The Annals of Probability
1994, Vol. 22, No. 2, 803–819

A NOTE ON INHOMOGENEOUS PERCOLATION

BY YU ZHANG

University of Colorado

Consider a special independent bond percolation model on Z^2, in which all bonds with vertices in the X axis are open with probability δ and closed with probability $1-\delta$, and all other bonds are open with probability p and closed with probability $1-p$. In this paper we show that no percolation occurs at p_c for any $\delta < 1$. The method allows us also to show no percolation at p_c in a more general inhomogeneous case.

1. Introduction and statement of results. Consider a general bond percolation model on Z^d. In the case of bond percolation each pair of neighboring sites in Z^d is thought of as defining a bond. Each bond $\{x,y\}$ is open with probability $p_{\{x,y\}}$ and closed with probability $1-p_{\{x,y\}}$ independently from bond to bond. An open (closed) path is a nearest-neighbor path on Z^d, all of whose bonds are open (closed). Write $\mathcal{C}$ for the set of vertices connected to the origin by open paths. For any collection A of vertices, $|A|$ denotes the cardinality of A. We shall say that percolation occurs if $|\mathcal{C}| = \infty$ with a positive probability. If $p_{\{x,y\}}$ is identical to a unique parameter p for all x and y, the corresponding percolation model is called the homogeneous percolation model and the corresponding probability measure on the configurations of open and closed bonds is denoted by P_p. Denote by

$$\theta(p) = P_p(|\mathcal{C}| = \infty)$$

the percolation probability. It is well known that there exists a critical point $0 < p_c < 1$ satisfying

$$\theta(p) = 0 \quad \text{if } p < p_c, \qquad (1.1)$$
$$\theta(p) > 0 \quad \text{if } p > p_c.$$

Later, Chayes, Chayes and Durrett [3] considered an inhomogeneous site percolation model. They use the inhomogeneous density $p_c + f(x)$ instead of p for each site $x \in Z^2$ and showed that whether or not percolation can occur depends on the function $f(x)$. After that, Campanino and Klein [2] and Madras, Schinazi and Schonmann [9] considered another kind of inhomogeneous model. For a fixed $x \in Z^{d-1}$, one can think of a percolation model (in dimension d) for which $p_{\{y,z\}} = p_x$ if $y = (v,x)$ and $z = (v+1,x)$, where $v \in Z$, and $p_{\{y,z\}} = p$ when y and z are neighbors but not of the preceding form. The corresponding probability measure on the configurations of open and closed bonds is denoted

Received June 1992.
AMS 1991 *subject classification.* 60K35.
Key words and phrases. Inhomogeneous percolation.

by P_{p,p_x}. Clearly, $P_{p,p_x} = P_p$ if and only if $p_x = p$. For simplicity of exposition, we shall assume that $x = 0$. Clearly, if $p_0 = 1$, percolation occurs for all p. Also, when $p_0 < 1$, it was proved in [9] and [1] that percolation cannot occur when $p < p_c$ and that percolation occurs when $p > p_c$. The question asked in [9] now is whether or not percolation occurs at p_c for $p_c < p_0 < 1$. A similar question was also asked in [9] for the contact process. When $2 < d \leq 47$, we even do not know whether percolation occurs or not in the homogeneous model at p_c. Hence it is likely to be more difficult to find some results in inhomogeneous models at the critical point when $d > 2$. When $d = 2$, by using some remarkable techniques in considering the dual lattice on Z^2, Harris [4] showed that no percolation occurs at $p \leq \frac{1}{2}$ for homogeneous percolation. Later, Kesten [5] showed that $p_c = \frac{1}{2}$ based on the work of Russo [10] and Seymour and Welsh [11]. Indeed, Russo, Seymour and Welsh (RSW) provided a principal tool for planar graphs. Define a left–right (respectively, top–bottom) open crossing of a rectangle B to be an open path in B which joins some vertex on the left (respectively, upper) side of B to some vertex on the right (respectively, lower) side of B but which uses no bond joining two vertices in the boundary of B. Similarly, we can define a left–right (respectively, top–bottom) closed crossing of a rectangle. Here we state their result for the Z^2 lattice:

RSW LEMMA. *If $p = \frac{1}{2}$, for any integer $k \geq 0$, there exists a constant $C(k) > 0$ such that*

$$\begin{aligned} &P_{1/2}\big(\exists \text{ a left–right open crossing in } [-kn, kn] \times [-n, n]\big) \\ &\quad = P_{1/2}\big(\exists \text{ a left–right closed crossing in } [-kn, kn] \times [-n, n]\big) \\ &\quad \geq C(k), \end{aligned} \tag{1.2}$$

for all integers $n > 0$.

To introduce the ideas behind the proof that there is no percolation at $p_c = \frac{1}{2}$, we need the knowledge of duality. We define Z^* as the dual graph of Z^2 with vertex set $\{v + (\frac{1}{2}, \frac{1}{2})\}$ and edges joining all pairs of vertices which are one unit apart. For any bond set $A \subset Z^2$, we write $A^* \subset Z^*$ for the corresponding dual bonds of A. For each bond $e^* \in Z^*$, we declare that $e^* \in Z^*$ is open or closed if e is open or closed. In other words, each open (closed) e^* crosses a corresponding open (closed) bond in Z^2. With this definition, we can obtain (see [6] for more details) that if there exists a closed circuit surrounding the origin in Z^*, then percolation cannot occur in Z^2. Therefore, to show no percolation occurs, the main work is to construct a closed circuit surrounding the origin in some finite set of Z^2. More precisely, set

$$A(n) = [-2n, 2n]^2 \backslash [-n, n]^2.$$

In a celebrated proof by Harris it was shown (see [4]) that for homogeneous percolation at p_c there exists a closed circuit in $A(n)$ with a probability bounded away from zero uniformly. Clearly, we do not care how many times the closed circuit crosses the X axis in the homogeneous model. However, if we want to use Harris' proof for inhomogeneous percolation, we have to restrict the number of intersections of the closed circuit and the X axis to be finite uniformly. Therefore, the difficulty is to construct a special closed circuit (for homogeneous model) in $A(n)$ which only intersects the X axis finitely many times with a positive probability. Fortunately, building on the work of [6], [8] and [12], we can demonstrate the following theorem.

THEOREM 1. *There exists a constant $\rho > 0$ which is independent of n such that*

$$P_{p_c}\big(\exists \text{ a closed circuit in } A(n) \text{ which only intersects the } X \text{ axis twice}\big) > \rho,$$

for all integers $n > 0$ when $d = 2$.

By using Theorem 1 and Harris' argument, we have the following corollary.

COROLLARY 1. *For any $p_0 < 1$, there is no percolation at $p = p_c$ when $d = 2$.*

REMARK 1. It is not very difficult to adapt the proof of Theorem 1 to show no percolation at p_c for $d = 2$ if the densities of bonds in both the X axis and the Y axis are changed.

