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LYAPUNOV FUNCTIONS FOR SEMIMARTINGALE REFLECTING BROWNIAN MOTIONS[1]

BY PAUL DUPUIS AND RUTH J. WILLIAMS[2]

Brown University and University of California, San Diego

We prove that a sufficient condition for a semimartingale reflecting Brownian motion in an orthant (SRBM) to be positive recurrent is that all solutions of an associated deterministic Skorokhod problem are attracted to the origin. To prove this result, we construct a Lyapunov function for the SRBM.

1. Introduction. Let d be a positive integer, let $S = \mathbb{R}_+^d \equiv \{x \in \mathbb{R}^d : x_i \geq 0, i = 1, \dots, d\}$, let $r^0, r^1, \dots, r^d \in \mathbb{R}^d$ and let Δ be a $d \times d$ nondegenerate covariance matrix. Let R denote the $d \times d$ matrix whose ith column is the vector r^i, $i = 1, \dots, d$. A semimartingale reflecting Brownian motion (SRBM) associated with the data $(S, r^0, r^1, \dots, r^d, \Delta)$ is defined precisely in the next section. Heuristically, such a process is a continuous stochastic process that has a certain semimartingale decomposition. It behaves like Brownian motion with a constant drift r^0 in the interior of S and is confined to the orthant S by instantaneous reflection (or "pushing") at the boundary of S, where the direction of reflection on the ith face $F^i \equiv \{x \in S: x_i = 0\}$ is given by r^i, $i = 1, \dots, d$. (At an intersection of faces, one may use any convex combination of the directions of reflection associated with the faces meeting there.)

In [21] it was shown that a necessary condition for the existence of an SRBM is that at each point on the boundary of S there is a direction of reflection that points back into the interior of S (see the next section, where this condition is formulated as the completely-$\mathcal{S}$ condition on the reflection matrix R). In [24] this necessary condition was shown to be sufficient for the existence of an SRBM and that in this case the SRBM is unique in law and has the strong Markov property. Our interest in SRBM's stems from the fact that they have been proposed as approximate models for multiclass open queueing networks under conditions of heavy traffic [8]. With this motivation in mind, it is natural to seek conditions that guarantee positive recurrence of SRBM's and methods for computing their stationary distributions. We will focus on the first problem here. (A characterization of stationary distributions for SRBM's is given in the paper by Dai and Kurtz [6], and a numerical algorithm which uses this characterization as its starting point has been proposed by Dai and Harrison [4], [5].)

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In this paper we show that if all solutions of a related *deterministic* Skorokhod problem are attracted to the origin, then the SRBM is positive recurrent. That is, we show that a sufficient condition for the SRBM to be positive recurrent is that all solutions of a simply related dynamical system are attracted to the origin. Our method of proof is to construct a smooth Lyapunov function for the SRBM. Besides its use in proving ergodicity, the Lyapunov function we construct can be used to obtain bounds on moments and path excursion estimates for the SRBM. It is also needed in the standard argument used to prove the convergence of functionals of the invariant measures associated to a sequence of processes that converge weakly to the SRBM. Finally, we note that our technique may be useful for establishing positive recurrence of other stochastic processes, given the stability of a related dynamical system.

Up until this time, few conditions for positive recurrence of SRBM's have been given, except when $d = 2$ or when the SRBM is an approximate model of a *single class* open queueing network. When $d = 2$, it follows from the work of Hobson and Rogers [10] (for $r^0 \neq 0$) and Williams [25] (for $r^0 = 0$) that an SRBM is positive recurrent if and only if

$$r_1^0 + r_1^2(r_2^0)^- < 0 \qquad \text{and} \qquad r_2^0 + r_2^1(r_1^0)^- < 0,$$

where the superscript minus denotes the negative part of a number and r^1 and r^2 have been normalized so that $r_1^1 = 1$ and $r_2^2 = 1$. When $R = I - P'$, where P is a transition matrix for a transient Markov chain on d states and P has zeros on its diagonal, Harrison and Williams [9] have shown that an SRBM is positive recurrent if and only if $R^{-1}r^0 < 0$, where the inequality is understood to hold for each component separately. In both of these cases, our sufficient condition can be seen to coincide with the known necessary and sufficient conditions. For the case treated in [9], the "stability" of the related dynamical system is established in Chen and Mandelbaum [3, Theorem 5.2]. It is natural to ask whcther our condition is actually necessary and sufficient, that is, if the SRBM is not positive recurrent, is there at least one solution of the associated deterministic Skorokhod problem that is not attracted to the origin? This is an interesting open problem.

Finally, we mention the work of Malyshev and co-workers [11, 15–17] (the recent works [11], [16] only became known to us during the course of our research), who have been working on problems and conjectures similar to ours, but for reflected random walks rather than reflected diffusion processes. While there appears to be some commonality in philosophy, there are differences in our assumptions and methods.

2. Problem formulation and statement of the main result.

DEFINITION 2.1 (Semimartingale reflecting Brownian motion). An SRBM associated with the data $(S, r^0, r^1, \ldots, r^d, \Delta)$ is a continuous $\{\mathcal{F}_t\}$-adapted d-dimensional process $z(\cdot)$, together with a family of probability measures $\{P_x,\ x \in S\}$, defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, such that, for each $x \in S$, under P_x,

$$z(t) = w(t) + r^0 t + Ry(t) \in S \quad \text{for all } t \geq 0,$$

where (i) $w(\cdot)$ is a d-dimensional Brownian motion $\{\mathcal{F}_t\}$-martingale with covariance matrix Δ such that $w(0) = x$ P_x-a.s., and (ii) y is an $\{\mathcal{F}_t\}$-adapted d-dimensional process such that P_x-a.s., for $i = 1, \dots, d$, (a) $y_i(0) = 0$, (b) y_i is continuous and nondecreasing and, (c) y_i can increase only when z is on the face F^i, that is, $\int_0^t 1_{\{z_i(s) \neq 0\}} dy_i(s) = 0$, for all $t \geq 0$.

TERMINOLOGY. For brevity, we shall sometimes simply refer to z as an SRBM, in which case the probability measures $\{P_x,\, x \in S\}$ are implicit.

DEFINITION 2.2 (Completely-$\mathcal{S}$). A principal submatrix of R is a square matrix obtained by deleting all rows and columns from R that are indexed by some possibly empty subset of $\{1, \dots, d\}$. The matrix R is said to be completely-$\mathcal{S}$ if for each principal submatrix $\widetilde{R}$ of R there exists a vector u with all components positive such that $\widetilde{R}u$ has all components positive.

It follows from [21] and [24] that the completely-$\mathcal{S}$ condition is necessary and sufficient for the existence of an SRBM associated with $(S, r^0, r^1, \dots, r^d, \Delta)$, and in this case the SRBM is unique in law.

A key to our proof of positive recurrence is the form of Itô's formula for an SRBM. Let $f \in C^2(\mathbb{R}^d)$, and define $Df(x)$ and $D^2 f(x)$ to be the gradient and Hessian of f at x. For a square matrix A, let $\operatorname{tr} A$ denote the trace of A. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^d$. If z is an SRBM as in Definition 2.1., then for each $x \in S$, P_x-a.s., for all $t \geq 0$,

$$f\big(z(t)\big) = f\big(z(0)\big) + \int_0^t \left(\tfrac{1}{2} \operatorname{tr}\left[D^2 f\big(z(s)\big)\Delta\right] + \left\langle Df\big(z(s)\big), r^0 \right\rangle \right) ds$$

$$+ \int_0^t \left\langle Df\big(z(s)\big), dw(s) \right\rangle + \sum_{i=1}^d \int_0^t \left\langle Df\big(z(s)\big), r^i \right\rangle dy_i(s).$$

For the following, let $C([0,\infty), \mathbb{R}^d)$ denote the set of continuous functions from $[0,\infty)$ into $\mathbb{R}^d$. Associated with the geometric data S and R is a version of the Skorokhod problem (SP).

DEFINITION 2.3 (Skorokhod problem). Let $\psi \in C([0,\infty), \mathbb{R}^d)$ with $\psi(0) \in S$. Then $(\phi, \eta) \in C([0,\infty), \mathbb{R}^d) \times C([0,\infty), \mathbb{R}^d)$ solves the SP for ψ (with respect to S and R) if the following hold:

(i) $\phi(t) = \psi(t) + R\eta(t) \in S$, for all $t \geq 0$;

(ii) η is such that, for $i = 1, \dots, d$, (a) $\eta_i(0) = 0$, (b) η_i is nondecreasing and (c) η_i can increase only when ϕ is on F^i, that is, $\int_0^t 1_{\{\phi_i(s) \neq 0\}}\, d\eta_i(s) = 0$, for all $t \geq 0$.

For $x \in \partial S$, we define

$$r(x) = \left\{ \sum_{i=1}^{d} q_i r^i : \sum_{i=1}^{d} q_i = 1,\ q_i \geq 0,\ \text{and}\ q_i > 0 \text{ only if } x_i = 0 \right\}. \tag{2.1}$$

Note that in the definition of the Skorokhod problem, the "pushing term" $R\eta$ ensures that ϕ remains in S; furthermore, this term only changes when ϕ is on ∂S and in this case the change points in the direction of an element of $r(\phi)$.

The above formulation of a Skorokhod problem is related to the problem of finding strong solutions of the stochastic equation defining an SRBM. Our formulation of the Skorokhod problem is the same as that in [2], [19] and [18]. It was shown in [2] and [19] that under the completely-S condition there is always a solution to the SP. Examples given in [2] and [18] show that uniqueness does not always hold.

