The Annals of Probability
1994, Vol. 22, No. 2, 764–802

COEXISTENCE IN THRESHOLD VOTER MODELS[1]

BY THOMAS M. LIGGETT

University of California, Los Angeles

The threshold voter models considered in this paper are special cases of the nonlinear voter models which were introduced recently by Cox and Durrett. They are spin systems on Z^d with transition rates

$$c(x,\eta) = \begin{cases} 1, & \text{if there is a } y \text{ with } \|x-y\| \le N \text{ and } \eta(x) \ne \eta(y), \\ 0, & \text{otherwise.} \end{cases}$$

This system is known to cluster if $N = d = 1$, and to coexist if $N \ge 4$ in one dimension and if N is reasonably large in other dimensions. Cox and Durrett conjectured that it coexists in all cases except $N = d = 1$. In this paper, we prove this conjecture. The proof is based on comparisons with threshold contact processes. The hard part of the proof consists of showing that the second nearest neighbor threshold contact process in one dimension with parameter 1 survives. The proof of this result is modeled after the proof by Holley and Liggett that the critical value of the basic contact process in one dimension is at most 2. By comparison with that proof, however, the fact that the interaction is not of nearest neighbor type presents substantial additional difficulties. In fact, part of the proof is computer aided.

1. Introduction. A d-dimensional spin system is a continuous time Markov process η_t on $\{0,1\}^S$, where $S = Z^d$, in which the configuration η changes its value at site $x \in Z^d$ from $\eta(x)$ to $1 - \eta(x)$ at a prescribed rate $c(x,\eta)$. Liggett [(1985), Chapters 3–7] treats various types of spin systems. When $c(x,\eta) = 0$ for $\eta \equiv 0$ and for $\eta \equiv 1$, the point masses on these two configurations are invariant for the system. A natural problem in this case is to determine whether there are any nontrivial invariant measures, that is, ones which are not mixtures of these two. If so, the system is said to coexist, since there is an equilibrium in which both opinions 0 and 1 coexist. If, on the other hand, $P\{\eta_t(x) \ne \eta_t(y)\} \to 0$ as $t \to \infty$ for every $x \ne y$ and every initial configuration, then the system is said to cluster.

The linear voter model has been studied for nearly two decades [see, e.g., Liggett (1985), Chapter 5]. One version of it has

$$c(x,\eta) = \sum_{\|y-x\| \le N} 1_{\{\eta(y) \ne \eta(x)\}}.$$

[Here and below, the norm is arbitrary, except for the normalization $\|x\| = 1$ for nearest neighbors of the origin; N is always taken to be an integer. The process

Received July 1992.
[1]Preparation of this paper was supported in part by NSF Grants DMS-86-01800 and 91-00725.
AMS 1991 *subject classification.* 60K35.
Key words and phrases. Spin systems, contact processes, coexistence, voter models.

is called linear because the transition rate at x is a linear function of the number of neighbors which disagree with $\eta(x)$.] The first thing one proves about this process is that it clusters if $d \leq 2$ and coexists if $d \geq 3$. In particular, this aspect of its behavior depends on the dimension of the lattice of sites, but not on the size of the neighborhood. Recently, Cox and Durrett (1991) discovered that certain nonlinear voter models can coexist even in one dimension. Among those of greatest interest are the threshold voter models, in which

$$c(x,\eta) = \begin{cases} 1, & \text{if there is a } y \text{ with } \|x-y\| \leq N \text{ and } \eta(x) \neq \eta(y), \\ 0, & \text{otherwise.} \end{cases}$$

Using comparisons with (nearest neighbor) contact processes, they showed that this process coexists in one dimension if $N \geq 4$, in two dimensions if $N \geq 2$ (when $\|\cdot\|$ is the l_∞ norm) or $N \geq 3$ (when $\|\cdot\|$ is the l_1 norm) and in three or more dimensions if $N \geq 1$. It is rather easy to show that the process clusters in one dimension if $N = 1$. [See Cox and Durrett (1991) and also Andjel, Liggett and Mountford (1992) for a more general result.] Based partly on computer simulations, Cox and Durrett conjectured that the threshold voter model coexists in all cases except $N = d = 1$. The purpose of this paper is to prove this conjecture:

THEOREM. *Suppose that $N \geq 1$, but that $(N, d) \neq (1,1)$. Then the threshold voter model on Z^d with parameter N coexists.*

The proof of this theorem is also based on contact process comparisons. However, we must consider threshold contact processes with the same neighborhood set as the one involved in the threshold voter model of interest, since the earlier reduction to nearest neighbor contact processes leads to too much loss of information. The threshold contact process on Z^d with parameters N and $\lambda \geq 0$ is the spin system with rates

$$c(x,\eta) = \begin{cases} \lambda, & \text{if } \eta(x) = 0 \text{ and } \eta(y) = 1 \text{ for some } \|x-y\| \leq N, \\ 0, & \text{if } \eta(x) = 0 \text{ and } \eta(y) = 0 \text{ for all } \|x-y\| \leq N, \\ 1, & \text{if } \eta(x) = 1. \end{cases}$$

This process is said to survive if it has an invariant measure other than the point mass on $\eta \equiv 0$.

The proof of the theorem is obtained by combining the following propositions. They are given in order of difficulty of proof, from easiest to hardest. A version of the first proposition was used by Cox and Durrett in their argument. In doing so, they used the complete convergence theorem for contact processes, which is based on the work of Bezuidenhout and Grimmett (1990). We will give a more elementary proof, which avoids the use of the complete convergence theorem.

PROPOSITION 1. *For any $N \geq 1$ and d, if the threshold contact process on Z^d with $\lambda = 1$ survives, then the threshold voter model (with the same N) on Z^d coexists.*

For a given λ, it is easy to see that the survival of the threshold contact process for one pair (N, d) implies survival of the threshold contact process for any other pair (N', d') for which $N' \geq N$ and $d' \geq d$. [See, e.g., Liggett (1985), Corollary 1.8 of Chapter 3.] Note that a similar comparison statement involving coexistence of threshold voter models, while a consequence of the above theorem, is not easy to prove directly. This is one reason for working with contact processes instead of voter models. In view of these remarks, once Proposition 1 is proved, we may restrict our attention to the threshold contact processes with $(N, d) = (2, 1)$ and $(N, d) = (1, 2)$, respectively (relative to the l_1 norm). Fortunately, the survival of the first implies the survival of the second:

PROPOSITION 2. *For any $\lambda \geq 0$, if the threshold contact process on Z^1 with $N = 2$ survives, then the threshold contact process on Z^2 with $N = 1$ survives.*

Now we may restrict our attention to the case $(N, d) = (2, 1)$ and $\lambda = 1$. In order to prove survival in this case, we follow the outline of the Holley–Liggett proof of survival of the (nearest neighbor) basic contact process in one dimension with parameter greater than or equal to 2 [see Liggett (1985), Theorem 1.33 of Chapter 6]. The idea is to take the renewal measure μ on $\{0, 1\}^Z$ corresponding to the probability density $f(n)$ on the positive integers, where $f(n)$ is determined by the requirement that

$$\frac{d}{dt}\mu_t\{\eta : \eta(k) = 0 \text{ for all } 1 \leq k \leq n\}\bigg|_{t=0} = 0,$$

for all $n \geq 1$. (Here μ_t is the distribution at time t of the threshold contact process with initial distribution μ.) Then one tries to show that

$$\mu_t\{\eta : \eta(k) = 0 \text{ for all } k \in A\} \downarrow$$

in t for every finite subset A of Z, and hence the process survives. Using the renewal property of μ, it is not hard to see that the above identities [which are the analogues of equations (1.20) of Chapter 6 of Liggett (1985)] in the present context are equivalent to

$$\text{(1.1)} \qquad \begin{aligned} &F(2) = \frac{1}{\lambda + 1}, \qquad F(3) = \frac{1}{(\lambda + 1)^2}, \qquad F(4) + F(5) = \frac{1}{\lambda(\lambda + 1)^2}, \\ &\sum_{k=1}^{n} F(k)F(n - k + 1) = 4\lambda F(n + 1) + 2\lambda F(n + 2) \quad \text{for } n \geq 4, \end{aligned}$$

where $F(n)$ are the tail probabilities,

$$F(n) = \sum_{k=n}^{\infty} f(k).$$

Note that (1.1) cannot be solved recursively, but that all the values of $F(n)$ can be computed recursively in terms of $F(4)$. The issue involved in finding a good renewal measure which can be used as an initial distribution for the threshold contact process in the proof of the theorem is whether the value of $F(4)$ can be chosen in such a way that the resulting $F(n)$ are the tail probabilities of a probability density on the positive integers with finite mean. For the statement of the next proposition, recall that the renewal sequence $u(n)$ associated with the density $f(n)$ is defined by $u(0) = 1$ and

$$u(n) = \sum_{k=1}^{n} f(k)u(n-k) \quad \text{for } n \geq 1. \tag{1.2}$$

PROPOSITION 3. *Take $\lambda = 1$. Suppose there exists a positive decreasing sequence $F(n)$ which satisfies $F(1) = 1, \Sigma_n F(n) < \infty$ and equations* (1.1) *for which the density is decreasing and the renewal sequence $u(n)$ satisfies the following inequalities*:

(a) $$u(n) \geq u(n+1) \quad \textit{for } n \geq 0;$$

(b) $$u(n-1) + u(n+1) \geq 2u(n) \quad \textit{for } n \geq 3.$$

Then the threshold contact process on Z^1 with $N = 2$ survives.

PROPOSITION 4. *If $\lambda = 1$, then there exists a positive decreasing sequence $F(n)$ which satisfies $F(1) = 1, \Sigma_n F(n) < \infty$ and equations* (1.1). *The corresponding density is decreasing and the renewal sequence $u(n)$ satisfies* (a) *and* (b) *of Proposition* 3.

REMARKS.

(a) A Mathematica computation indicates that a bounded solution $F(n)$ of (1.1) exists for all $\lambda \geq 0.985\ldots$, but not for smaller λ's. Thus this technique just barely works. Simulations by Buttel, Cox and Durrett (1993) suggest that the critical value for this system is about 0.81.

(b) One interesting aspect of the proof of Proposition 4 is that it is computer aided. We were not able to give an analytic proof of the complete result. Note that Property 3(a) follows from Property 3(b), except for a few easily checked values of n. For Property 3(b), we give an analytic proof for $n \geq 1000$, but we have to resort to computer calculations for smaller values of n.

Propositions 1 and 2 are proved in the next section. Propositions 3 and 4 are proved in Sections 3 and 4, respectively. At the end of Section 4, there is a discussion of the computer work involved in the proof. Each reader will have to decide whether a proof of this type can be regarded as entirely rigorous. This is an issue which will almost certainly arise with increasing frequency in the future.

2. The comparisons. In this section, we give the proofs of the first two propositions. Inequalities between probability measures on $\{0,1\}^S$ refer to stochastic monotonicity. [See, e.g., Liggett (1985), Section 2 of Chapter 2.] Convergence of measures means weak convergence.

PROOF OF PROPOSITION 1. Suppose that the threshold contact process with $\lambda = 1$ (and arbitrary N and d) survives, and let ν be its upper invariant measure. By assumption, ν concentrates on configurations with infinitely many 1's. Let $S_c(t), S_v(t)$ and $S_i(t)$ be the semigroups corresponding to the threshold contact process with $\lambda = 1$, threshold voter model (with the same N and d) and the independent flip process [with $c(x,\eta) \equiv 1$], respectively. Finally, let $\nu_{1/2}$ be the product measure with density $\frac{1}{2}$. We then have the following elementary facts [see, e.g., Liggett (1985), Corollary 1.7 of Chapter 3, for the comparisons]:

$$\nu = \nu S_c(t) \leq \nu S_i(t), \tag{2.1}$$

$$\nu S_i(t) \to \nu_{1/2} \tag{2.2}$$

and

$$\nu = \nu S_c(t) \leq \nu S_v(t). \tag{2.3}$$

Combining (2.1) and (2.2) gives

$$\nu \leq \nu_{1/2}. \tag{2.4}$$

Combining (2.3) and (2.4) gives

$$\nu \leq \nu_{1/2} S_v(t).$$

Thus every weak limit ν^* of Cesaro averages of $\nu_{1/2}S_v(t)$ is stochastically larger than ν, and hence puts all of its mass on configurations with infinitely many 1's. Since $\nu_{1/2}S_v(t)$ is invariant under the operation of interchanging 0's and 1's, ν^* puts all of its mass on configurations with infinitely many zeros as well. Therefore ν^* is a nontrivial invariant measure for the threshold voter model [see Liggett (1985), Proposition 1.8 of Chapter 1], and hence there is coexistence. □

PROOF OF PROPOSITION 2. [This proof is based on the Holley–Liggett proof that the critical value for the basic contact process in d dimensions is less than or equal to $2/d$. See Theorem 4.1 and its corollary in Chapter 6 of Liggett (1985).] We need to compare the threshold contact process η_t on Z, where k has neighbors $k-2, k-1, k+1$ and $k+2$ with the threshold contact process ζ_t on Z^2, where (m,n) has neighbors $(m-1,n)$, $(m+1,n)$, $(m,n-1)$ and $(m,n+1)$ (and the same λ). Define the mapping π: $Z^2 \to Z$ by $\pi(m,n) = m + 2n$. Then the four neighbors of (m,n) in Z^2 map onto the four neighbors of $\pi(m,n)$ in Z. This property permits one to couple the two processes together in order to maintain the relation $\eta_t \leq \pi(\zeta_t)$, thus proving that survival of η_t implies survival of ζ_t. To construct the coupling, associate a $k \in Z$ such that $\eta(k) = 1$ with any of the $(m,n) \in Z^2$ such that $\pi(m,n) = k$ and $\zeta(m,n) = 1$, letting the exponential times for $1 \to 0$ at the associated sites be the same. For sites k with $\eta(k) = 0$ such that some neighbor j satisfies $\eta(j) = 1$, let (m, n) be the site associated with j, and then associate k with any neighbor of (m, n). Again, couple the $0 \to 1$ transitions at the associated sites. □

REMARK. Following a talk on this paper in Zurich, Professor Dobrushin asked whether the approach used in the proof of Proposition 2 above could be applied to other lattices, such as the triangular or hexagonal lattices in two dimensions. The answer is yes in the case of the lattice in which each site has six neighbors. To see this, we make a comparison between the threshold contact process on this lattice and the threshold contact process on Z^1 with $N = 3$. The analogue of the mapping π used above is given in pictorial form below:

	+5		0	
+7		+2		−3
	+4		−1	
+6		+1		−4
	+3		−2	
+5		0		−5

This mapping preserves the neighborhood structure, as required in the proof. This approach does not work in case each site has three neighbors. It may be necessary to prove survival for the two-dimensional system directly, rather than relying on a comparison with one dimension.

