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A CHARACTERIZATION OF STOPPING TIMES

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Let R be a random time in $\mathcal{F}_\infty$, the terminal element of a filtration $\mathcal{F}_t$ satisfying the usual hypotheses. It is shown that if optional sampling holds at R for all bounded martingales, then R is optional. If $\mathcal{F}_t$ is the natural pseudo-path filtration of a measurable process X_t, then R is optional if (and only if) the conditional distribution of $X_{R+\cdot}$ given $\mathcal{F}_R$ is Z_R, where Z_t is an optional version of the conditional distribution of $X_{t+\cdot}$ given $\mathcal{F}_t$.

1. Introduction and description of results. Let $\mathcal{F}_t$, $0 \le t < \infty$, be a right-continuous filtration containing all P-nullsets on a complete probability space $(\Omega, \mathcal{F}, P)$. A *random time* R is simply any random variable with values in $[0, \infty]$. If $\{R \le t\} \in \mathcal{F}_t$ for all t, then R is a *stopping time* or *optional time*, and *the past up to time* R may be defined as

$$\mathcal{F}_R = \{A \in \mathcal{F}\colon A \wedge \{R \le t\} \in \mathcal{F}_t \text{ for all } t\}. \tag{1.1}$$

If R is not a stopping time, treatment of the past and future of a process at time R tends to be problematical. The above definition does not give a σ-field. It may be replaced by $\{A \in \mathcal{F}\colon$ for all t there exists $A_t \in \mathcal{F}_t$ such that $A \cap \{R \le t\} = A_t \cap \{R \le t\}\}$, but even if R is the end of an optional set, this extension is by no means the only reasonable possibility. Indeed, for such R it equals

$$\mathcal{F}_R^+ \doteq \sigma\{Y(R);\ Y(t) \text{ a bounded, progressively measurable process}\}, \tag{1.2}$$

whereas the usual extension of (1.1) is

$$\mathcal{F}_R \doteq \sigma\{Y(R);\ Y(t) \text{ a bounded, optional process}\} \tag{1.3}$$

(see [6], XX, 26, where $\{U < s\}$ may be replaced by $\{U \le s\}$ in view of $\mathcal{F}_{s+} = \mathcal{F}_s$). If R is a stopping time (1.1) to (1.3) are all equivalent (see [4], IV, 68, and [3], Theorem 20). But if, for example, R is the last exit time from 0 before time 1 for a standard Brownian motion B with natural filtration $\mathcal{F}_t$, then $\mathcal{F}_R^+ = \sigma(\operatorname{sgn} B(1)) \vee \mathcal{F}_R$.

General classes of R other than stopping times abound in the literature, for example the "regular birth times" of Pittenger [9] and the "honest" times of Barlow [1], of which the above R is an example. Another example (in which, however, $\mathcal{F}_R^+ = \mathcal{F}_R$) is $R \doteq \operatorname{argmax} B_t^o \doteq \sup\{s \le t\colon B_s^o = \max_{r \le t} B_r^o\}$, where B^o is B absorbed at -1. Relative to the natural filtration, this R is honest [it is the end of the optional set $\{(t, w)\colon B_t^o = \max_{s \le t} B_s^o\}$], but it is not a regular birth time

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because B^o_{R+t}, while Markovian given $\mathcal{F}_R$, has a semigroup dependent explicitly on $\mathcal{F}_R$. Namely, it is well known that $B^o_R - B^o_{R+t}$ has, given $\mathcal{F}_R$, the semigroup of a three-dimensional Bessel process starting at 0 and absorbed at $1+B^o_R$, so that $-B^o_{R+t}$ has the semigroup of a translated Bessel process absorbed at 1, whereas the Bessel semigroup does not commute with translation.

In general, if R is not a stopping time, treatment of X_{R+t} for adapted processes X is relatively difficult. It thus becomes of interest to identify properties of its behavior which characterize stopping times (so that one knows the meaning of the stopping time concept in terms of the future of processes). Here, two special cases come immediately to mind. If X_t is a right-continuous, left-limited (r.c.l.l.) uniformly integrable *martingale*, say $X_t = E(H \mid \mathcal{F}_t)$, $H \in L^1(\mathcal{F})$, and if R is a stopping time, then, given $\mathcal{F}_R$, the process X_{R+t} is a martingale, as a consequence of the optional sampling theorem of Doob. This property does not characterize stopping times, even if we assume that $\mathcal{F}_t$ is the natural filtration of X: take $R \doteq \frac{1}{2}$ (the first jump time of a bilateral Poisson process stopped at time 1). It does become characteristic (this is our first main result) if we require it for *all* such martingales [note that in the preceding example optional sampling fails at R for the martingale $X_t^2 - t$]. In fact, we shall prove [Theorem 2.1(a)] that R is optional if (and only if)

