The Annals of Probability
1994, Vol. 22, No. 3, 1429–1472

GAMBLER'S RUIN AND THE FIRST EXIT POSITION OF RANDOM WALK FROM LARGE SPHERES

BY PHILIP S. GRIFFIN[1] AND TERRY R. MCCONNELL[2]

Syracuse University

Dedicated to the memory of William Pruitt

Let T_r be the first time a sum S_n of nondegenerate i.i.d. random vectors in $\mathbb{R}^d$ leaves the sphere of radius r in some given norm. We characterize, in terms of the distribution of the individual summands, the following probabilistic behavior: $S_{T_r}/\|S_{T_r}\|$ has no subsequential weak limit supported on a closed half-space. In one dimension, this result solves a very general form of the gambler's ruin problem. We also characterize the existence of degenerate limits and obtain analogous results for triangular arrays along any subsequence $r_k \to \infty$. Finally, we compute the limiting joint distribution of $(\|S_{T_r}\| - r,\ S_{T_r}/\|S_{T_r}\|)$.

1. Introduction. We investigate distributional properties of the position at which a d-dimensional random walk first exits a sphere of radius r in some arbitrary norm. Specific results include an analytic characterization of when the distribution does not (and does) asymptotically concentrate on a half-space as the radius approaches ∞. Specialized to one dimension, this is a gambler's ruin problem for general sums of i.i.d. random variables. (The word "analytic" refers to conditions involving only the distribution of individual summands, as opposed to probabilistic conditions which may involve the entire sample path.) Under this condition we go on to compute the limiting distribution of the overshoot (when it exists).

Let $X, X_1, X_2, \dots$ be a sequence of nondegenerate, independent and identically distributed $\mathbb{R}^d$-valued random vectors and set $S_n = \sum_{j=1}^n X_j$. Let $\|\ \|$ be an arbitrary norm on $\mathbb{R}^d$ and define $T_r = \min\{n: \|S_n\| > r\}$. The overshoot $\|S_{T_r}\| - r$, one of the main objects of study in one-dimensional renewal theory, is typically studied in two quite distinct cases: the first when $EX > 0$ and the second when $EX = 0$. In trying to extend some of the results in the latter case to multidimensions, we found [4] that the following condition is of fundamental importance:

(*E*) The family $S_{T_r}/\|S_{T_r}\|$ has no subsequential limit supported on a closed half-space.

Received May 1992.
[1]Supported in part by NSF Grant DMS-91-00811.
[2]Supported in part by NSF Grand DMS-89-00503.
AMS 1991 *subject classifications.* Primary 60J15, 60G50; secondry 60K05.
Key words and phrases. Random walk, multidimensional renewal theory, overshoot, gambler's ruin.

For example, one proves in classical renewal theory that, for every $p > 0$, if $X \in L^{2+p}$ and $EX = 0$, then the overshoot is bounded in L^p, that is, $\sup_{r>0} E(|S_{T_r}| - r)^p < \infty$. The converse, however, is false. In [4] we showed that in any dimension and with any norm, L^p-boundedness of the overshoot together with (E) is equivalent to $X \in L^{2+p}$ and $EX = 0$. Our aim in this paper is twofold: first to give an analytic characterization of condition (E) and second, under this condition, to compute the joint limiting distribution of $(\|S_{T_r}\| - r, S_{T_r}\|S_{T_r}\|^{-1})$ whenever it exists.

To describe our results we must introduce some notation. For $r > 0$ set

$$G(r) = P(\|X\| > r), \qquad K(r) = r^{-2}E(\|X\|^2;\ \|X\| \le r),$$
$$M(r) = r^{-1}E(X;\ \|X\| \le r), \qquad h(r) = G(r) + K(r) + \|M(r)\|.$$

In the case of one-dimensional random variables, we also introduce

$$G_+(r) = P(X > r), \qquad G_-(r) = P(-X > r), \qquad J(r) = -G_-(r) + M(r) + G_+(r).$$

To characterize (E) in one dimension, even along a subsequence, it suffices to characterize (E_+) along an arbitrary subsequence r_k, where (E_+) is the condition

$$(E_+) \qquad P(S_{T_{r_k}} > 0) \to 1 \quad \text{as } k \to \infty.$$

This is a very general gambler's ruin problem. We provide two solutions to this: one directly in terms of the functions defined above and the other in terms of Laplace transforms. The former has the advantage of being easier to verify and of providing, perhaps, more insight into the behavior of the random walk, while the latter is more concise and is easier to prove.