DEFINITION 2.4. We say that a path $\phi \in C([0,\infty),\mathbb{R}^d)$ is attracted to the origin if for any $\varepsilon > 0$ there exists $T < \infty$ such that $t \geq T$ implies $|\phi(t)| \leq \varepsilon$.

Our main result relates stability properties of solutions of the SP when $\psi(t) = x + r^0 t$ to positive recurrence of associated SRBM's.

DEFINITION 2.5. An SRBM $z(\cdot)$ is said to be positive recurrent if for each closed set A in S having positive Lebesgue measure we have $E_x[\tau_A] < \infty$, for all $x \in S$, where $\tau_A = \inf\{t \geq 0 : z(t) \in A\}$ and E_x denotes expectation under P_x.

THEOREM 2.6. *Assume that the matrix $R = (r^1,\dots,r^d)$ satisfies the completely-S condition. Let $z(\cdot)$ be an SRBM associated with $(S, r^0, r^1, \dots, r^d, \Delta)$. Suppose that the ϕ component of all solutions of the SP for unreflected paths $\psi(\cdot)$ of the form $\psi(t) = x + r^0 t, t \geq 0, x \in S$, is attracted to the origin. Then the process $z(\cdot)$ is positive recurrent and it has a unique stationary distribution.*

In order to prove the positive recurrence we will construct a Lyapunov function. The properties that we require such a function $W(\cdot)$ to have are as follows.

1. $W(\cdot) \in C^2(S \setminus \{0\})$.
2. Given $N < \infty$, there is an $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $W(x) \geq N$.
3. Given $\varepsilon > 0$, there is an $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $\|D^2 W(x)\| \leq \varepsilon$.
4. There exists $c > 0$ such that

$$\begin{aligned} \langle DW(x), r^0 \rangle &\leq -c \quad \text{for all } x \in S \setminus \{0\}, \\ \langle DW(x), r \rangle &\leq -c \quad \text{for all } r \in r(x),\ x \in \partial S \setminus \{0\}, \end{aligned}$$

The new feature here (as compared with the situation for processes without reflecting boundaries) is the presence of multiple constraints on the gradient of W for points on the boundary. The existence of such a Lyapunov function is established in the next section. Given such a function and the form of Itô's formula for an SRBM, the proof of positive recurrence and existence of a stationary distribution is quite straightforward [12, 13]. Uniqueness follows also by a standard argument once Lebesgue measure has been established as a reference measure [9]. For completeness, we provide a sketch of the proof of positive recurrence and existence and uniqueness of a stationary distribution in the appendix.

3. Construction of the Lyapunov function. We will construct the Lyapunov function in a series of propositions. To begin, we introduce some notation. Define $\mathrm{In}(x) = \{i: x_i = 0\}$, and let Λ be the collection of all subsets of $\{1, 2, \dots, d\}$. The function $\mathrm{In}(x)$ partitions S according to the equation $\mathrm{In}(x) = \lambda$, for $\lambda \in \Lambda$, into the interior ($\lambda = \emptyset$), the relative interiors of faces of codimension 1, and so on. Let e_i denote the unit vector in the ith coordinate direction. The set of inward normals to S at $x \in S$ is defined to equal $\{\sum_{i \in \mathrm{In}(x)} q_i e_i : q_i \geq 0, i \in \mathrm{In}(x); \sum_{i \in \mathrm{In}(x)} q_i > 0\}$. This set is empty if $\mathrm{In}(x) = \emptyset$. For any $\lambda \in \Lambda$, all x satisfying $\mathrm{In}(x) = \lambda$ have the same set of inward normals to S at x. In the development given below we will often abuse terminology and refer to ϕ alone as the solution to the SP for a given ψ.

PROPOSITION 3.1. *Assume that the matrix $R = (r^1, r^2, \dots, r^d)$ satisfies the completely-S condition. Then the following conclusions hold.*

(i) *For each $\lambda \in \Lambda$, $\lambda \neq \emptyset$, there exists a vector d^λ such that*

$$d^\lambda \in r(x) \quad \text{for all } x \in S \text{ such that } \mathrm{In}(x) = \lambda, \tag{3.1}$$

$$\langle d^\lambda, e_i \rangle > 0 \quad \text{for all } i \in \lambda. \tag{3.2}$$

(ii) *For each $\lambda \in \Lambda$, $\lambda \neq \emptyset$, there exists a vector n_λ which is an inward normal at each point $x \in \partial S$ satisfying $\mathrm{In}(x) = \lambda$, such that $\langle n_\lambda, r^i \rangle > 0$, for all $i \in \lambda$.*

PROOF. The first statement is actually just a restatement of the completely-$\mathcal{S}$ condition. The second statement is a direct consequence of the fact that R is completely-$\mathcal{S}$ if and only if R' is completely-$\mathcal{S}$, where the prime denotes transpose [21]. □

We next describe a vector field $v(\cdot)$ on $\mathbb{R}^d$. Let $\lambda(x) = \{i: \langle x, e_i \rangle \leq 0\}$, and define

$$v(x) = \begin{cases} r^0, & \text{for } x \in S, \\ d^{\lambda(x)}, & \text{for } x \notin S. \end{cases}$$

Note that $v(\cdot)$ possesses a radial homogeneity: $v(\alpha x) = v(x)$ for any x and any $\alpha > 0$. The definition of $v(\cdot)$ is illustrated in Figure 1 for the two-dimensional case. The vector $d^{\{1,2\}}$ has the nice property of pointing toward S at all points x for which $\lambda(x)$ equals $\{1, 2\}$, $\{1\}$ or $\{2\}$.

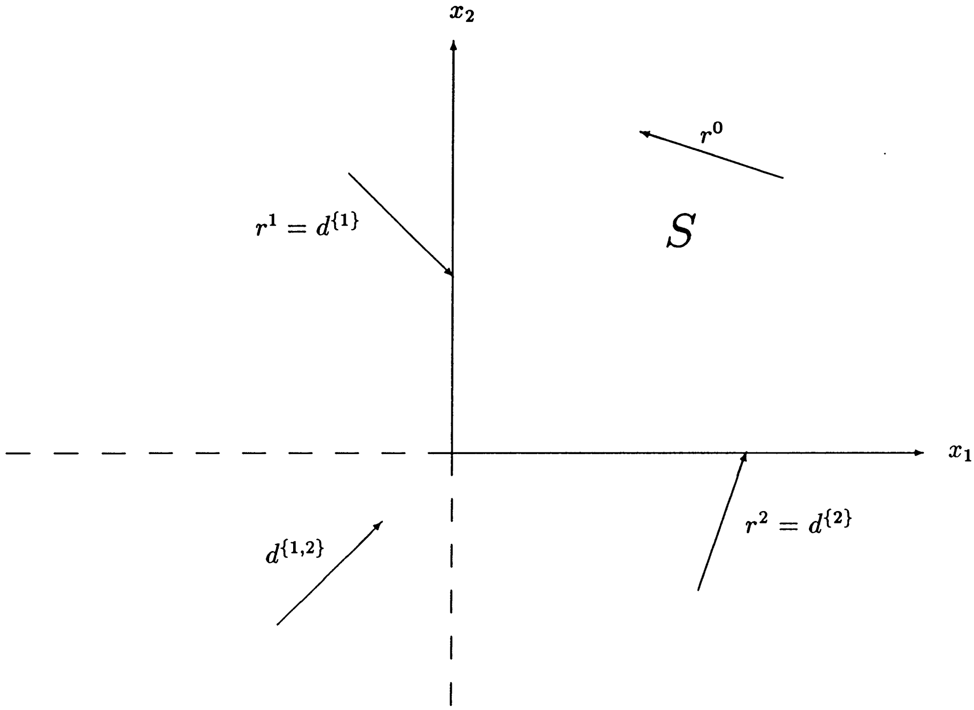

FIG. 1.

We next define a family of smooth set-valued functions on $\mathbb{R}^d$. In the definition we will be careful to preserve the radial homogeneity property of $v(\cdot)$. We first define mollified versions of $v(\cdot)$. Let $\rho: \mathbb{R}^d \to \mathbb{R}$ be a function such that

$$\rho(\cdot) \in C^\infty(\mathbb{R}^d), \qquad \operatorname{supp} \rho(\cdot) \subset \{x: |x| \le 1\} \quad \text{and} \quad \int_{\mathbb{R}^d} \rho(x)\, dx = 1, \tag{3.3}$$

where $\operatorname{supp} \rho(\cdot)$ denotes the support of $\rho(\cdot)$. For $b > 0$ define $c(b) = [\int_{\mathbb{R}^d} \rho(x/b)\, dx]^{-1}$. The mollified versions of $v(\cdot)$ are defined for $x \neq 0$ by

$$v^a(x) = c(a|x|) \int_{\mathbb{R}^d} \rho\left(\frac{x-y}{a|x|}\right) v(y)\, dy, \qquad a > 0.$$

For $A \subset \mathbb{R}^d$ we define the distance $d(x,A) = \inf\{|x-y|: y \in A\}$. When A is a closed convex cone, that is, a closed convex set with the property that $x \in A$ implies $\alpha x \in A$ for all $\alpha \ge 0$, the distance function has the following radial property: $d(\alpha x, A) = \alpha d(x,A)$, for all $x \in \mathbb{R}^d$ and $\alpha \ge 0$. It should also be noted that $d(x,A)$ is a Lipschitz continuous function of $x \in \mathbb{R}^d$.

