented paths that pass through every level of G. Let $X(\sigma)$ be independent, identically distributed random variables with common law μ and $B \subseteq \mathbb{R}^n$.*

Then

$$1 - P(B;G) \geq (1 - \mu^n(B))^{K(G)}.$$

If B is an upwardly closed set, then both the conjecture and the inequality (1) follow easily from the FKG inequality. In the case where $n = 2$ (G is bipartite), Conjecture 2 is due to Sidorenko [16, Conjecture 5.2] in the form of an analytic inequality; Sidorenko has proved several related analytic inequalities on graphs in [14], [15] and [16] including the conjecture itself for the special cases where G is acyclic, a single cycle or sufficiently small.

The remainder of the paper is devoted to an in-depth application of this theorem to first-passage percolation, which we now describe. Let Γ be a spherically symmetric infinite tree with growth function f, and let $\{X(v)\}$ be a collection of independent exponential random variables of mean 1. (Our results hold for a much more general class distributions described later.) Think of $X(v)$ as a transit time across the edge connecting v to its parent. Define

$$S(v) = \sum_{\rho < w \leq v} X(w) \quad \text{and} \quad M_n = \min_{|v|=n} S(v).$$

In the context of first-passage percolation, $S(v)$ is the passage time from the root to the vertex v and M_n is the first passage time to the nth generation of Γ. Say that an explosion occurs if the increasing sequence M_n is bounded.

An explosion is a tail event, so for a given tree the probability of an explosion is 0 or 1. It is natural to try to determine when the probability is 0 and when it is 1. This problem was brought to our attention by Enrique Andjel [2] in reference to uniqueness proofs for particle systems via graphical representations. For arbitrary trees, such questions are difficult to settle without leaving open some "critical" case (see, e.g., [11]). As observed by Andjel, it is elementary, for the case of spherically symmetric trees, to obtain conditions for explosion or nonexplosion which are almost sharp. Indeed, if

$$\liminf_{n \to \infty} \left[(1/n!) \prod_{i=1}^{n} f(i) \right]^{1/n} = C < \infty.$$

then a simple Borel–Cantelli argument shows there is almost surely no explosion before time $(2C)^{-1}$, from which it follows easily that with probability 1, no explosion occurs at all. On the other hand, the condition $\sum f(n)^{-1} < \infty$ implies that the greedy algorithm finds an explosion. Thus, in particular, there is no explosion when $f(n) = n$ but there is one when $f(n) = n \ln(n)^{1+\varepsilon}$. We show that the summability condition is sharp, the statement being a little more complicated in the case where f is not an increasing function.

THEOREM 3. *Let Γ be a spherically symmetric tree with growth function f that is nondecreasing. The probability of an explosion is 0 or 1 according to whether $\sum_{n=1}^{\infty} f(n)^{-1}$ is infinite or finite. Furthermore, if the sum is infinite and $f(n)$ is unbounded as well, then $M_N / \sum_{n=1}^{N} f(n)^{-1}$ converges almost surely to e^{-1}.*

In most integral tests in probability theory, some regularity condition is imposed. An unusual feature of the problem considered here is that it permits a criterion (Theorem 4) valid for arbitrary growth functions f. The condition in Theorem 3 always suffices for explosion but simple examples [$f(2n - 1) = 1$, $f(2n) = 2^n$] show it is not necessary. For any $f: \mathbb{N} \to \mathbb{N}$, define a function $\tilde{f}$: $\mathbb{N} \to \mathbb{R}^+$ recursively by

$$(2) \qquad \tilde{f}(n + 1) = \sup\left\{a : a^m \prod_{i=1}^{n} \tilde{f}(i) \leq \prod_{i=1}^{n+m} f(i) \text{ for all } m \geq 1\right\}.$$

In particular,

$$\tilde{f}(1) = \inf_{m \geq 1} \left(\prod_{i=1}^{m} f(i)\right)^{1/m}.$$

It is easy to see that the function $\tilde{f}$ is always nondecreasing and coincides with f when f is nondecreasing.

THEOREM 4. *Let Γ be a spherically symmetric tree with growth function f, labeled as before by independent exponentials $X(v)$ of mean 1. Then the probability of an explosion is 0 or 1 according to whether $\sum_{n=1}^{\infty} \tilde{f}(n)^{-1}$ is infinite or finite. Furthermore, if the sum is infinite and $\tilde{f}$ is unbounded, then*

$$\limsup_{N \to \infty} M_N \Big/ \sum_{n=1}^{N} \tilde{f}(n)^{-1} = e^{-1}.$$

REMARKS.

