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DISTINGUISHING A SEQUENCE OF RANDOM VARIABLES FROM A RANDOM TRANSLATE OF ITSELF

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Let $\mathbf{X} = \{X_k\}$ be an i.i.d. real random sequence, let $\varepsilon = \{\varepsilon_k\}$ be a Rademacher sequence independent of $\mathbf{X}$ and let $\mathbf{a} = \{a_k\}$ be a deterministic real sequence. The aim of this paper is to prove that the mutual absolute continuity of probability measures induced by $\{X_k\}$ and $\{X_k + a_k\varepsilon_k\}$ implies $\mathbf{a} \in \ell_4$. This is a generalization of a result of Shepp.

1. Introduction. Let $\mathbf{X} = \{X_k\}$ be an i.i.d. real random sequence, let $\mathbf{Y} = \{Y_k\}$ be an independent random sequence which is also independent of $\mathbf{X}$ and let $\mu_\mathbf{X}$ and $\mu_{\mathbf{X+Y}}$ be the probability measure on the sequence space $\mathbf{R}^\mathbf{N}$ induced by $\mathbf{X}$ and $\mathbf{X} + \mathbf{Y} = \{X_k + Y_k\}$, respectively. Then by Kakutani's dichotomy theorem [1] $\mu_{\mathbf{X+Y}}$ and $\mu_\mathbf{X}$ are either mutually absolutely continuous (denoted by $\mu_{\mathbf{X+Y}} \sim \mu_\mathbf{X}$) or singular (denoted by $\mu_{\mathbf{X+Y}} \perp \mu_\mathbf{X}$).

Denote the distribution of X_1 by λ and the density of λ by f if it exists. Furthermore, if $f(x)$ is an absolutely continuous function, denote the Radon–Nikodym derivative of $f(x)$ by $f'(x)$ and define

$$\mathbf{I}_1(f) = \int_{-\infty}^{+\infty} \frac{f'(x)^2}{f(x)}\, dx,$$

and if $f'(x)$ is an absolutely continuous function, denote the Radon–Nikodym derivative of $f'(x)$ by $f''(x)$ and define

$$\mathbf{I}_2(f) = \int_{-\infty}^{+\infty} \frac{f''(x)^2}{f(x)}\, dx.$$

When $\mathbf{Y}$ is a deterministic sequence $\mathbf{y} = \{y_k\}$, Shepp [6] proved the following theorem, which has many applications.

THEOREM 1.

(i) $\mu_{\mathbf{X+y}} \sim \mu_\mathbf{X}$ *implies* $\mathbf{y} \in \ell_2$.
(ii) *if* $\mathbf{I}_1(f) < +\infty$, *then* $\mathbf{y} \in \ell_2$ *implies* $\mu_{\mathbf{X+y}} \sim \mu_\mathbf{X}$.

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(iii) *If* $\mu_{\mathbf{X}+\mathbf{y}} \sim \mu_{\mathbf{X}}$ *for every* $\mathbf{y} \in \ell_2$, *then* $\mathbf{I}_1(f) < +\infty$.

Sato and Watari [5] proved the following theorem.

THEOREM 2. *If* $\mathbf{I}_2(f) < +\infty$ *and* $\mathbf{Y}$ *is symmetric, then* $\mathbf{Y} \in \ell_4$ *a.s. implies* $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$.

In this paper we investigate the case $\mathbf{Y} = \mathbf{a}\varepsilon = \{a_k\varepsilon_k\}$, where $\mathbf{a} = \{a_k\}$ is a real sequence and $\varepsilon = \{\varepsilon_k\}$ is a Rademacher sequence. Okazaki [3] proved the following theorem.

THEOREM 3. *If* $\mu_{\mathbf{X}+\mathbf{a}\varepsilon} \sim \mu_{\mathbf{X}}$ *for every* $\mathbf{a} \in \ell_4$, *then* $\mathbf{I}_2(f) < +\infty$.

The aim of this paper is to prove the following theorem, which was proved by Sato [4] under the additional assumption that a twice continuously differentiable f exists.

THEOREM 4. $\mu_{\mathbf{X}+\mathbf{a}\varepsilon} \sim \mu_{\mathbf{X}}$ *implies* $\mathbf{a} \in \ell_4$.