Next let $g: \mathbb{R} \to [0,1]$ be a smooth function that satisfies $g(z) = 1$, for $z \in [0, 1/2]$, and $g(z) = 0$, for $z \in [1, \infty)$. Recall that F^i denotes the "face" of ∂S that is orthogonal to e_i: $F^i = \{x \in \partial S: x_i = 0\}$. For each $i = 1, 2, \ldots, d$, $a > 0$ and $x \neq 0$, we define

$$v_i^a(x) = g\left(\frac{d(x,F^i)}{a|x|}\right)r^i + \left[1 - g\left(\frac{d(x,F^i)}{a|x|}\right)\right]v^a(x)$$

and also

$$v_0^a(x) = g\left(\frac{d(x,S)}{a|x|}\right)r^0 + \left[1 - g\left(\frac{d(x,S)}{a|x|}\right)\right]v^a(x).$$

Note that the radial homogeneity has been preserved in the definitions of the $v_i^a(x)$, $i = 0, 1, \dots, d$, and that these functions are Lipschitz continuous on any closed set that does not contain the origin.

For $A \subset \mathbb{R}^d$ let $\operatorname{conv} A$ denote the closure of the convex hull of A. We then define set-valued functions $K^a(x)$, for $x \neq 0$ by,

$$K^a(x) = \operatorname{conv}\{v_i^a(x),\ i = 0, 1, 2, \dots, d\}.$$

The definition of $K^a(0)$ is actually unimportant, and we simply use $K^a(0) = \{r^0\}$. We also define a set-valued function $K(x)$ by stipulating that $v \in K(x)$ if and only if there exist sequences $\{a_n\}, \{x_n\}, \{v_n\}$ such that $a_n \downarrow 0$, $x_n \to x$, and $v_n \to v$ as $n \to \infty$, and $v_n \in K^{a_n}(x_n)$ for each n.

We note the following important properties of $K^a(\cdot)$ and $K(\cdot)$. The properties follow from the construction of the $K^a(\cdot)$ and the definition of $K(\cdot)$.

PROPOSITION 3.2.

(i) *For each $a > 0$, the set-valued function $K^a(\cdot)$ is the convex hull of $d + 1$ vector-valued functions that are locally Lipschitz continuous on $\mathbb{R}^d \setminus \{0\}$.*

(ii) *For each $a > 0$, $\alpha > 0$ and $x \in \mathbb{R}^d$, $K^a(\alpha x) = K^a(x)$.*

(iii) *If $d(x,S) \leq a|x|/2$, then $r^0 \in K^a(x)$. If $d(x,F^i) \leq a|x|/2$ and $x \neq 0$, then $r^i \in K^a(x)$.*

(iv) *Let $\lambda_1(x) = \{i: x_i < 0\}$. If $d(x,S) > 0$, then $K(x)$ is contained in the convex hull of $\{d^\lambda: \lambda_1(x) \subset \lambda\}$. See Figure 1.*

(v) *If $x \in S^0$, the interior of S, then $K(x) = \{r^0\}$, and if $x \in \partial S \setminus \{0\}$, then $K(x)$ equals*

$$\operatorname{conv}\Big(\{r^0\} \cup \{r^i: i \in \operatorname{In}(x)\} \cup \{d^\lambda: \lambda \subset \operatorname{In}(x)\}\Big) = \operatorname{conv}\Big(\{r^0\} \cup \{r^i: i \in \operatorname{In}(x)\}\Big).$$

(vi) *$K(\cdot)$ is radially homogeneous on $\mathbb{R}^d \setminus \{0\}$ in the sense that if $x \neq 0$ and $v \in K(x)$, then $v \in K(\alpha x)$ for all $\alpha > 0$.*

(vii) *$K(x)$ is an upper-semicontinuous function of $x \in \mathbb{R}^d \setminus \{0\}$ in the sense that $x_n \to x$, $v_n \to v$, $v_n \in K(x_n) \Rightarrow v \in K(x)$.*

By a solution to a differential inclusion of the form $\dot{\phi} \in H(\phi)$ we mean an absolutely continuous function $\phi: [0, \infty) \to \mathbb{R}^d$ such that $\dot{\phi}(t) \in H(\phi(t))$, for a.e. t. We consider the solutions to a differential inclusion as taking values in $C([0, \infty), \mathbb{R}^d)$. We can use any metric on this space under which convergence of functions is equivalent to uniform convergence of the corresponding restrictions on each compact subset of $[0, \infty)$.

We consider solutions to the differential inclusion $\dot{\phi} \in K^a(\phi)$. The remarks following the definition of the Skorokhod problem suggest that this differential inclusion can be viewed as a perturbed version of the SP with $\psi(t) = x + r^0 t$. In particular, one should compare the properties of $K(x)$ given in Proposition 3.2(iv) and (v) with the set-valued function $r(x)$ defined in (2.1). The approximating problems are in some respects easier to work with because $K^a(x)$ is smooth in $x \neq 0$. This connection is made more precise in the following result.

PROPOSITION 3.3. *Let ϕ^a_γ, $\gamma \in \Gamma(a)$, index the set of solutions to*

$$\dot{\phi}^a \in K^a(\phi^a), \qquad \phi^a(0) = x^a,$$

where $\{x^a,\ a \in (0,1]\}$ is any bounded set in $\mathbb{R}^d$. We then have the following conclusions:

(i) *The set $\{\phi^a_\gamma(\cdot): \gamma \in \Gamma(a),\ a \in (0,1]\}$ is precompact.*

(ii) *Suppose a indexes a sequence of numbers in $(0,1]$ that converges to zero. For each a in the sequence, let $\gamma(a) \in \Gamma(a)$. Suppose that $\phi^a_{\gamma(a)} \to \phi$ and $x^a \to x$ as $a \to 0$.* (a) *If $x \in S$, then modulo a rescaling of time (this point is clarified in the proof) ϕ is a solution to the SP for the path $\psi(t) = x + r^0 t$.* (b) *If $x \notin S$, then $\phi(\tau) \in S$ for some $\tau < \infty$. Furthermore, modulo a rescaling of time, $\phi(\cdot + \tau)$ is a solution to the SP for the unreflected path $\psi(t) = \phi(\tau) + r^0 t$.*

REMARKS. Our main use of the proposition will be to show that if all solutions to the SP for $\psi(t) = x + r^0 t$ are attracted to the origin, then so are the solutions to the perturbed system for small $a > 0$. Essentially the same proof as that below can be used to show that solutions to the SP [for $\psi(t) = x + r^0 t$] form a precompact set if their initial conditions lie in a bounded set and that the limit of any convergent sequence of solutions to the SP is also a solution to the SP.

PROOF OF PROPOSITION 3.3. Part (i) follows from Ascoli's theorem and the fact that the sets $K^a(x)$ are uniformly bounded over all x and a. Note that this uniform boundedness also tells us that the limits of the $\phi^a_{\gamma(a)}$ as $a \to 0$ are Lipschitz continuous and, therefore, absolutely continuous.

We next consider (a) in part (ii). As a first step, assuming that $\phi^a_{\gamma(a)} \to \phi$ as $a \to 0$, we prove that

$$\dot{\phi}(t) \in K\big(\phi(t)\big) \quad \text{for a.e. } t \geq 0.$$

This is basically a direct consequence of the property

$$a_n \to 0, \qquad x_n \to x, \qquad v_n \to v, \qquad v_n \in K^{a_n}(x_n) \quad \Rightarrow \quad v \in K(x).$$

For the proof, fix any $T < \infty$. Define the measures $\mu^a(\cdot)$ on the Borel subsets of

$\mathbb{R}^d \times \mathbb{R}^d \times [0,T]$ by

$$\mu^a(A \times B \times C) = \int_C I_A\big(\phi^a_{\gamma(a)}(s)\big) I_B\big(\dot{\phi}^a_{\gamma(a)}(s)\big)\, ds,$$

for Borel sets A, B and C. The boundedness of $K^a(x)$ in x and a implies that $\{\mu^a(\cdot), a \in (0,1]\}$ is tight. Suppose that a subsequence of a's tending to zero has been extracted such that both $\mu^a(\cdot)$ and $\phi^a_{\gamma(a)}(\cdot)$ converge. Let $\mu(\cdot)$ and $\phi(\cdot)$ denote the respective limits. For simplicity of notation we will retain a as the index. For $\varepsilon > 0$, we define

$$S_\varepsilon = \left\{ (x,\beta) \colon \beta \in \bigcup_{|y-x|\le\varepsilon} \{\tilde{\beta} \colon d(\tilde{\beta}, K(y)) \le \varepsilon\} \right\}.$$

We first claim that $\mu^a(S^c_\varepsilon \times [0,T]) = 0$ for all sufficiently small $a > 0$, where the superscript c stands for complement. Since the sets $K^a(x)$ are uniformly bounded for all x and a and $\{x^a,\ a \in (0,1]\}$ is bounded, there is an $M > 0$ such that $|\phi^a_{\gamma(a)}(s)| \le M$ for all $s \in [0,T]$ and a. It follows that it suffices to show $\{(x,\beta)\colon |x| \le M,\ \beta \in K^a(x)\} \cap S^c_\varepsilon = \emptyset$ whenever $a > 0$ is sufficiently small. If the last equality were not true, then by the boundedness of $K^a(x)$ there would exist sequences $x_n \to x$, $\beta_n \to \beta$ and $a_n \to 0$ such that $\beta_n \in K^{a_n}(x_n)$ and $(x_n, \beta_n) \in S^c_\varepsilon$. From the definition of $K(x)$ we have $\beta \in K(x)$, while $(x_n,\beta_n) \in S^c_\varepsilon$ implies $d(\beta, K(x)) \ge \varepsilon$, a contradiction.