$$E(H \mid \mathcal{F}_R) = H_R \quad \text{on } \{R < \infty\} \text{ for all } H \in b(\mathcal{F}), \tag{1.4}$$

where H_t denotes an r.c.l.l. version of $E(H \mid \mathcal{F}_t)$. We remark that the choice of "past" $\mathcal{F}_R$ is not critical—the same result holds with $\mathcal{F}_R^+$ in place of $\mathcal{F}_R$. Furthermore, an analogous characterization holds from the left: (1.4) with $\mathcal{F}_{R-}$ and H_{R-} characterizes the previsible (or predictable) stopping times R.

Second, if X_t is a strong Markov process on a Lusin space $(E, \mathcal{E})$, relative to the natural augmented filtration $\mathcal{F}_t$ and a Borel family P^x, $x \in E$, of transition probabilities (see [10], I, Section 6] for definitions), then, for any initial distribution μ,

$$P^\mu(X_{R+\cdot} \in A \mid \mathcal{F}_R) = P^{X_R}(A), \qquad A \in \mathcal{F}^0 \text{ over } \{R < \infty\} \tag{1.5}$$

for all stopping times R. An interesting result of Pittenger [9], Corollary (4.3), says that if X is a *right process*, (1.5) for all μ characterizes stopping times. This situation seems quite specialized until we remark that with any measurable process X_t there is associated its prediction process Z_t^X, which is a realization of a Borel right process ([7], Definition 6.3). In more detail, let $\Omega' = \{\text{all } (E, \mathcal{E})\text{-valued, measurable paths } w'\}$ and $\mathcal{F}'_t = \sigma\{\int_0^s f(w'(u))\,du;\ s < t, f \in b(\mathcal{E})\}$, $\mathcal{F}' = \mathcal{F}'_\infty$. Borrowing a term from [4], IV, (4), we call $\mathcal{F}'_t$ the *pseudo-path filtration* on Ω'. Let $(M, \mathcal{M})$ be the Lusin space of probability measures on $(\Omega', \mathcal{F}')$. Then, for any measurable process X_t on $(E, \mathcal{E})$, Z_t^X is the unique (up to indistinguishability) $(M, \mathcal{M})$-valued process satisfying

(1.6)
(a) $P(X_{t+\cdot} \in A \mid \mathcal{F}_t^X) = Z_t^X(A)$, $\qquad A \in \mathcal{F}'$.
(b) Z_t^X is r.c.l.l. in a suitable topology on M (equivalently, Z_t^X is $\mathcal{F}_t^X$-optional), where $\mathcal{F}_t^X \doteq X^{-1}(\mathcal{F}'_{t+})$ augmented by all P-nullsets.

It follows that $Z_t^X(A)$ is the $\mathcal{F}_t^X$-optional projection of $I_{\{X(t+\cdot)\in A\}}$, $A \in \mathcal{F}'$. In particular, if R is a stopping time,

$$P(X_{R+\cdot} \in A \mid \mathcal{F}_R^X) = Z_R^X(A) \quad \text{on } \{R < \infty\}, \qquad A \in \mathcal{F}'. \tag{1.7}$$

Our main result for measurable processes is that (1.7) characterizes stopping times R. We note that the direct extension of Pittenger's result to this situation would only yield that R is a stopping time if and only if, given $\mathcal{F}_R$, Z_{R+t}^X is Markovian with the same transition function as Z_t^X (besides the need to show that a single initial distribution μ suffices). This is obviously a poorer characterization than (1.7), which does not invoke the Markov property of Z^X.

As in the case of (1.4), there is an analogous characterization from the left: R is previsible if and only if (1.7) holds with $\mathcal{F}_{R-}^X$ and Z_{R-}^X. Finally, we show that Pittenger's result (with fixed μ) follows from (1.7), which in turn gives the alternative characterization of R in terms of the Markov property of Z_{R+t}^X.