3. Proof of Proposition 3. Initially, we let λ be general, and we only take it to be 1 later in the section, when the expressions which appear would otherwise be too cumbersome. Let μ be the renewal measure corresponding to the probability density $f(n)$ with tail probabilities $F(n)$, whose existence is given by the hypothesis of the proposition. For any finite set A of integers, let

$$Q(A) = -\frac{1}{\mu\{\eta: \eta(0) = 1\}} \frac{d}{dt} \mu_t\{\eta: \eta(k) = 0 \text{ for all } k \in A\}\bigg|_{t=0},$$

where μ_t is the distribution at time t of the threshold contact process in one dimension with $N = 2$ and initial distribution μ. Recall that equation (1.1), which determines $F(n)$, is just the statement that $Q(A) = 0$ for all connected sets A. The first part of the argument is contained in the following lemma, and is essentially the same for many types of contact processes.

LEMMA 3.1. *If $Q(A) \geq 0$ for all finite $A \subset Z$, then the threshold contact process survives.*

PROOF. This threshold contact process has a (coalescing) dual. [See Liggett (1985), Section 4 of Chapter 3, for the definition.] The dual is the Markov chain A_t on the set of finite subsets of Z which has transitions

$$A \to A\backslash\{k\} \quad \text{at rate 1 for each } k \in A,$$

and

$$A \to A \cup \{k-2, k-1, k+1, k+2\} \quad \text{at rate } \lambda \text{ for each } k \in A.$$

The duality relation gives

$$\begin{aligned}\mu_{t+s}&\{\eta: \eta(k) = 0 \text{ for all } k \in A\}\\ &= \sum_B P^A(A_t = B)\mu_s\{\eta: \eta(k) = 0 \text{ for all } k \in B\}.\end{aligned}$$

Differentiating with respect to s, setting $s = 0$ and using the nonnegativity of $Q(B)$, we see that

$$\mu_t\{\eta: \eta(k) = 0 \text{ for all } k \in A\}$$

is nonincreasing in t for every A. Therefore, it cannot tend to 1 as $t \uparrow \infty$, so the process survives. □

Next we must prove that $Q(A) \geq 0$ for all finite A. In order to do so, it is necessary to find a useful expression for it, which incorporates the fact that $F(n)$ satisfies (1.1). Because of the renewal property, the summands which appear in the expression for $Q(A)$ can be expressed in terms of the following conditional probabilities:

$$\begin{aligned}L_A(k) &= \mu\{\eta: \eta(j) = 0 \text{ for all } j \in A \cap (-\infty, k) \mid \eta(k) = 1\}\\ R_A(k) &= \mu\{\eta: \eta(j) = 0 \text{ for all } j \in A \cap (k, \infty) \mid \eta(k) = 1\}.\end{aligned}$$

Using the renewal property again and a decomposition according to the location of the first one to the left (respectively, right) of k, it is not hard to see that these functions satisfy the following relations:

$$L_A(k) = \sum_{j<k, j \notin A} L_A(j) f(k-j); \tag{3.2}$$

$$R_A(k) = \sum_{j>k, j \notin A} R_A(j) f(j-k). \tag{3.3}$$

[See e.g., Liggett (1985), equation (1.27) of Chapter 6.]

To begin the computation, apply the generator of the process to the indicator function of the set $\{\eta: \eta(k) = 0 \text{ for all } k \in A\}$ and integrate with respect to μ to obtain

$$Q(A) = \lambda \sum_{\substack{j<k<l\\ k \in A, j, l \notin A\\ |k-j| \leq 2 \text{ or } |l-k| \leq 2}} f(l-j) L_A(j) R_A(l) - \sum_{k \in A} L_A(k) R_A(k). \tag{3.4}$$

In order to use the fact that $F(n)$ satisfies (1.1), rewrite the second sum in (3.4) in the following form, using (3.2) and (3.3):

$$\begin{aligned}\sum_{k\in A} L_A(k)R_A(k) &= \sum_{\substack{j<k<l\\ k\in A,\, j,l\notin A}} L_A(j)f(k-j)R_A(l)f(l-k)\\ &= \sum_{\substack{j<l\\ j,l\notin A}} L_A(j)R_A(l)\sum_{j<k<l} f(k-j)f(l-k)\\ &\quad-\sum_{\substack{j<k<l\\ j,k,l\notin A}} L_A(j)f(k-j)R_A(l)f(l-k).\end{aligned} \tag{3.5}$$

Strictly speaking, the right-hand side above is the difference of two divergent sums. Here and below, such expressions are to be interpreted according to the following convention: Identical summands which appear in the two sums are to be cancelled before the summations are performed. Note that after this cancellation, the remaining sums are convergent.

The convolution equation (1.1) for $F(n)$ can be rewritten as a convolution equation for $f(n)$ for use in the first sum on the right-hand side of (3.5):

$$\begin{aligned} f^2(1) &= (\lambda+2)f(2) - 2\lambda F(3),\\ 2f(1)f(2) &= (2\lambda+2)f(3) - 2\lambda f(4) - 2\lambda f(5),\\ 2f(1)f(3)+f^2(2) &= (3\lambda+2)f(4) - 2\lambda f(5) - 2\lambda f(6),\\ \sum_{k=1}^{n} f(k)f(n-k+1) &= (4\lambda+2)f(n+1) - 2\lambda f(n+2) - 2\lambda f(n+3)\\ &\qquad\qquad \text{for } n\geq 4.\end{aligned} \tag{3.6}$$

Therefore, the first term on the right-hand side of (3.5) is equal to

$$\begin{aligned} &f^2(1)\sum_{j,j+2\notin A} L_A(j)R_A(j+2) + 2f(1)f(2)\sum_{j,j+3\notin A} L_A(j)R_A(j+3)\\ &\quad+\left[2f(1)f(3)+f^2(2)\right]\sum_{j,j+4\notin A} L_A(j)R_A(j+4)\\ &\quad+\sum_{\substack{j,l\notin A\\ l\geq j+5}} L_A(j)R_A(l)\left\{(4\lambda+2)f(l-j) - 2\lambda f(l-j+1) - 2\lambda f(l-j+2)\right\}.\end{aligned}$$

Now use (3.2) and (3.3) on the last sum above, adding and subtracting the summands corresponding to small values of $l-j$, to rewrite the first term on the right-hand side of (3.5) as

$$\begin{aligned}
&\{4\lambda + 2 + 2\lambda f(1) + 2\lambda f(2)\} \sum_{j \notin A} L_A(j) R_A(j) + 2\lambda f(1) \sum_{j, j-1 \notin A} L_A(j) R_A(j-1) \\
&\quad - \lambda \sum_{j \notin A} \{L_A(j) R_A(j-1) \\
&\qquad + L_A(j) R_A(j-2) + L_A(j+1) R_A(j) + L_A(j+2) R_A(j)\} \\
&\quad + \{2\lambda f(2) + 2\lambda f(3) - (4\lambda + 2) f(1)\} \sum_{j, j+1 \notin A} L_A(j) R_A(j+1) \\
&\quad + \{2\lambda f(3) + 2\lambda f(4) - (4\lambda + 2) f(2) + f^2(1)\} \sum_{j, j+2 \notin A} L_A(j) R_A(j+2) \\
&\quad + \{2\lambda f(4) + 2\lambda f(5) - (4\lambda + 2) f(3) + 2 f(1) f(2)\} \sum_{j, j+3 \notin A} L_A(j) R_A(j+3) \\
&\quad + \{2\lambda f(5) + 2\lambda f(6) - (4\lambda + 2) f(4) + 2 f(1) f(3) + f^2(2)\} \sum_{j, j+4 \notin A} L_A(j) R_A(j+\cdot
\end{aligned} \tag{3.7}$$

Note that the expressions in brackets above could be simplified using (3.6). They are being left in the more cumbersome form for the time being in order to facilitate checking the computations up to this point. Using (3.2) and (3.3) again, we see that the second sum on the right-hand side of (3.5) is

$$\sum_{k \notin A} L_A(k) R_A(k). \tag{3.8}$$

The first expression on the right-hand side of (3.4) can be written as

$$\begin{aligned}
&\lambda \sum_{\substack{k<l \\ k \in A;\, k-2, l \notin A}} f(l-k+2) L_A(k-2) R_A(l) \\
&\quad + \lambda \sum_{\substack{k<l \\ k \in A;\, k-1, l \notin A}} f(l-k+1) L_A(k-1) R_A(l) \\
&\quad + \lambda \sum_{\substack{j<k \\ k \in A;\, j, k+1 \notin A}} f(k-j+1) L_A(j) R_A(k+1) \\
&\quad + \lambda \sum_{\substack{j<k \\ k \in A;\, j, k+2 \notin A}} f(k-j+2) L_A(j) R_A(k+2) \\
&\quad - \lambda f(3) \sum_{k \in A;\, k-2, k+1 \notin A} L_A(k-2) R_A(k+1) \\
&\quad - \lambda f(4) \sum_{k \in A;\, k-2, k+2 \notin A} L_A(k-2) R_A(k+2) \\
&\quad - \lambda f(2) \sum_{k \in A;\, k-1, k+1 \notin A} L_A(k-1) R_A(k+1) \\
&\quad - \lambda f(3) \sum_{k \in A;\, k-1, k+2 \notin A} L_A(k-1) R_A(k+2).
\end{aligned} \tag{3.9}$$

Next, use (3.2) and (3.3) to reexpress the first four terms in (3.9) as

$$
\begin{aligned}
&\lambda \sum_{k\in A,\, k-2\notin A} L_A(k-2)R_A(k-2) \\
&\quad + \lambda \sum_{k\in A,\, k-1\notin A} L_A(k-1)R_A(k-1) \\
&\quad + \lambda \sum_{k\in A,\, k+2\notin A} L_A(k+2)R_A(k+2) \\
&\quad + \lambda \sum_{k\in A,\, k+1\notin A} L_A(k+1)R_A(k+1) \\
&\quad - \lambda f(1) \sum_{k\in A;\, k-1,k-2\notin A} L_A(k-2)R_A(k-1) \\
&\quad - \lambda f(1) \sum_{k\in A;\, k+1,k+2\notin A} L_A(k+1)R_A(k+2).
\end{aligned} \tag{3.10}
$$

So, combining

$$(3.10) + \bigl(\text{negative terms in } (3.9)\bigr) + (3.8) - (3.7),$$

using (3.6) to simplify the coefficients of some of the sums in (3.7) and using the relation $f(1) = \lambda F(2)$ [which follows from (1.1)], we obtain the following expression for $Q(A)$:

$$
\begin{aligned}
&\sum_{k\notin A} \bigl\{\lambda L_A(k)\bigl[R_A(k-2)+R_A(k-1)\bigr] + \lambda\bigl[L_A(k+2)+L_A(k+1)\bigr]R_A(k)\bigr\} \\
&\quad -\bigl[1+2\lambda f(1)+2\lambda f(2)\bigr]\sum_{k\notin A} L_A(k)R_A(k) \\
&\quad + 2\lambda\,\bigl[f(1)+F(4)\bigr] \sum_{k,k+1\notin A} L_A(k)R_A(k+1) \\
&\quad - \lambda \sum_{k,k+1\notin A} \{2f(1)L_A(k+1)R_A(k) + L_A(k)R_A(k) + L_A(k+1)R_A(k+1)\} \\
&\quad - \lambda \sum_{k,k+2\notin A} \{L_A(k)R_A(k) + L_A(k+2)R_A(k+2) \\
&\qquad\qquad -2[f(2)+F(5)]L_A(k)R_A(k+2)\} \\
&\quad + \lambda \sum_{k,k+1,k+2\notin A} \bigl\{f(1)L_A(k)R_A(k+1) \\
&\qquad\qquad + f(1)L_A(k+1)R_A(k+2) + f(2)L_A(k)R_A(k+2)\bigr\} \\
&\quad + \lambda f(3) \sum_{k,k+1,k+3\notin A} L_A(k)R_A(k+3) + \lambda f(3) \sum_{k,k+2,k+3\notin A} L_A(k)R_A(k+3) \\
&\quad + \lambda f(4) \sum_{k,k+2,k+4\notin A} L_A(k)R_A(k+4).
\end{aligned} \tag{3.11}
$$

In showing that $Q(A) \geq 0$, we will need to use some monotonicity and convexity properties of the functions L_A and R_A. These are given in the next lemma.

LEMMA 3.12. *The functions L_A and R_A satisfy the following inequalities*:

(a)
$$L_A(k) \geq L_A(k-1)\big[1 - u(1)1_{\{k-1\in A\}}\big].$$

In particular,

$$L_A(k) \geq L_A(k-1)$$

if $k-1 \notin A$.