We next claim that S_ε is closed. To prove this, let $x_n \to x$ and $\beta_n \to \beta$ with $(x_n, \beta_n) \in S_\varepsilon$. Then there exist y_n with $|y_n - x_n| \le \varepsilon$ such that $d(\beta_n, K(y_n)) \le \varepsilon$. By extracting a convergent subsequence, we can assume $y_n \to y$, with $|x-y| \le \varepsilon$. Since the upper semicontinuity of $x \to K(x)$ implies $d(\beta, K(y)) \le \liminf_n d(\beta_n, K(y_n))$ whenever $y_n \to y$ and $\beta_n \to \beta$, we have $d(\beta, K(y)) \le \varepsilon$, and thus $(x,\beta) \in S_\varepsilon$.

Using the convergence of μ^a to μ and the fact that S_ε is closed, we now see that $\mu(S_\varepsilon \times [0,T]) = T$, for all $\varepsilon > 0$. We next assert that $\cap_{\varepsilon>0} S_\varepsilon = \{(x,\beta)\colon \beta \in K(x)\}$. The inclusion $\{(x,\beta)\colon \beta \in K(x)\} \subset \cap_{\varepsilon>0} S_\varepsilon$ is obvious. To prove the reverse inclusion, assume $(x,\beta) \in \cap_{\varepsilon>0} S_\varepsilon$. Then for each $n < \infty$ there is a y_n such that $|x - y_n| \le 1/n$ and $d(\beta, K(y_n)) \le 1/n$. Sending $n \to \infty$ and using the upper semicontinuity of $K(\cdot)$ shows that $\beta \in K(x)$. If we now combine this representation with the fact that $\mu(S_\varepsilon \times [0,T]) = T$, for all $\varepsilon > 0$, we see that $\mu(\cap_{\varepsilon>0} S_\varepsilon \times [0,T]) = \mu(\{(x,\beta,t)\colon \beta \in K(x)\}) = T$, and therefore $\mu(\{(x,\beta,t)\colon \beta \notin K(x)\}) = 0$.

Since $\mathbb{R}^{2d}$ and $\mathbb{R}$ are Polish spaces and $\mu(\mathbb{R}^d \times \mathbb{R}^d \times [0,s]) = s$, for each $s \in [0,T]$, an argument similar to that used to establish the existence of regular conditional probability distributions (cf. Ethier and Kurtz [7], page 502) shows that there is a kernel $\mu_s(\cdot)$ such that for each $s \in [0,T]$, μ_s is a probability measure on the Borel subsets of $\mathbb{R}^d \times \mathbb{R}^d$, for each pair of Borel sets $A, B \in \mathbb{R}^d$, $s \to \mu_s(A \times B)$ is a Borel-measurable function on $[0,T]$, and for any Borel sets $A, B \in \mathbb{R}^d$ and $C \subset [0,T]$ we have

$$\mu(A \times B \times C) = \int_C \mu_s(A \times B)\, ds.$$

Since $\mu(\{(x,\beta,t):\beta\notin K(x)\})=0$, we know $\mu_s(\{(x,\beta):\beta\notin K(x)\})=0$ for a.e. s. It is also easy to show from the convergence $\phi^a \to \phi$ that $\mu(\cup_{s\in[0,T]}[\{\phi(s)\}\times\mathbb{R}^d\times\{s\}]) = T$, and therefore $\mu_s(\{\phi(s)\}\times\mathbb{R}^d) = 1$, for a.e. $s \in [0,T]$. Thus if $(\phi^a_{\gamma(a)}, \mu^a) \to (\phi,\mu)$ in the appropriate product topology, these facts together with the representation

$$\phi^a_{\gamma(a)}(t) = \int_{\mathbb{R}^d\times\mathbb{R}^d\times[0,t]} \beta\mu^a(dx\times d\beta\times ds) + x^a$$

imply

$$\dot\phi(t) = \int_{\{\phi(s)\}\times K(\phi(s))} \beta\mu_s(dx\times d\beta) \in K\big(\phi(t)\big) \qquad \text{for a.e. } t\in[0,T].$$

Since T was arbitrary, it now follows that $\dot\phi(t)\in K(\phi(t))$, for a.e. $t\in[0,\infty)$.

We next prove that if $\phi(\cdot)$ is any Lipschitz continuous function that satisfies $\phi(0)\in S$ and $\dot\phi(t)\in K(\phi(t))$ for a.e. t, then $\phi(t)\in S$, for all $t\ge 0$. We will prove the existence of $\delta>0$ such that

$$\frac{d}{dt}\left[\min_{i\in\{1,\dots,d\}}\langle\phi(t),e_i\rangle\right] \ge \delta, \tag{3.4}$$

for a.e. t such that $\phi(t)\notin S$. This clearly implies $\phi(t)\in S$, for all $t\ge 0$. The argument uses the properties of the vectors d^λ described in Proposition 3.1 in an essential way. For each $c > 0$, define a translated version of S by $S_c = \{y = x - c(1,1,\dots,1): x\in S\}$. For any $y\notin S$, let $c>0$ be such that $y\in\partial S_c$ and let $\lambda_y = \{i: y_i = -c\}$. According to Proposition 3.2, $K(y)$ is contained in the convex hull of $\{d^\lambda: \lambda_y\subset\lambda\}$ (see Figure 2). Since $\langle d^\lambda, e_i\rangle > 0$ whenever $\lambda_y\subset\lambda$ and $i\in\lambda_y$, there exists $\delta>0$ such that $\langle v,e_i\rangle\ge\delta$, for all $v\in K(y)$, all $i\in\lambda_y$ and all $y\notin S$.

Because the mapping $x\to\min_{i\in\{1,\dots,d\}}\langle x,e_i\rangle$ is Lipschitz continuous, the composed function $\min_{i\in\{1,\dots,d\}}\langle\phi(t),e_i\rangle$ is absolutely continuous. Let t be such that this composed function and $\phi(\cdot)$ are differentiable at t and $y\equiv\phi(t)\notin S$. The definition of λ_y implies the existence of $\varepsilon>0$ such that $\langle\phi(s),e_i\rangle > -c$, for all $s\in[t,t+\varepsilon]$, $i\notin\lambda_y$. Thus, by choosing $\varepsilon>0$ perhaps even smaller, we can ensure that $\min_{i\in\{1,\dots,d\}}\langle\phi(s),e_i\rangle = \min_{i\in\lambda_y}\langle\phi(s),e_i\rangle$, for all $s\in[t,t+\varepsilon]$. Since $\phi_i(t) = -c$ for all $i\in\lambda_y$, we also have

$$\min_{i\in\lambda_y}\langle\phi(s),e_i\rangle - \min_{i\in\lambda_y}\langle\phi(t),e_i\rangle = \min_{i\in\lambda_y}\langle\phi(s)-\phi(t),e_i\rangle. \tag{3.5}$$

Define $v_b = (\phi(t+b)-\phi(t))/b$. Since $\phi(\cdot)$ is Lipschitz continuous, we can assume the existence of a subsequence of $\{v_b,\ b>0\}$ that converges to a limit v as $b\downarrow 0$. For simplicity we retain b as the index of this convergent subsequence. We have

$$v_b = \frac{1}{b}\int_0^b \dot\phi(t+s)\,ds \quad\text{and}\quad \dot\phi(t+s)\in K\big(\phi(t+s)\big) \quad \text{for a.e. } s\in[0,b].$$

Thus the upper semicontinuity and convexity of $K(\cdot)$ imply $v\in K(y)$. Together with (3.5) this implies

$$\frac{d}{dt}\left[\min_{i\in\{1,\dots,d\}}\langle\phi(t),e_i\rangle\right] \ge \lim_{b\downarrow 0}\min_{i\in\lambda_y}\langle v_b,e_i\rangle = \min_{i\in\lambda_y}\langle v,e_i\rangle.$$

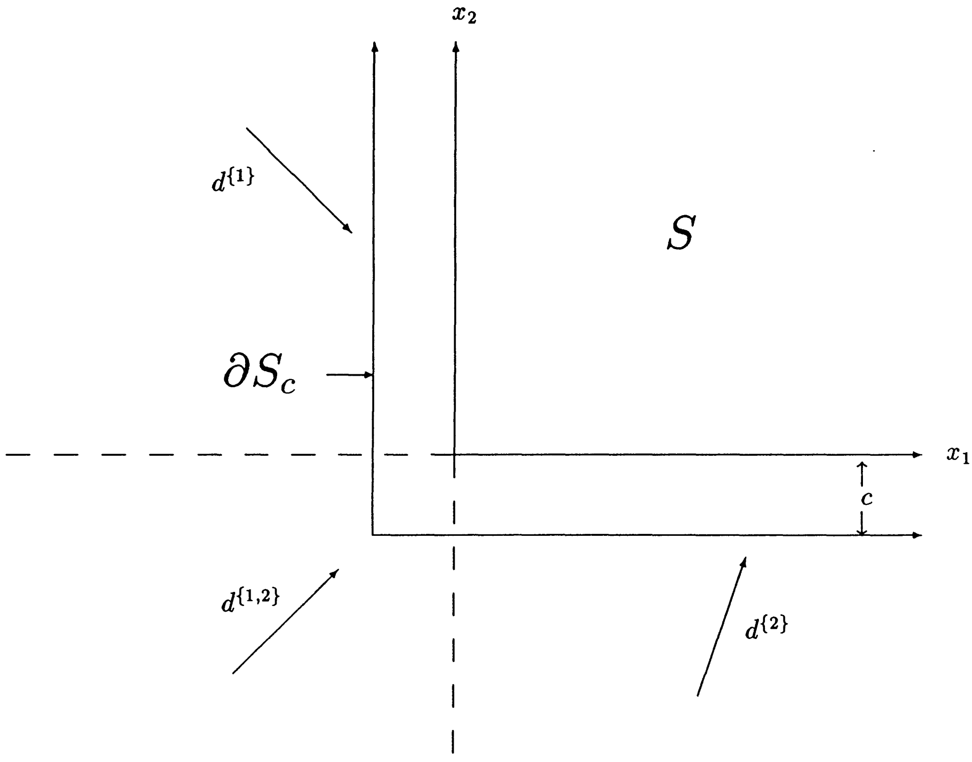

FIG. 2.