The proof, based on [2], is given in Section 2. These Theorems 2, 3 and 4 complete a generalization of Theorem 1 to the case $\mathbf{Y} = \mathbf{a}\varepsilon$ and then a natural question arises. If $\mathbf{Y}$ is symmetric and $\lim_k \mathbf{Y}_k = 0$ a.s., then does $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ imply $\mathbf{Y} \in \ell_4$ a.s.? The answer is no, and in Section 3 we shall give a counterexample.

2. Proof of Theorem 4. Assume $\mu_{\mathbf{X}+\mathbf{a}\varepsilon} \sim \mu_{\mathbf{X}}$. Then for every $k \in \mathbf{N}$ we have $\mu_{X_k+a_k\varepsilon_k} \sim \mu_{X_k}$, where $\mu_{X_k+a_k\varepsilon_k}$ and μ_{X_k} are the distribution of $X_k + a_k\varepsilon_k$ and X_k, respectively. Without loss of generality, we may assume $a_k \geq 0$ for every $k \in \mathbf{N}$. Define

$$p_k(x) = \frac{d\mu_{X_k+a_k\varepsilon_k}}{d\mu_{X_k}}(x)$$

and set $A_k = \{x \in \mathbf{R}: p_k(x) - 1 < 1\}$. Then by Theorem 2 of Kitada and Sato [2] we have

$$\text{(K.1)} \qquad \sum_k \int_{A_k^C} \big(p_k(x) - 1\big)\, d\lambda(x) < +\infty,$$

$$\text{(K.2)} \qquad \sum_k \int_{A_k} \big(p_k(x) - 1\big)^2\, d\lambda(x) < +\infty.$$

LEMMA 1. $\mathbf{a} \in \ell_\infty$.

PROOF. Assume $\limsup_k a_k = +\infty$. Then there exists a subsequence $\{a_{k(n)}\}$ such that

$$\sum_k \Big(1 - \lambda\Big(\big[-\tfrac{1}{2}a_{k(n)}, \tfrac{1}{2}a_{k(n)}\big]\Big)\Big) < +\infty;$$

define $\Gamma = \{\mathbf{x} = \{x_k\} \in \mathbf{R}^{\mathbf{N}}: 2|x_{k(n)}| \le a_{k(n)}, n \in \mathbf{N}\}$. Then we have $\mu_{\mathbf{X}}(\Gamma) > 0$, but

$$\mu_{\mathbf{X}+\mathbf{a}\varepsilon}(\Gamma) = \prod_n \tfrac{1}{2}\Big\{\lambda\Big(\big[-\tfrac{1}{2}a_{k(n)}, \tfrac{1}{2}a_{k(n)}\big] + a_{k(n)}\Big) + \lambda\Big(\big[-\tfrac{1}{2}a_{k(n)}, \tfrac{1}{2}a_{k(n)}, \big] - a_{k(n)}\Big)\Big\} = 0,$$

which is a contradiction. □

LEMMA 2. $\lim_k a_k = 0$.

PROOF. Assume that $\limsup_k a_k = b > 0$. Then there exists a subsequence $\{a_{k(n)}\}$ such that $b = \lim_n a_{k(n)}$. On the other hand (K.1) and (K.2) imply

$$\lim_k \int_{-\infty}^{+\infty} \big|p_k(x) - 1\big| d\lambda(x) = 0.$$

Therefore for every $t \in \mathbf{R}$ we have

$$\begin{aligned} 0 &= \lim_k \int_{-\infty}^{+\infty} (p_k(x) - 1)e^{itx} d\lambda(x) \\ &= \lim_k \left(\int_{+\infty}^{+\infty} e^{itx} d\mu_{X_k + a_k\varepsilon_k}(x) - \int_{-\infty}^{+\infty} e^{itx} d\mu_{X_k}(x) \right) \\ &= \lim_k (\cos ta_k - 1)\widetilde{\lambda}(t), \end{aligned}$$

where $\widetilde{\lambda}(t)$ is the characteristic function of λ, so that

$$(\cos tb - 1)\widetilde{\lambda}(t) = \lim_n (\cos ta_{k(n)} - 1)\widetilde{\lambda}(t) = 0, \qquad t \in \mathbf{R},$$

which is a contradiction. □

PROOF OF THEOREM 4. Inequalities (K.1) and (K.2) imply

$$\sum_k \left\{ \left(\int_{A_k^C} (p_k(x) - 1)\, d\lambda(x) \right)^2 + \int_{A_k} (p_k(x) - 1)^2 d\lambda(x) \right\} < +\infty.$$