THEOREM 1.1. *We have*

$$P(S_{T_{r_k}} > 0) \to 1 \tag{1.1}$$

if and only if the following two conditions hold:

$$\sup_{\lambda>0} \lim_{k\to\infty} \sup \frac{G_-(\lambda r_k)}{h(\lambda r_k)} = 0, \tag{1.2a}$$

$$\inf_{\lambda>0} \lim_{k\to\infty} \inf \frac{J(\lambda r_k)}{h(\lambda r_k)} \ge \frac{1}{2}; \tag{1.2b}$$

if and only if

$$\lim_{k\to\infty} r_k \alpha(r_k) = \infty, \tag{1.3}$$

where $\alpha(r_k) = \inf\{\lambda > 0\colon E \exp(\lambda(-3r_k \vee X \wedge 3r_k)) \ge 1\}$ *and* $\inf(\emptyset) = \infty$.

We would like to emphasize that while the first condition looks more complicated, it is in fact much easier to apply. This is well illustrated by the examples given at the end of Section 3. The extension of Theorem 1.1 to triangular arrays and to multidimensions can be found in Section 3, 4 and 5.

The other main result of this paper yields, under condition (E), the limiting joint distribution of $(\|S_{T_r}\| - r, S_{T_r}\|S_{T_r}\|^{-1})$. We recall ([4], Theorem 1.3) that, under (E), tightness of the overshoot distributions already implies that the X_i have mean 0 and finite second moments. Thus we may assume $E\|X\|^2 < \infty$ and $EX = 0$ without loss of generality.

Let ω_r be the distribution of $S_{T_r}\|S_{T_r}\|^{-1}$ on $\partial B_1 = \{x \in \mathbb{R}^d: \|x\| = 1\}$. A standard application of Donsker's invariance principle (sketched in [4], Lemma 4.1) shows that $\omega_r \to_w \omega$, where ω is the exit distribution on ∂B_1 of a Brownian motion process. (ω is the standard harmonic measure on ∂B_1 if the convariance matrix of X_1 is the identity.)

The limiting behavior of the other marginal, $\|S_{T_r}\| - r$, is more complicated. Recall from classical renewal theory (see, e.g., Chapter 11 of [3]) that if Z_i are i.i.d. nonnegative, nonlattice random variables with $0 < \mu = E(Z_1) < \infty$, then $\sum_{j=1}^{T_r} Z_j - r$ converges in distribution as $r \to \infty$ to a random variable Z^* satisfying

$$P(Z^* \le x) = \frac{1}{\mu}\int_0^x \big(1 - F(s)\big)\, ds,$$

where F is the common distribution function of the Z_i.

As observed, for example, by Lai [8], this may be extended to nontrivial mean 0 random variables, Z_i with *finite variance* as follows: let $L, L_1, L_2, \dots$ be i.i.d. random variables having the same distribution as the first (strict) ladder height of $S_n = \sum_{j=1}^n Z_j$. Then, by a classical result of Spitzer [12], $E(L_1) < \infty$. Letting $T_r^+ = \min\{n : S_n > r\}$ and applying the above renewal theory to the ladder height process, we have

$$\lim_{r\to\infty} P\big(S_{T_r^+} - r \in I\big) = P(L^* \in I) = \frac{1}{E(L)}\int_I \big[1 - F_+(s)\big]\, ds \tag{1.4}$$

for any interval I, where F_+ is the distribution of L_1. A similar result holds in the lattice case; if $\delta\mathbb{Z}$ is the minimal lattice supporting X, then, for any $0 \le \gamma < \delta$,

$$\lim_{n\to\infty} P\big(S_{T^+_{n\delta+\gamma}} - (n\delta+\gamma) \in I\big) = P(L^* \in I) = \frac{1}{E(L)}\int_I \big[1 - F_+(s)\big]\, d\mu(s), \tag{1.4a}$$

where L is again a strict ladder height variable and $\mu(A) = \delta|A \cap \delta\mathbb{Z}|$.