2. Theorems and proofs. We begin with the characterization by optional sampling. As in Section 1, $\mathcal{F}_t$ is a filtration satisfying the usual hypotheses ($\mathcal{F}_{0-} = \mathcal{F}_0$), R is an arbitrary random time, $\mathcal{F}_R$ is defined by (1.3) and

$$\mathcal{F}_{R-} \doteq \sigma\{Y(R);\ Y(t) \text{ a bounded, previsible process}\}. \tag{2.1}$$

THEOREM 2.1. (a) *Suppose that, for all* $H \in b(\mathcal{F})$,

$$E(H \mid \mathcal{F}_R) = H_R \quad \text{on } \{R < \infty\},$$

where H_t *is an r.c.l.l. version of* $E(H \mid \mathcal{F}_t)$. *Then* R *is a stopping time.*

(b) *Suppose that for all* $H \in b(\mathcal{F})$, $E(H \mid \mathcal{F}_{R-}) = H_{R-}$ *on* $\{R < \infty\}$ (*where* $H_{0-} = H_0$). *Then* R *is previsible.*

PROOF. These results are consequences, with $A_t = I_{\{t \geq R\}}$, of the following, which is of independent interest.

LEMMA 2.1. *Let* A_t *be an r.c. increasing process defined on* $(\Omega, \mathcal{F}, P)$ *and such that* $EA_t < \infty$ *for each* t.

(a) *If* $E(HA_t) = E\int_{[0,t]} H_s\, dA_s$; $0 \leq t$, $H \in b(\mathcal{F})$, *then* A_t *is adapted to* $\mathcal{F}_t$ (*i.e.*, A_t *is optional*).

(b) *If* $E(HA_t) = E\int_{[0,t]} H_{s-}\, dA_s$; $0 \leq t$, $H \in b(\mathcal{F})$, *then* A_t *is previsible.*

PROOF. (a) Replacing H by H_t (stopping the martingale at t), we also have

$$\begin{aligned} E(H_t A_t) &= E\int_{[0,t]} H_{s\wedge t}\, dA_s \\ &= E\int_{[0,t]} H_s\, dA_s. \end{aligned}$$

Hence $E(HA_t) = E(H_t A_t)$, and it follows that, setting $A'_t = E(A_t \mid \mathcal{F}_t)$,

$$E(HA_t) = E(H_t A'_t) = E(HA'_t).$$

This means that the measures $A_t \cdot dP$ and $A'_t \cdot dP$ are equal, hence $A_t = A'_t$, P-a.s. It follows that $A_t \in \mathcal{F}_t$, as asserted.

(b) If we repeat the preceding argument under the assumption of (b), we can only show that $A_t \in \mathcal{F}_{t-}$. However, setting $K_s = H1_{[0,r]}(s)$, with previsible projection ${}^pK_s = H_{s-}1_{[0,r]}(s)$, the assumption of (b) gives

$$E\int_0^\infty K_s\,dA_s = E\int_0^\infty {}^pK_s\,dA_s \quad \text{for each } r > 0. \tag{2.2}$$

Now the set of such K is closed under products, and the class of all $K \in b(\mathcal{R}^+ \times \mathcal{F})$ for which (2.2) holds is a vector space, closed under monotone bounded limits, and generates $\mathcal{R}^+ \times \mathcal{F}$. It follows by a monotone class theorem ([4], I, 21) that (2.2) holds for all K. According to [5], VI, (59.2), this proves that A_t is previsible, finishing the proof of Lemma 2.1, and hence Theorem 2.1. □

We turn next to the characterization by conditional future (1.7) and its left counterpart. We consider a (Borel) measurable process X_t (no continuity assumption is made), its pseudo-path filtration $\mathcal{F}_t^X$, $\mathcal{F}^X = \mathcal{F}_\infty^X$, and its prediction process Z_t^X as in (1.6). We recall that, for $f \in b(\mathcal{F}')$, $Z_t^X f$ and $Z_{t-}^X f$ are the optional and previsible projections of $f(X_{t+\cdot})$. We also recall the splicing operators on $\Omega' \times R^+ \times \Omega' \to \Omega'$, given by

$$(w'_1/t/w'_2)_s \doteq \begin{cases} w'_1(s), & s < t, \\ w'_2(s-t), & s \geq t. \end{cases}$$

In the sequel $X(w)$ will denote the element $X_\cdot(w)$ of Ω'. The mapping φ: $(t, w, w') \mapsto (X(w)/t/w')$ is $(\mathcal{P} \otimes \mathcal{F}', \mathcal{F}')$ measurable ($\mathcal{P}$ denotes the previsible σ-field relative to $\mathcal{F}_t^X$). In fact, for $r \geq 0, f \in b(\mathcal{E})$,

$$\int_0^r f\big(X(w)/t/w'\big)(s)\,ds = \int_0^{r\wedge t} f\big(X_s(w)\big)\,ds + \int_0^{(r-t)^+} f\big(w'(s)\big)\,ds,$$

the first term is previsible in (t, w), and the second is measurable in (t, w').