(i) It is usually easy to calculate $\tilde{f}$ from f. Informally, if you graph $\sum_{j=1}^{n} \ln(f(j))$ against n and consider the convex hull of the region above this graph, its boundary is the graph of $\sum_{j=1}^{n} \ln \tilde{f}(j)$. For instance, if the two sequences $\{f(2n - 1)\}$ and $\{f(2n)\}$ are nondecreasing with $f(2n - 1) < f(2n)$ for all n, then

$$\tilde{f}(2n - 1) = \tilde{f}(2n) = (f(2n - 1)f(2n))^{1/2}.$$

(ii) Theorem 4 is proved by comparing the tree Γ to a tree with nondecreasing growth function, which is where the application of Theorem 1 is needed.

These theorems are proved in Section 3. Versions where the variables $\{X(v)\}$ are not exponential are given in Section 4. Results completely analogous to Theorems 3 and 4 hold for distributions G satisfying a power law near zero, i.e., $\lim_{t \downarrow 0} G(t)/t^{\alpha}$ is finite and positive for some $\alpha > 0$. Obtaining sharp criteria for explosion to occur that are valid for arbitrary transit-time distributions G seems more delicate, although we cannot find a G for which the natural criterion (12) in the last section fails.

2. Domination between trees.

PROOF OF THEOREM 1. We start by establishing the theorem under the additional assumption that Γ' is spherically symmetric. This is the only case that is used in the proof of Theorem 4.

Let μ be any probability measure on $\mathbb{R}$ and let D be a Borel set in $\mathbb{R}^n$. If $b_1, \ldots, b_n$ are the cardinalities of the generations of a spherically symmetric tree [i.e., the growth function is $f(i) = b_i/b_{i-1}$], let the vertices index i.i.d. random variables $X(v)$ with common law μ and let $\Psi(b_1, \ldots, b_n; D)$ denote the probability that all paths $\rho, v_1, \ldots, v_n$ in the tree satisfy $(X(v_1), \ldots, X(v_n)) \in D$. Passing to complements in the definition of domination, we must show that

$$\Psi(b_1, \ldots, b_n; D) \leq \Psi(b_1', \ldots, b_n'; D)$$

whenever both are defined and $b_i \geq b_i'$ for all i.

The key to doing this is the following recursive relation, obtained by conditioning on the variables $X(v)$ for $|v| = 1$:

$$(3) \qquad \Psi(b_1, \ldots, b_n; D) = \left[\int \Psi\left(\frac{b_2}{b_1}, \ldots, \frac{b_n}{b_1}; D/v_1 \right) d\mu(x_i) \right]^{b_1},$$

where for $D \subseteq \mathbb{R}^n$ and $(x_1, \ldots, x_k) \in \mathbb{R}^k$ the notation $D/(x_1, \ldots, x_k)$ is used for the cross section of D given by

$$\left\{ (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k} : (x_1, \ldots, x_n) \in D \right\}.$$

Observe that relation (3) together with the initial condition $\Psi(b; D) = \mu(D)^b$, for $D \subseteq \mathbb{R}$, uniquely determines Ψ and in fact remains a valid inductive definition when the arguments b_i are positive reals, not necessarily integral. Call this extension Ψ as well, since it agrees with the old Ψ on integral arguments. We verify by induction on n that, for any $D \subseteq \mathbb{R}_n$, the function $\Psi(b_1, \ldots, b_n; D)$ is nonincreasing in each of its n arguments.

This is clear for $n = 1$, so fix $n > 1$ and observe the fact that Ψ is nonincreasing in $b_2, \ldots, b_n$ follows directly from the induction hypothesis and (3). It remains to check that $\Psi(b_1, \ldots, b_n) \leq \Psi(b_1', b_2, \ldots, b_n)$ when $b_1 \geq b_1'$. Rewriting this as

$$\left[\int \Psi\left(\frac{b_2}{b_1}, \ldots, \frac{b_n}{b_1}; D/x_1 \right) d\mu(x_1) \right]^{b_1} \leq \left[\int \Psi\left(\frac{b_2}{b_1'}, \ldots, \frac{b_n}{b_1'}; D/x_1 \right) d\mu(x_1) \right]^{b_1'},$$

we see it is just Hölder's inequality $\int h(x_1) d\mu(x_1) \leq [\int h(x_1)^r d\mu(x_1)]^{1/r}$ with $r = b_1/b_1'$ applied to the function

$$h(x_1) = \left[\int \Psi\left(\frac{b_3}{b_2}, \ldots, \frac{b_n}{b_2}; D/(x_1, x_2) \right) d\mu(x_2) \right]^{b_2/b_1}.$$

[For $n = 2$ take $h(x_1) = \mu(D/x_1)^{b_2/b_1}$.]