Now, as we discuss in greater detail in Section 6, it is possible to choose, for ω-a.e. $\theta \in \partial B_1$, a unique linear functional ℓ_θ^* satisfying $\ell_\theta^*(\theta) = 1 = \|\ell_\theta^*\|$. [The hyperplane $\{z: \ell_\theta^*(z) = 1\}$ is a support plane of B_1 at θ.] Let L_θ^* be a random variable satisfying (1.4) or (1.4a) for the choice $Z_j = \ell_\theta^*(X_j)$. Then we shall prove the following result.

THEOREM 1.2. *Assume (E). Then $(\|S_{T_r}\| - r, S_{T_r}/\|S_{T_r}\|)$ converges in the sense of weak convergence of probability measures on $\mathbb{R}_+ \times \partial B_1$ if and only if $EX =$*

0, $E\|X\|^2 < \infty$ *and* $\omega(\theta\colon \ell^*_\theta(X)$ *is lattice*$) = 0$. *In that case,*

$$\lim_{r\to\infty} P\left(\|S_{T_r}\| - r \in dx,\ \frac{S_{T_r}}{\|S_{T_r}\|} \in d\theta\right) = P(L^*_\theta \in dx)\omega(d\theta). \tag{1.5}$$

As a corollary we have the following analogue of (1.4).

COROLLARY 1.3. *For any interval* $I \subset [0,\infty)$, *we have*

$$\lim_{r\to\infty} P\big(\|S_{T_r}\| - r \in I\big) = \int_{\partial B_1} \omega(d\theta)P(L^*_\theta \in I).$$

In the case of the Euclidean norm in dimensions $d \ge 2$, $\ell^*_\theta(X) = \langle X, \theta\rangle$ is nonlattice except for at most a countable set of θ, even when X itself is lattice. (Here $\langle\ ,\ \rangle$ denotes the usual inner product on $\mathbb{R}^d$.) Thus the hypothesis $\omega(\theta\colon \ell^*_\theta(X)$ is lattice$) = 0$ is satisfied. In general, this need not be true. For example, in two dimensions suppose the unit sphere is the square with vertices at $(\pm 1, \pm 1)$ and the distribution of X is supported on the corners. Here the limit in Theorem 1.2 does not exist. This will be discussed further in Remark 6.7.

The plan of the paper is as follows. We give the proof of Theorem 1.1 [except for (1.3)] and related results in Section 3, after presenting some preliminaries in Section 2. Section 4 contains the extension of Theorem 1.1 to higher dimensions. In Section 5 we present the proof of the equivalence of (1.1) and (1.3) and the analogue of this equivalence in higher dimensions. Section 6 is devoted to the proof of Theorem 1.2. Finally, in the Appendix we present the proof of an analytic result which is related to condition (1.2b).

2. Preliminaries. The purpose of this section is to collect notation, definitions and results which will be used repeatedly in later sections.

We reserve the symbol $\|\ \|_E$ for the usual Euclidean norm on $\mathbb{R}^d$ and $\|\ \|$ for a given (arbitrary) norm. The open ball of radius r in norm $\|\ \|$ centered at 0 will be denoted by B_r, and its boundary by ∂B_r. The Euclidean unit sphere in $\mathbb{R}^d$ is S^{d-1}. All norms on $\mathbb{R}^d$ are equivalent, so there is a positive constant ρ such that

$$\rho^{-1}\|x\|_E \le \|x\| \le \rho\|x\|_E, \qquad x \in \mathbb{R}^d. \tag{2.1}$$

This constant ρ will appear in many later formulas.

We shall often use a "rounded" version, $\widehat{X}$, of a random vector X, defined by

$$\widehat{X} = \begin{cases} X, & \text{if } \|X\| \le 3r, \\ 3r\dfrac{X}{\|X\|}, & \text{otherwise.} \end{cases} \tag{2.2}$$

The choice of $r > 0$ will be clear from the context. Another way to effect such a rounding is to change the underlying probability measure from P to $\widehat{P}$. For reasons of flexibility and convenience, we shall use both notations. The advantage of using the number 3 (or any number larger than 2) in (2.2) is that T_r has the same distribution under P as it has under $\widehat{P}$. The same is true of the exit position, $S_{T_r}/\|S_{T_r}\|$, in one dimension; even in higher dimensions this holds in an approximate sense. (See Lemmas 4.2 and 4.3.)