(b)
$$\begin{aligned} &2L_A(k) - L_A(k-1) - L_A(k+1) \\ &\quad\geq \big[u(3)+u(1)-2u(2)\big]L_A(k-2)1_{\{k-2\in A\}} \\ &\qquad - \big[2u(1)-u(2)\big]L_A(k-1)1_{\{k-1\in A\}} + u(1)L_A(k)1_{\{k\in A\}}. \end{aligned}$$

In particular,

$$L_A(k-1) + L_A(k+1) \leq 2L_A(k)$$

if $k-2, k-1, k \notin A$.

(c)
$$\begin{aligned} &L_A(m+2) + \big[f(1)-f(2)\big]L_A(m) + \big[f(2)-f(3)\big]L_A(m-1) \\ &\quad \leq \big[1+f(1)\big]L_A(m+1) \end{aligned}$$

if $m-1, m, m+1 \notin A$.

(a′)
$$R_A(k) \geq R_A(k+1)\big[1 - u(1)1_{\{k+1\in A\}}\big].$$

In particular,

$$R_A(k) \geq R_A(k+1)$$

if $k+1 \notin A$.

(b′)
$$\begin{aligned} 2R_A(k) - R_A(k+1) - R_A(k-1) &\\ \geq \big[u(3)+u(1)-2u(2)\big]\, &R_A(k+2)1_{\{k+2\in A\}} \\ - \big[2u(1)-u(2)\big]&R_A(k+1)1_{\{k+1\in A\}} \\ + u(1)&R_A(k)1_{\{k\in A\}}. \end{aligned}$$

In particular,

$$R_A(k-1) + R_A(k+1) \leq 2R_A(k)$$

if $k, k+1, k+2 \notin A$.

(c′)
$$R_A(m-2)+\left[f(1)-f(2)\right]R_A(m)+\left[f(2)-f(3)\right]R_A(m+1) \le \left[1+f(1)\right]R_A(m-1)$$

if $m-1, m, m+1 \notin A$.

PROOF. Statements (a), (b), (a′) and (b′) follow from the monotonicity and convexity properties of the renewal sequence $u(n)$ which are part of the hypothesis of Proposition 3, together with the following two relations:

$$\begin{aligned}1-L_A(k) &= \mu\{\eta\colon \eta(j)=1 \text{ for some } j\in A\cap(-\infty,k) \mid \eta(k)=1\}\\ &= \sum_{j<k, j\in A} u(k-j)L_A(j);\\ 1-R_A(k) &= \mu\{\eta\colon \eta(j)=1 \text{ for some } j\in A\cap(k,\infty) \mid \eta(k)=1\}\\ &= \sum_{j>k, j\in A} u(j-k)R_A(j).\end{aligned}$$

These relations follow from the renewal property and a decomposition of the event of interest according to the site in A which is furthest from k for which the configuration takes the value 1. To check the first statement in (a), for example, write

$$\begin{aligned}&L_A(k)-L_A(k-1)\\ &\quad= \sum_{j<k-1, j\in A} u(k-j-1)L_A(j) - \sum_{j<k, j\in A} u(k-j)L_A(j)\\ &\quad= \sum_{j<k-1, j\in A} \left[u(k-j-1)-u(k-j)\right] L_A(j)\\ &\qquad -u(1)L_A(k-1)1_{\{k-1\in A\}}.\end{aligned}$$

For the first statement in part (b), write

$$\begin{aligned}&2L_A(k)-L_A(k-1)-L_A(k+1)\\ &\quad= \sum_{j<k-1, j\in A} u(k-j-1)L_A(j) + \sum_{j<k+1, j\in A} u(k-j+1)L_A(j)\\ &\qquad -2\sum_{j<k, j\in A} u(k-j)L_A(j)\\ &\quad= \sum_{j<k-1, j\in A} \left[u(k-j-1)+u(k-j+1)-2u(k-j)\right]L_A(j)\\ &\qquad + \left[u(2)-2u(1)\right]L_A(k-1)1_{\{k-1\in A\}} + u(1)L_A(k)1_{\{k\in A\}}.\end{aligned}$$

For part (c), first use the renewal property to write

$$\begin{aligned} L_A(m) &= u(1)L_A(m-1) + L_{A\cup\{m-1\}}(m),\\ L_A(m+1) &= u(2)L_A(m-1) + u(1)L_{A\cup\{m-1\}}(m) + L_{A\cup\{m,m-1\}}(m+1),\\ L_A(m+2) &= u(3)L_A(m-1) + u(2)L_{A\cup\{m-1\}}(m)\\ &\quad + u(1)L_{A\cup\{m,m-1\}}(m+1) + L_{A\cup\{m+1,m,m-1\}}(m+2). \end{aligned}$$

Therefore part (c) is equivalent to the nonnegativity of

$$L_{A\cup\{m,m-1\}}(m+1) - L_{A\cup\{m+1,m,m-1\}}(m+2),$$

which is equal to

$$\sum_{k<m-1}\big[f(m-k+1) - f(m-k+2)\big]L_A(k).$$

This in turn is nonnegative because $f(n)$ is decreasing. □

At this point, we take $\lambda = 1$ to simplify the expressions which we must consider. For future reference, we record the values of $F(n), f(n)$ and $u(n)$ for small values of n in this case. The $F(n)$'s are obtained from (1.1). This determines the $f(n)$'s, and then the $u(n)$'s are obtained from (1.2). To get started, define $\beta = F(4)$;

$$\begin{aligned} &F(2) = \tfrac{1}{2}, \quad F(3) = \tfrac{1}{4}, \quad F(4) = \beta, \quad F(5) = \tfrac{1}{4} - \beta,\\ &F(6) = 3\beta - \tfrac{3}{8}, \quad F(7) = \tfrac{33}{32} - \tfrac{13}{2}\beta,\\ &f(1) = \tfrac{1}{2}, \quad f(2) = \tfrac{1}{4}, \quad f(3) = \tfrac{1}{4} - \beta, \quad f(4) = 2\beta - \tfrac{1}{4},\\ &f(5) = \tfrac{5}{8} - 4\beta, \quad f(6) = \tfrac{19}{2}\beta - \tfrac{45}{32},\\ &u(1) = \tfrac{1}{2}, \quad u(2) = \tfrac{1}{2}, \quad u(3) = \tfrac{5}{8} - \beta, \quad u(4) = \tfrac{5}{16} + \beta,\\ &u(5) = \tfrac{15}{16} - \tfrac{13}{4}\beta. \end{aligned} \tag{3.13}$$

The next task is to rearrange (3.11) so it is a sum over maximal intervals in the complement of A, since we will show that $Q(A) \geq 0$ by showing that the sum over each such interval is greater than or equal to 0. For most terms in (3.11) there is only one natural choice of a maximal interval with which to associate it. In some cases, however, there is more than one natural choice, and in these cases we must make some decisions. The first guiding principle is that a term which involves site k either in the argument of L_A or R_A or in the constraint in the summation can only be assigned to an interval $\{m, m+1, \ldots, n-1, n\}$ if $k \in \{m-2, m-1, \ldots, n+1, n+2\}$. The remaining ambiguity occurs when two adjacent intervals in the complement of A have a single point of A between them, say, $m \in A$ and $m-1, m+1 \notin A$. Then we assign terms which have not been already determined by the above principle to the two intervals according to Table 1.

TABLE 1

Left interval	Right interval
$L_A(m+1)R_A(m-1)$	$L_A(m+1)R_A(m-1)$
$L_A(m)R_A(m-1)$	$L_A(m+1)R_A(m)$
$-L_A(m+1)R_A(m+1)$	$-\frac{5}{2}L_A(m+1)R_A(m+1)$
$-\frac{5}{2}L_A(m-1)R_A(m-1)$	$-L_A(m-1)R_A(m-1)$
$\left(\frac{1}{2}-\beta\right)L_A(m-1)R_A(m+1)$	$\left(\frac{1}{2}-\beta\right)L_A(m-1)R_A(m+1)$

Now we begin to check the nonnegativity of the contributions to (3.11) which are associated with a given maximal interval $\{m, m+1, \dots, n-1, n\}$ in the complement of A. Thus we assume that $m-1, n+1 \in A$ and $m, m+1, \dots, n-1, n \notin A$. It is necessary to consider a number of cases, depending on the size of this interval and on whether $m-1$ and/or $n+1$ are isolated points in A.

CASE 1 ($m = n; m-2, m+2 \in A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are

$$L_A(m)R_A(m-2) + L_A(m)R_A(m-1) + L_A(m+2)R_A(m) + L_A(m+1)R_A(m) - \tfrac{5}{2}L_A(m)R_A(m). \tag{3.14}$$

To check that this is nonnegative, we need to apply Lemma 3.12 to a set which has more points in its complement than A does. Let

$$L(k) = L_{A\setminus\{m+1\}}(k) \quad \text{and} \quad R(k) = R_{A\setminus\{m-1\}}(k).$$

Then $L(m) = L_A(m)$ and $L(m+1) = L_A(m+1)$. Writing

$$\{\eta: \eta(k) = 0\, \forall\, k \in A \cap (-\infty, m]\} = \{\eta: \eta(k) = 0\, \forall\, k \in A \cap (-\infty, m+1]\} \cup \{\eta: \eta(m+1) = 1, \eta(k) = 0\, \forall\, k \in A \cap (-\infty, m]\},$$

and using the renewal property and (3.13), we see that

$$L(m+2) = L_A(m+2) + \tfrac{1}{2}L_A(m+1).$$

Similarly, $R(m) = R_A(m), R(m-1) = R_A(m-1)$ and

$$R(m-2) = R_A(m-2) + \tfrac{1}{2}R_A(m-1).$$

Using these relations, (3.14) becomes

$$L(m)R(m-2) + \tfrac{1}{2}L(m)R(m-1) + L(m+2)R(m) + \tfrac{1}{2}L(m+1)R(m) - \tfrac{5}{2}L(m)R(m). \tag{3.15}$$

By parts (a) and (a′) of Lemma 3.12,

$$L(m) \le L(m+1) \le L(m+2) \quad \text{and} \quad R(m) \le R(m-1) \le R(m-2).$$

Therefore, it is clear that (3.15) is nonnegative.

CASE 2 ($m = n; m-2 \notin A$ and $m+2 \in A$). (The case $m = n, m-2 \in A$ and $m+2 \notin A$ is similar.) From (3.11), we see that the contributions to $Q(A)$ from this interval are

$$\begin{aligned} &L_A(m)R_A(m-2) + L_A(m)R_A(m-1) + L_A(m+2)R_A(m) + L_A(m+1)R_A(m) \\ &\quad - L_A(m-2)R_A(m-2) - \tfrac{5}{2}L_A(m)R_A(m) + \left(\tfrac{1}{2} - \beta\right)L_A(m-2)R_A(m). \end{aligned} \tag{3.16}$$

This time we let

$$L(k) = L_{A \setminus \{m-1, m+1\}}(k) \quad \text{and} \quad R(k) = R_{A \setminus \{m-1, m+1\}}(k). \tag{3.17}$$

Partitioning the appropriate events as before and using (3.13) again, we now obtain the following relations:

$$\begin{aligned} L_A(m-2) &= L(m-2), \\ L_A(m-1) &= L(m-1), \\ L_A(m) &= L(m) - \tfrac{1}{2}L(m-1), \\ L_A(m+1) &= L(m+1) - \tfrac{1}{2}L(m-1), \\ L_A(m+2) &= L(m+2) - \tfrac{1}{2}L(m+1) - \left(\tfrac{3}{8} - \beta\right)L(m-1); \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} R_A(m+2) &= R(m+2), \\ R_A(m+1) &= R(m+1), \\ R_A(m) &= R(m) - \tfrac{1}{2}R(m+1), \\ R_A(m-1) &= R(m-1) - \tfrac{1}{2}R(m+1), \\ R_A(m-2) &= R(m-2) - \tfrac{1}{2}R(m-1) - \left(\tfrac{3}{8} - \beta\right)R(m+1). \end{aligned} \tag{3.19}$$

Application of Lemma 3.12 to $A \setminus \{m-1, m+1\}$ now gives the following inequalities:

$$\begin{gathered} 0 \le L(m-2) \le L(m-1) \le L(m) \le L(m+1) \le L(m+2), \\ L(m-1) + L(m+1) \le 2L(m), \\ L(m) + L(m+2) \le 2L(m+1); \\ 0 \le R(m+1) \le R(m) \le R(m-1) \le R(m-2), \\ R(m-2) + R(m) \le 2R(m-1), \\ \tfrac{2}{3}R(m-2) + \tfrac{1}{6}R(m) + \tfrac{2}{3}\beta R(m+1) \le R(m-1). \end{gathered} \tag{3.20}$$

Our task is now to show that (3.16) is nonnegative whenever L_A and R_A are related to L and R by (3.18) and (3.19), and L and R satisfy (3.20). Initially, we will regard the L's and R's as independent variables in the following differentiations, with the L_A's and R_A's being functions of them. As we proceed through the proof, we will find that we can impose relations among the L's and R's. Begin by computing

$$\begin{aligned}\frac{\partial(3.16)}{\partial L(m-2)} &= -R_A(m-2) + \left(\frac{1}{2} - \beta\right)R_A(m)\\ &= -R(m-2) + \frac{1}{2}R(m-1) + \left(\frac{1}{2} - \beta\right)R(m) + \left(\frac{1}{8} - \frac{1}{2}\beta\right)R(m+1)\\ &\le -\beta R(m) + \left(\frac{1}{8} - \frac{1}{2}\beta\right)R(m+1),\end{aligned}$$

by (3.20). Using (3.20) again, we see that this is nonpositive, since by (3.13), $f(4) \ge 0$ and $f(5) \ge 0$ imply that

$$\tfrac{1}{8} \le \beta \le \tfrac{5}{32}. \tag{3.21}$$

So, (3.16) is a decreasing function of $L(m-2)$. Therefore, for the remainder of the argument, we may assume by (3.20) that $L(m-2) = L(m-1)$, since if (3.16) is nonnegative when $L(m-2) = L(m-1)$, it will also be nonnegative when $L(m-2) \le L(m-1)$.