This proves Theorem 1 for spherically symmetric trees. To obtain the general case requires the following lemma.

LEMMA 5. *For any Borel set $D \subseteq \mathbb{R}^n$, the function $\Psi(b_1, \ldots, b_n; D)$ defined by (3) is log-convex on the positive orthant of $\mathbb{R}^n$.*

PROOF. Assume that $\mu^n(D) > 0$, since otherwise $\Psi(\cdots; D)$ is identically zero. For $n = 1$, $\ln \Psi(\cdot; D)$ is linear. Proceeding by induction, fix $n > 1$. Relation (3) shows that $\ln \Psi(\cdots; D)$ is homogeneous of degree 1:

$$\ln \Psi(\lambda b_1, \ldots, \lambda b_n; D) = \lambda \ln \Psi(b_1, \ldots, b_n; D).$$

Hence, to prove convexity it suffices to verify that $\ln \Psi(1, b_2, b_3, \ldots, b_n; D)$ is a convex function of the positive variables $b_2, \ldots, b_n$. However,

$$\Psi(1, b_2, \ldots, b_n; D) = \int \Psi(b_2, \ldots, b_n; D/x_1) \, d\mu(x_1)$$

and the sum or integral of log-convex functions is log-convex (see, e.g., [3], pages 7–10). $\square$

PROOF OF THEOREM 1 (Completed). We verify by induction that, for any tree T of depth n with generation cardinalities $|T_i| = b_i$ for $i = 1, \ldots, n$ and for any $D \subseteq \mathbb{R}^n$, the probability $\phi(T; D)$ that all paths $\rho, w_1, \ldots, w_n$ in T satisfy $(X(w_1), \ldots, X(w_n)) \in D$ is at least $\Psi(b_1, \ldots, b_n; D)$, where as usual, $X(v)$ are i.i.d. with common law μ, and Ψ is defined by (3).

For every w in the first generation T_1 of T, let $T(w)$ denote the subtree $\{\rho\} \cup \{u \in T: u \geq w\}$. Let $b_i(w)$ denote the cardinality of the ith generation of $T(w)$. [In particular, $b_1(w) = b_0(w) = 1$.] Fix $D \subseteq \mathbb{R}^n$. By the induction hypothesis,

$$\phi(T(w); D) \geq \int \Psi(b_2(w), \ldots, b_n(w); D/x_1) \, d\mu(x_1)$$

$$= \Psi(1, b_2(w), \ldots, b_n(w); D).$$

Therefore, since $\sum_{w \in T_1} b_i(w) = b_i$ for $i \geq 1$, using the log-convexity established in the previous lemma gives

$$\phi(T; D) = \prod_{w \in T_1} \phi(T(w); D)$$

$$\geq \prod_{w \in T_1} \Psi(1, b_2(w), \ldots, b_n(w); D)$$

$$\geq \Psi\left(1, \frac{b_2}{b_1}, \ldots, \frac{b_n}{b_1}; D\right)^{b_1}$$

$$= \Psi(b_1, \ldots, b_n; D). \qquad \square$$

Lemma 5 yields a simple description of domination between trees of height 2.

PROPOSITION 6.   *Let Γ and Γ' be trees of height 2. For each vertex in Γ_1, count its children and order the numbers so obtained in a decreasing sequence $n_1 \geq n_2 \geq \cdots \geq n_b$. Similarly, obtain a sequence $n'_1 \geq n'_2 \geq \cdots \geq n'_b$ from Γ', appending zeros to one of the sequences if necessary so that both have the same length b. Then Γ dominates Γ' if and only if*

$$(4) \qquad \sum_{i>k} n_i \geq \sum_{i>k} n'_i \quad \text{for every } k \geq 0.$$

REMARK.   Thinking of the numbers n_i as a partition of $\Sigma n_i = |\Gamma_2|$, this is the order obtained by combining the usual majorization order of partitions (in the reverse direction) with the inclusion order (Young's lattice).

PROOF OF PROPOSITION 6.   We start by showing that the condition (4) implies domination. Assume without loss of generality that

$$(5) \qquad \sum_{i=1}^{b} n_i = \sum_{i=1}^{b} n'_i,$$

since otherwise we could increase n'_1, thereby obtaining a tree dominating Γ', while condition (4) would remain unaffected.