Since $v \in K(y)$, this implies (3.4).

There is one more fact that is needed before we can show that the function ϕ is a solution to the SP. Suppose that $\phi(\cdot)$ is a Lipschitz continuous function and that $\dot{\phi}(t) \in K(\phi(t))$ for a.e. $t > 0$. Since, for all $x \in S$, $K(x)$ is contained in the convex hull of the vectors r^0 and $\{r^i : i \in \mathrm{In}(x)\}$. (Proposition 3.2), by the argument given in the Appendix, there exist measurable functions $q_i: \mathbb{R}_+ \to \mathbb{R}_+$, $i = 0, 1, \ldots, d$, such that, for a.e. t,

$$\dot{\phi}(t) = q_0(t) r^0 + \sum_{i \in \mathrm{In}(\phi(t))} q_i(t) r^i, \tag{3.6}$$

and

$$q_0(t) + \sum_{i \in \mathrm{In}(\phi(t))} q_i(t) = 1. \tag{3.7}$$

The property that is needed is that there exists $c > 0$ such that $q_0(t) \geq c$ for a.e. t. Fix any $\lambda \in \Lambda$, $\lambda \neq \emptyset$, and let $L = \{x \in S: \mathrm{In}(x) = \lambda\}$. For any Borel subset

$M \subset [0,T] \cap \{t: \phi(t) \in L\}$, define

$$l(M) = \int_M I_{\{\phi(t)\in L\}} \dot{\phi}(t)\, dt.$$

We claim that in an almost-everywhere sense the derivative $I_{\{\phi(t)\in L\}} \dot{\phi}(t)$ always lies in the smallest linear space containing L. This fact follows from the following one dimensional version: If ϕ: $[0,T] \to \mathbb{R}$ is an absolutely continuous function, then $I_{\{\phi(t)=0\}} \dot{\phi}(t) = 0$, for a.e. t. A proof of the last statement is outlined in [7], pages 334–335.

It follows from the preceding paragraph that $\langle l(M), e_i \rangle = 0$, for $i \in \lambda$, and therefore $\langle l(M), n_\lambda \rangle = 0$ (recall that the vectors n_λ are in the positive cone generated by e_i, $i \in \lambda$, as described in Proposition 3.1). Owing to the fact that $\langle r^i, n_\lambda \rangle > 0$, for $i \in \lambda$, there are $a > 0$ and $b > 0$ (not depending on T) such that, for a.e. t for which $q_0(t) \le a$ and $\phi(t) \in L$,

$$\left\langle n_\lambda, \dot{\phi}(t) - q_0(t) r^0 \right\rangle \ge b > 0.$$

Now suppose that given any $c \in (0,a)$ there is a set $M \subset [0,T] \cap \{t: \phi(t) \in L\}$ of positive measure such that on this set $q_0(t) \le c$ almost surely. Then a.e. on M,

$$\left\langle n_\lambda, \dot{\phi}(t) \right\rangle = \left\langle n_\lambda, \dot{\phi}(t) - q_0(t) r^0 \right\rangle + \left\langle n_\lambda, q_0(t) r^0 \right\rangle \ge b - c.$$

Therefore if $c < b$, we contradict $\langle l(M), n_\lambda \rangle = 0$. Using the facts that the elements of Λ are finite in number and $q_0(t) = 1$ for a.e. t such that $\phi(t) \in S^0$, it follows that there exists $c > 0$ such that $q_0(t) \ge c$ for a.e. t.

Now let $a(s) = \int_0^s q_0(\tau)\, d\tau$. We claim that $\phi(a^{-1}(t))$ is a solution to the SP for $\psi(t) = x + r^0 t$, where $a(a^{-1}(t)) = t$. Note that, for a.e. t,

$$\begin{aligned} \frac{d}{dt}\phi\left(a^{-1}(t)\right) &= \frac{1}{q_0\left(a^{-1}(t)\right)} \dot{\phi}\left(a^{-1}(t)\right) \\ &= r^0 + \sum_{i \in \mathrm{In}\left(\phi\left(a^{-1}(t)\right)\right)} \frac{q_i\left(a^{-1}(t)\right)}{q_0\left(a^{-1}(t)\right)} r^i. \end{aligned}$$

Thus, setting

$$\eta_i(t) = \int_0^t 1_{\left\{\phi_i\left(a^{-1}(s)\right)=0\right\}} \frac{q_i\left(a^{-1}(s)\right)}{q_0\left(a^{-1}(s)\right)}\, ds,$$

we obtain the desired result. This completes the proof of (a) in part (ii) of the proposition.

Finally we consider (b) in part (ii) of the proposition. The first statement follows from the fact that (3.4) holds for a.e. t such that $\phi(t) \notin S$, while the second statement in (b) follows from part (ii)(a) of the proposition. □

The next result needed is the following.

PROPOSITION 3.4. *Assume that all solutions of the SP for ψ of the form $\psi(t) = x + r^0 t$, $x \in S$, are attracted to the origin. Then the following conclusions hold*:

(i) *Given $c > 0$, there exist $r > 0$ and $a_0 > 0$ such that, for all $a \in (0, a_0)$,*

$$\dot{\phi} \in K^a(\phi) \quad \textit{and } \left|\phi(0)\right| \le r$$

imply

$$\left|\phi(t)\right| \le c \quad \textit{for all } t \ge 0.$$

(ii) *Given $r > 0$ and $R < \infty$, there exist $T < \infty$ and $a_0 > 0$ such that, for all $a \in (0, a_0)$,*

$$\dot{\phi} \in K^a(\phi) \quad \textit{and } \left|\phi(0)\right| \le R$$

imply

$$\left|\phi(t)\right| \le r \quad \textit{for some } t \le T.$$

PROOF. We first note that part (ii) follows from Proposition 3.3 by an elementary argument by contradiction. Suppose that we now apply part (ii) of the proposition with $R = 1$ and $r = \frac{1}{2}$. Define

$$C = \sup_{a \in (0, a_0)} \sup_{\phi \in K^a(\phi), |\phi(0)| \le 1} \sup_{0 \le t \le \tau_\phi} \left|\phi(t)\right|,$$

where $\tau_\phi = \inf\{t: |\phi(t)| \le \frac{1}{2}\}$. By again using an argument by contradiction, it follows that $C < \infty$ whenever $a_0 > 0$ is sufficiently small. Then, by radial homogeneity of K^a, part (i) of the proposition follows with this choice of a_0 and $r = c/C$. □

Henceforth we assume that the hypothesis of Proposition 3.4 holds. We are now ready to begin the construction of our Lyapunov function. The first step adapts an idea due to Massera [20]. We will prove that the stability of solutions of the differential inclusion $\dot{\phi} \in K^a(\phi)$, for small $a > 0$, implies the existence of a function $V^a(\cdot)$ that is nearly the Lyapunov function we seek. Owing to the fact that we are dealing with a differential inclusion, it is appropriate to define $V^a(\cdot)$ as the maximum value function for a certain deterministic optimal control problem. Our first step will be to define this function and to show that it has some of the desired properties. However, we can only show that the function $V^a(\cdot)$ is Lipschitz continuous. Thus, in order to make the function suitable to serve as a Lyapunov function for an SRBM, a mollification is needed. This will comprise the second step of the construction.

Let $g: \mathbb{R} \to [0, 1]$ be in $C^\infty(\mathbb{R})$ and also satisfy $g(z) = 0$, for $z \in (-\infty, 1]$, $g(z) = 1$, for $z \in [2, \infty)$, and $dg(z)/dz \ge 0$, for all z. Define $k(x) = g(|x|)$.