Assuming $L(m-2) = L(m-1)$, we next compute

$$\begin{aligned}\frac{\partial(3.16)}{\partial L(m-1)} &= -\frac{3}{2}R_A(m-2) - \frac{1}{2}R_A(m-1) + \frac{7}{8}R_A(m)\\ &= -\frac{3}{2}R(m-2) + \frac{1}{4}R(m-1) + \frac{7}{8}R(m) + \left(\frac{3}{8} - \frac{3}{2}\beta\right)R(m+1)\\ &\le -\tfrac{3}{8}R(m) + \left(\frac{3}{8} - \frac{3}{2}\beta\right)R(m+1),\end{aligned}$$

by (3.20). Another application of (3.20) implies that the above partial derivative is nonpositive. Therefore (3.16) is a decreasing function of $L(m-1)$; so for the rest of this case, we may assume by (3.20) that

$$L(m-2) = L(m-1) = 2L(m) - L(m+1).$$

Using this relation and regarding $L(k)$ for $k = m, m+1, m+2$ as independent variables, compute

$$\begin{aligned}\frac{\partial(3.16)}{\partial L(m)} &= -2R_A(m-2) - \frac{3}{4}R_A(m)\\ &= -2R(m-2) + R(m-1) + \frac{3}{4}R(m) + \left(\frac{9}{8} - 2\beta\right)R(m+1)\\ &\le -R(m-2) + \left(\frac{3}{8} - 2\beta\right)R(m+1),\\ &\le 0,\end{aligned}$$

where the inequalities follow from (3.20). Therefore, (3.16) is a decreasing function of $L(m)$, so we may assume by (3.20) that

$$L(m) = 2L(m+1) - L(m+2)$$
$$\text{and} \quad L(m-2) = L(m-1) = 3L(m+1) - 2L(m+2).$$

To complete the consideration of Case 2, we must now show that the following holds: (3.16) is nonnegative whenever $0 \le L(m+1) \le L(m+2)$; $L(m-2), L(m-1)$ and $L(m)$ are nonnegative and have the values given above; and the R's satisfy (3.20). Since (3.16) is [via (3.18)] linear in $L(m+1)$ and $L(m+2)$, it suffices to show the nonnegativity at the two extreme points, which by homogeneity we may take as

$$L(m+1) = L(m+2) = 1 \tag{3.22}$$

and

$$L(m+1) = 2, \qquad L(m+2) = 3. \tag{3.23}$$

In case (3.22), the expression in (3.16) is equal to

$$\begin{aligned} &-\tfrac{1}{2}R_A(m-2) + \tfrac{1}{2}R_A(m-1) - \tfrac{1}{8}R_A(m) \\ &\quad = -\tfrac{1}{2}R(m-2) + \tfrac{3}{4}R(m-1) - \tfrac{1}{8}R(m) - \tfrac{1}{2}\beta R(m+1), \end{aligned}$$

which is nonnegative by (3.20).

In case (3.23), the expression in (3.16) is

$$\begin{aligned} &= R_A(m-2) + R_A(m-1) + \tfrac{3}{2}R_A(m) \\ &= R(m-2) + \tfrac{1}{2}R(m-1) + \tfrac{3}{2}R(m) - \left(\tfrac{13}{8} - \beta\right)R(m+1), \end{aligned}$$

which is nonnegative by (3.20). Thus we conclude that (3.16) is nonnegative in Case 2.

CASE 3 ($m = n$; $m-2, m+2 \notin A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are

$$\begin{aligned} &L_A(m)R_A(m-2) + L_A(m)R_A(m-1) + L_A(m+2)R_A(m) + L_A(m+1)R_A(m) \\ &\quad - L_A(m-2)R_A(m-2) - \tfrac{5}{2}L_A(m)R_A(m) - L_A(m+2)R_A(m+2) \\ &\quad + \left(\tfrac{1}{2} - \beta\right)L_A(m-2)R_A(m) + \left(\tfrac{1}{2} - \beta\right)L_A(m)R_A(m+2) \\ &\quad + \left(2\beta - \tfrac{1}{4}\right)L_A(m-2)R_A(m+2). \end{aligned} \tag{3.24}$$

Making the same definition as in (3.17), the functions L and R again satisfy (3.18) and (3.19). Since now $m+2 \notin A$, Lemma 3.12 generates additional inequalities:

$$
(3.25)\quad
\begin{aligned}
&0 \le L(m-2) \le L(m-1) \le L(m) \le L(m+1) \le L(m+2),\\
&L(m-1) + L(m+1) \le 2L(m),\\
&L(m) + L(m+2) \le 2L(m+1),\\
&\tfrac{2}{3}L(m+2) + \tfrac{1}{6}L(m) + \tfrac{2}{3}\beta L(m-1) \le L(m+1);\\
&0 \le R(m+2) \le R(m+1) \le R(m) \le R(m-1) \le R(m-2),\\
&R(m-1) + R(m+1) \le 2R(m),\\
&R(m-2) + R(m) \le 2R(m-1),\\
&\tfrac{2}{3}R(m-2) + \tfrac{1}{6}R(m) + \tfrac{2}{3}\beta R(m+1) \le R(m-1).
\end{aligned}
$$

We proceed in much the same way as in the previous case. First compute

$$
\begin{aligned}
\frac{\partial(3.24)}{\partial L(m-2)} &= -R_A(m-2) + \left(\frac{1}{2} - \beta\right) R_A(m) + \left(2\beta - \frac{1}{4}\right) R_A(m+2)\\
&= -R(m-2) + \frac{1}{2}R(m-1) + \left(\frac{1}{2} - \beta\right) R(m) + \left(\frac{1}{8} - \frac{1}{2}\beta\right) R(m+1)\\
&\quad + \left(2\beta - \frac{1}{4}\right) R(m+2) \le -\beta R(m) + \left(\frac{3}{2}\beta - \frac{1}{8}\right) R(m+1),
\end{aligned}
$$

where the inequality follows from (3.25). This is nonpositive by (3.25) and (3.21). So, we may assume from now on that $L(m-2) = L(m-1)$. By symmetry, we may also assume that $R(m+2) = R(m+1)$.

Next, compute

$$
\begin{aligned}
\frac{\partial(3.24)}{\partial L(m-1)} &= -\frac{3}{2}R_A(m-2) - \frac{1}{2}R_A(m-1) + \frac{7}{8}R_A(m) + \left(\frac{3}{2}\beta - \frac{1}{8}\right) R_A(m+2)\\
&= -\frac{3}{2}R(m-2) + \frac{1}{4}R(m-1) + \frac{7}{8}R(m) + \frac{1}{4}R(m+1),
\end{aligned}
$$

which is nonpositive by (3.25). Therefore (3.24) is a decreasing function of $L(m-1)$, so we may assume henceforth that

$$
\begin{aligned}
L(m-2) &= L(m-1)\\
&= \min\left\{2L(m) - L(m+1), \frac{3}{2\beta}L(m+1) - \frac{1}{4\beta}L(m) - \frac{1}{\beta}L(m+2)\right\}.
\end{aligned}
$$

By symmetry, we may also assume that

$$
\begin{aligned}
R(m+2) &= R(m+1)\\
&= \min\left\{2R(m) - R(m+1), \frac{3}{2\beta}R(m-1) - \frac{1}{4\beta}R(m) - \frac{1}{\beta}R(m-2)\right\}.
\end{aligned}
$$

At this point, we may assume by homogeneity that $L(m+2) = 1$. Then the first half of the inequalities in (3.25) become

$$
L(m+1) \le 1, \qquad L(m) + 1 \le 2L(m+1), \qquad L(m+1) \le 2L(m),
$$
$$
4 + L(m) \le 6L(m+1).
$$

These determine a quadrilateral in the $(L(m), L(m+1))$ plane with vertices at $(\frac{4}{11}, \frac{8}{11})$, $(\frac{1}{2}, \frac{3}{4})$, $(\frac{1}{2}, 1)$ and $(1, 1)$. The corresponding values of $L(m-2) = L(m-1)$ are 0 in the first three cases and 1 in the last case. By (3.18), the values of $(L_A(m-2), L_A(m-1), L_A(m), L_A(m+1), L_A(m+2))$, in the four cases are then $(0, 0, \frac{4}{11}, \frac{8}{11}, \frac{7}{11})$, $(0, 0, \frac{1}{2}, \frac{3}{4}, \frac{5}{8})$, $(0, 0, \frac{1}{2}, 1, \frac{1}{2})$ and $(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{8} + \beta)$. By symmetry, these are also the extreme points of the set of possible values of the vector $(R_A(m+2), R_A(m+1), R_A(m), R_A(m-1), R_A(m-2))$. The value of (3.24) obtained by taking one extreme point from one set of vectors and another from the other are given in the following 4×4 matrix:

$$\begin{matrix} \frac{80}{121} & \frac{8}{11} & \frac{17}{22} & \frac{2}{11} \\ \frac{8}{11} & \frac{3}{4} & \frac{13}{16} & 0 \\ \frac{17}{22} & \frac{13}{16} & \frac{7}{8} & \frac{3}{16} \\ \frac{2}{11} & 0 & \frac{3}{16} & 0 \end{matrix}$$

Since all the entries are nonnegative and expression (3.24) is bilinear, it follows that (3.24) is nonnegative whenever the inequalities (3.25) are satisfied. This completes the consideration of Case 3.

We must now consider maximal intervals $\{m, \dots, n\}$ in A^c of length greater than 1. In each of Cases 4–9, we take

$$L(k) = L_{A\setminus\{n+1\}}(k) \quad \text{and} \quad R(k) = R_{A\setminus\{m-1\}}(k).$$

Then, as before,

$$L_A(n+2) = L(n+2) - \tfrac{1}{2}L(n+1) \quad \text{and} \quad L_A(k) = L(k) \quad \text{for } k \le n+1. \tag{3.26}$$

Similarly,

$$\begin{aligned} R_A(m-2) &= R(m-2) - \tfrac{1}{2}R(m-1), \\ R_A(k) &= R(k) \quad \text{for } k \ge m-1. \end{aligned} \tag{3.27}$$

These functions satisfy the following inequalities, by Lemma 3.12:

$$0 \le L(m) \le L(m+1) \le \cdots \le L(n+2); \tag{3.28}$$

$$\begin{aligned} L(k-1) + L(k+1) &\le 2L(k) \quad \text{for } m+2 \le k \le n+1, \\ 2L(m+1) - L(m) - L(m+2) &\ge \left(\tfrac{1}{8} - \beta\right) L(m-1); \end{aligned} \tag{3.29}$$

$$0 \le R(n) \le R(n-1) \le \cdots \le R(m-2); \tag{3.30}$$

$$\begin{aligned} R(k-1) + R(k+1) &\le 2R(k) \quad \text{for } m-1 \le k \le n-2, \\ 2R(n-1) - R(n) - R(n-2) &\ge \left(\tfrac{1}{8} - \beta\right) R(n+1); \end{aligned} \tag{3.31}$$

$$L(m-1) \le 2L(m) \quad \text{and} \quad R(n+1) \le 2R(n); \tag{3.32}$$

$$L(m-2) \le L(m-1) \quad \text{and} \quad \tfrac{1}{2}L(m-1) + L(m+1) \le 2L(m) \tag{3.33}$$

if $m-2 \notin A$;

$$R(n+2) \le R(n+1) \quad \text{and} \quad \tfrac{1}{2}R(n+1) + R(n-1) \le 2R(n) \tag{3.34}$$

if $n+2 \notin A$.

Cases 4–6 correspond to intervals of length 2, while Cases 7–9 correspond to longer intervals.

CASE 4 ($n = m+1$; $m-2, n+2 \in A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are [using (3.26) and (3.27)]

$$\begin{aligned} &L(m)R(m-2) + L(m+1)R(m-1) + L(m+2)R(m) + L(m+3)R(m+1) \\ &\quad + \tfrac{1}{2}L(m)R(m-1) + L(m+1)R(m) + \tfrac{1}{2}L(m+2)R(m+1) \\ &\quad - \tfrac{7}{2}L(m)R(m) - \tfrac{7}{2}L(m+1)R(m+1) + (1+2\beta)L(m)R(m+1). \end{aligned}$$

By (3.28) and (3.30), this is greater than or equal to

$$3L(m+1)R(m) + (1+2\beta)L(m)R(m+1) - 2L(m)R(m) - 2L(m+1)R(m+1).$$

Using (3.28) and (3.30) again, we see that this is nonnegative.

CASE 5 ($n = m+1$; $m-2 \notin A$, $m+3 \in A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are [using (3.26) and (3.27)]

$$\begin{aligned} &L(m)R(m-2) + L(m+1)R(m-1) + L(m+2)R(m) + L(m+3)R(m+1) \\ &\quad + \tfrac{1}{2}L(m)R(m-1) + L(m+1)R(m) + \tfrac{1}{2}L(m+2)R(m+1) \\ &\quad - \tfrac{7}{2}L(m)R(m) - \tfrac{7}{2}L(m+1)R(m+1) + (1+2\beta)L(m)R(m+1) \\ &\quad + \left(\tfrac{1}{2}-\beta\right)L(m-2)R(m) + \left(\tfrac{1}{4}-\beta\right)L(m-2)R(m+1) \\ &\quad - L(m-2)R(m-2) + \tfrac{1}{2}L(m-2)R(m-1). \end{aligned}$$

By (3.28) and (3.30), this is greater than or equal to

$$\begin{aligned} &L(m)\left[R(m-2) + \tfrac{3}{2}R(m-1) - \tfrac{3}{2}R(m) - (1-2\beta)R(m+1)\right] \\ &\quad + L(m-2)\left[-R(m-2) + \tfrac{1}{2}R(m-1) + \left(\tfrac{1}{2}-\beta\right)R(m)\right. \\ &\qquad \left. + \left(\tfrac{1}{4}-\beta\right)R(m+1)\right]. \end{aligned} \tag{3.35}$$

Since $L(m-2) \le 2L(m)$ by (3.32) and (3.33), the expression in (3.35) will be nonnegative if

$$R(m-2) + \tfrac{3}{2}R(m-1) - \tfrac{3}{2}R(m) - (1-2\beta)R(m+1) \ge 0$$

and

$$-R(m-2)+\tfrac{5}{2}R(m-1)-\left(\tfrac{1}{2}+2\beta\right)R(m)-\tfrac{1}{2}R(m+1)\geq 0.$$

The first of these follows from (3.30). For the second, use (3.30) and (3.31) to conclude that the expression is greater than or equal to

$$\tfrac{1}{4}R(m-2)+\left(\tfrac{1}{4}-2\beta\right)R(m).$$

The nonnegativity of this follows easily from (3.30) and (3.21).