Conditions (4) and (5) together imply that the vector $(n'_i\colon i = 1,\ldots,b)$ *majorizes* the vector $(n_i\colon i = 1,\ldots,b)$ in the sense that the latter is a convex combination of permutations of the former (see [9], Theorem 47). To show that Γ dominates Γ', it suffices to verify that for any $D \subseteq \mathbb{R}^2$,

$$(6) \qquad \prod_{i=1}^{b} \Psi(1,n_i;D) \leq \prod_{i=1}^{b} \Psi(1,n'_i;D).$$

Fixing D, let $h(n_1,\ldots,n_b)$ denote the left-hand side of (6). Clearly h is invariant under permutations of its arguments. By Lemma 5 it is a product of log-convex functions and hence log-convex. Since $(n_1,\ldots,n_b)$ is a weighted average of permutations of $(n'_1,\ldots,n'_b)$, the inequality (6) follows.

For the converse, assume Γ dominates Γ'. Let $r \geq 1$. For $0 < \varepsilon < 1$, take

$$D_\varepsilon = ([0,\varepsilon^r] \times [0,1]) \cup ([0,1] \times [0,\varepsilon]).$$

If $\{X(v)\colon v \in \Gamma\}$ are independent and uniform on $[0,1]$, then the probability that $(X(v_1),X(v_2)) \in D$ for every path ρ, v_1, v_2 lies between $\prod_{i=1}^{b}\varepsilon^{\min(n_i,r)}$ and $\prod_{i=1}^{b}(2\varepsilon)^{\min(n_i,r)}$. When ε is sufficiently small, the assumption that Γ dominates Γ' forces

$$\sum_{i=1}^{b} \min(n_i,r) \geq \sum_{i=1}^{b} \min(n'_i,r).$$

Choosing $r = n'_k$ yields

$$rk + \sum_{i>k} n_i \geq \sum_{i=1}^{b} \min(n'_i,r) = rk + \sum_{i>k} n'_i,$$
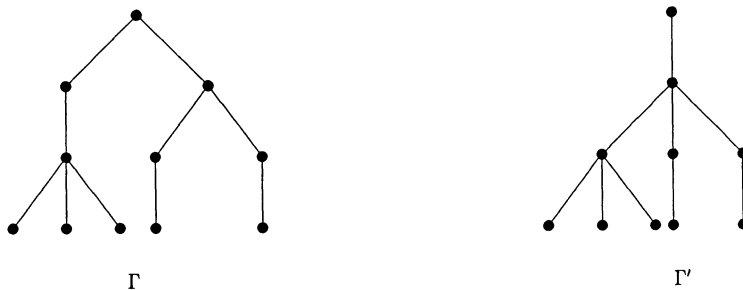
proving (4). $\square$

Γ   Γ′

FIG. 1.

We end this section with some remarks and questions about domination.

1. Already for trees of height 3 the domination order is somewhat mysterious. Consider the trees Γ and Γ′ in Figure 1, where Γ′ is obtained from Γ by gluing together the vertices in the first generation. Intuitively, it seems that Γ should dominate Γ′, but this is not the case. Let

$$D = ([0, 1/2] \times [0, 1] \times [0, 2/3]) \cup ([1/2, 1] \times [0, 1/2] \times [0, 1])$$

and let $X(v)$ be uniform on $[0, 1]$. The probability that all paths in Γ have $(X(v_1), X(v_2), X(v_3)) \in D$ is $1075/7776$, while the corresponding probability for Γ′ is only $998/7776$.

2. To verify that a tree Γ dominates another tree Γ′, it suffices to consider the case in which the i.i.d. variables $X(v)$ are uniform on $[0, 1]$, since other variables can be written as functions of these. Along the same lines, Theorem 1 may be seen to sharpen a result from [5]. There it was shown that any tree of height n with b_n vertices in the nth generation is dominated by the tree consisting of b_n disjoint paths of length n from the root. The notion of domination in [5] is for tree-indexed Markov chains; we omit the easy proof that this is an equivalent definition of domination.

3. A counterexample of the type given in Figure 1 cannot occur if the set D is restricted to being an upwardly closed subset of $[0, 1]^n$ (i.e., $\mathbf{x} \in D$ and $\mathbf{y} \geq \mathbf{x}$ coordinatewise imply $\mathbf{y} \in D$). These subsets occur naturally in percolation problems. Domination for upwardly closed sets subsumes gluing, as may be shown using the FKG inequality. Is this partial order any more tractable for trees of height greater than 2?