REMARK 2. A consequence of the proof of Theorem 1 is the following refined version of the RSW lemma:

$$P_{p_c}\big(\exists \text{ a left–right open crossing in } [-kn, kn] \times [-n, n] \text{ which only intersects the } Y \text{ axis once}\big) > C(k),$$

for all integers $n > 0$, where k is a positive integer and $C(k)$ is a positive constant which only depends on k.

REMARK 3. Van den Berg and Kesten [12] proved the following result:

$$P_{p_c}\big(\text{the origin is connected by an open path to the boundary of } [-n, n]^2\big) \geq Cn^{-1/2}. \tag{1.3}$$

Later, Kesten [7] improved the lower boundary of (1.3) to $Cn^{-1/3}$. Here, by our Lemma 4, we can obtain a slightly stronger result than (1.3) by a different approach as follows:

$$P_{p_c}\big(\text{the origin is connected to the boundary of } [-n, n]^2 \text{ by an open path with bonds in } [-n, n] \times (0, n)\big) \geq Cn^{-1/2}.$$

REMARK 4. It can be seen that the symmetry of the Z^2 lattice plays an important role in the proof of Theorem 1. However, if we consider the contact process or the oriented percolation, we cannot take advantage of symmetry as much as we did in the Z^2 lattice. Therefore, the corresponding question for the contact process or the oriented percolation is still open.

2. Proofs.

PROOF OF COROLLARY 1 FROM THEOREM 1. Let

$$S(n) = A(n) + \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \quad \text{and} \quad \{y \in A\} = \{(x,y): x \in (-\infty, \infty),\ y \in A\}.$$

Define each bond in $S(n)$ to be open with a homogeneous density p_c. It follows from Theorem 1 that there is a closed circuit in $S(n)$ which only intersects $\{y = \frac{1}{2}\}$ twice with probability ρ. We select such a closed circuit and denote by v_1 and v_2 the two intersection vertices. Then the existence of such a closed circuit can also be written as the following event:

$$\begin{aligned} E_n = \Big\{ &\exists \text{ two vertices } v_1 \in \left[-2n + \tfrac{1}{2}, -n + \tfrac{1}{2}\right] \times \left\{\tfrac{1}{2}\right\} \text{ and} \\ &v_2 \in \left[n + \tfrac{1}{2}, 2n + \tfrac{1}{2}\right] \times \left\{\tfrac{1}{2}\right\} \text{ such that } v_1 \text{ and } v_2 \text{ are connected by} \\ &\text{two closed paths } \Gamma_1 \text{ and } \Gamma_2 \text{ on } Z^* \text{ with } \Gamma_1 \subset S(n) \cap \left\{y > \tfrac{1}{2}\right\} \text{ and} \\ &\Gamma_2 \subset S(n) \cap \left\{y < \tfrac{1}{2}\right\} \text{ except } v_1 \text{ and } v_2 \Big\}. \end{aligned}$$

Denote

$$\begin{aligned} \overline{S}(n) = &\left[-2n + \tfrac{1}{2}, 2n + \tfrac{1}{2}\right] \\ &\times \left[-2n + \tfrac{1}{2}, 2n + \tfrac{1}{2}\right] \setminus \left[-n + \tfrac{1}{2}, n + \tfrac{1}{2}\right] \times \left[-n + \tfrac{1}{2}, n + \tfrac{1}{2}\right]. \end{aligned}$$

Now we define the bonds in $\overline{S}(n) \cap \{y = 0\}^*$ to be open with a density p_0 and the other bonds in $\overline{S}(n)$ to be open with a density p_c. Define

$$\begin{aligned} W_n = \Big\{ &\exists \text{ two closed bonds } \left\{\left(a, -\tfrac{1}{2}\right), \left(a, \tfrac{1}{2}\right)\right\} \text{ and } \left\{\left(b, -\tfrac{1}{2}\right), \left(b, \tfrac{1}{2}\right)\right\} \text{ on} \\ &\overline{S}(n) \cap \{y = 0\}^* \text{ with } a \in \left[-2n + \tfrac{1}{2}, -n + \tfrac{1}{2}\right] \text{ and } b \in \left[n + \tfrac{1}{2}, 2n + \tfrac{1}{2}\right] \\ &\text{such that there exist two closed paths } \Gamma_1 \text{ and } \Gamma_2 \text{ with bonds on} \\ &\overline{S}(n) \cap \left\{y > \tfrac{1}{2}\right\} \text{ and } \overline{S}(n) \cap \left\{y < -\tfrac{1}{2}\right\} \text{ which connect} \\ &\left(a, \tfrac{1}{2}\right) \text{ to } \left(b, \tfrac{1}{2}\right) \text{ and } \left(a, -\tfrac{1}{2}\right) \text{ to } \left(b, -\tfrac{1}{2}\right), \text{respectively} \Big\}. \end{aligned}$$

Clearly, by Theorem 1,

$$P_{p_c,p_0}(W_n) = P_{p_c}(E_n)(1 - p_0)^2 \geq \rho(1 - p_0)^2.$$

In addition, there exists a closed circuit on $\overline{S}(n)$ if W_n occurs. Therefore,

$$\begin{aligned} P_{p_c,p_0}\left(|\mathcal{C}| = \infty\right) &\leq \prod_{n=1}^{\infty} P_{p_c,p_0}\left(\text{no closed circuit in } \overline{S}(n)\right) \\ &\leq \prod_{n=1}^{\infty} \left(1 - (1 - p_0)^2 \rho\right) = 0. \end{aligned}$$

Hence Corollary 1 is proved. □

We now turn to the proof of Theorem 1. Note that Theorem 1 only involves the critical case for the homogeneous percolation. We shall always consider the homogeneous percolation from now on, and we abbreviate the measure P_{p_c} to P. In addition, throughout the proof of Theorem 1, C or C_i will always stand for a strictly positive finite constant, whose value is of no significance to us. In fact the value of C or C_i may change from appearance to appearance. Before the proof of Theorem 1, we first give some lemmas.

LEMMA 1. *There exist constants $\varepsilon > 0$ and C such that*

$$P\left(\exists \text{ two disjoint closed paths on } [-n,n]\times[0,n] \text{ from } [-k,k]\times\{0\} \text{ to } B_n\right)$$
$$\le C\left(\frac{n}{k}\right)^{-\varepsilon} P^2\left(\exists \text{ a closed path on } [-n,n]\times[0,n] \text{ from } [-k,k]\times\{0\} \text{ to } B_n\right)$$

for any integers $n \ge k > 0$, where $B_n = ([-n,n]\times\{n\})\cup(\{-n\}\times[0,n])\cup(\{n\}\times[0,n])$.