CASE 6 ($n=m+1$; $m-2, m+3 \notin A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are [using (3.36) and (3.37)]

$$\begin{aligned} &L(m)R(m-2)+L(m+1)R(m-1)+L(m+2)R(m)+L(m+3)R(m+1)\\ &\quad+\tfrac{1}{2}L(m)R(m-1)+L(m+1)R(m)+\tfrac{1}{2}L(m+2)R(m+1)\\ &\quad-\tfrac{7}{2}L(m)R(m)-\tfrac{7}{2}L(m+1)R(m+1)+(1+2\beta)L(m)R(m+1)\\ &\quad+\left(\tfrac{1}{2}-\beta\right)L(m-2)R(m)+\left(\tfrac{1}{4}-\beta\right)L(m-2)R(m+1)\\ &\quad-L(m-2)R(m-2)+\tfrac{1}{2}L(m-2)R(m-1).\\ &\quad+\left(\tfrac{1}{2}-\beta\right)L(m+1)R(m+3)+\left(\tfrac{1}{4}-\beta\right)L(m)R(m+3)\\ &\quad-L(m+3)R(m+3)+\tfrac{1}{2}L(m+2)R(m+3). \end{aligned} \tag{3.36}$$

Constraints (3.28) become

$$0\leq L(m)\leq L(m+1)\leq L(m+2)\leq L(m+3). \tag{3.37}$$

The first constraint in (3.29) becomes

$$L(m+1)+L(m+3)\leq 2L(m+2), \tag{3.38}$$

while the second combines with the second in (3.33), to give

$$\left(\tfrac{3}{2}-4\beta\right)L(m)+L(m+2)\leq\left(\tfrac{9}{4}-2\beta\right)L(m+1). \tag{3.39}$$

Finally, the two parts of (3.33) combine to give

$$L(m-2)+2L(m+1)\leq 4L(m). \tag{3.40}$$

Similar constraints hold for the R's.

The coefficient of $L(m+2)$ in (3.36) is nonnegative, so by (3.38) we may assume that $2L(m+2)=L(m+1)+L(m+3)$. After doing so, the coefficient of $L(m+3)$ in (3.36) is

$$\tfrac{5}{4}R(m+1)+\tfrac{1}{2}R(m)-\tfrac{3}{4}R(m+3).$$

By the analogues for the R's of (3.37) and (3.40), this is nonnegative. Therefore, by (3.37), we may assume that $L(m+3) = L(m+1)$. Similar reductions can be made by symmetry, so at this point we may assume that $L(m+1) = L(m+2) = L(m+3) = 1$ and $R(m-2) = R(m-1) = R(m) = 1$, and that the following inequalities are satisfied:

$$\tfrac{1}{4}L(m-2) + \tfrac{1}{2} \le L(m) \le 1 \tag{3.41}$$

and

$$\tfrac{1}{4}R(m+3) + \tfrac{1}{2} \le R(m+1) \le 1. \tag{3.42}$$

After these reductions are made, the coefficient of $L(m)$ in (3.36) becomes

$$(1+2\beta)R(m+1) + \left(\tfrac{1}{4} - \beta\right)R(m+3) - 2,$$

which is less than or equal to

$$(2-2\beta)R(m+1) - \tfrac{5}{2} + 2\beta \le 0,$$

by (3.42). Therefore, by (3.41), we may take $L(m) = 1$, and by symmetry, $R(m+1) = 1$. With these choices, (3.36) becomes

$$2\beta + \left(\tfrac{1}{4} - 2\beta\right)\left[L(m-2) + R(m+3)\right].$$

This is nonnegative by (3.41), (3.42) and (3.21).

CASE 7 ($n > m+1$; $m-2, n+2 \in A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are [using (3.26) and (3.27)]

$$\begin{aligned}
&\sum_{k=m-1}^{n+1} L(k+1)R(k-1) + \sum_{k=m+1}^{n-1} L(k+1)R(k-1) \\
&\quad + \sum_{k=m+1}^{n} L(k)R(k-1) + \tfrac{1}{2}L(m)R(m-1) + \tfrac{1}{2}L(n+1)R(n) \\
&\quad - \tfrac{9}{2}\sum_{k=m}^{n} L(k)R(k) - \sum_{k=m+1}^{n-2} L(k)R(k) - \sum_{k=m+2}^{n-1} L(k)R(k) \\
&\quad + \left(\tfrac{3}{2} + 2\beta\right)\sum_{k=m}^{n-1} L(k)R(k+1) + \tfrac{1}{2}\sum_{k=m+1}^{n-2} L(k)R(k+1) \\
&\quad + \left(\tfrac{5}{4} - 2\beta\right)\sum_{k=m+1}^{n-1} L(k-1)R(k+1) \\
&\quad + \left(\tfrac{1}{2} - 2\beta\right)\sum_{k=m+1}^{n-2} L(k-1)R(k+2) \\
&\quad + \left(2\beta - \tfrac{1}{4}\right)\sum_{k=m+2}^{n-2} L(k-2)R(k+2).
\end{aligned} \tag{3.43}$$

By the bilinearity of this expression, in order to show that it is nonnegative whenever (3.28) and (3.30) are satisfied, it is enough to show that it is nonnegative when

$$L(k) = 1_{\{k \geq l\}} \quad \text{and} \quad R(k) = 1_{\{k \leq r\}}, \tag{3.44}$$

where $m \leq l \leq n+2$ and $m-2 \leq r \leq n$. All terms in (3.43) are nonnegative unless $m \leq l \leq r \leq n$, so we assume this throughout. With these choices, (3.43) becomes

$$\begin{aligned}
&(r-l+3) + \big[\min(n-1, r+1) - \max(m+1, l-1) + 1\big] \\
&\quad + \big[\min(n, r+1) - \max(m+1, l) + 1\big] + \tfrac{1}{2}1_{\{l=m\}} + \tfrac{1}{2}1_{\{r=n\}} - \tfrac{9}{2}(r-l+1) \\
&\quad - \big[\min(n-2, r) - \max(m+1, l) + 1\big]^+ - \big[\min(n-1, r) - \max(m+2, l) + 1\big]^+ \\
&\quad + \big(\tfrac{3}{2} + 2\beta\big)(r-l) + \tfrac{1}{2}\big[\min(n-2, r-1) - \max(m+1, l) + 1\big]^+ \\
&\quad + \big(\tfrac{5}{4} - 2\beta\big)(r-l-1)^+ + \big(\tfrac{1}{2} - 2\beta\big)(r-l-2)^+ + \big(2\beta - \tfrac{1}{4}\big)(r-l-3)^+.
\end{aligned}$$

If $r = l$, this becomes $1 + \frac{1}{2}1_{\{m<r<n\}}$. If $r = l+1$, it becomes 2β. If $r = l+2$ it becomes $2\beta - \frac{1}{4}$. If $r \geq l+3$, it is zero. Thus in all cases it is nonnegative.

Case 8 ($n > m+1$; $m-2 \notin A$, $n+2 \in A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are [using (3.26) and (3.27)]

$$\begin{aligned}
&\sum_{k=m-1}^{n+1} L(k+1)R(k-1) + \sum_{k=m+1}^{n-1} L(k+1)R(k-1) \\
&\quad + \sum_{k=m+1}^{n} L(k)R(k-1) + \tfrac{1}{2}L(m)R(m-1) + \tfrac{1}{2}L(n+1)R(n) \\
&\quad - \tfrac{9}{2}\sum_{k=m}^{n} L(k)R(k) - \sum_{k=m+1}^{n-2} L(k)R(k) - \sum_{k=m+2}^{n-1} L(k)R(k) \\
&\quad + \big(\tfrac{3}{2} + 2\beta\big)\sum_{k=m}^{n-1} L(k)R(k+1) + \tfrac{1}{2}\sum_{k=m+1}^{n-2} L(k)R(k+1) \\
&\quad + \big(\tfrac{5}{4} - 2\beta\big)\sum_{k=m+1}^{n-1} L(k-1)R(k+1) + \big(\tfrac{1}{2} - 2\beta\big)\sum_{k=m+1}^{n-2} L(k-1)R(k+2) \\
&\quad + \big(2\beta - \tfrac{1}{4}\big)\sum_{k=m+2}^{n-2} L(k-2)R(k+2) \\
&\quad + \big(\tfrac{1}{2} - \beta\big)L(m-2)R(m) + \big(\tfrac{1}{4} - \beta\big)L(m-2)R(m+1) \\
&\quad + \tfrac{1}{2}L(m-2)R(m-1) \\
&\quad + \big(2\beta - \tfrac{1}{4}\big)L(m-2)R(m+2) - L(m-2)R(m-2).
\end{aligned} \tag{3.45}$$

By (3.32) and (3.33), $0 \leq L(m-2) \leq 2L(m)$, so we need to check the nonnegativity of (3.45) at the two extreme points. If $L(m-2) = 0$, then (3.45) is the same as (3.43), so we are back to the previous case. So, we may take $L(m-2) = 2L(m)$. The coefficient of $R(m-1)$ in (3.45) is nonnegative, so by the first constraint in (3.31) with $k = m-1$ we may assume that $2R(m-1) = R(m-2) + R(m)$. After doing so, the coefficient of $R(m-2)$ is

$$\tfrac{5}{4}L(m) + \tfrac{1}{2}L(m+1) - L(m-2) \geq -\tfrac{11}{4}L(m) + \tfrac{5}{2}L(m+1),$$

where the inequality comes from (3.33). The right-hand side above is nonnegative by (3.28). Therefore, by (3.30) we may assume that $R(m-2) = R(m-1) = R(m)$. Now we can proceed as in Case 7, letting $L(k)$ and $R(k)$ be defined by (3.44) for $k \geq m$, where $m \leq l \leq r \leq n$, and checking the nonnegativity of (3.45) in each case. The result of the computation is the following expression for (3.45):

$$\tfrac{3}{2}1_{\{r=l\}} - \left(\tfrac{1}{2} + 2\beta\right)1_{\{r=l=m\}} - \tfrac{1}{2}1_{\{r=l=n\}}$$
$$+2\beta 1_{\{r=l+1\}} - \left(4\beta - \tfrac{1}{2}\right)1_{\{r=m+1,\,l=m\}} + \left(2\beta - \tfrac{1}{4}\right)1_{\{r=l+2\}}.$$

This is nonnegative by (3.21).

CASE 9 ($n > m+1$; $m-2, n+2 \notin A$). From (3.11), we see that the contributions to $Q(A)$ from this interval are [using (3.26) and (3.27)]

$$\sum_{k=m-1}^{n+1} L(k+1)R(k-1) + \sum_{k=m+1}^{n-1} L(k+1)R(k-1)$$
$$+ \sum_{k=m+1}^{n} L(k)R(k-1) + \tfrac{1}{2}L(m)R(m-1) + \tfrac{1}{2}L(n+1)R(n)$$
$$- \tfrac{9}{2}\sum_{k=m}^{n} L(k)R(k) - \sum_{k=m+1}^{n-2} L(k)R(k) - \sum_{k=m+2}^{n-1} L(k)R(k)$$
$$+ \left(\tfrac{3}{2} + 2\beta\right)\sum_{k=m}^{n-1} L(k)R(k+1) + \tfrac{1}{2}\sum_{k=m+1}^{n-2} L(k)R(k+1)$$
$$+ \left(\tfrac{5}{4} - 2\beta\right)\sum_{k=m+1}^{n-1} L(k-1)R(k+1) + \left(\tfrac{1}{2} - 2\beta\right)\sum_{k=m+1}^{n-2} L(k-1)R(k+2)$$
$$+ \left(2\beta - \tfrac{1}{4}\right)\sum_{k=m+2}^{n-2} L(k-2)R(k+2)$$
$$+ \left(\tfrac{1}{2} - \beta\right)L(m-2)R(m) + \left(\tfrac{1}{4} - \beta\right)L(m-2)R(m+1)$$
$$+ \tfrac{1}{2}L(m-2)R(m-1) + \left(2\beta - \tfrac{1}{4}\right)L(m-2)R(m+2) - L(m-2)R(m-2)$$
$$+ \left(\tfrac{1}{2} - \beta\right)L(n)R(n+2) + \left(\tfrac{1}{4} - \beta\right)L(n-1)R(n+2) + \tfrac{1}{2}L(n+1)R(n+2)$$
$$+ \left(2\beta - \tfrac{1}{4}\right)L(n-2)R(n+2) - L(n+2)R(n+2).$$

This case is handled much like the previous one. The reduction which was made at the left boundary in the previous case is now made at both boundaries before computing the above expression explicitly in case L and R are given by (3.44). The result of the computation is the following expression:

$$\tfrac{3}{2}1_{\{r=l\}} - \left(\tfrac{1}{2}+2\beta\right)1_{\{r=l=m\}} - \left(\tfrac{1}{2}+2\beta\right)1_{\{r=l=n\}} + 2\beta 1_{\{r=l+1\}}$$
$$- \left(4\beta-\tfrac{1}{2}\right)1_{\{r=m+1,l=m\}} - \left(4\beta-\tfrac{1}{2}\right)1_{\{l=n-1,r=n\}} + \left(2\beta-\tfrac{1}{4}\right)1_{\{r=l+2\}}.$$

This is nonnegative by (3.21).