PROPOSITION 3.5. *For each $x \in \mathbb{R}^d$, define*

$$V^a(x) = \sup \int_0^\infty k\big(\phi(t)\big)\,dt,$$

where the supremum is over all solutions to

$$\dot{\phi} \in K^a(\phi), \qquad \phi(0) = x.$$

Then there exist $a_0 > 0$ and $r_0 > 0$ such that, for all $a \in (0, a_0)$, $V^a(x) = 0$ for $|x| \le r_0$, $V^a(\cdot)$ is finite and locally Lipschitz continuous on $\mathbb{R}^d$ and

$$\big\langle DV^a(x), u\big\rangle \le -1,$$

for almost every x such that $|x| \ge 2$ and every $u \in K^a(x)$. In fact,

$$\max_{u \in K^a(x)} \big\langle DV^a(x), u\big\rangle + k(x) = 0 \quad \textit{for a.e. } x \in \mathbb{R}^d. \tag{3.8}$$

PROOF. It follows from Proposition 3.4 that there are $a_0 > 0$ and $r_0 > 0$ such that, for all $a \in (0, a_0)$, all solutions to $\dot{\phi} \in K^a(\phi)$ with $|\phi(0)| \le r_0$ satisfy $|\phi(t)| \le 1$, for all $t \ge 0$, and hence $V^a(x) = 0$, for $|x| \le r_0$. Furthermore, given $R > 2$, there are $T < \infty$ and $a_0 > 0$ such that Proposition 3.4 (ii) holds with $r = r_0$. It follows that $0 \le V^a(x) < \infty$, for all x such that $|x| \le R$ whenever $a > 0$ is sufficiently small. However, the radial homogeneity of $K^a(\cdot)$ actually guarantees that $0 \le V^a(x) < \infty$ for all x, whenever $a > 0$ is sufficiently small (not depending on x).

We next prove the Lipschitz property. Fix $x \in \mathbb{R}^d \setminus \{0\}$, and let $R = |x| + 1$. Pick a_0 and r according to Proposition 3.4(i) with $c = 1$, and then choose a_0 smaller if necessary and T such that part (ii) holds for this choice of r. It will then follow from Proposition 3.4 that, for $a \in (0, a_0)$, all solutions to

$$\dot{\phi} \in K^a(\phi), \qquad \phi(0) = y, \tag{3.9}$$

satisfying $|y| \le R$ also satisfy $|\phi(t)| \le 1$, for $t \ge T$. In particular, this means that $\int_0^\infty k(\phi(t))\,dt = \int_0^T k(\phi(t))\,dt$ whenever ϕ solves (3.9) and $|y| \le R$.

For the given x and $a \in (0, a_0)$, choose ϕ which solves (3.9) with $y = x$ and also satisfies

$$V^a(x) \le \int_0^\infty k\big(\phi(t)\big)\,dt + \varepsilon.$$

Recall that the sets $K^a(x)$ are locally Lipschitz in x in the sense described in Proposition 3.2. Fix any $\delta \in (0, 1)$. By a modification of the classical ordinary differential equations argument based on Gronwall's inequality, we may prove the existence of a constant $C < \infty$ such that, given y satisfying $|y - x| \le \delta$, there is a solution ϕ^y to

$$\dot{\phi}^y \in K^a(\phi^y), \quad \phi^y(0) = y \tag{3.10}$$

satisfying

$$\sup_{0\le t\le T} \left|\phi(t) - \phi^y(t)\right| \le C|x-y|. \tag{3.11}$$

Let C' be the Lipschitz constant of $k(\cdot)$. Given y, choose ϕ^y satisfying (3.10) and (3.11). Since $|y| \le |x| + 1 \le R$,

$$\begin{aligned} V^a(x) - V^a(y) &\le \int_0^\infty k\big(\phi(t)\big)\,dt + \varepsilon - \int_0^\infty k\big(\phi^y(t)\big)\,dt \\ &= \int_0^T \Big[k\big(\phi(t)\big) - k\big(\phi^y(t)\big)\Big]\,dt + \varepsilon \\ &\le C'CT|x-y| + \varepsilon. \end{aligned}$$

Sending $\varepsilon \to 0$ and exploiting symmetry, we conclude that $V^a(\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^d$.

Now let x be a point at which $V^a(\cdot)$ is differentiable. We wish to show that

$$\max_{u\in K^a(x)} \langle DV^a(x), u\rangle = -k(x).$$

Fix $u \in K^a(x)$. Using the Lipschitz property of $K^a(\cdot)$, there exist $\delta > 0$, $c < \infty$ and $u(y)$, defined for all y satisfying $|x-y| \le \delta$, such that

$$u(y) \in K^a(y) \quad \text{and} \quad \big|u(y) - u\big| \le c|y-x|.$$

Clearly $u(y)$ can be chosen to be Lipschitz continuous in y. For each $\varepsilon > 0$ and $y \in \mathbb{R}^d$, let $\phi^{\varepsilon,y}(\cdot)$ be an ε-optimal trajectory that starts at y:

$$\begin{aligned} &\dot\phi^{\varepsilon,\,y}(t) \in K^a\big(\phi^{\varepsilon,\,y}(t)\big), \qquad \phi^{\varepsilon,\,y}(0) = y, \\ &V^a(y) \le \int_0^\infty k\big(\phi^{\varepsilon,\,y}(s)\big)\,ds + \varepsilon. \end{aligned}$$

We define a controlled trajectory that starts at x as follows. For $t \in [0,\gamma]$, where $\gamma > 0$ is chosen small enough to guarantee that $|\phi(s) - x| \le \delta$ for $s \in [0,\gamma]$, we let

$$\dot\phi(t) = u\big(\phi(t)\big), \qquad \phi(0) = x.$$

Suppose $\phi(\gamma) = y$. Then we define $\phi(t)$, for $t > \gamma$, to equal $\phi^{\varepsilon,\,y}(t-\gamma)$. There is a constant $C_1 < \infty$ such that $|\phi(t) - \phi(0)| \le C_1 t$. Together with the Lipschitz property of $u(\cdot)$ this gives $|u(\phi(t)) - u| \le Ct$, for some $C < \infty$ and all $t \in [0,\gamma]$. Integrating $\dot\phi = u(\phi)$ gives

$$|\phi(\gamma) - \phi(0) - \gamma u| \le C\gamma^2/2.$$

By the definition of $V^a(x)$,

$$V^a(x) \ge \int_0^\infty k\big(\phi(s)\big)\,ds \ge \int_0^\gamma k\big(\phi(s)\big)\,ds + V^a\big(\phi(\gamma)\big) - \varepsilon.$$

Sending $\varepsilon \to 0$ and using the continuity of $k(\cdot)$, we have

$$V^a(x) \geq \gamma k(x) + V^a\big(x + \gamma u + o(\gamma)\big) + o(\gamma).$$

Since $V^a(\cdot)$ is differentiable at x, this implies

$$\big\langle DV^a(x), u\big\rangle \leq -k(x).$$

Next let $\gamma_n = 1/n$, and let $\varepsilon_n = o(\gamma_n)$ (e.g., $\varepsilon_n = 1/n^2$). The uniform boundedness of the sets $K^a(y)$ in $y \in \mathbb{R}^d$ implies the boundedness of the vectors $[\phi^{\varepsilon_n,x}(\gamma_n) - x]/\gamma_n$. Thus we can extract a convergent subsequence (also indexed by n), with limit u, say. The continuity of $K^a(\cdot)$ guarantees that $u \in K^a(x)$. We have

$$\begin{aligned} V^a(x) &\leq \int_0^\infty k\big(\phi^{\varepsilon_n,x}(s)\big)\,ds + \varepsilon_n \\ &\leq \int_0^{\gamma_n} k\big(\phi^{\varepsilon_n,x}(s)\big)\,ds + \varepsilon_n + V^a\big(\phi^{\varepsilon_n,x}(\gamma_n)\big). \end{aligned}$$

Using

$$\phi^{\varepsilon_n,x}(\gamma_n) = x + u\gamma_n + o(\gamma_n),$$

we have

$$V^a(x) \leq k(x)\gamma_n + \varepsilon_n + V^a(x) + \big\langle DV^a(x), u\big\rangle\gamma_n + o(\gamma_n).$$

Dividing by γ_n and letting $n \to \infty$ gives

$$\big\langle DV^a(x), u\big\rangle \geq -k(x),$$

which completes the proof. □

For the remainder of the proof we will fix $a_0 > 0$ such that Proposition 3.5 holds and $a \in (0, a_0)$.

PROPOSITION 3.6. *The function $V^a(x)$ grows at least linearly, that is, there exist constants $C_1 > 0$ and C_2 such that $V^a(x) \geq C_1|x| - C_2$.*

PROOF. Let $B = \sup_x \sup_{u \in K^a(x)} |u|$. Then we have the lower bound $(|x| - 2)/B$ on the time at which any solution to the differential inclusion that starts at x reaches the set $\{y : |y| \leq 2\}$. The proposition now follows from the fact that $k(y) \geq 0$, for $y \in \mathbb{R}^d$, and $k(y) = 1$ whenever $|y| > 2$. □

In the next proposition we consider in a more precise fashion the rate of growth of $V^a(\cdot)$ in the radial direction. In particular, we show that for each fixed $x \neq 0$, $V^a(\alpha x)$ is a monotonic function of $\alpha > 0$. Note that the radial homogeneity of $K^a(\cdot)$ has not been preserved by $V^a(\cdot)$ since $k(\cdot)$ is not constant. Unfortunately, this seems unavoidable.