Before the proof of our Lemma 1, we need an inequality for disjoint occurrence of two events. More precisely, let

$$\Omega = \Omega' = \{0,1\}^{\tau},$$

for some bond subset τ on Z^2. A typical point w (w') of Ω (Ω') is a sequence $\{w(e)\}_{e\in\tau}(\{w'(e)\}_{e\in\tau})$. The value $w(e) = 1$ (0) corresponds to e being open (closed). The measure P is the product measure on Ω with

$$P\left(w(e)=1\right) = \tfrac{1}{2} = P\left(w(e)=0\right) \quad \text{for } e\in\tau.$$

We define P' in the same way as the product measure on Ω' with

$$P'\left(w'(e)=1\right) = \tfrac{1}{2} = P'\left(w'(e)=0\right) \quad \text{for } e\in\tau.$$

For any event $B\subset\Omega$ write B' for its copy in Ω', that is,

$$B' = \{w'\in\Omega' \colon \exists\, w\in B \text{ such that } w'(e)=w(e) \text{ for all } e\in\tau\}. \tag{2.1}$$

For a fixed $\overline{w}\subset\Omega$ and $K\subset\tau$, $[\overline{w}]_K$ denotes the cylinder

$$[\overline{w}]_K = \{w\in\Omega\colon w(e)=\overline{w}(e),\ e\in K\}.$$

Similarly, for fixed $\overline{w}'\in\Omega'$,

$$[\overline{w}']_K = \{w'\in\Omega'\colon w'(e)=\overline{w}'(e),\ e\in K\}.$$

We say that two events A and B occur disjointly if $A\circ B$ occurs, where

$$A\circ B := \{w\in\Omega\colon \exists\, K,L\subset\tau \text{ such that } K\cap L=\emptyset \text{ and } [w]_K\subset A,\ [w]_L\subset B\}. \tag{2.2}$$

This terminology should be reasonably intuitive; we interpret $[w]_K\subset A$ as A occurs because of the coordinates of w in K; $A\circ B$ then is the event that A and

B occur because of the disjoint sets of coordinates. We define in a similar way for two events A in Ω and B' in Ω' the event $A \circ B'$ in $\Omega \times \Omega'$ by

$$A \circ B' = \{(w, w') \in \Omega \times \Omega: \exists\, K, L \subset \tau \text{ such that } K \cap L = \emptyset \text{ and } [w]_K \subset A,\ [w']_L \subset B'\}. \tag{2.3}$$

With these definitions and interpretations, the following lemma was proved in [8].

LEMMA 2. *If A and B are increasing events of Ω, each depending on finitely many coordinates only, and B' is the copy of B in Ω' as defined in* (2.1), *then*

$$P(A \circ B) \leq P \times P'(A \circ B') \tag{2.4}$$

(*here $P \times P'$ is the product measure of P and P' on $\Omega \times \Omega'$*).

The following inequality of van den Berg and Kesten (BK) is implied by Lemma 2 directly:

$$P(A \circ B) \leq P(A)P(B),$$

for any increasing events A and B.

PROOF OF LEMMA 1 FROM LEMMA 2. We follow the same method of [8], Proposition 1. When $n \leq 2^6 k$, Lemma 1 is implied by the RSW lemma directly. When $n \geq 2^6 k$, we write H_i for the half-square $[-2^i, 2^i] \times [0, 2^i]$. An Ω-closed (Ω'-open) path is a path $(v_0, e_1, \ldots, e_n, v_n)$ on Z^2 with $w(e_i) = 1$ [$w'(e_i) = 1$] for $0 \leq i \leq n$. A half-circuit surrounding H_i in H_j is a path from $\{0\} \times [-2^j, -2^i]$ to $\{0\} \times [2^i, 2^j]$ in $H_j \backslash H_i$. By the RSW lemma and the FKG inequality, we have

$$P(\exists \text{ a closed half-circuit on } Z^2 \text{ surrounding } H_n \text{ in } H_{n+1}) \geq \delta. \tag{2.5}$$

Now let

$$\mathcal{E}_j = \{\exists \text{ two } \Omega\text{-closed half-circuits, one surrounding } H_{3j-1} \text{ in } H_{3j} \text{ and another surrounding } H_{3j+1} \text{ in } H_{3j+2}\} \tag{2.6}$$

for every j, and let

$$J = \{j: k \leq 2^{3j-1} \leq 2^{3j+2} \leq n \text{ and } \mathcal{E}_j \text{ occurs}\}$$

(note that $n \geq 2^6 k$). The events $\mathcal{E}'_j$ and J' are defined in the same way for Ω'-closed circuits, and we further define

$$N = \text{cardinality of } J \cap J'.$$

Our first step will be the easy estimate

$$P \times P'\Big(N \leq C_1 \log \frac{n}{k}\Big) \leq \frac{1}{2}, \tag{2.7}$$

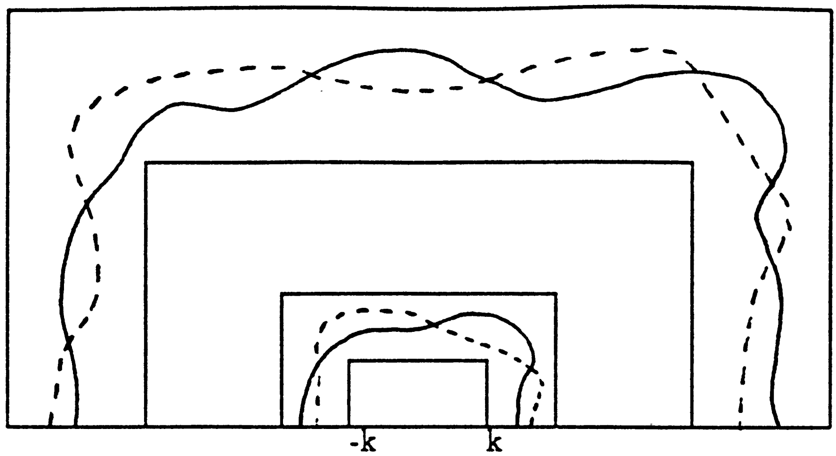

FIG. 1. *Illustration of $\mathcal{E}_j \cap \mathcal{E}_j'$. The solid (dashed) half-circuits are $\mathcal{G}_{3j}$ $\left(\mathcal{G}_{3j}'\right)$ and $\mathcal{D}_{3j+2}$ $\left(\mathcal{D}_{3j+2}'\right)$.*

for some suitable C_1 and n sufficiently large. Actually, (2.7) is easy by virtue of (2.5) and independence of the w and w'. We now take

$$A = B = \{[-k,k] \times \{0\} \text{ is connected to } B_n \text{ in } [-n,n] \times [0,n] \text{ by a closed path}\}, \tag{2.8}$$

for some k and n. By (2.7) and the FKG inequality,

$$P \times P'(A \circ B') \le 2P \times P'\left(A \circ B' \text{ and } N \ge C_1 \log \frac{n}{k}\right). \tag{2.9}$$

We shall estimate (2.9) by conditioning on J and J' and on certain half-circuits in H_{3j} and H_{3j+2}, $j \in J \cap J'$. When $\mathcal{E}_j$ occurs, let $\mathcal{G}_{3j}$ be the innermost Ω-closed half-circuit surrounding H_{3j-1} in H_{3j}, and let $\mathcal{D}_{3j+2}$ be the outermost Ω-closed half-circuit surrounding H_{3j+1} in H_{3j+2}. Define $\mathcal{G}_{3j}'$ and $\mathcal{D}_{3j+2}'$ in a similar way as extremal Ω'-closed half-circuits when $\mathcal{E}_j'$ occurs (see Figure 1). The existence of such innermost half-circuits and outermost half-circuits can be demonstrated by the method of [5], Lemma 1 or [6], Proposition 2.3. For any half-circuit G defined above, G and a segment of the X axis form a Jordan curve $\widehat{G}$. Define

$$G^\circ = \text{interior of } \widehat{G}, \qquad G^e = \text{exterior of } \widehat{G}, \qquad \overline{G} = G^\circ \cup \widehat{G}.$$