3. Exponential transit times and nondecreasing growth functions.

PROOF OF THEOREM 3. When $\sum_{n=1}^{\infty} f(n)^{-1} < \infty$, use the greedy algorithm to select the (a.s. unique) random path $\rho = v_0, v_1, v_2, \ldots$ for which $X(v(n))$ is minimal among $\{X(w) \colon w \text{ is a child of } v_{n-1}\}$. Then

$$\mathbf{E}X(v_n) = \mathbf{E}[X(v_n)|v_{n-1}] = f(n)^{-1},$$

being the minimum of $f(n)$ standard exponentials. Hence $\mathbf{E}\sum_{n=1}^{\infty} X(v_n) < \infty$, so in particular, $\sum_{n=1}^{\infty} X(v_n) < \infty$ almost surely.

The more interesting case is when $\sum_{n=1}^{\infty} f(n)^{-1} = \infty$. In this case, consider the weighted sums

$$S^*(v) = \sum_{\rho < w \leq v} Y(w),$$

where $Y(w) = f(|w|)X(w)$. Define

$$M_n^* = \min_{|v|=n} S^*(v).$$

For any vertex $w \neq \rho$ and any $\lambda > 0$, we have

$$\mathbf{E}[e^{-\lambda Y(w)}] = \frac{1}{1 + \lambda f(|w|)}.$$

Multiplying these along a path yields

$$\mathbf{E}[e^{-\lambda S^*(v)}] = \prod_{\rho < w \leq v} \frac{1}{1 + \lambda f(|w|)} < \lambda^{-|v|} \sum_{j=1}^{|v|} f(j)^{-1},$$

and summing over Γ_n gives $\mathbf{E}[\sum_{v \in \Gamma_n} \exp(-\lambda S^*(v))] < \lambda^{-n}$. Now, for any ε with $0 < \varepsilon < 1$, Markov's inequality implies that

$$\mathbf{P}\left[\exists\, v \in \Gamma_n : S^*(v) \leq \frac{n}{\lambda} \ln((1-\varepsilon)\lambda)\right] < (1-\varepsilon)^n.$$

By Borel–Cantelli, it follows that $M_n^* \leq (n/\lambda)\ln((1-\varepsilon)\lambda)$ finitely often almost surely; choosing $\lambda = e$ and letting $\varepsilon \to 0$ leads to

$$(7) \qquad\qquad \liminf_{n \to \infty} \frac{1}{n} M_n^* \geq e^{-1} \quad \text{almost surely.}$$

Deriving a lower estimate for M_n from (7) requires no probability theory. Choose $\varepsilon > 0$; with probability 1 there exists an integer N_ε such that $M_k^* \geq k(e^{-1} - \varepsilon)$ for all $k \geq N_\varepsilon$. For any path $\{v_k : 0 \leq k \leq n\}$ starting from $v_0 = \rho$, summation by parts yields

$$\sum_{k=1}^{n} X(v_k) = \sum_{k=1}^{n} f(k)^{-1}[S^*(v_k) - S^*(v_{k-1})]$$

$$= S^*(v_n) f(n+1)^{-1} + \sum_{k=1}^{n} S^*(v_k)\left[f(k)^{-1} - f(k+1)^{-1}\right]$$

$$> S^*(v_n) f(n+1)^{-1} + (e^{-1} - \varepsilon) \sum_{k=1}^{n} k\left[f(k)^{-1} - f(k+1)^{-1}\right]$$

$$- C(N_\varepsilon),$$

where $C(N_\varepsilon)$ depends only on N_ε and f. The last inequality is the only place

we use the assumption that f is nondecreasing. Summing by parts again,

$$\sum_{k=1}^{n} X(v_k) > (e^{-1} - \varepsilon) \sum_{j=1}^{n} f(j)^{-1} - C(N_\varepsilon)$$

$$+ f(n+1)^{-1}\left[S^*(v_n) - n(e^{-1} - \varepsilon) \right]$$

$$\geq (e^{-1} - \varepsilon) \sum_{j=1}^{n} f(j)^{-1} - C(N_\varepsilon).$$

Thus M_n is also greater than the right-hand side of the last inequality, which easily implies that almost surely

$$\liminf_{n \to \infty} M_n \bigg/ \sum_{j=1}^{n} f(j)^{-1} \geq e^{-1},$$

as long as the series $\Sigma f(j)^{-1}$ diverges. This completes the proof of the explosion criterion.

To show that

$$\limsup_{n \to \infty} M_n \bigg/ \sum_{j=1}^{n} f(j)^{-1} \leq e^{-1}$$

when the denominator tends to infinity and f is unbounded, it suffices to exhibit, for every $\varepsilon > 0$, an infinite path in Γ along which the condition

$$(8) \qquad S(v) \bigg/ \sum_{j=1}^{|v|} f(j)^{-1} \leq e^{-1} + \varepsilon$$

fails only finitely often. This may be accomplished by a branching process argument as in [12]; the reader is referred there for greater detail.