PROPOSITION 3.7. *Let $x \neq 0$ be a point at which $V^a(\cdot)$ is differentiable. Then*

$$\langle DV^a(x), x/|x| \rangle \geq V^a(x)/|x|.$$

PROOF. Let $\phi^{\varepsilon,x}(\cdot)$ be defined as in the proof of Proposition 3.5. Thus

$$V^a(x) \leq \int_0^\infty k\big(\phi^{\varepsilon,x}(t)\big)\,dt + \varepsilon.$$

Owing to the radial homogeneity of $K^a(x)$, for all $c > 0$, the function $\theta^c(t) = (1+c)\phi^{\varepsilon,x}(t/(1+c))$ satisfies

$$\dot{\theta}^c(t) \in K^a\big(\theta^c(t)\big), \quad \theta^c(0) = (1+c)x.$$

Thus $\theta^c(\cdot)$ can serve as a candidate path in the supremization that defines $V^a((1+c)x)$. The monotonicity of $k(\cdot)$ in the radial direction implies that the cost along the path $\theta^c(\cdot)$ satisfies

$$\begin{aligned}\int_0^\infty k\big(\theta^c(t)\big)\,dt &= (1+c)\int_0^\infty k\big((1+c)\phi^{\varepsilon,x}(t)\big)\,dt \\ &\geq (1+c)\int_0^\infty k\big(\phi^{\varepsilon,x}(t)\big)\,dt.\end{aligned}$$

Thus $V^a((1+c)x) \geq (1+c)(V^a(x) - \varepsilon)$. Since $\varepsilon > 0$ is arbitrary $V^a((1+c)x) \geq (1+c)V^a(x)$, or

$$V^a\big((1+c)x\big) - V^a(x) \geq cV^a(x).$$

Sending $c \downarrow 0$ gives the desired bound. $\square$

Recall the mollification function $\rho(\cdot)$ and its properties as stated in (3.3). For each $b \in (0,1]$, we define a smoother version of $V^a(\cdot)$ via

$$V^{a,b}(x) = c(b)\int \rho\Big(\frac{x-y}{b}\Big)V^a(y)\,dy.$$

Recall from Proposition 3.2 that $r^0 \in K^a(x)$ whenever $d(x,S) \leq a|x|/2$ and $r^i \in K^a(x)$ whenever $d(x,F^i) \leq a|x|/2$, $x \neq 0$. By combining the fact that the support of ρ is contained in the unit ball with Proposition 3.5, we see that there exists $M < \infty$ which depends on the fixed value of a but which is independent of $b \in (0,1]$ such that

$$\begin{aligned}\langle DV^{a,b}(x), r^0\rangle &\leq -1 \quad \text{for all } x \in S \text{ and } |x| \geq M,\\ \langle DV^{a,b}(x), r\rangle &\leq -1 \quad \text{for all } r \in r(x),\ x \in \partial S,\ |x| \geq M.\end{aligned}$$

We now fix such an M.

The function $V^{a,b}(\cdot)$ is nearly the Lyapunov function we seek. The only remaining difficulty is in proving $\|D^2V^{a,b}(x)\| \to 0$ as $|x| \to \infty$. Our third and final step in the construction of $W(\cdot)$ is intended to circumvent this problem, and incidentally it will allow us to remove the restriction that $|x| \geq M$ in the inequalities immediately above.

According to Propositions 3.6 and 3.7 there are $M_1 \in [M,\infty)$ and $c \in (0,\infty)$ such that $\langle DV^a(x), x/|x|\rangle \geq c$ whenever $|x| \geq M_1 - 1$. Let $U = \sup_{x:\, |x|\leq M_1} V^a(x)$, and choose $M_2 \in [M_1,\infty)$ such that $|x| \geq M_2$ implies $V^a(x) \geq U + 3$. Thus we have the following properties:

1. $V^a(x) \leq U$, for $|x| \leq M_1$;
2. $\{x: V^a(x) = U+2\} \subset \{x: M_1 < |x| < M_2\}$;
3. $\langle DV^a(x), x/|x|\rangle \geq c$ whenever $|x| \geq M_1 - 1$.

Part of the conclusion of Proposition 3.5 is that $V^a(\cdot)$ is locally Lipschitz continuous. It follows from this and the properties of $V^a(\cdot)$ just listed that for sufficiently small $b \in (0,1]$,

1. $V^{a,b}(x) \leq U+1$, for $|x| \leq M_1$,
2. $\{x: V^{a,b}(x) = U+2\} \subset \{x: M_1 < |x| < M_2\}$,
3. $\langle DV^{a,b}(x), x/|x|\rangle \geq c/2$ on the set $\{x: M_1 \leq |x| \leq M_2\}$.

Fix such a value of b.

Define the set $L = \{x: V^{a,b}(x) \leq U+2\}$. (Since a and b are both fixed at this point, we drop them from any new notation.) Thus ∂L is a level set of the function $V^{a,b}(\cdot)$, and the outward normal to L satisfies all of the desired properties of the gradient of $V^{a,b}(\cdot)$. The fact that $DV^{a,b}(x) \neq 0$ on ∂L implies that ∂L is C^2. We will build the desired Lyapunov function by requiring that all of its level sets be multiples of L. Thus $W(\cdot)$ is defined by the equation

$$\{x: W(x) \leq l\} = \{lx : x \in L\}.$$

Note that $x \in \partial L$ implies $V^{a,b}(x) = U+2$ and also $|x| \in (M_1, M_2)$. Since $V^{a,b}(x) \leq U+1$ whenever $|x| \leq M_1$ and $\langle DV^{a,b}(x), x/|x|\rangle \geq c/2$ whenever $|x| \in [M_1, M_2]$, we conclude that L is "star-shaped" with respect to the origin, that is, if y is any point in L, then the relative interior of the line segment connecting the origin to y is contained in the interior of L. Thus, for each $x \in \mathbb{R}^d$, there is a unique $l \in [0,\infty)$ such that $x \in \partial(lL)$, and therefore W is well defined.

Together with the fact that ∂L is C^2, this implies that the function $W(\cdot)$ is in $C^2(\mathbb{R}^d \setminus \{0\})$. A proof is as follows. Fix $x \neq 0$. Let $\alpha \in \partial L$ and $\eta \in (0,\infty)$ be such that $\eta\alpha = x$. Let v be the outward normal to ∂L at α. Define $\widetilde{e}_d = v$, and suppose that $\{\widetilde{e}_i,\ i = 1,\dots,d\}$ forms an orthonormal basis for $\mathbb{R}^d$. Since ∂L is C^2, there exists an open neighborhood N_α of α and a C^2 function g such that $\beta \in N_\alpha \cap \partial L$ implies $\beta_d = g(\beta_1,\dots,\beta_{d-1})$, where $(\beta_1,\dots,\beta_d)$ is the representation of β with respect to the coordinate system defined by $\{\widetilde{e}_i,\ i = 1,\dots,d\}$. Note that $g_{\beta_i}(\alpha_1,\dots,\alpha_{d-1}) = 0$, for $i = 1,\dots,d-1$. Let N_x be an open neighborhood of x with the property that $y \in N_x$ only if there is $\beta(y) \in N_\alpha \cap \partial L$ and $\eta(y) \in (0,\infty)$ such that $y = \eta(y)\beta(y)$. The existence of such a neighborhood follows from $\langle v, \alpha\rangle > 0$.

Of course $W(y) = \eta(y)$ for $y \in N_x$ and $F(\eta(y), y) = 0$, where

$$F(\eta, y) = \frac{1}{\eta} y_d - g\left(\frac{1}{\eta} y_1, \dots, \frac{1}{\eta} y_{d-1}\right).$$

Since $F_\eta(\eta(x), x) = -x_d/\eta(x_d)^2 \neq 0$, the implicit function theorem implies $\eta(y)$ is C^2 in an open neighborhood of x. Since $x \neq 0$ is arbitrary, $W(\cdot) \in C^2(\mathbb{R}^d \backslash \{0\})$.

By construction the function satisfies

$$W(\alpha x) = \alpha W(x) \quad \text{for } \alpha > 0,\ x \neq 0.$$

Thus for given $\varepsilon > 0$ there exists $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $\|D^2 W(x)\| \leq \varepsilon$. We conclude that the function $W(\cdot)$ satisfies all of the properties required of the Lyapunov function.

APPENDIX

COMPLETION OF THE PROOF OF THEOREM 2.6. Let $W(\cdot)$ be a function possessing the properties listed after the statement of Theorem 2.6. Choose $r < \infty$ such that $\frac{1}{2}\text{tr}\,[D^2 W(x)\Delta] + \langle DW(x), r^0\rangle \leq -\varepsilon/2$, for some $\varepsilon > 0$, and $W(x) \geq 0$ whenever $|x| \geq r$ and $x \in S$. Define $\tau_r = \inf\{t\colon |z(t)| \leq r\}$. If $|x| \leq r$, then $E_x[\tau_r] < \infty$ is automatic. For the remainder of the proof we assume $|x| > r$. For $n > r$, define $\tau_r^n = \inf\{t\colon |z(t)| \notin (r, n)\}$. Then Itô's formula implies that, for each $t \geq 0$,

$$E_x\big[W\big(z(t \wedge \tau_r^n)\big)\big] - W(x) \leq -\frac{\varepsilon}{2} E_x[t \wedge \tau_r^n],$$

and therefore

$$E_x[t \wedge \tau_r^n] \leq 2W(x)/\varepsilon.$$

Sending $t \to \infty$ gives $E_x[\tau_r^n] \leq 2W(x)/\varepsilon$. Since $\tau_r^n \uparrow \tau_r$ as $n \to \infty$, it follows that $E_x[\tau_r] \leq 2W(x)/\varepsilon < \infty$.