It follows from the method of [5], Lemma 1 or [6], Proposition 2.3, that, conditionally on $\mathcal{E}_j$, $\mathcal{G}_{3j}$ and $\mathcal{D}_{3j+2}$, the families

$$\left\{w(b): b \in \mathcal{D}_{3j+2}^\circ \cap \mathcal{G}_{3j}^e\right\} \text{ and } \left\{w(b): b \in \mathcal{G}_{3j}^\circ \cup \mathcal{D}_{3j+2}^e\right\} \cap \left\{w'(b): b \in Z^2\right\} \tag{2.10}$$

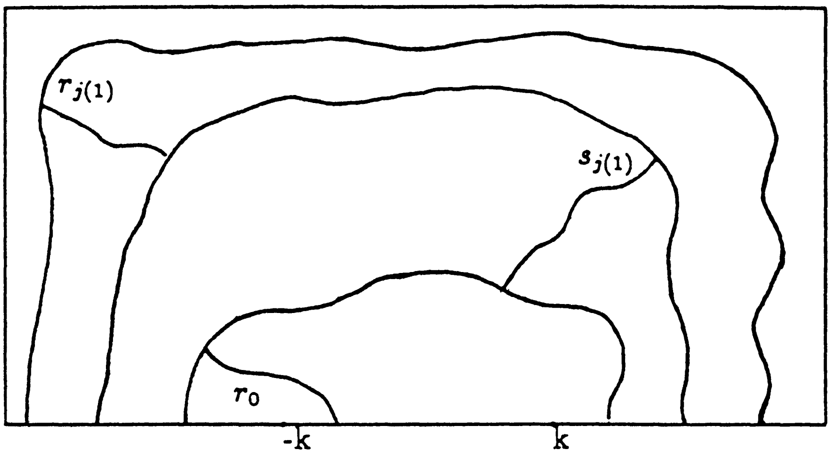

FIG. 2. $s_{j(i)}$ *and* $r_{j(i)}$ *are closed paths which connect* $\mathcal{G}_{3j(i)}$ *and* $\mathcal{D}_{3j(i)+2}$, *and* $\mathcal{G}_{3j(i+1)}$ *and* $\mathcal{D}_{3j(i)+2}$

are independent. Moreover, the conditional distribution of the family in (2.10) is equal to the unconditional distribution P. Now condition on the set of indices J and the closed half-circuits $\mathcal{G}_{3j}$ and $\mathcal{D}_{3j+2}$, $j \in J$. Let J consist of $j(1) < j(2) < \cdots < j(v)$. Then A occurs if and only if there exists the following collection of Ω-closed paths (see Figure 2):

(2.11) an Ω-closed path r_0 from $[-k,k] \times \{0\}$ to $\mathcal{G}_{3j(1)}$ that lies in $\mathcal{G}^{\circ}_{3j(1)}$ except for its endpoint on $\mathcal{G}_{3j(1)}$;

(2.12) an Ω-closed path $s_{j(i)}$ from $\mathcal{G}_{3j(i)}$ to $\mathcal{D}_{3j(i)+2}$ that lies in $\mathcal{G}^{e}_{3j(i)} \cap \mathcal{D}^{\circ}_{3j(i)+2}$ except for its endpoints on $\mathcal{G}_{3j(i)}$ and $\mathcal{D}_{3j(i)+2}$, $i = 1, \dots, v$;

(2.13) an Ω-closed path $r_{j(i)}$ from $\mathcal{D}_{3j(i)+2}$ to $\mathcal{G}_{3j(i+1)}$ that lies in $\mathcal{G}^{\circ}_{3j(i+1)} \cap \mathcal{D}^{e}_{3j(i)+2}$ except for its endpoints on $\mathcal{G}_{3j(i+1)}$ and $\mathcal{D}_{3j(i)+2}$, $i = 1, \dots, v-1$;

(2.14) an Ω-closed path $r_{j(v)}$ from $\mathcal{D}_{3j(v)+2}$ to B_n that lies in $[-n,n] \times [0,n] \cap \mathcal{D}^{e}_{3j(v)+2}$ except for its endpoint on $\mathcal{D}_{3j(v)+2}$.

It is obvious that such Ω-closed paths must exist for A to occur. In the opposite direction, once such paths exist, they can be connected by pieces of the Ω-closed circuits $\mathcal{G}_{3j}$ and $\mathcal{D}_{3j+2}$ to make an Ω-closed path from a vertex in $[-k,k] \times \{0\}$ to B_n (see Figure 2). In exactly the same way, we see that B' occurs if and only if there exist Ω'-closed paths $r'_{j'(i)}$ and $s'_{j'(i)}$ as in (2.11)–(2.14) with $\mathcal{G}$ and $\mathcal{D}$ replaced by $\mathcal{G}'$ and $\mathcal{D}'$ and $j(i)$ by $j'(i)$. For $A \circ B'$ to occur, we must be able to pick the $\{r_{j(i)}, s_{j(i)}\}$ disjoint from the $\{r'_{j(i)}, s'_{j(i)}\}$. We shall only insist on s_j being disjoint from the s'_j when $j \in J \cap J'$. This then leads to the following inequality whenever the cardinality of $J \cap J'$ is at least $C_1 \log(n/k)$:

$$\begin{aligned} &P \times P'\Big(A \circ B' \text{ and } N \geq C_1 \log \frac{n}{k} \Big| J, J', \mathcal{G}_{3j}, \mathcal{D}_{3j+2}, j \in J,\ \mathcal{G}'_{3j'}, \mathcal{D}'_{3j'+2}, j' \in J'\Big) \\ (2.15)\quad &\leq P \times P'\Big(r_0,\ r_j \text{ and } s_j \text{ exist as in (2.11)–(2.14) and their analogues} \\ &\qquad r'_0,\ r'_j \text{ and } s'_j \text{ exist in such a way that } s_j \text{ is disjoint from } s'_j \\ &\qquad \text{when } j \in J \cap J' \Big| J, J', \mathcal{G}_{3j}, \mathcal{D}_{3j+2}, j \in J,\ \mathcal{G}'_{3j'},\ \mathcal{D}'_{3j'+2}, j' \in J'\Big). \end{aligned}$$