For maximum generality, and since it involves no extra work, we have formulated many of our results in terms of limits along particular sequences of radii $r_k \to \infty$ and in terms of "triangular arrays" of random variables; that is, we allow the distribution of the summands to vary with k, as well as the radii. Thus, in the expression $P^k(S_{T_{r_k}} > 0)$, $S_{T_{r_k}}$ represents the position of $S_n = \sum_{i=1}^n X_i$ at first exit from $[-r_k, r_k]$, where $(X_i)_{i=1}^\infty$ is an i.i.d. sequence under the probability measure P^k and E^k will denote expectation with respect to P^k. We will use subscripts on E and P to indicate the starting point of the random walk. For example, under P_a we have $P_a(S_0 = a) = 1$. If no subscript is given it is assumed that the random walk starts from the origin.

We will sometimes attach a subscript to the functions defined in Section 1 to indicate their dependence on a random variable X; thus G_X, K_X, and so on. Also, for $\theta \in S^{d-1}$, we define G_θ (resp. $G_{\theta\pm}$) by $G_\theta(r) = P(|\langle X, \theta\rangle| > r)$ [resp. $P(\pm\langle X, \theta\rangle > r)$], where $\langle\ ,\ \rangle$ denotes the usual Euclidean inner product. The subscript θ on the other functions will have a similar meaning. We use $\widehat{G}$, $\widehat{K}$, and so forth as shorthand for $G_{\hat{X}}$, $K_{\hat{X}}$, and so on. Numerical superscripts will be used for clarity when the P^k vary. Thus h^k is computed from the distribution of X_1 under P^k. In the composite expressions $\widehat{h}_\theta$, and so on, which will appear in the characterization of (E) and related phenomena in higher dimensions (cf. Section 4), it is important to emphasize that rounding is to be done *before* projection; thus $\widehat{h}_\theta(r)$ is computed from the distribution of $\langle \widehat{X}, \theta\rangle$, where $\widehat{X}$ is given by (2.2).

We will need a version of Wald's identity, which we state here for ease of reference. Let $S_n = X_1 + \cdots + X_n$ be a sum of i.i.d. random vectors satisfying $E\|X_1\| < \infty$. Let $(\mathcal{G}_n)_{n=1}^\infty$ be an increasing sequence of σ-fields such that $(X_1, X_2, \dots, X_n)$ is $\mathcal{G}_n$-measurable and X_{n+1} is independent of $\mathcal{G}_n$. Then, for any $\mathcal{G}_n$-stopping time T such that $ET < \infty$, we have

$$ES_T = (EX_1)ET. \tag{2.3}$$

(See, e.g., [1], page 137.)

We shall also frequently use the following very useful estimates due to Pruitt [10]: there is a constant $c > 0$ depending only on the norm and dimension such that

$$\frac{1}{ch(r)} \le ET_r \le \frac{c}{h(r)}. \tag{2.4}$$

Also, for any $r > 0$ and any $n \in \mathbb{N}$,

$$P(T_r > n) \le \frac{c}{nh(r)} \tag{2.5}$$

and

$$P(T_r \le n) \le cnh(r). \tag{2.6}$$

(Actually, Pruitt's results are given in terms of the Euclidean norm. The extension to general norms is relatively straightforward; see Section 2 of [4].).

The function h satisfies a useful doubling property; there is an absolute constant $c > 0$ such that

$$\frac{1}{c}h(r) \le h(2r) \le ch(r). \tag{2.7}$$

The same result holds for another useful function, Q, defined by

$$Q(r) = G(r) + K(r).$$

See [10] and [11] for this and other properties of the functions h and Q.

The quantity $J(r)/h(r)$, which appears on the left-hand side of (1.2b), may now be understood as an estimate for $E\widehat{S}_{T_r}/r$; combine (2.3), (2.4) and (2.7).