Next let A be an arbitrary closed set in S having positive Lebesgue measure. We may assume without loss of generality that $A \subset \{x\colon |x| \leq r\}$. According to the previous paragraph $E_x[\tau_r]$ is uniformly bounded as x ranges over any fixed compact set. It is proved in Lemma 7.9 of [9] that $P_x(z(1) \in A) \equiv p_A(x) > 0$, for all $x \in S$. (Although Lemma 7.9 in [9] is proved only for a special class of SRBM's, the proof carries over since Lemma 7.2 in [9] holds for all SRBM's [24] and the only other property needed is that an SRBM behaves like a Brownian motion in the interior of S.) Therefore, for sufficiently large $M(x) < \infty$, we have

$$P_x\Big(z(1) \in A \text{ and } |z(t)| \leq M(x) \text{ for all } t \in [0,1]\Big) \geq p_A(x)/2.$$

Let $\{x_i,\ i \in \mathbb{N}\}$ be a sequence in S that converges to x. Since the weak limit of the sequence of SRBM's with initial conditions x_i is the SRBM with initial condition

x, we can argue by contradiction to establish the existence of $M \in [r, \infty)$ and $p_A > 0$ such that

$$P_x\Big(z(1) \in A \text{ and } |z(t)| \le M \text{ for all } t \in [0,1]\Big) \ge p_A,$$

for all x satisfying $|x| \le r$. A standard argument that uses stopping times then yields the following upper bound for arbitrary $x \in S$:

$$E_x[\tau_A] \le \frac{1}{p_A}\left(2 + \sup_{v:|v|\le M} E_v[\tau_r]\right) + E_x[\tau_r].$$

We next consider the existence and uniqueness of the stationary distribution. Fix any $x \in S$ and let $z(\cdot)$ be the SRBM with this initial condition. Given the existence of the Lyapunov function, it follows from the proof of Theorem 1 on page 146 of [14] that the set of measures $\{\mu_t,\ t \in [0,\infty)\}$ defined by $\mu_t(B) = P_x\{z(t) \in B\}$ is tight (although the proof in [14] is for diffusions, it carries over directly to our case). Hence the set of normalized occupation measures $\{\bar{\mu}_t,\ t \in (0,\infty)\}$ defined by $\bar{\mu}_t(B) = \int_0^t \mu_s(B)\,ds/t$, for $t > 0$, is also tight. By a standard calculation (e.g., equation (9.5) on page 240 of [7]) any weak limit as $t \to \infty$ of a sequence from the set $\{\bar{\mu}_t,\ t \in (0,\infty)\}$ is a stationary distribution. Thus we obtain existence of stationary distributions. Uniqueness now follows by the same argument as in Section 7 of [9] which shows that Lebesgue measure is a reference measure for SRBM's. □

PROOF OF THE EXISTENCE OF MEASURABLE FUNCTIONS q_i SATISFYING (3.6) AND (3.7) The following proof is due to Avi Mandelbaum. While there may be other means of proving the result, we feel that this method of proof, especially Lemma 4.1, may be of independent interest.

LEMMA 4.1. *Let A be an $i \times j$ matrix ($i \ge 1$, $j \ge 1$), $B = \{b \in \mathbb{R}^i : Ax = b$, for some $x \in \mathbb{R}^j_+\}$, and let F be the set-valued mapping defined on B by $F(b) = \{x \in \mathbb{R}^j_+ : Ax = b\}$, $b \in B$. Then F is lower semicontinuous, that is, if $\{b^n\}$ is a sequence in B converging to $b^0 \in B$, and $x^0 \in F(b^0)$, then there exists $x^n \in F(b^n)$ for each n, such that $\{x^n\}$ converges to x^0.*

PROOF. Let x^0, b^0 and $\{b^n\}$ be as in the definition of lower semicontinuity above. Now (cf. [23], Theorem 5, page 119), there are finitely many extreme points $e_1, \dots, e_k$ of $F(b^0)$, solutions $d_1, \dots, d_m$ of $Ad_l = 0$, $d_l \in \mathbb{R}^j_+$, and non-negative real numbers $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m$ such that $\sum_{l=1}^k \alpha_l = 1$ and

$$x^0 = \sum_{l=1}^k \alpha_l e_l + \sum_{l=1}^m \beta_l d_l.$$

If for each $l \in \{1, \dots, k\}$, we can construct a sequence $\{e_l^n\}$ such that $e_l^n \in F(b^n)$ for each n and $e_l^n \to e_l$ as $n \to \infty$, then, by convexity,

$$x^n \equiv \sum_{l=1}^k \alpha_l e_l^n + \sum_{l=1}^m \beta_l d_l \in F(b^n)$$

and clearly $x^n \to x^0$ as $n \to \infty$. In other words, it suffices to assume that x^0 is an extreme point of $F(b^0)$. Assuming this, by relabeling the coordinates if necessary, we have that $x^0 = (x_1^0, \dots, x_r^0, x_{r+1}^0, \dots, x_j^0)$, where $x_1^0 > 0, \dots, x_r^0 > 0$, $x_{r+1}^0 = \cdots = x_j^0 = 0$ and the first r columns of A are linearly independent (cf. [23], Theorem 1, page 114). Let C be the matrix consisting of these first r columns of A, and let D denote the matrix consisting of the remaining columns of A. Let e denote the vector of all 1's in $\mathbb{R}_+^{j-r}$. Consider the following linear program:

$$\begin{gathered} \min\langle e, \delta\rangle \\ \text{subject to} \\ C\gamma + D\delta = b^n, \\ \gamma \geq 0, \ \delta \geq 0. \end{gathered}$$

Since $F(b^n)$ is nonempty, each of these linear programs is feasible and each has an optimal solution, say $x^n = (\gamma^n, \delta^n) \in F(b^n)$. Consider also the linear program

$$\begin{gathered} \min\langle e, \delta\rangle \\ \text{subject to} \\ C\gamma + D\delta = b^0, \\ \gamma \geq 0, \ \delta \geq 0. \end{gathered}$$

Then $\gamma^0 = (x_1^0, \dots, x_r^0)$, $\delta^0 = 0$ provides a solution of this linear program. Moreover, since $\langle e, \delta^0\rangle = 0$, the optimal value for this linear program is zero and (γ^0, δ^0) is an optimal solution. By a sensitivity result for linear programs [22, Theorem 10.5], the sequence of optimal values of the linear programs depending on n converges to the optimal value for the linear program immediately above, that is, $\langle e, \delta^n\rangle \to \langle e, \delta^0\rangle = 0$, as $n \to \infty$. However, $\delta^n \geq 0$, and so $\delta^n \to 0$. Thus,

$$C\gamma^n = b^n - D\delta^n \to b^0 = C\gamma^0 \quad \text{as } n \to \infty.$$

Since the columns of C are linearly independent, we may multiply the above by a left inverse for C to conclude that $\gamma^n \to \gamma^0$ as $n \to \infty$. Thus, $x^n = (\gamma^n, \delta^n) \to (\gamma^0, \delta^0) = x^0$ and $x^n \in F(b^n)$, as desired. □

For $\lambda \in \Lambda$ fixed, let $k = |\lambda|$, let $\widetilde{A}^\lambda$ be the $d \times (k+1)$ matrix whose columns are given by the vectors $r^0, \{r^i\}_{i\in\lambda}$ and let A^λ be the $(d+1) \times (k+1)$ matrix obtained by adding a $(d+1)$st row to $\widetilde{A}^\lambda$ that contains all one's. Let $Q^\lambda = \{q = (q_0, q_1, \dots, q_k) \in \mathbb{R}_+^{k+1}\}$ and let $B^\lambda = \{A^\lambda q : q \in Q^\lambda\}$. By the above lemma, the set-valued map F^λ defined on B^λ by $F^\lambda(b) = \{q \in Q^\lambda : A^\lambda q = b\}$, $b \in B^\lambda$, is lower semicontinuous and hence by Michael's selection theorem [1, Theorem 1.11.1, page 82], there exists a continuous selection function $f^\lambda : B^\lambda \to Q^\lambda$ from F^λ. Given ϕ as described above equation (3.6), for all t such that $\text{In}(\phi(t)) = \lambda$

and $\dot{\phi}(t) \in K(\phi(t))$, define

$$q_0(t) = f_0^\lambda \begin{pmatrix} \dot{\phi}(t) \\ 1 \end{pmatrix},$$

$$q_i(t) = \begin{cases} f_{j_i}^\lambda \begin{pmatrix} \dot{\phi}(t) \\ 1 \end{pmatrix}, & \text{for } i \in \lambda, \\ 0, & \text{for } i \notin \lambda, \end{cases}$$

where j_i denotes the position in an ordered λ of the element $i \in \lambda$. In particular, the $(j_i + 1)$th column of the matrix $\widetilde{A}^\lambda$ is r^i.

Once λ is allowed to range over all of Λ, the q_i, $i = 0, \dots, d$, will be defined for a.e. t and they can be defined to be say zero on the remaining exceptional set of t's. It is then readily verified that, for each i, $q_i: \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function and that (3.6)–(3.7) hold for a.e. t. □

NOTE ADDED IN PROOF. Stimulated by Theorem 2.6, J. G. Dai has proved an analogue of this result for queuing networks (see "On positive Harris recurrence of multiclass queuing networks: a unified approach via fluid limit models," to appear in *Annals of Applied Probability*). More precisely, Dai has shown that stability of fluid limits associated with a queuing network implies positive recurrence for a Markov process which describes the dynamics of the network. His approach is different from ours in the sense that he does not construct a Lyapunov function. Dai and other authors have been using his result to determine sufficient conditions for stability of multiclass networks with feedback (see the Proceedings of the 1994 IMA Workshop on Stochastic Networks for more details).

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LEFSCHETZ CENTER FOR DYNAMICAL SYSTEMS
DIVISION OF APPLIED MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RHODE ISLAND 02912

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO
9500 GILMAN DRIVE
LA JOLLA, CALIFORNIA 92093-0112