4. Proof of Proposition 4. Throughout this section, we take $\lambda = 1$. Once we have let $\beta = F(4)$, equations (1.1) can be solved recursively for $F(n)$ for $n \geq 2$. The first few values are given in (3.13). As can be seen from those values, the requirement that $F(n)$ be positive and decreasing imposes conditions on β. We need to show that there is a choice of β such that the corresponding solution has the required properties. Our first task is to show that there is a choice of β which makes the solution bounded. To do so, define the generating function $\phi(x)$ of the sequence $F(n)$ by

$$\phi(x) = \sum_{n=1}^{\infty} F(n)x^n. \tag{4.1}$$

Multiply (1.1) by x^{n+1} and sum for $n \geq 4$, to get

$$\sum_{l,m\geq 1;\, l+m\geq 5} F(l)F(m)x^{l+m} = 4\sum_{n=5}^{\infty} F(n)x^n + \frac{2}{x}\sum_{n=6}^{\infty} F(n)x^n.$$

Using (4.1) and the values of $F(n)$ for $1 \leq n \leq 5$ from (3.13), this becomes

$$\phi^2(x) - \left(4 + \frac{2}{x}\right)\phi(x) + 2 + 5x + \frac{3}{2}x^2 + 2\beta x^3 + \left(2\beta - \frac{1}{4}\right)x^4 = 0.$$

Solving this quadratic for $\phi(x)$ gives the following expression, after some algebra:

$$\phi(x) = \frac{1 + 2x - \sqrt{P(x)}}{x},$$

where

$$P(x) = 1 + 4x + 2x^2 - 5x^3 - \tfrac{3}{2}x^4 - 2\beta x^5 - \left(2\beta - \tfrac{1}{4}\right)x^6. \tag{4.2}$$

[The negative sign in front of the root is chosen, since otherwise $\phi(x)$ would be unbounded near the origin.]

Next, regard $P(x) = 0$ and $P'(x) = 0$ as two equations in the two unknowns x and β:

$$1 + 4x + 2x^2 - 5x^3 - \tfrac{3}{2}x^4 + \tfrac{1}{4}x^6 = 2\beta x^5(1+x) \tag{4.3}$$

and

$$4 + 4x - 15x^2 - 6x^3 + \tfrac{3}{2}x^5 = 2\beta x^4(5 + 6x).$$

Eliminating β gives

$$x^6 + 12x^5 + 66x^4 + 8x^3 - 104x^2 - 88x - 20 = 0.$$

This polynomial is negative at 0 and positive at -1, so it has a root in $(-1, 0)$. Mathematica gives this root as $x_0 = -0.4254654735\ldots$. Using this in (4.3) gives

$$\beta = \frac{x_0^6 - 6x_0^4 - 20x_0^3 + 8x_0^2 + 16x_0 + 4}{8x_0^5(x_0 + 1)} = 0.1497729\ldots.$$

With this definition of x_0 and β, $P(x)$ has a double root at x_0. Therefore, it can be factored as

$$P(x) = (x - x_0)^2(a_0 - a_1x - a_2x^2 - a_3x^3 - a_4x^4),$$

where $a_0 = 5.524\ldots$, $a_1 = 3.871\ldots$, $a_2 = 1.272\ldots$, $a_3 = 0.257\ldots$ and $a_4 = 0.0495\ldots$. Since $a_0 > |a_1| + |a_2| + |a_3| + |a_4|$, the image of the closed unit disk in the complex plane under $a_0 - a_1z - a_2z^2 - a_3z^3 - a_4z^4$ lies entirely in the right half-plane. Therefore, for this choice of β (which we use from now on), $\phi(z)$ is analytic in a neighborhood of the closed unit disk, and hence $F(n)$ decays exponentially rapidly (so, in particular, it is bounded).

In order to show that $F(n)$ is decreasing, we will need an explicit expression for $F(n)$. To obtain one, use the expansion

$$\sqrt{1-t} = 1 - 2\sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!}\left(\frac{1}{4}t\right)^n$$

to write

$$\begin{aligned}
&\sqrt{a_0 - a_1x - a_2x^2 - a_3x^3 - a_4x^4} \\
&\quad = \sqrt{a_0}\left[1 - 2\sum_{n=1}^{\infty} \frac{(2n-2)}{n!(n-1)!}\left(\frac{a_1x + a_2x^2 + a_3x^3 + a_4x^4}{4a_0}\right)^n\right] \\
&\quad = \sqrt{a_0}\left[1 - 2\sum_{\substack{j+k+l+m\geq 1\\ j,k,l,m\geq 0}} \frac{(2j+2k+2l+2m-2)!}{(j+k+l+m-1)!j!k!l!m!}\right. \\
&\qquad\qquad \left.\times \frac{a_1^j a_2^k a_3^l a_4^m}{(4a_0)^{j+k+l+m}} x^{j+2k+3l+4m}\right].
\end{aligned} \tag{4.4}$$

This suggests that we define

$$S(n) = \sum_{\substack{j+2k+3l+4m=n\\ j,k,l,m\geq 0}} \frac{(2j+2k+2l+2m-2)!}{(j+k+l+m-1)!j!k!l!m!} \frac{a_1^j a_2^k a_3^l a_4^m}{(4a_0)^{j+k+l+m}}.$$

Using (4.1) and the expressions for $P(x)$ and $\phi(x)$ above, we can then write

$$F(n) = 2\sqrt{a_0}S(n) + 2S(n+1) \tag{4.5}$$

for $n \geq 1$. Thus, in order to show that $F(n)$ is decreasing, for example, it is enough to show that $S(n)$ is decreasing.

Before doing so, we derive a relationship between the renewal sequence $u(n)$ and $F(n)$, since we will need to check some of its properties. Equation (1.2), which relates $u(n)$ to $f(n)$ [or, equivalently, to $F(n)$], is nonlinear. However, it is possible to use the fact that $f(n)$ satisfies (3.6) to replace this by a linear relation. In order to do this, take $n \geq 4$ and use (1.2) twice, (3.6) and (3.13) to write

$$\begin{aligned} u(n) &= \sum_{k=1}^{n} f(k)u(n-k) \\ &= f(n) + \sum_{k=1}^{n-1} f(k) \sum_{j=1}^{n-k} f(j)u(n-k-j) \\ &= f(n) + \sum_{i=0}^{n-2} u(i) \sum_{k=1}^{n-i-1} f(k)f(n-k-i) \\ &= f(n) + \tfrac{1}{4}u(n-2) + \tfrac{1}{4}u(n-3) + \left(\tfrac{5}{16} - \beta\right) u(n-4) \\ &\quad + \sum_{i=0}^{n-5} u(i)\left[6f(n-i) - 2f(n-i+1) - 2f(n-i+2)\right]. \end{aligned}$$

Using (1.2) again and (3.13), and then simplifying, one obtains

$$\begin{aligned} f(n) &= 2u(n+2) + u(n+1) - \tfrac{13}{2}u(n) + (2+2\beta)u(n-1) \\ &\quad + \left(\tfrac{5}{4} - 2\beta\right) u(n-2) + \left(\tfrac{1}{2} - 2\beta\right) u(n-3) + \left(2\beta - \tfrac{1}{4}\right) u(n-4). \end{aligned} \tag{4.6}$$

Since $f(n) = F(n) - F(n+1)$, this can be rewritten [assuming for the time being that $f(n) \geq 0$, so that the limit of $u(n)$ exists by the renewal theorem] as

$$\begin{aligned} F(n) &= -2u(n+1) - 3u(n) + \tfrac{7}{2}u(n-1) \\ &\quad + \left(\tfrac{3}{2} - 2\beta\right) u(n-2) + \tfrac{1}{4}u(n-3) + \left(2\beta - \tfrac{1}{4}\right) u(n-4), \end{aligned} \tag{4.7}$$

for $n \geq 4$. Letting $\Delta(n) = u(n) - u(n+1)$ for $n \geq 0$, (4.7) can be rewritten as

$$\begin{aligned} F(n) &= 2\Delta(n) + 5\Delta(n-1) + \tfrac{3}{2}\Delta(n-2) + 2\beta\Delta(n-3) \\ &\quad + \left(2\beta - \tfrac{1}{4}\right)\Delta(n-4). \end{aligned} \tag{4.8}$$

Write (4.8) as $\mathbf{F} = M\boldsymbol{\Delta}$, where $\mathbf{F}$ and $\boldsymbol{\Delta}$ are the column vectors with entries $F(n)$ for $n \geq 1$ and $\Delta(n)$ for $n \geq 0$, respectively, and M is the matrix with entries

$m_{k,j}$, for $k, j \geq 1$, given by the following:

$$m_{1,1} = 2; \qquad m_{2,1} = \tfrac{3}{2} - 4\beta; \qquad m_{3,1} = \tfrac{1}{2} - 2\beta;$$

$$m_{k,k+1} = 2 \quad \text{for } k \geq 1; \qquad m_{k,k} = 5 \quad \text{for } k \geq 2; \qquad m_{k,k-1} = \tfrac{3}{2} \quad \text{for } k \geq 3;$$

$$m_{k,k-2} = 2\beta \quad \text{for } k \geq 4; \qquad m_{k,k-3} = 2\beta - \tfrac{1}{4} \quad \text{for } k \geq 4;$$

and all other $m_{k,j} = 0$. We wish to invert this matrix, in order to be able to solve (4.8) for $\Delta(n)$. We will look for an inverse P with entries $p_{k,j}$ of the form

$$p_{k,j} = (-1)^{j-k}\left[c_k p^j - b_{k-j}\right],$$

where $b_k = 0$ if $k \leq 0$ and p is the unique root of absolute value smaller than one of the polynomial

$$2 - 5z + \tfrac{3}{2}z^2 - 2\beta z^3 + \left(2\beta - \tfrac{1}{4}\right)z^4.$$

(Mathematica gives $p = 0.45748\ldots$.) Writing out $PM = I$ gives the following equations:

$$4c_1(1 - 2p) = p \quad \text{and} \quad 2b_1 = 1; \tag{4.9}$$

$$\begin{aligned} 4c_k \frac{1-2p}{p} &= 2b_{k-1} - \left(\frac{3}{2} - 4\beta\right)b_{k-2} + \left(\frac{1}{2} - 2\beta\right)b_{k-3} \\ &\quad - \left(2\beta - \frac{1}{4}\right)b_{k-4} \quad \text{for } k > 1; \end{aligned} \tag{4.10}$$

$$2b_{n+1} - 5b_n + \frac{3}{2}b_{n-1} - 2\beta b_{n-2} + \left(2\beta - \frac{1}{4}\right)b_{n-3} = 0 \quad \text{for } n \geq 1. \tag{4.11}$$

Next we need to solve (4.11). Let p^{-1}, q, w and $\overline{w}$ be the roots of the polynomial

$$2z^4 - 5z^3 + \tfrac{3}{2}z^2 - 2\beta z + 2\beta - \tfrac{1}{4}. \tag{4.12}$$

(Mathematica gives the following approximate values: $p^{-1} = 2.18586$, $q = 0.244625$, $w = 0.0347572 + 0.212417i$). Then the solution to (4.11) is of the form

$$b_n = Ap^{-n} + Bq^n + \Re\left(Cw^n\right), \tag{4.13}$$

for $n \geq -2$, where A and B are real and C is complex. These coefficients are chosen so that (4.11) is satisfied for small values of n. (Mathematica gives the following approximate values for them: $A = 0.2634$, $B = -0.1729$, $C = -0.0905 + 0.1427i$.) Using this expression for b_n in (4.10) gives

$$c_k = Ap^{-k} + Dq^k + \Re\left(Ew^k\right),$$

for $k > 1$, where

$$(4.14)\quad D = \frac{Bp(q-2)}{1-2p} \approx 1.633 \quad \text{and} \quad E = \frac{Cp(w-2)}{1-2p} \approx 0.7937 - 1.612i.$$

In this computation, we have used several times the fact that p^{-1}, q and w are roots of (4.12). Now, we can complete the computation of

$$(4.15)\quad p_{k,j} = A(-p)^{j-k} + D(-q)^k(-p)^j + (-p)^j\Re\big[E(-w)^k\big] \quad \text{if } 1 < k \le j,$$

and

$$(4.16)\quad \begin{aligned} p_{k,j} &= D(-p)^j(-q)^k - B(-q)^{k-j} \\ &\quad + \Re\big[E(-p)^j(-w)^k - C(-w)^{k-j}\big], \quad \text{if } 1 \le j < k. \end{aligned}$$

Letting $p_{k,0} = 0$ and using (4.5), we obtain the following for $k \ge 1$:

$$(4.17)\quad \begin{aligned} \Delta(k) &= \sum_{j=1}^{\infty} p_{k+1,j}F(j) \\ &= 2\sum_{j=1}^{\infty}\big[\sqrt{a_0}\,p_{k+1,j} + p_{k+1,j-1}\big]S(j). \end{aligned}$$

When we use the expressions for $p_{k,j}$ given in (4.15) and (4.16) to evaluate the right-hand side of (4.17), there will be terms which have a factor of A, terms which have a factor of B and so on. We will compute these separately. The ones which have a factor of D are

$$\begin{aligned} &-2pD\sqrt{a_0}S(1)(-q)^{k+1} + 2D\big(\sqrt{a_0} - p^{-1}\big)(-q)^{k+1}\sum_{j=2}^{\infty}(-p)^jS(j) \\ &\quad = GD(-q)^{k+1}, \end{aligned}$$

where

$$G = -2p\sqrt{a_0}S(1) + 2\big(\sqrt{a_0} - p^{-1}\big)\sum_{j=2}^{\infty}(-p)^jS(j).$$

Similarly, the terms in (4.17) which have a factor of E are $G\Re\big[E(-w)^{k+1}\big]$.