REMARK 2.1. On several occasions, and in several different contexts, we shall use the following straightforward consequence of the strong Markov property, whose proof we omit. If $\tau_{\varepsilon,k}$ denotes the first passage time of S_n from $[-\varepsilon r, kr]$, then

$$P\big(S_{\tau_{\varepsilon,k}} > 0\big) \ge \Big(1 - P\big(S_{T_{\varepsilon r}} < 0\big)^{\varepsilon/(\varepsilon+1)}\Big)^k.$$

In particular, it follows easily from this that, for any $\lambda > 0$ and sequence $r_k \to \infty$,

$$P\big(S_{T_{r_k}} > 0\big) \to 1 \quad \Rightarrow \quad P\big(S_{T_{\lambda r_k}} > 0\big) \to 1$$

as $k \to \infty$.

We shall use $k(1), k(2), \dots$ to denote nested subsequences of the natural numbers. Thus $k(1)$ will stand for a strictly increasing sequence of integers greater than or equal to 1 (in place of the more cumbersome $n_{k(1)}$), $k(2)$ denotes a subsequence of $k(1)$ and so forth. The symbol $I\{\ \}$ will denote the indicator function of a set, and c will stand for an absolute constant which may change from one use to the next.

3. Condition (E_+) and gambler's ruin. In this section we will give a necessary and sufficient condition, in terms of the elementary functions G, K and M, for (E_+) to hold in one dimension along any given subsequence (see Theorem 3.9). Along the way we also prove a triangular array version of this result. This will be needed in the multidimensional case.

The first two results of this section will be used in the proof of necessity in Theorem 3.9 and the next four in the proof of sufficiency. We begin by observing that essentially the same proof as Proposition 3.1 in [4] shows that, for any positive Borel function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$,

$$E\varphi(S_{T_r}) = \int_{\|x\| \le r} E\big(\varphi(X+x);\ \|X+x\| > r\big)\, dU_r(x).$$

Here dU_r is the occupation measure defined by

$$\int_A dU_r = E\left(\sum_{n=0}^{T_r - 1} I_A(S_n)\right).$$

Our first result is stated in the multidimensional case, since we will need it in that form later.

PROPOSITION 3.1. *There is a universal constant $c > 0$, depending only on dimension, such that, for all $r > 0$ and all $\theta \in S^{d-1}$,*

$$P\Big(\big\langle S_{T_r}, \theta\big\rangle < -\rho r\Big) \ge c\frac{G_{\theta-}(2\rho r)}{h(r)}.$$

PROOF. Let $\varphi(x) = I\{\langle x, \theta\rangle < -\rho r\}$. Since $\{\langle X + x, \theta\rangle < -\rho r\} \subset \{\|X + x\| > r\}$, we have

$$\begin{aligned}
P\Big(\big\langle S_{T_r}, \theta\big\rangle < -\rho r\Big) &= \int_{\|x\| \le r} P\big(\langle X+x, \theta\rangle < -\rho r\big)\, dU_r(x)\\
&\ge \int_{\|x\| \le r} P\big(\langle X, \theta\rangle < -2\rho r\big)\, dU_r(x)\\
&= G_{\theta-}(2\rho r) ET_r\\
&\ge c\frac{G_{\theta-}(2\rho r)}{h(r)}
\end{aligned}$$

by (2.4). □

PROPOSITION 3.2. *There is a universal constant $c > 0$ such that, for all $r > 0$,*

$$P\big(S_{T_r} < 0\big) \ge \frac{1}{4} - c\frac{J(2r)}{h(2r)}. \tag{3.1}$$

PROOF. Fix $\lambda > 2$ and let

$$\begin{aligned}
\widetilde{X}_i &= (-\lambda r) \vee X_i \wedge (\lambda r),\\
\widetilde{S}_n &= \sum_{i=1}^{n} \widetilde{X}_i,\\
\widetilde{T}_r &= \min\big\{n\colon |\widetilde{S}_n| > r\big\}.
\end{aligned}$$

Then $T_r = \widetilde{T}_r$ and $\{S_{T_r} < 0\} = \{\widetilde{S}_{T_r} < 0\}$. By Wald's identity (2.3),

$$E\widetilde{X}_1 ET_r = E\widetilde{S}_{T_r} \geq rP(\widetilde{S}_{T_r} > 0) - (\lambda+1)rP(\widetilde{S}_{T_r} < 0)$$
$$= r - (\lambda+2)rP(\widetilde{S}_{T_r} < 0).$$