When $J \cap J'$ contains fewer than $C_1 \log(n/k)$ indices, then the left-hand side of (2.15) is zero. Then by the independence statement (2.10) and its analogue for primed quantities, the right-hand side of (2.15) can be written as the product of the following factors:

$$(2.16)\quad \begin{aligned} &P\big(\text{every } r \text{ required by (2.11), (2.13) and (2.14)} \\ &\quad \text{exists} \mid J, \mathcal{G}_{3j}, \mathcal{D}_{3j+2},\ j \in J\big); \end{aligned}$$

$$(2.17)\quad \prod_{j \in J \setminus J'} P\big(s_j \text{ exists as required by (2.12)} \mid J, \mathcal{G}_{3j}, \mathcal{D}_{3j+2}\big);$$

$$(2.18)\quad \begin{aligned} &P\big(\text{every } r' \text{ required by the analogues of (2.11), (2.13)} \\ &\quad \text{and (2.14) exists} \mid J', \mathcal{G}'_{3j'}, \mathcal{D}'_{3j'+2}, j' \in J'\big); \end{aligned}$$

$$(2.19)\quad \begin{aligned} \prod_{j' \in J' \setminus J} &P'\big(s'_{j'} \text{ exists as required by the analogue} \\ &\qquad \text{of (2.12)} \mid J', \mathcal{G}'_{3j'}, \mathcal{G}'_{3j'+2}\big); \end{aligned}$$

$$(2.20)\quad \begin{aligned} \prod_{j \in J' \cap J} &P \times P'\big(s_j \text{ and } s'_j \text{ exist and can be chosen} \\ &\qquad \text{disjoint} \mid J, J', \mathcal{G}_{3j}, \mathcal{D}_{3j+2}, \mathcal{G}'_{3j}, \mathcal{D}'{}_{3j+2}\big). \end{aligned}$$

By a straight adaptation of the method of Lemma 4 in [8] we can prove the following estimate for some constant $0 \leq \lambda < 1$:

$$\begin{aligned} &P \times P'\big(s_j \text{ and } s'_j, \text{exist and can be chosen disjoint} \mid J, J', \mathcal{G}_{3j}, \mathcal{D}_{3j+2}, \mathcal{G}'_{3j}, \mathcal{D}'_{3j+2}\big) \\ (2.21)\quad &\leq \lambda P\big(s_j \text{ as required in (2.12) exists} \mid J, \mathcal{G}_{3j}, \mathcal{D}_{3j+2}\big) \\ &\quad \times P'\big(s'_j \text{ as required by the analogue of (2.12) exists} \mid J', \mathcal{G}'_{3j}, \mathcal{D}'_{3j+2}\big). \end{aligned}$$

We then obtain

$$
\begin{aligned}
&P \times P'\Big(A \circ B' \text{ and } N \geq C_1 \log \frac{n}{k} \Big| J, J', \mathcal{G}_{3j}, \mathcal{D}_{3j+2},\\
&\qquad\qquad\qquad\qquad\qquad\qquad j \in J,\ \mathcal{G}'_{3j'}, \mathcal{D}'_{3j'+2}, j' \in J'\Big)\\
&\quad \leq [\text{product of (2.16)–(2.19)}]\lambda^{C_1 \log(n/k)}\\
&\qquad \times \prod_{j \in J \cap J'} P\big(s_j \text{ as required in (2.12) exists} \mid J, \mathcal{G}_{3j}, \mathcal{D}_{3j+2}\big)\\
&\qquad \times \prod_{j \in J \cap J'} P\big(s'_j \text{ as required by the analogue}\\
&\qquad\qquad\qquad \text{of (2.12) exists} \mid J', \mathcal{G}'_{3j}, \mathcal{D}'_{3j+2}\big)\\
&\quad = \lambda^{C_1 \log(n/k)} P\big(A \mid J, \mathcal{G}_{3j}, \mathcal{D}_{3j+2}, j \in J\big) P'\big(B' \mid J', \mathcal{G}'_{3j'}, \mathcal{D}'_{3j'+2}, j' \in J'\big).
\end{aligned} \tag{2.22}
$$

Taking expectations with respect to J, J', and all the $\mathcal{G}$, $\mathcal{D}$, $\mathcal{G}'$ and $\mathcal{D}'$, we finally get

$$
P\{A \circ B\} \leq 2\lambda^{C_1 \log(n/k)} P\{A\} P'\{B'\} = 2\left(\frac{n}{k}\right)^{-C_1|\log \lambda|} P^2(A). \tag{2.23}
$$

Hence Lemma 1 is proved by (2.23). □

LEMMA 3. *There is a constant C such that*

$$
\begin{aligned}
&P(\exists \textit{ a closed path from } [u, u+k] \times \{0\} \textit{ to } B_n \textit{ in } [-n, n] \times [0, n])\\
&\quad \leq CP(\exists \textit{ a closed path from } [u, u+k] \times \{0\} \textit{ to } [-n, n] \times \{n\}\\
&\qquad\qquad \textit{in } [-n, n] \times [0, n]),
\end{aligned} \tag{2.24}
$$

for all integers n, k and u with $n \geq 2k > 0$ and $u \in [-n/2, n/2-k]$. In particular,

$$
\begin{aligned}
&P(\exists \textit{ a closed path from } (u, 0) \textit{ to } B_n \textit{ with bonds in } [-n, n] \times [0, n])\\
&\quad \leq CP(\exists \textit{ a closed path from } (u, 0) \textit{ to } [-n, n] \times \{n\} \textit{ with bonds}\\
&\qquad\qquad \textit{in } [-n, n] \times (0, n]),
\end{aligned} \tag{2.25}
$$

for all integers u with $u \in [-n/2, n/2]$.

PROOF. We write $\mathcal{F}_n$ for the event that there exists a closed half-circuit in $[-n, n] \times [0, n] \backslash [-n/2, n/2] \times [0, n/2]$, and we write D_n for the event that there exists a top–bottom closed crossing in $[-n/2, n/2] \times [n/2, n]$. Clearly,

$$
\{\exists \text{ a closed path from } [u, u+k] \text{ to } [-n, n] \times \{n\} \text{ in } [-n, n] \times [0, n]\}
$$

must occur if

$$
\{\exists \text{ a closed path from } [u, u+k] \text{ to } B_n \text{ in } [-n, n] \times [0, n]\} \cap \mathcal{F}_n \cap D_n
$$

occurs. By the RSW lemma,

$$P(\mathcal{F}_n) \geq C_1 \quad \text{and} \quad P(D_n) \geq C_1, \tag{2.26}$$

for some constant C_1. Then Lemma 3 is implied by (2.26) and the FKG inequality. □

If l_n^* is a path on the dual of $[-n,n] \times [0,n]$ from v^* to B_n^* for some vertex

$$v^* \in \left[-\frac{n+1}{2}, \frac{n-1}{2}\right] \times \left\{\frac{1}{2}\right\},$$

then l_n may not be a connected path on Z^2. However, l_n divides $[-n,n] \times [0,n]$ into two parts. In other words, any connected path from part 1 to part 2 in $(-n,n) \times (0,n)$ has at least a common bond with l_n. Actually, l_n is called a cut set. With this geometric knowledge and Lemmas 1 and 3, we have the following corollary.