The following result is known (see [8], Theorem 2), but we shall need its proof as well as the result itself.

LEMMA 2.2. *Let $h \in b(\mathcal{F}')$, $H = h(X)$. Then the processes H_t and H_{t-} of Theorem* 2.1 *are indistinguishable from h_t and $\overline{h}_t$, where for $t \geq 0$, $w \in \Omega$, $h_t(w)$ and $\overline{h}_t(w)$ denote the integrals of $h((X(w)/t/\cdot)) = h(\varphi(t, w, \cdot\,))$ with respect to $Z_t^X(w)$ and $Z_{t-}^X(w)$, respectively.*

PROOF. For $g(t, w, w') = u(t, w)v(w')$, $u \in b(\mathcal{P})$, $v \in b(\mathcal{F}')$, the optional and previsible projections of $g(t, w, X_{t+\cdot}(w))$ are the integrals of $g(t, w, \cdot\,)$ with respect to $Z_t^X(w)$ and $Z_{t-}^X(w)$, respectively, and this extends to all $g \in b(\mathcal{P} \otimes \mathcal{F}')$ by a monotone class argument. Lemma 2.2 is proved by taking $g = h \cdot \varphi$, since then $H(w) = g(t, w, X_{t+\cdot}(w))$ for all t. □

THEOREM 2.2. *Let R be a random time on $(\Omega, \mathcal{F}^X, P)$.*

(a) *Suppose that $P(X_{R+\cdot} \in A \mid \mathcal{F}_R^X) = Z_R^X(A)$ on $\{R < \infty\}$ for $A \in \mathcal{F}'$. Then R is a stopping time relative to $\mathcal{F}_t^X$.*

(b) *Suppose that $P(X_{R+\cdot} \in A \mid \mathcal{F}_{R-}^X) = Z_{R-}^X(A)$ on $\{R < \infty\}$, where $\mathcal{F}_{0-}^X \doteq \mathcal{F}_0^X$ and $\mathcal{Z}_{0-}^X \doteq \mathcal{Z}_0^X$.*

Then R is previsible.

PROOF. For $h \in b(\mathcal{F}')$, $H = h(X)$, the identity $H(w) = h(\varphi(R(w), w, X_{R(w)+\cdot}(w)))$ on $\{R < \infty\}$ and the same argument as for Lemma 2.2 show that under (a) one has $E(H \mid \mathcal{F}_R) = h_R$ on $\{R < \infty\}$. But $h_R = H_R$ a.s. on $\{R < \infty\}$ by Lemma 2.2. The equality $E(H \mid \mathcal{F}_R) = H_R$ on $\{R < \infty\}$ extends to all $H \in b(\mathcal{F}^+)$, since $\mathcal{F}^X = X^{-1}(\mathcal{F}')$ up to null sets. Thus Theorem 2.2(a) follows from Theorem 2.1(a); similarly, Theorem 2.2(b) follows from Theorem 2.1(b). □

REMARK. The converses to Theorems 2.2(a) and (b) are straightforward. See, for example, [7], Chapter 1.

We turn finally to an extension of Corollary (4.3) of Pittenger [9], which has been the catalyst for this paper. It may be derived directly from Theorem 2.1, but the proof given here, based on Theorem 2.2, also illustrates the use of Z_t^X when X is a right-continuous or an r.c.l.l. process.