The terms in (4.17) which have a factor of A, B or C are

$$\begin{aligned} &2A\big(\sqrt{a_0} - p^{-1}\big)\sum_{j=k+2}^{\infty}(-p)^{j-k-1}S(j) \\ &\quad - 2B\big(\sqrt{a_0} - q\big)\sum_{j=1}^{k+1}(-q)^{k+1-j}S(j) + 2BS(1)(-q)^{k+1} \\ &\quad - 2\Re C\big(\sqrt{a_0} - w\big)\sum_{j=1}^{k+1}(-w)^{k+1-j}S(j) + 2\Re CS(1)(-w)^{k+1}. \end{aligned}$$

In this computation, we have used the relation $A + B + \Re C = 0$, which follows from (4.13) with $n = 0$.

Next, we compute G. By (4.4) and the expression given above for G,

$$G = -2S(1) + \left(\sqrt{a_0} - p^{-1}\right)\left(\sqrt{a_0} - \sqrt{a_0 + a_1 p - a_2 p^2 + a_3 p^3 - a_4 p^4}\right)\Big/\sqrt{a_0}.$$

To evaluate the right-hand side of this expression, write

$$P(-p) = (p + x_0)^2 (a_0 + a_1 p - a_2 p^2 + a_3 p^3 - a_4 p^4).$$

From the definition of p,

$$2 - 5p + \tfrac{3}{2} p^2 - 2\beta p^3 + \left(2\beta - \tfrac{1}{4}\right) p^4 = 0.$$

Therefore, by (4.2),

$$\begin{aligned} P(-p) &= 1 - 4p + 2p^2 + 5p^3 - \tfrac{3}{2} p^4 + 2\beta p^5 - \left(2\beta - \tfrac{1}{4}\right) p^6 \\ &= (1 - 2p)^2. \end{aligned}$$

Combining these two expressions for $P(-p)$ yields

$$a_0 + a_1 p - a_2 p^2 + a_3 p^3 - a_4 p^4 = \left(\frac{1 - 2p}{p + x_0}\right)^2.$$

Therefore

$$G = -2S(1) + \left(\sqrt{a_0} - p^{-1}\right)\left(1 - \frac{1 - 2p}{(p + x_0)\sqrt{a_0}}\right).$$

Now, $S(1) = a_1/(4a_0)$, $a_0 x_0^2 = 1$ and $2a_0 x_0 + a_1 x_0^2 = -4$, so that $S(1) = \frac{1}{2}\sqrt{a_0} - 1$. Therefore, a short computation yields

$$G = 4 - 2p^{-1}.$$

From the definitions of D and E, we conclude that

$$GD = 2(2 - q)B \quad \text{and} \quad GE = 2(2 - w)C.$$

Combining the above expressions for the various parts of (4.17) then leads to the following:

$$\begin{aligned} &\Delta(k) - \Delta(k+1) \\ &\quad = 2B(2 - q)(1 + q)(-q)^{k+1} + 2\Re C(2 - w)(1 + w)(-w)^{k+1} \\ &\qquad + 2A\left(1 + p^{-1}\right)\left(\sqrt{a_0} - p^{-1}\right) \sum_{j=k+3}^{\infty} (-p)^{j-k-1} S(j) \\ &\qquad + 2\left[A\left(1 - p\sqrt{a_0}\right) + B\left(\sqrt{a_0} - q\right) + \Re C\left(\sqrt{a_0} - w\right)\right] S(k+2) \qquad (4.18) \\ &\qquad - 2B(1 + q)\left(\sqrt{a_0} - q\right) \sum_{j=1}^{k+1} (-q)^{k+1-j} S(j) + 2B(1 + q) S(1)(-q)^{k+1} \\ &\qquad - 2\Re C(1 + w)\left(\sqrt{a_0} - w\right) \sum_{j=1}^{k+1} (-w)^{k+1-j} S(j) + 2\Re C(1 + w) S(1)(-w)^{k+1}. \end{aligned}$$

We must show that this expression is nonnegative for $k \geq 2$. In order to do so, we need some information about the behavior of the sequence $S(j)$. The required result is contained in Lemma 4.19. By (4.5) this result also implies the monotonicity of $F(n)$, which had been left unproved earlier.

LEMMA 4.19. *Let $\sigma \approx 0.990075$ be a root of the polynomial*

$$a_0x^4 - a_1x^3 - a_2x^2 - a_3x - a_4$$

[so that $P(\sigma^{-1}) = 0$]. Then

$$0 \leq \sigma - \frac{S(n+1)}{S(n)} \leq \frac{7.58}{n}.$$

Before proving the lemma, we will use it to complete the verification of the nonnegativity of (4.18). The idea is to show first that it is nonnegative for large k, and then to compute the left-hand side for the remaining finitely many values of k. So, using the lemma, compute

$$\begin{aligned}
L &= \lim_{k\to\infty} \frac{\Delta(k) - \Delta(k+1)}{S(k+2)} \\
&= 2A(1+p^{-1})(\sqrt{a_0} - p^{-1}) \sum_{i=0}^{\infty} (-p)^{i+2}\sigma^{i+1} \\
&\quad - 2B(1+q)(\sqrt{a_0} - q) \sum_{i=0}^{\infty} (-q)^i \sigma^{-i-1} - 2\Re C(1+w)(\sqrt{a_0} - w) \sum_{i=0}^{\infty} (-w)^i \sigma^{-i-1} \\
&\quad + 2A(1 - p\sqrt{a_0}) + 2B(\sqrt{a_0} - q) + 2\Re C(\sqrt{a_0} - w) \\
&= -2(1-\sigma)\left(\frac{A(p\sqrt{a_0} - 1)}{1 + p\sigma} + \frac{B(\sqrt{a_0} - q)}{\sigma + q} + \Re\frac{C(\sqrt{a_0} - w)}{\sigma + w} \right) \approx 0.0075.
\end{aligned}$$

This implies that (4.18) is positive for large k. Next we need to know how large k must be.

Iterating the inequalities in the lemma gives

$$\left(\sigma - \frac{7.58}{n}\right)^m \leq \frac{S(n+m)}{S(n)} \leq \sigma^m. \tag{4.20}$$

This implies that

$$\frac{S(n+m)}{S(n)} \geq \frac{1}{2^m},$$

for $n \geq 16$. To include all n, we use the inequality

$$\frac{S(n)}{S(n+m)} \leq \frac{S(1)}{S(16)} 2^m. \tag{4.21}$$

Now take $k \geq 35$. Then (4.18), (4.20) and (4.21) imply that

$$
\begin{aligned}
&\frac{\Delta(k)-\Delta(k+1)}{S(k+2)}\\
&\quad \geq \frac{2B(2-q+S(1))(1+q)}{S(16)}(2q)^{k+1} - \frac{2\left|C(2-w+S(1))(1+w)\right|}{S(16)}\left(2|w|\right)^{k+1}\\
&\qquad + 2A(p+1)\left(p\sqrt{a_0}-1\right)\left[\frac{\sigma-7.58/(k+2)}{1-p^2\left(\sigma-7.58/(k+2)\right)^2} - \frac{p\sigma^2}{1-p^2\sigma^2}\right]\\
&\qquad + 2A\left(1-p\sqrt{a_0}\right) + 2B\left(\sqrt{a_0}-q\right) + 2\Re C\left(\sqrt{a_0}-w\right) - 2B(1+q)\left(\sqrt{a_0}-q\right)\\
&\qquad \times\left[\sigma\frac{1-(q/\sigma)^{22}}{\sigma^2-q^2} - q\frac{1-\left(q/\left[\sigma-7.58/(k-19)\right]\right)^{20}}{\left(\sigma-7.58/(k-19)\right)^2-q^2}\right]\\
&\qquad - 2\Re C(1+w)\left(\sqrt{a_0}-w\right)\\
&\qquad \times\left[\sigma\frac{1-(w/\sigma)^{22}}{\sigma^2-w^2} - w\frac{1-\left(w/\left[\sigma-7.58/(k-19)\right]\right)^{20}}{\left(\sigma-7.58/(k-19)\right)^2-w^2}\right]\\
&\qquad + 4B(1+q)\left(\sqrt{a_0}-q\right)\frac{S(1)}{S(16)}\frac{(2q)^{21}}{1-2q}\\
&\qquad - 4\left|C(1+w)\left(\sqrt{a_0}-w\right)\right|\frac{S(1)}{S(16)}\frac{(2|w|)^{21}}{|1-2w|}.
\end{aligned}
$$

In using (4.20) above, note that the overall coefficient of $S(j)$ in (4.18) for $j \leq k+1$ has the same sign as $(-q)^{k+1-j}$, since $|w| \leq q$ and

$$\left|C(1+w)\left(\sqrt{a_0}-w\right)\right| \leq |B|(1+q)\left(\sqrt{a_0}-q\right).$$

Therefore, one of the bounds in (4.20) can be used for even i and the other for odd i.

After discarding all the terms above which have powers greater than or equal to 20, the remaining expression is increasing in k. Using the explicit values which we have for the constants, it is easy to check that the expression is positive if $k = 1000$. Therefore (4.18) is positive for $k \geq 1000$. The final step is to show that it is positive for $2 \leq k \leq 1000$. We have computed $\Delta(k) - \Delta(k+1)$ for all these values of k, and in fact it always is positive. In the computation, we use the recursion (3.6) to compute the density $f(n)$ for $n \leq 1000$, and then (4.6) to compute $u(n)$ for these n. This computation is discussed further at the end of this section.

PROOF OF LEMMA 4.19. For nonnegative integers j, k, l, m, not all zero, let

$$c(j,k,l,m) = \frac{(2j+2k+2l+2m-2)!}{(j+k+l+m-1)!j!k!l!m!} \frac{a_1^j a_2^k a_3^l a_4^m}{(4a_0)^{j+k+l+m}}.$$

By the definition of $S(n)$,

$$S(n) = \sum_{j+2k+3l+4m=n} c(j,k,l,m). \tag{4.22}$$

Take $a, b, c, d > 0$, and write

$$\begin{aligned} S(n+1) &= \sum_{j+2k+3l+4m=n+1} c(j,k,l,m)\frac{aj+bk+cl+dm}{aj+bk+cl+dm} \\ &= \sum_{j+2k+3l+4m=n+1, j>0} c(j,k,l,m)\frac{aj}{aj+bk+cl+dm} \\ &\quad + \sum_{j+2k+3l+4m=n+1, k>0} c(j,k,l,m)\frac{bk}{aj+bk+cl+dm} \\ &\quad + \sum_{j+2k+3l+4m=n+1, l>0} c(j,k,l,m)\frac{cl}{aj+bk+cl+dm} \\ &\quad + \sum_{j+2k+3l+4m=n+1, m>0} c(j,k,l,m)\frac{dm}{aj+bk+cl+dm}. \end{aligned}$$

Using the form of $c(j,k,l,m)$ and then making a change of variable in the sum, the first sum on the right-hand side above can be written as

$$\begin{aligned} &\frac{aa_1}{2a_0} \sum_{j+2k+3l+4m=n+1, j>0} c(j-1,k,l,m)\frac{2j+2k+2l+2m-3}{aj+bk+cl+dm} \\ &\quad = \frac{aa_1}{2a_0} \sum_{j+2k+3l+4m=n} c(j,k,l,m)\frac{2j+2k+2l+2m-1}{aj+bk+cl+dm+a}. \end{aligned}$$

Similarly, the second sum is

$$\begin{aligned} &\frac{ba_2}{a_1} \sum_{j+2k+3l+4m=n+1, k>0} c(j+1,k-1,l,m)\frac{1}{aj+bk+cl+dm} \\ &\quad = \frac{ba_2}{a_1} \sum_{j+2k+3l+4m=n} c(j,k,l,m)\frac{j}{aj+bk+cl+dm-a+b}. \end{aligned}$$

Arguing in an analogous way for the other two sums, we obtain the following expression:

$$S(n+1) = \sum_{j+2k+3l+4m=n} c(j,k,l,m)d(j,k,l,m), \tag{4.23}$$

where

$$d(j,k,l,m) = \frac{aa_1}{2a_0}\frac{2j+2k+2l+2m-1}{aj+bk+cl+dm+a} + \frac{ba_2}{a_1}\frac{j}{aj+bk+cl+dm-a+b} + \frac{ca_3}{a_2}\frac{k}{aj+bk+cl+dm-b+c} + \frac{da_4}{a_3}\frac{l}{aj+bk+cl+dm-c+d}.$$

Note that the limits of this expression as $j \to \infty$ with the other variables fixed, as $k \to \infty$ with the other variables fixed, and so on, are

$$\text{(4.24)} \qquad \frac{a_1}{a_0} + \frac{ba_2}{aa_1}, \quad \frac{aa_1}{ba_0} + \frac{ca_3}{ba_2}, \quad \frac{aa_1}{ca_0} + \frac{da_4}{ca_3} \quad \text{and} \quad \frac{aa_1}{da_0}.$$

Our choice of σ makes it possible to choose a, b, c and d in such a way that each of these expressions has the value σ. The values are $a \approx 1.413$, $b \approx 1.244$ and $c \approx 1.194$ if we choose $d = 1$. Using the fact that each of the expressions in (4.24) has the value σ, we see that

$$\frac{d(j,k,l,m)}{\sigma} = \frac{j+k+l+m-\frac{1}{2}}{aj+bk+cl+dm+a} + \frac{(a-1)j}{aj+bk+cl+dm-a+b} + \frac{(b-1)k}{aj+bk+cl+dm-b+c} + \frac{(c-1)l}{aj+bk+cl+dm-c+d}.$$