Therefore we have $\sum_k (a_k)^4 B_k < +\infty$, where

$$B_k = \left(\int_{A_k^C} \frac{p_k(x) - 1}{(a_k)^2} d\lambda(x) \right)^2 + \int_{A_k} \left(\frac{p_k(x) - 1}{(a_k)^2} \right)^2 d\lambda(x).$$

We shall show $\liminf_k B_k > 0$. Assume that $\liminf_k B_k = 0$. Then there exists a subsequence $\{B_{k(n)}\}$ such that $\lim_n B_{k(n)} = 0$. Then we have

$$\lim_n \left\{ \left(\int_{A_{k(n)}^C} \frac{p_{k(n)}(x) - 1}{(a_{k(n)})^2} d\lambda(x) \right)^2 + \int_{A_{k(n)}} \left(\frac{p_{k(n)}(x) - 1}{(a_{k(n)})^2} \right)^2 d\lambda(x) \right\} = 0,$$

so that

$$D_n(x) = \frac{1}{a_{k(n)})^2}\left(p_{k(n)}(x) - 1\right)$$

converges to 0 in $\mathbf{L}_1(d\lambda)$. For every infinitely differentiable function φ with compact support we have

$$\begin{aligned}
\tfrac{1}{2}\int_{-\infty}^{+\infty} &\left(\varphi(x + a_{k(n)}) + \varphi(x - a_{k(n)})\right) d\lambda(x) \\
&= \int_{-\infty}^{+\infty} \varphi(x)p_{k(n)}(x)\, d\lambda(x) \\
&= \int_{-\infty}^{+\infty} \varphi(x)\left(1 + (a_{k(n)})^2 D_n(x)\right) d\lambda(x),
\end{aligned}$$

so that

$$\begin{aligned}
0 &= \lim_n \int_{-\infty}^{+\infty} \varphi(x)D_n(x)\, d\lambda(x) \\
&= \lim_n \int_{-\infty}^{+\infty} \frac{1}{2(a_{k(n)})^2}\left(\varphi(x + a_{k(n)}) + \varphi(x - a_{k(n)}) - 2\varphi(x)\right) d\lambda(x) \\
&= \int_{-\infty}^{+\infty} \varphi''(x)\, d\lambda(x).
\end{aligned}$$

Therefore the second derivative in the distribution sense of λ vanishes and λ is a linear function, which is a contradiction. □

3. A counterexample. Let $\mathbf{X} = \{X_k\}$ be a standard Gaussian sequence, and define an independent random sequence $\mathbf{Y} = \{Y_k\}$ independent of $\mathbf{X}$ by

$$\begin{gathered}
\mathbf{P}\left(Y_k = k^{-1/5}\right) = \mathbf{P}\left(Y_k = -k^{-1/5}\right) = (k + 32)^{-1/5}, \\
\mathbf{P}(Y_k = 0) = 1 - 2(k + 32)^{-1/5}, \qquad k \in \mathbf{N}.
\end{gathered}$$

Then, obviously, $\mathbf{Y}$ is symmetric and $\lim_k Y_k = 0$ a.s., and since

$$\sum_k \mathbf{E}\left[(Y_k)^2 \colon |Y_k| \le 1\right]^2 = 2\sum_k k^{-1/5}(k + 32)^{-3/5} < +\infty,$$

by Theorem 9 of [2] we have $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$. However, since

$$\sum_k \mathbf{E}\left[(Y_k)^4 \colon |Y_k| \le 1\right] = 2\sum_k k^{-4/5}(k + 32)^{-1/5} = +\infty,$$

we have, by the Kolmogorov three series theorem,

$$\sum_k (Y_k)^4 = +\infty \quad \text{a.s.}$$

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