COROLLARY 2. *There exist $\varepsilon > 0$ and C such that*

$$\begin{aligned} &P\big(\exists \textit{ closed bond sets } l_1 \textit{ and } l_2 \textit{ in } [-n,n] \times [0,n] \textit{ with } l_1 \cap l_2 = \emptyset \\ &\quad \textit{such that } l_1 \textit{ is a path from } [v_1, v_1+k] \times \{0\} \textit{ to } B_n \textit{ and} \\ &\quad l_2^* \textit{ is a path from } [v_1 - \tfrac{1}{2}, v_1 + k - \tfrac{1}{2}] \times \{\tfrac{1}{2}\} \textit{ to } B_n^*\big) \\ &\quad \leq C\Big(\frac{n}{k}\Big)^{-\varepsilon} P^2\big(\exists \textit{ a closed path in } [n,n] \times [0,n] \textit{ from} \\ &\qquad\qquad [0,k] \times \{0\} \textit{ to } B_n\big), \end{aligned} \tag{2.27}$$

for all integers n, k and v_1 with $n \geq 2k > 0$, and $v_1 \in [-n/2, n/2 - k]$.

PROOF. By the method of Lemma 1,

$$\begin{aligned} &P\big(\exists \text{ closed bond sets } l_1 \text{ and } l_2 \text{ in } [-n,n] \times [0,n] \text{ with } l_1 \cap l_2 = \emptyset \text{ such that} \\ &\quad l_1 \text{ is a path from } [v_1, v_1+k] \times \{0\} \text{ to } B_n \text{ and} \\ &\quad l_2^* \text{ is a path from } [v_1 - \tfrac{1}{2}, v_1 + k - \tfrac{1}{2}] \times \{\tfrac{1}{2}\} \text{ to } B_n^*\big) \\ &\quad \leq C_1\Big(\frac{n}{k}\Big)^{-\varepsilon} P^2\big(\exists \text{ a closed path in } [-n,n] \times [0,n] \text{ from} \\ &\qquad\qquad [v_1, v_1+k] \times \{0\} \text{ to } B_n\big) \\ &\quad \leq C_2\Big(\frac{n}{k}\Big)^{-\varepsilon} P^2\big(\exists \text{ a closed path in } [-n,n] \times [0,n] \text{ from} \\ &\qquad\qquad [v_1, v_1+k] \times \{0\} \text{ to } [-n,n] \times \{n\}\big) \\ &\qquad\qquad\qquad (\text{by Lemma 3 for some constant } C_2) \\ &\quad \leq C_2\Big(\frac{n}{k}\Big)^{-\varepsilon} P^2\big(\exists \text{ a closed path in } [-n,n] \times [0,n] \text{ from} \\ &\qquad\qquad [0,k] \times \{0\} \text{ to } B_n\big) \quad (\text{by translation invariance}). \end{aligned} \tag{2.28}$$

Therefore, Corollary 2 is proved by (2.28). □

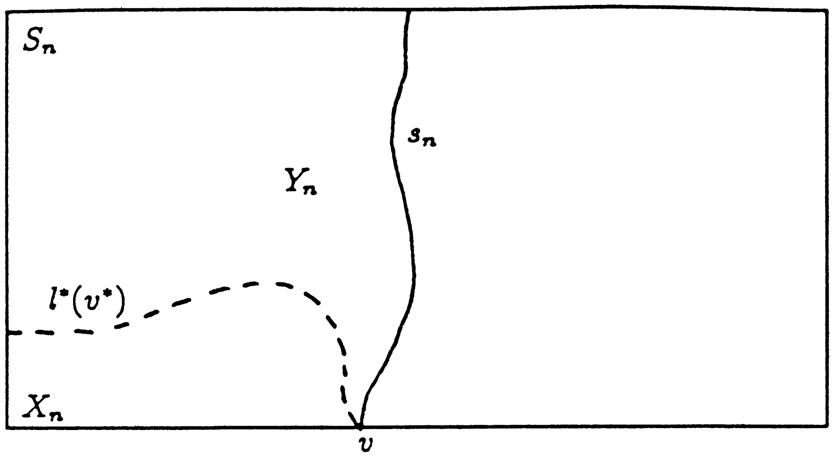

FIG. 3. *The solid path is the leftmost open path and the dashed path is the closed path with the smallest area X_n.*

With Corollary 2 and some techniques of [12], we shall show the following lemma.

LEMMA 4. *There are constants $\varepsilon > 0$ and C such that*

$$C\left(\frac{k}{n}\right)^{1/2-\varepsilon} \le P(\exists \text{ a closed path on } [-n,n] \times [0,n] \text{ from } [0,k] \times \{0\} \text{ to } B_n),$$

for all integers $n \ge 2k > 0$.

PROOF. Let T_n be the event that there exist a top–bottom closed crossing in $[-n/2, 0] \times [0,n]^*$, a left–right closed crossing in $[-n, 0] \times [0,n]^*$ and a bottom–top open crossing in $[0, n/2] \times [0,n]$. By the RSW lemma and the FKG inequality, we have

$$P(T_n) > C_3, \tag{2.29}$$

for some C_3. Since Z^2 is a planar graph, we can define the leftmost open crossing, denoted s_n, in $[-n,n] \times [0,n]$ if there exists a bottom–top open crossing (see a similar definition in [6]). We define S_n to be the left region surrounded by s_n and the boundary of $[-n,n] \times [0,n]$. If T_n occurs, it can be seen that one end vertex of s_n has to stay on $[-n/2, n/2] \times \{0\}$. We write $v = (v_1, 0)$ for this vertex. By the definition of the leftmost open crossing, there also exists another closed path in the dual of Z^2 from $v^* = (v_1 - \frac{1}{2}, \frac{1}{2})$ to B_n^*. Similarly, we can find a closed path $l^*(v^*)$ from v^* to B_n^* such that the region $X_n^* \subset S_n^*$ surrounded by $l^*(v^*)$ and the boundary of $[-n,n] \times [0,n]^*$ is minimized (see Figure 3). The existence of such a path $l^*(v^*)$ can be demonstrated by the method of Proposition 2.3 in [6]. Hence $l(v^*)$ divides $[-n,n] \times [0,n]$ into two parts. One is X_n, and we write Y_n for the other. We denote by $\mathcal{M}(Y_n)$ the event that there exists an open path from v to B_n in Y_n. It follows from the method of Proposition 2.3 in [6] that, conditionally on the existence of $l(v^*)$, the families $\{w(e): e \in X_n\}$ and $\{w(e): e \in Y_n\}$ are independent. Therefore,

$$\begin{aligned} C_3 \le P(T_n) &\le \sum_{\Gamma} P\left(l(v^*) = \Gamma,\ \mathcal{M}(Y_n)\right) \\ &\le \sum_{\Gamma} P\left(l(v^*) = \Gamma\right) P\left(\mathcal{M}(Y_n)\right) \\ &= \sum_{\Gamma} P\left(l(v^*) = \Gamma\right) P(\exists \text{ a closed path from } v \text{ to } B_n \text{ in } Y_n) \\ &\qquad\qquad (\text{note that } p_c = \tfrac{1}{2}) \\ &\le P\Bigg(\exists \text{ closed bond sets } l_1 \text{ and } l_2 \text{ in } [-n,n] \times [0,n] \\ &\qquad \text{with } l_1 \cap l_2 = \emptyset \text{ such that } l_1 \text{ is a path from} \\ &\qquad (v_1, 0) \text{ to } B_n \text{ and } l_2^* \text{ is a path from} \\ &\qquad \left(v_1 - \frac{1}{2}, \frac{1}{2}\right) \text{ to } B_n^* \text{ for some } (v_1, 0) \in \left[-\frac{n}{2}, \frac{n}{2}\right] \times \{0\}\Bigg), \end{aligned} \tag{2.30}$$