THEOREM 2.3. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t^\mu, X_t, \theta_t, P^x)$ be a family of processes on a Lusin space $(E, \mathcal{E})$, where P^x on $\mathcal{F}^0$ is assumed $\mathcal{E}$-measurable, $\mathcal{F}_t^\mu$ being the right-continuous P^μ-augmentation of the minimal filtration $\mathcal{F}_t^0$ (see [10], 3.3, for these definitions, and for the shift θ_t).*

(a) *If X_t is right-continuous and the strong Markov property holds, then $R \in \mathcal{F}_\infty^\mu$ is a stopping time if (and only if)*

$$P^\mu(\theta_R^{-1} B \mid \mathcal{F}_R^\mu) = P^{X_R}(B) \quad \text{on } \{R < \infty\}, \qquad B \in \mathcal{F}^0.$$

(b) *If X_t is r.c.l.l. and the moderate Markov property holds (for the definition, see [2], Section 2.4), then R is previsible if (and only if)*

$$P^\mu(\theta_R^{-1} B \mid \mathcal{F}_{R-}^\mu) = P^{X_{R-}}(B) \quad \text{on } \{R < \infty\}, \qquad B \in \mathcal{F}^0.$$

PROOF. We remark first that if X_t is right-continuous, then X^{-1} maps $\mathcal{F}'$ onto $\mathcal{F}^0$. Indeed, for $f \in C^b(E)$, we have

$$f(X_t) = \frac{d^+}{dt^+} \int_0^t f(X_s)\, ds \in X^{-1}(\mathcal{F}'_\infty) \quad \text{for } 0 \le t,$$

so that X_t is $X^{-1}(\mathcal{F}'_\infty)$-measurable, while $\int_0^t f(X_s)\,ds$ is $\mathcal{F}_t^0$-measurable by a Riemann sum approximation [and this extends to $f \in b(\mathcal{E})$ by a monotone class argument]. From this we see that $\mathcal{F}_t^\mu = \mathcal{F}_t^X$. Then by the remark following Theorem 2.2 we have for stopping times $T < \infty$ and $A \in \mathcal{F}'$

$$\begin{aligned} P^{X_T}\big(X^{-1}(A)\big) &= P^\mu\Big(\theta_T^{-1}\big(X^{-1}(A)\big) \,\big|\, \mathcal{F}_T^\mu\Big) \\ &= P^\mu\big(X_{T+\cdot} \in A \,\big|\, \mathcal{F}_T^X\big) \\ &= Z_T^X(A), \quad P\text{-a.s.} \end{aligned}$$

Similarly, it follows that if X is r.c.l.l. and T is previsible, then $P^\mu\{X_{T-}^X(A) = P^{X_{T-}}(X^{-1}(A))\} = 1$. But since both $Z_t^X(A)$ and $P^{X_t}(X^{-1}(A))$ are optional, it follows by the optional section theorem that $P^\mu\{Z_t^X(A) = P^{X_t}(X^{-1}(A))$ for all $t\} = 1$. Similarly, by the previsible section theorem, we have $Z_{t-}^X(A) = P^{X_{t-}}(X^{-1}(A))$ for all t under the assumptions of (b), and since $X^{-1}(\mathcal{F}') = \mathcal{F}^0$, Theorem 2.3 follows from Theorem 2.2. □

As a concluding remark, we obtain the following result.

COROLLARY 2.3. *Under the conditions of Theorem 2.2, R is a stopping time if (and only if), given $\mathcal{F}_R^X$, Z_{R+t}^X is Markovian with the same transition function Q^z as Z_t^X. Moreover, R is previsible if (and only if), given $\mathcal{F}_{R-}^X$, $Z_{R+\cdot}^X$ is Markovian with initial distribution $Q^{Z_{R-}^X}\{Z_0^X \in dz\}$ and transition function Q^z.*

PROOF. It is known (see [7], *loc. cit.*) that Z_t^X is r.c.l.l., with the strong and moderate Markov properties relative to Q^z (Q^z does not depend on X, although that is irrelevant here). Likewise, it is known that for given P the natural augmented filtration of Z_t^X is $\mathcal{F}_t^X$. We need to apply Theorem 2.3 to $(\Omega, \mathcal{F}, \mathcal{F}_t^X, Z_t^X, Q^z)$, where the role of μ is assumed by the P-distribution of Z_0^X. There is one minor difficulty; namely, the translation operators (if any) of X_t are not inherited by Z_t^X. Here one may transfer Z_t^X to a canonical "prediction space" Ω_Z (as in [7], 2.3), but a simpler expedient is to check that Theorem 2.3 remains true without translation operators, if we replace $\theta_R^{-1}(B)$ by $\{X_{R+\cdot} \in A\}$, for $A \in \mathcal{F}'$ such that $B = X^{-1}(A)$, and replace B by $\{X_\cdot \in A\}$ on the right in (a) and (b). Indeed, this only makes the proof a little simpler, and the result follows. □

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