Let k be any positive integer. We prune the tree Γ to obtain a random subtree Γ' as follows. First, take $\rho \in \Gamma'$; next, for $v \in \Gamma'$ and w a child of v in Γ, let $w \in \Gamma'$ if and only if $X(w)$ has one of the k least values among $\{X(u): u$ is a child of v in $\Gamma\}$. Then Γ' is a tree which is eventually k-ary. Again, let $Y(v) = f(|v|)X(v)$ with partial sums $S^*(v) = \Sigma_{\rho < w \leq v} Y(w)$ and M_n^* as before. For $v \in \Gamma'$, the variables $\{Y(w): w \in \Gamma'$ is a child of $v\}$ are the first k order statistics of $f(|v|)$ independent exponentials of mean $f(|v|)$. As $|v| \to \infty$, we have assumed that also $f(|v|) \to \infty$, so these order statistics are converging weakly to the first k hits of a mean 1 Poisson process. [This is an easy consequence of the usual Poisson convergence theorem and the fact that the common distribution of the original random variables $X(v)$ has a density of 1 at the origin.] In particular, for every $\varepsilon > 0$ and $v \in \Gamma'$, the joint distribution of

$$\{Y(w): w \in \Gamma', w \text{ is a child of } v\}$$

is stochastically dominated by the first k hits of a mean $1 - \varepsilon$ Poisson process.

In other words, conditional on the selection of the vertices belonging to the subtree Γ', we can couple the process $\{Y(v) \colon v \in \Gamma'\}$ to a process $\{Z(v) \colon v \in \Gamma'\}$ for which the values of Z over children of v are independent for different v and each distributed as a uniform random permutation of the first k order statistics of a mean $1 - \varepsilon$ Poisson process whenever $f(|v| + 1) \geq k$; the coupling will satisfy $Z(v) > Y(v)$ for all v such that $|v| > R$ for some R depending on k and ε.

If Z is a random variable distributed as a uniform random selection of one of the first k hits of a mean $1 - \varepsilon$ Poisson process, then

$$\mathbf{E}e^{-\lambda Z} = k^{-1} \sum_{i=1}^{k} \left(1 + \frac{\lambda}{1 - \varepsilon}\right)^{-i} = \frac{1 - \varepsilon}{k\lambda}\left[1 - \left(1 + \frac{\lambda}{1 - \varepsilon}\right)^{-k}\right].$$

Choose $k = k(\varepsilon)$ so that this is at least $(1 - 2\varepsilon)/k\lambda$ for all $\lambda > \frac{1}{10}$, say. The rate function for Z satisfies

$$m_Z(a) \overset{\text{def}}{=} \inf_{\lambda > 0} e^{\lambda a} \mathbf{E}e^{-\lambda Z}$$

$$\geq \frac{1 - 2\varepsilon}{k} \inf_{\lambda > 0} e^{\lambda a} \min\left\{\frac{1}{\lambda}, 10\right\}$$

$$\geq \frac{1 - 2\varepsilon}{k} ea,$$

for $0 < a < 10/e$. In particular, if we fix $a = (1 - 3\varepsilon)^{-1}e^{-1}$, then $m_Z(a) > k^{-1}$.

Now the proof of (2.8) in [12, page 1237] or the main result of [7] shows that there is some integer L for which the following branching process is supercritical: $v \in \Gamma'$ begets $w \in \Gamma'$ if $|w| = |v| + L$ and $L^{-1}\sum_{v < u \leq w} Z(u) \leq a$. This implies that for such an L there exists almost surely a path $\rho, v_1, v_2, \ldots$ for which

(9) $$\frac{1}{L} \sum_{i=jL+1}^{jL+L} Y(v_i) \leq a \quad \text{for all but finitely many } j.$$

From (9) we infer that

$$\sum_{i=jL+1}^{jL+L} X(v_i) \leq aLf(jL + 1)^{-1} \leq a \sum_{i=jL-L+1}^{jL} f(i)^{-1}$$

for all but finitely many j, and hence

$$S(v_n) \leq a \sum_{i=1}^{n} f(i)^{-1} + O(1).$$

Since $a = (1 - 3\varepsilon)^{-1}e^{-1}$, letting $\varepsilon \to 0$ completes the proof. $\square$