where the sum is taken over all possible sets Γ such that Γ^* on the dual of $[-n,n], \times [0,n]$ is such a path from $[(-n-1)/2, (n-1)/2] \times \{\frac{1}{2}\}$ to B_n^*. By convention we assume that n/k is an integer, otherwise we can always use $\lfloor n/k \rfloor$ instead of n/k. Now we divide $[-n/2, n/2] \times \{0\}$ into n/k segments with equal length k and denote these segments by $L_1, \dots, L_{n/k}$. Hence if T_n occurs, the leftmost open crossing has to go from one of these segments to the B_n. It follows from Corollary 2 that the right-hand side of (2.30) is less than

$$\text{constant} \left(\frac{n}{k}\right)^{1-\varepsilon} P^2\left(\exists \text{ a closed path from } [0,k] \times \{0\} \text{ to } B_n\right). \tag{2.31}$$

Hence Lemma 4 is proved by (2.30) and (2.31). □

PROOF OF THEOREM 1. For any v on the X axis, denote by $\mathcal{V}_n(v)$ the event that there exist two closed paths from v to B_n and $\overline{B}_n$ with bonds in $[-n,n] \times (0,n]$ and $[-n,n] \times [-n,0)$, respectively, where $\overline{B}_n = (\{-n\} \times [-n,0]) \cup ([-n,n] \times \{-n\}) \cup (\{n\} \times [-n,0])$, that is, $\overline{B}_n$ is a reflection of B_n along the X axis. Let $I(v)$ be the indicator function of $\mathcal{V}_n(v)$, and let $K_n = [n/2, n/2] \times \{0\}$. By Lemma 3 and translation invariance, there exists a constant C_1 such that

$$P(\mathcal{V}_n(u)) \le C_1 P(T_n(u)) \le C_1 P(\mathcal{V}_n(v)), \tag{2.32}$$

for any n and $v, u \in K_n$, where $T_n(u)$ is the event that there exist two closed paths from u to $[-n,n] \times \{n\}$ and $[-n,n] \times \{-n\}$ with bonds in $[-n,n] \times (0,n]$ and $[-n,n] \times [-n,0)$, respectively. Now we need to estimate

$$P\left(\left\{\sum_{v \in K_n} I(v)\right\} > 0\right) \ge C, \tag{2.33}$$

for some constant C. To estimate (2.33), let us first estimate the second moment of $\sum_{v\in K_n} I(v)$. Obviously,

$$
\begin{aligned}
&E\Big(\sum_{v\in K_n} I(v)\Big)^2\\
&\quad= E\sum_{v,u\in K_n} I(u)I(v)\\
(2.34)\qquad &\quad= E\sum_{v\in K_n} I(v)\sum_{k=1}^{n}\sum_{\|u-v\|=k} I(u)\\
&\quad= E\sum_{v\in K_n} I(v)\sum_{k=4}^{n}\sum_{\|u-v\|=k} I(u) + E\sum_{v\in K_n} I(v)\sum_{k=1}^{3}\sum_{\|u-v\|=k} I(u)\\
&\quad= I + II.
\end{aligned}
$$

By convention we assume that $k/4$ is an integer, otherwise we can use $\lfloor k/4\rfloor$ instead of $k/4$. Let us estimate term I in (2.34). If $\mathcal{V}_n(v)\cap\mathcal{V}_n(u)$ occurs with $\|u-v\| = k$ for $k\geq 4$, then there exist two closed paths r_1 and r_2 from u and v to B_n with bonds in $[-n,n]\times(0,n]$, and another two closed paths r_3 and r_4 from u and v to $\overline{B}_n$ with bonds in $[-n,n]\times(-n,0]$. Note that each pair of paths may intersect each other. Suppose that r_1 and r_2 intersect each other. Then there exists the innermost closed half-circuit M which connects u to v with bonds in $[-n,n]\times(0,n]$, and M is also connected to B_n by a closed path with bonds in $M^e\cap[-n,n]\times(0,n]$. Once such a circuit and path exist, it can be seen that there exist two disjoint closed paths on $[-n,n]\times(0,n]$ either from u to B_n and from v to $v+B_{k/2}$ or from v to B_n and from u to $u+B_{k/2}$. Clearly, we have the same situation if r_1 and r_2 do not intersect. A repetition of the argument above with r_1 and r_2 replaced by r_3 and r_4 shows that there also exist two disjoint closed paths on $[-n,n]\times[-n,0)$ either from u to $\overline{B}_n$ and from v to $v+\overline{B}_{k/2}$ or from v to $\overline{B}_n$ and from u to $u+\overline{B}_{k/2}$. By (2.32) and the BK inequality,

$$
(2.35)\qquad P(\mathcal{V}_n(v)\cap\mathcal{V}_n(u)\text{ with }\|u-v\|=k)\leq C_1^2P(\mathcal{V}_n(u))P\Big(\mathcal{V}_{k/2}((0,0))\Big),
$$

for any $k\geq 4$ and $u,v\in K_n$. Hence, by (2.34) and (2.35),

$$
(2.36)\qquad I\leq C_1^2\sum_{v\in K_n} P(\mathcal{V}_n(v))\sum_{k=4}^{n}\sum_{\|u-v\|=k} P\Big(\mathcal{V}_{k/2}((0,0))\Big).
$$

Now let $\mathcal{G}_n$ be the event that there exists a closed half-circuit in $[-n,n]\times[-n,0]\backslash[-n/2,n/2]\times[-n/2,0]$, that is, the symmetric event corresponding to $\mathcal{F}_n$ reflected through the X axis (see Figure 4). It follows from (2.26) and the FKG inequality that

$$
(2.37)\qquad P\Big(\mathcal{V}_{k/2}((0,0))\Big)\leq C_2P\Big(\mathcal{V}_{k/2}((0,0))\cap\mathcal{F}_{k/2}\cap\mathcal{G}_{k/2}\Big),
$$

for some constant C_2 and $k \geq 4$. For $k \geq 4$, let $\mathcal{J}_{k,n}$ denote the event that there exist two closed paths in Z^2, one from $[-k/4, k/4] \times \{k/4\}$ to B_n in $[-n,n] \times [k/4, n]$ and the other from $[-k/4, k/4] \times \{-k/4\}$ to $\overline{B}_n$ in $[-n,n] \times [-n, -k/4]$. Obviously, if $\mathcal{V}_{k/2}((0,0)) \cap \mathcal{F}_{k/2} \cap \mathcal{G}_{k/2} \cap \mathcal{J}_{k,n}$ occurs, then $\mathcal{V}_n((0,0))$ occurs (see Figure 4). Hence,