Therefore,

$$\text{(4.25)} \qquad \frac{[\sigma - d(j,k,l,m)](aj+bk+cl+dm+a)}{\sigma} = a + \frac{1}{2} - \frac{(a-1)(2a-b)j}{aj+bk+cl+dm-a+b} - \frac{(b-1)(a+b-c)k}{aj+bk+cl+dm-b+c} - \frac{(c-1)(a+c-d)l}{aj+bk+cl+dm-c+d}.$$

Using

$$0 \le \frac{j}{aj+bk+cl+dm-a+b} \le \frac{1}{b},$$

two similar inequalities and the values of a, b, c, d and σ, we see from the above expression that

$$\text{(4.26)} \qquad 0 \le \sigma - d(j,k,l,m) \le \frac{1.894}{aj+bk+cl+dm+a}.$$

Using this in (4.23) and recalling (4.22), we obtain the required result. □

It now remains to prove that the density $f(n)$ is decreasing. We begin by combining (4.22) and (4.23) to obtain

$$\text{(4.27)} \qquad S(n) - S(n+1) = \sum_{j+2k+3l+4m=n} [1 - d(j,k,l,m)]c(j,k,l,m).$$

Replacing n by $n+1$ and applying the technique used in proving Lemma 4.19, we can write

$$S(n+1) - S(n+2) = \sum_{j+2k+3l+4m=n} D(j,k,l,m)c(j,k,l,m), \tag{4.28}$$

where

$$\begin{aligned} D(j,k,l,m) &= \sigma\left[1 - d(j+1,k,l,m)\right]\frac{j+k+l+m-\frac{1}{2}}{aj+bk+cl+dm+a} \\ &\quad + \sigma(a-1)j\frac{1-d(j-1,k+1,l,m)}{aj+bk+cl+dm-a+b} \\ &\quad + \sigma(b-1)k\frac{1-d(j,k-1,l+1,m)}{aj+bk+cl+dm-b+c} \\ &\quad + \sigma(c-1)l\frac{1-d(j,k,l-1,m+1)}{aj+bk+cl+dm-c+d}. \end{aligned} \tag{4.29}$$

Considering (4.5), (4.27) and (4.28), it is clear that we need to show that

$$D(j,k,l,m) \le 1 - d(j,k,l,m).$$

While this inequality probably holds for all choices of j,k,l,m, its verification would involve a lot of algebra. On the other hand, it is clearly true for sufficiently large values of the arguments, since the limit of the right-hand side is $1-\sigma$ by (4.26), while the limit of the left-hand side is $\sigma(1-\sigma)$ by (4.26) and (4.29). Therefore our strategy will be to estimate the rate of convergence in order to prove the inequality beyond a certain point, and then rely on the computer to verify $f(n) \ge f(n+1)$ for the remaining finite number of cases. So, using (4.29), write

$$\begin{aligned} &1 - d(j,k,l,m) - D(j,k,l,m) \\ &\quad = \left[1 - d(j,k,l,m)\right]^2 \\ &\qquad + \sigma\frac{j+k+l+m-\frac{1}{2}}{aj+bk+cl+dm+a}\left[d(j+1,k,l,m) - d(j,k,l,m)\right] \\ &\qquad + \frac{\sigma(a-1)j}{aj+bk+cl+dm-a+b}\left[d(j-1,k+1,l,m) - d(j,k,l,m)\right] \\ &\qquad + \frac{\sigma(b-1)k}{aj+bk+cl+dm-b+c}\left[d(j,k-1,l+1,m) - d(j,k,l,m)\right] \\ &\qquad + \frac{\sigma(c-1)l}{aj+bk+cl+dm-c+d}\left[d(j,k,l-1,m+1) - d(j,k,l,m)\right]. \end{aligned} \tag{4.30}$$

Using (4.26), the first term on the right-hand side of (4.30) can be bounded below by $(1-\sigma)^2$. For the second term, use (4.25) to write

$$
\begin{aligned}
&\big[d(j+1,k,l,m)-d(j,k,l,m)\big]\,[aj+bk+cl+dm+a]\\
&\quad\geq -\frac{\sigma(a-1)(2a-b)aj}{(aj+bk+cl+dm-a+b)(aj+bk+cl+dm+b)}\\
&\qquad -\frac{\sigma(b-1)(a+b-c)ak}{(aj+bk+cl+dm-b+c)(aj+bk+cl+dm+a-b+c)}\\
&\qquad -\frac{\sigma(c-1)(a+c-d)al}{(aj+bk+cl+dm-c+d)(aj+bk+cl+dm+a-c+d)}\\
&\quad\geq -a\sigma\frac{(a-1)(2a-b)j+(b-1)(a+b-c)k+(c-1)(a+c-d)l}{(aj+bk+cl+dm-c+d)(aj+bk+cl+dm+a-c+d)}\\
&\quad\geq -\frac{0.66}{aj+bk+cl+dm-c+d}.
\end{aligned}
$$

Arguing similarly for the other terms on the right-hand side of (4.30), we obtain

$$
\begin{aligned}
&\big[d(j-1,k+1,l,m)-d(j,k,l,m)\big]\,[aj+bk+cl+dm+b]\\
&\quad\geq -\frac{0.97}{aj+bk+cl+dm-2a+2b},\\
&\big[d(j,k-1,l+1,m)-d(j,k,l,m)\big]\,[aj+bk+cl+dm+a-b+c]\\
&\quad\geq -\frac{0.45}{aj+bk+cl+dm-2b+2c}
\end{aligned}
$$

and

$$
\begin{aligned}
&\big[d(j,k,l-1,m+1)-d(j,k,l,m)\big]\,[aj+bk+cl+dm+a-c+d]\\
&\quad\geq -\frac{0.68}{aj+bk+cl+dm-2c+2d}.
\end{aligned}
$$

Using these bounds in (4.30) gives

$$1-d(j,k,l,m)-D(j,k,l,m)\geq(1-\sigma)^2-\frac{1}{(aj+bk+cl+dm-c+d)^2}.$$

This is nonnegative provided that

$$aj+bk+cl+dm-c+d\geq(1-\sigma)^{-1}.$$

If $j+2k+3l+4m=n$, this requires that $n\geq 434$. Computation of $f(n)$ for smaller n completes the proof that $f(n)$ is decreasing.

Since the results of this section rely heavily on the results of computation, we list in Table 2, some sample values of the density $f(n)$ and renewal sequence $u(n)$.

The complete table for $n\leq 999$ has been provided to the Editor of this journal and will be provided on request to any interested reader. The computation was performed on a Macintosh Quadra, using Mathematica 2.0.4. In using (3.6) to compute $f(n)$, the value of β must be used. Since β is defined in terms of a

TABLE 2

n	$f(n)$	$u(n)$	$\Delta(n) - \Delta(n+1)$
001	0.500000000000	0.500000000	0.0247729291433
002	0.250000000000	0.500000000	0.0118187874301
003	0.100227070856	0.475227070	0.0014191928538
004	0.049545858286	0.462272929	0.0036847060256
005	0.025908283426	0.450737980	0.0015215456146
276	0.000000233542	0.383035925	0.0000000259411
277	0.000000229683	0.383034225	0.0000000255132
278	0.000000225893	0.383032552	0.0000000250929
279	0.000000222172	0.383030903	0.0000000246803
280	0.000000218518	0.383029280	0.0000000242751
501	0.000000008530	0.382923072	0.0000000009507
502	0.000000008416	0.382922998	0.0000000009380
503	0.000000008304	0.382922925	0.0000000009255
504	0.000000008194	0.382922853	0.0000000009132
505	0.000000008084	0.382922782	0.0000000009011
751	0.000000000354	0.382917453	0.0000000000395
752	0.000000000350	0.382917450	0.0000000000390
753	0.000000000346	0.382917447	0.0000000000386
754	0.000000000341	0.382917443	0.0000000000381
755	0.000000000337	0.382917440	0.0000000000376
999	0.000000000019	0.382917185	0.0000000000021

root of a certain polynomial, we must approximate its value when performing the computation. Since there is only one value of β for which the solution to (3.6) is even bounded, and our approximation will not be that value, it is clear that there is the potential for large errors to occur. When combined with the possible accumulation of round-off errors, some care must be taken to ensure that the computation is rigorous. We conclude the paper with several remarks concerning this point.

We carried out the computation several times, with different degrees of accuracy in the value of β used, and the corresponding numbers of digits carried throughout. In four trials, we used 200, 400, 500 and 650 digits of accuracy, respectively. It was clear from the behavior of the output that, with 200-digit accuracy, the results were not correct for $n \geq 530$. (The results oscillated, became negative, etc.) However, the results for the other three levels of accuracy were identical (to each other) up to $n = 999$. These are the ones reported above. In addition, an unusually responsible referee reports carrying out the computation using Mathematica 2.1 on both a DOS machine and a NeXT with an initial accuracy of 400 digits. His results were "identical to the author's." He also did them using Maple V on a NeXT, but that computation "self-destructed" soon after $n = 500$.

While probably completely convincing to most people, the approach described above may not be regarded as entirely rigorous (and, of course, it is not). There are at least three ways to proceed in order to make the computation rigorous.

One is to use the IntervalAnalysis Package of Mathematica 2.1. This was not done because the author did not have access to this version of Mathematica. A second approach is to carry out an error analysis, which would include both the error in the value of β and the round-off error. This was carried out by the author, with the conclusion that 650-digit accuracy is sufficient. (Hence the choice of this level of accuracy above.) We will not present this analysis in detail here, because we prefer the third alternative on grounds of elegance.

This approach begins with the observation that β, as defined at the beginning of Section 4, is a rational function of x_0, and x_0 is the root of a polynomial of degree 6 with integer coefficients. Therefore, β itself is a root of another sixth-degree polynomial with integer coefficients. In this case, this polynomial turns out to be

$$1{,}600{,}000\beta^6 + 15{,}959{,}552\beta^5 + 53{,}702{,}464\beta^4 \\ - 8{,}798{,}080\beta^3 - 2{,}956{,}744\beta^2 + 722{,}760\beta - 40{,}609.$$

Therefore, all the $f(n)$'s and $u(n)$'s can be written explicitly and with perfect accuracy as fifth-degree polynomials in β with rational coefficients. After doing this computation with perfect accuracy, one can compute β as accurately as necessary to obtain the desired accuracy in the values of the $f(n)$'s and $u(n)$'s. There is a practical difficulty in that the numerators and denominators involved in the rational coefficients are extremely large integers, and their storage and computation requires large amounts of memory. In the case of the author's Quadra, the memory limits were reached at about $n = 700$, so another computer had to be used. The computation described below took approximately 60 hours on a Sun 4 Sparc using Mathematica, version 2.0. Writing the polynomials in the form

$$10^{974}f(999) = -f_0 + f_1\beta - f_2\beta^2 - f_3\beta^3 + f_4\beta^4 - f_5\beta^5$$

and

$$Nu(999) = -u_0 + u_1\beta - u_2\beta^2 - u_3\beta^3 + u_4\beta^4 - u_5\beta^5,$$

one finds that the coefficients are all positive integers, with $f_0, \dots, f_5$ having 1415, 1442, 1422, 1423, 1423 and 1423 digits, respectively, and $N, u_0, \dots, u_5$ having 1448, 1888, 1894, 1895, 1895, 1896 and 1894 digits, respectively. Therefore, the rational coefficients in the polynomial expressions for $f(999)$ and $u(999)$ are bounded by 10^{469}. We then approximated β to 500 digits and evaluated these polynomials, with results which again agree with those given above. The coefficients themselves have too many digits to be included explicitly in this paper, but they will be shared with any interested reader.

Even with the efforts reported above, the computations do, of course, rely on the assumption that Mathematica is doing what it is supposed to be doing. This is impossible to check, since the routines are not public domain (and if they were, could one then be confident of their correctness?). Therefore, we conclude by stating explicitly what we are assuming is happening. First, we assume that integer operations are done correctly. This is easy to check for small integers, but conceivably something could go wrong with the large integers which come up in this computation. Second, we assume that when we solve the sixth-degree polynomial for β to 500-digit accuracy (with NSolve $[P(\beta) == 0, \beta, 500]$), Mathematica really provides the required accuracy. Finally, we assume that there is minimal loss of accuracy in computing the first five powers of β. (Note that we allowed an extra 31 digits of accuracy for this purpose). The author's own conclusion about all of this is that the chance that anything is wrong with the computation is significantly smaller than the chance that there is an error in the "rigorous" proof in Section 3, with all its cases and pencil-and-paper computations.

REFERENCES

ANDJEL, E. D., LIGGETT, T. M. and MOUNTFORD, T. (1992). Clustering in one dimensional threshold voter models. *Stochastic Process Appl.* **42** 73–90.

BEZUIDENHOUT, C. and GRIMMETT, G. (1990). The critical contact process dies out. *Ann. Probab.* **18** 1462–1482.

BUTTEL, L., COX, J. T. and DURRETT, R. (1993). Estimating the critical values of stochastic growth models. *J. Appl. Probab.* **30** 455–461.

COX J. T. and DURRETT R. (1991). Nonlinear voter models. In *Random Walks, Brownian Motion and Interacting Particle Systems: Festschrift in Honor of Frank Spitzer* (R. Durrett and H. Kesten, eds.) 189–201. Birkhäuser, Boston.

DURRETT, R. and STEIF, J. E. (1993). Fixation results for threshold voter systems. *Ann. Probab.* **21** 232–247.

LIGGETT, T. M. (1985). *Interacting Particle Systems*. Springer, Berlin.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CALIFORNIA 90024