4. Exponential transit times, arbitrary growth functions.

PROOF OF THEOREM 4. Consider first the case $\Sigma \tilde{f}(n)^{-1} = \infty$. If

$$\liminf_{n \to \infty} \left(\prod_{i=1}^{n} f(i) \right)^{1/n} < \infty,$$

then clearly there is no explosion, since utilizing the gamma distribution shows that $P(M_n < C) \le (C^n/n!)\prod_{i=1}^{n} f(i)$. Henceforth we assume that

$$\liminf_{n \to \infty} \left(\prod_{i=1}^{n} f(i) \right)^{1/n} = \infty,$$

which is equivalent to $\tilde{f}(n) \to \infty$. Under this assumption, for every $n \ge 1$ there exists an $m \ge 1$ such that

$$\tilde{f}(n+1)^m \prod_{i=1}^{n} \tilde{f}(i) = \prod_{i=1}^{n+m} f(i).$$

Consequently, there is an increasing sequence $\{n(k) : k = 1, 2, 3, \ldots\}$ for which

$$\prod_{i=1}^{n(k)} \tilde{f}(i) = \prod_{i=1}^{n(k)} f(i),$$

for all k.

To establish the lower bound on $\limsup M_n / \Sigma_{i=1}^{n} \tilde{f}(i)^{-1}$, and hence also that no explosion occurs, use the proof of Theorem 3 but with $Y(v) = \tilde{f}(|v|)X(v)$. As in the proof of that theorem, the partial sums $S^*(v) = \Sigma_{\rho < w \le v} Y(w)$ satisfy

$$\mathbf{E}[e^{-\lambda S^*(v)}] \le \lambda^{-|v|} \prod_{i=1}^{|v|} \tilde{f}(i),$$

for $\lambda > 0$. When $|v| = n(k)$, we can substitute f for $\tilde{f}$ on the right-hand side. It follows for $\varepsilon > 0$, that with probability 1 the inequality

$$M_{n(k)}^* \le \frac{n(k)}{\lambda} \ln((1-\varepsilon)\lambda)$$

holds for only finitely many k. Taking $\lambda = e$ we get, as before,

$$\liminf_{k \to \infty} M_{n(k)} \bigg/ \sum_{i=1}^{n(k)} \tilde{f}(i)^{-1} \ge e^{-1};$$

so, in particular,

$$\limsup_{n \to \infty} M_n \bigg/ \sum_{i=1}^{n} \tilde{f}(i)^{-1} \ge e^{-1}.$$

Next, let us bound this lim sup from above, still assuming that $\Sigma \tilde{f}(n)^{-1} = \infty$ and $\tilde{f}(n) \to \infty$. Let $g(n)$ denote the integer part of $\tilde{f}(n)$, let Γ' be a spherically

symmetric tree with growth function g, and let $\{X(w)\}$ be independent, identically distributed random variables. Apply Theorem 3 to Γ'. Examining the proof at (8), we see that for any $\varepsilon > 0$ there exists almost surely some path $\rho, v_1, v_2, \ldots$ satisfying

$$(10) \qquad\qquad X(v_n) \leq (e^{-1} + \varepsilon) \sum_{i=1}^{n} g(i)^{-1},$$

for all but finitely many n. From Theorem 1 it follows that a path satisfying (10) exists almost surely in Γ as well as Γ'. Since $\sum_{i=1}^{n} g(i)^{-1}/\sum_{i=1}^{n} \tilde{f}(i)^{-1} \to 1$, this implies that almost surely

$$\limsup_{n \to \infty} M_n \bigg/ \sum_{i=1}^{n} \tilde{f}(i)^{-1} \leq e^{-1}.$$

Finally, in the case when $\sum \tilde{f}(n)^{-1}$ converges, define g and Γ' as above and apply Theorem 3 to conclude that, for some fixed $L > 0$,

$$\liminf_{n \to \infty} P\big[\exists\ w \in \Gamma'_n\colon S'(w) \leq L\big] > 0.$$

This together with Theorem 1 and the zero–one law for explosions proves that the probability of an explosion is 1. $\square$

5. Other transit-time distributions. Theorems 3 and 4 hold without change when the i.i.d. transit times $X(v)$ have any distribution function G for which $G(t)/t = 1 + o(1)$ as $t \to 0$. More generally, we have the following.