$$
\begin{aligned}
\mathrm{I} &\leq C_1^2 \sum_{v\in K_n} P(\mathcal{V}_n(v)) \sum_{k=4}^{n} \sum_{\|u-v\|=k} P\Big(\mathcal{V}_{k/2}((0,0))\Big) \\
&\leq C_1^2 C_2 \sum_{v\in K_n} P(\mathcal{V}_n(v)) \sum_{k=4}^{n} 2P\Big(\mathcal{V}_{k/2}((0,0)) \cap \mathcal{F}_{k/2} \cap \mathcal{G}_{k/2}\Big) \quad \text{[by (2.37)]} \\
&\leq C_1^2 C_2 C_3 \sum_{v\in K_n} P(\mathcal{V}_n(v)) \sum_{k=4}^{n} \left(\frac{n}{k}\right)^{(1-2\varepsilon)} \\
&\qquad\qquad \times 2P\Big(\mathcal{V}_{k/2}((0,0)) \cap \mathcal{F}_{k/2} \cap \mathcal{G}_{k/2} \cap \mathcal{J}_{k,n}\Big)
\end{aligned}
$$

(by Lemma 4 and the FKG inequality for some constant C_3)

$$
\begin{aligned}
&\leq 2C_1^2 C_2 C_3 \sum_{v\in K_n} P(\mathcal{V}_n(v)) \sum_{k=4}^{n} \left(\frac{n}{k}\right)^{(1-2\varepsilon)} P\Big(\mathcal{V}_n((0,0))\Big) \qquad (2.38) \\
&\leq 2C_4 C_1^2 C_2 C_3 \sum_{v\in K_n} P(\mathcal{V}_n(v)) n P\Big(\mathcal{V}_n((0,0))\Big) \\
&\qquad\qquad \left[\text{note that } \sum_{k=4}^{n} \left(\frac{n}{k}\right)^{(1-2\varepsilon)} \leq C_4 n \text{ for some } C_4\right] \\
&\leq 2C_4 C_1^3 C_2 C_3 \sum_{v\in K_n} \sum_{u\in K_n} P(\mathcal{V}_n(v)) P(\mathcal{V}_n(u)) \quad \text{[by (2.32)]} \\
&= 2C_4 C_1^3 C_2 C_3 E\left(\left\{\sum_{v\in K_n} I(v)\right\}\right) E\left(\left\{\sum_{v\in K_n} I(v)\right\}\right).
\end{aligned}
$$

Clearly,

$$
II \leq C_5 E\left(\left\{\sum_{v\in K_n} I(v)\right\}\right) E\left(\left\{\sum_{v\in K_n} I(v)\right\}\right), \tag{2.39}
$$

for some constant C_5. It follows from (2.34), (2.38) and (2.39) that

$$
E\left(\sum_{v\in K_n} I(v)\right)^2 \leq C\left(E\left\{\sum_{v\in K_n} I(v)\right\}\right)^2, \tag{2.40}
$$

for some constant C. It also follows from Lemma 4 that

$$
E\left(\sum_{v\in K_n} I(v)\right) \geq C_1, \tag{2.41}
$$

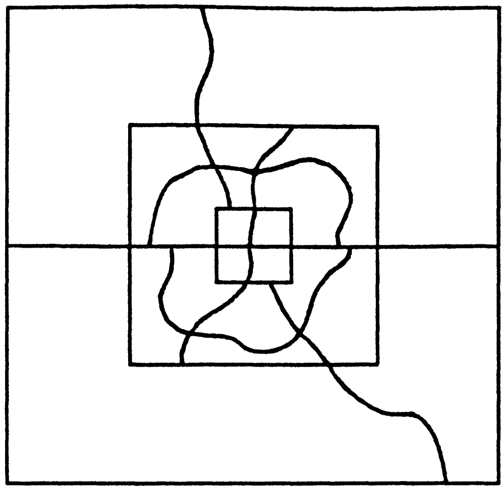

FIG. 4. *The event* $\mathcal{V}_{k/2}((0,0)) \cap \mathcal{F}_{k/2} \cap \mathcal{G}_{k/2} \cap \mathcal{J}_{k,n}$.

for some constant C_1. Therefore, (2.33) is implied by Schwarz's inequality, (2.40) and (2.41). It follows from (2.33) that

$$P\big(\exists \text{ a top–bottom closed crossing in } [-n,n]^2 \text{ which only intersects the } X \text{ axis once}\big) \geq C. \tag{2.42}$$

By translation invariance, (2.42), the RSW lemma and the FKG inequality, there exists $\rho > 0$ such that

$$P\big(\exists \text{ a closed circuit in } A(n) \text{ which only intersects the } X \text{ axis twice}\big) > \rho. \tag{2.43}$$

Therefore, Theorem 1 is proved by (2.43). □

Acknowledgments. The author would like to thank R. Schonmann, G. Morrow and R. Schinazi for many useful discussions. He would also like to thank the referee for carefully reading this manuscript and for numerous helpful comments and typographical corrections.

REFERENCES

[1] BARSKY, D., GRIMMETT, G. and NEWMAN, G. (1991). Percolation in half-space: equality of critical densities and continuity of the percolation probability. *Probab. Theory Related Fields* **90** 111–148.

[2] CAMPANINO, M. and KLEIN, A. (1991). Decay of two-point functions for $d+1$ dimensional percolation, Ising and Potts models with d-dimensional disorder. *Comm. Math. Phys.* **135** 489–497.

[3] CHAYES, J., CHAYES, L. and DURRETT, R. (1987). Inhomogeneous percolation problems and incipient infinite clusters. *J. Phys. A* **20** 1521–1530.

[4] HARRIS, T. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.* **56** 13–20.

[5] KESTEN, H. (1980). The critical probability of bond percolation on square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.* **74** 41–59.

[6] KESTEN, H. (1982). *Percolation Theory for Mathematicians.* Birkhäuser, Boston.

[7] KESTEN, H. (1987). Scaling relations for 2D-percolation. *Comm. Math. Phys.* **109** 109–156.

[8] KESTEN, H. and ZHANG, Y. (1987). Strict inequalities for some critical exponents in 2D-percolation. *J. Statist. Phys.* **46** 1031–1056.

[9] MADRAS, N., SCHINAZI, R. and SCHONMANN, R. (1992). On the critical behavior of the contact process in deterministic inhomogeneous environment. *Ann. Probab.* To appear.

[10] RUSSO, L. (1978). A note on percolation. *Z. Wahrsch. Verw. Gebiete* **43** 39–48.

[11] SEYMOUR, P. and WELSH. D. (1978). Percolation probability on the square lattice. *Ann. Discrete Math.* **3** 227–245.

[12] VAN DEN BERG, J. and KESTEN, H. (1985). Inequalities with application to percolation and reliability. *J. Appl. Probab.* **22** 556–569.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO
COLORADO SPRINGS, COLORADO 80933