PROPOSITION 7. *Let Γ be an infinite spherically symmetric tree with growth function f. Suppose $\{X(u)\colon v \in \Gamma\}$ are i.i.d., with their common distribution function G satisfying*

$$\lim_{t \to 0} G(t)t^{-\alpha} = c > 0,$$

for some $\alpha > 0$. With $S(v)$ and M_n defined as throughout, we then have the following analogues of Theorems 3 and 4:

(i) *If f is nondecreasing, then there is an explosion ($\sup M_n < \infty$) if and only if $\sum f(n)^{-1/\alpha}$ converges. If the sum diverges, then*

$$\lim_{n \to \infty} M_n \bigg/ \sum_{j=1}^{n} f(j)^{-1/\alpha} = \alpha e^{-1}[c\Gamma(1 + \alpha)]^{-1/\alpha}.$$

(ii) *For a general growth function f, define $\tilde{f}$ as in the preface to Theorem 4. There is almost surely an explosion if and only if $\sum \tilde{f}(n)^{-1/\alpha}$ converges. If the sum diverges, then, with probability 1,*

$$\limsup_{n \to \infty} M_n \bigg/ \sum_{j=1}^{n} \tilde{f}(j)^{-1/\alpha} = \alpha e^{-1}[c\Gamma(1 + \alpha)]^{-1/\alpha}.$$

PROOF. We only discuss the modifications needed in the proof for the exponential case. For (i), first assume that $\Sigma f(n)^{-1/\alpha}$ converges. For every vertex $v \in \Gamma_{n-1}$, the number of its children $w \in \Gamma_n$ for which

$$(11) \qquad X(w) \le \left[\frac{c}{2}f(n)\right]^{-1/\alpha}$$

converges in distribution to a Poisson with mean 2. Comparison with a Galton–Watson process shows that (11) holds along an entire infinite path with positive probability, and this implies almost sure explosion.

Next, assume that $\Sigma f(n)^{-1/\alpha} = \infty$ and that $f(n) \to \infty$. Given $\varepsilon > 0$, choose $\delta > 0$ so that $G(t) < (c + \varepsilon)t^\alpha$ for $0 < t < \delta$. Estimate the moment generating function

$$
\begin{aligned}
\mathbf{E}[e^{-\lambda X(v)}] &= \int_0^\infty e^{-\lambda t}\, dG(t) \\[4pt]
&= \int_0^\infty G(t)\lambda e^{-\lambda t}\, dt \\[4pt]
&\le \int_0^\delta (c + \varepsilon)t^\alpha \lambda e^{-\lambda t}\, dt + e^{-\lambda\delta} \\[4pt]
&\le (c + \varepsilon)\lambda^{-\alpha}\Gamma(1 + \alpha) + e^{-\lambda\delta} \\[4pt]
&\le (c + 2\varepsilon)\lambda^{-\alpha}\Gamma(1 + \alpha),
\end{aligned}
$$

for large positive λ. Define $S^*(v) = \Sigma_{\rho < w \le v} f(|w|)^{1/\alpha}X(w)$ and $M_n^* = \min_{|v|=n} S^*(v)$. Since $f(n) \to \infty$ it follows that

$$
E[\exp(-\lambda S^*(v))] \le A(\lambda)\prod_{j=1}^{|v|} (c + 2\varepsilon)\lambda^{-\alpha}f(j)^{-1}\Gamma(1 + \alpha),
$$

where $A(\lambda) > 0$, and hence

$$
E\left[\sum_{|v|=n} \exp(-\lambda S^*(v))\right] \le A(\lambda)\big[(c + 2\varepsilon)\lambda^{-\alpha}\Gamma(1 + \alpha)\big]^n.
$$

As in the proof of Theorem 3, optimizing over λ yields

$$
\liminf_{n\to\infty} \frac{1}{n}M_n^* \ge e^{-1}\alpha\big[c\Gamma(1 + \alpha)\big]^{-1/\alpha};
$$

consequently, with probability 1,

$$
\liminf_{n\to\infty} M_n \bigg/ \sum_{j=1}^n f(j)^{-1/\alpha} = \alpha e^{-1}\big[c\Gamma(1 + \alpha)\big]^{-1/\alpha}
$$

The rest of the proof proceeds as in Theorem 3.

Finally, the only change in the proof of (ii) is to note that the operation $f \mapsto \tilde{f}$ commutes with taking powers. $\square$

QUESTION. Is there a simple explosion criterion for arbitrary transit time distributions? At least when the growth function f is nondecreasing and G is

strictly monotone and continuous, it seems possible that an explosion occurs if and only if

$$(12) \qquad \sum_{n=1}^{\infty} G^{-1}\left(\frac{1}{f(n)}\right) < \infty,$$

where G^{-1} is the inverse function to G. The technique used to obtain Proposition 7 is powerful enough to verify this criterion for a slightly more general class of distributions, but the general case has eluded us.

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