Thus, by (2.4) and (2.7),

$$P(\widetilde{S}_{T_r} < 0) \geq \frac{1}{\lambda+2} - \frac{c}{(\lambda+2)r}\frac{E\widetilde{X}}{h(2r)}$$
$$= \frac{1}{\lambda+2} - \frac{c\lambda}{(\lambda+2)}\left(\frac{J(\lambda r)}{h(2r)}\right).$$

By right continuity we can let $\lambda \downarrow 2$ to obtain (3.1). □

LEMMA 3.3. *There is a universal constant* $c > 0$ *such that, for all* $\varepsilon > 0$, $\delta > 0$ *and* $r > 0$,

$$P(X_i < -\varepsilon r \text{ for some } i \leq T_{\delta r}) \leq c\frac{G_-(\varepsilon r)}{h(\delta r)}.$$

PROOF. Using (2.3) and (2.4),

$$P(X_i < -\varepsilon r \text{ for some } i \leq T_{\delta r}) = P\left(\sum_{i=1}^{T_{\delta r}} I(X_i < -\varepsilon r) \geq 1\right)$$
$$\leq E\left(\sum_{i=1}^{T_{\delta r}} I(X_i < -\varepsilon r)\right)$$
$$= G_-(\varepsilon r)ET_{\delta r}$$
$$\leq \frac{cG_-(\varepsilon r)}{h(\delta r)}.$$ □

By (2.7), it immediately follows that, for all $\varepsilon > 0$, $\delta > 0$ and $r > 0$,

$$P(X_i < -\varepsilon r \text{ for some } i \leq T_{\delta r}) \leq c_{\varepsilon\delta^{-1}}\frac{G_-(\varepsilon r)}{h(\varepsilon r)}, \tag{3.2}$$

where $c_{\varepsilon\delta^{-1}} > 0$ depends only on the ratio $\varepsilon\delta^{-1}$.

LEMMA 3.4. *Let* $\delta > 0$ *and* $\varepsilon \in (0, \delta)$. *Then there exist a universal constant* $1 \geq c > 0$ *and a constant* $c_{\varepsilon\delta^{-1}}$, *which depends only on the ratio* $\varepsilon\delta^{-1}$, *such that, for all* $r > 0$,

$$P(S_{T_{\delta r}} > 0) \geq \frac{1 + cJ(\varepsilon r)/h(\varepsilon r)}{2 + \varepsilon\delta^{-1}} - c_{\varepsilon\delta^{-1}}\left(\frac{G_-(\varepsilon r)}{h(\varepsilon r)}\right), \tag{3.3}$$

provided $J(\varepsilon r) \geq 0$.

PROOF. By the homogeneity of (3.3), it suffices to consider the case $\delta = 1$.

Let

$$\widetilde{X}_i = (-\varepsilon r) \vee X_i \wedge (\varepsilon r),$$
$$\widetilde{S}_n = \sum_1^n \widetilde{X}_i,$$
$$\widetilde{T}_r = \min\{n: |\widetilde{S}_n| > r\}.$$

Then

$$P(S_{T_r} > 0) \geq P(\widetilde{S}_{\tilde{T}_r} > 0) - P(X_i < -\varepsilon r \text{ for some } i \leq T_r) = I - II.$$

To estimate I, by Wald's identity,

$$\begin{aligned} E\widetilde{X}_1 E\widetilde{T}_r &= E\widetilde{S}_{\tilde{T}_r} \\ &\leq (1+\varepsilon) r P(\widetilde{S}_{\tilde{T}_r} > 0) - rP(\widetilde{S}_{\tilde{T}_r} < 0) \\ &= (2+\varepsilon) r P(\widetilde{S}_{\tilde{T}_r} > 0) - r. \end{aligned}$$

We may assume $\varepsilon r J(\varepsilon r) = E\widetilde{X}_1 \geq 0$. Thus

$$\begin{aligned} P(\widetilde{S}_{\tilde{T}_r} > 0) &\geq \frac{1 + r^{-1} E\widetilde{X}_1 E\widetilde{T}_r}{2+\varepsilon} \\ &\geq \frac{1 + c\varepsilon J(\varepsilon r)/\widetilde{h}(r)}{(2+\varepsilon)} \end{aligned}$$

by (2.4), where

$$\begin{aligned} \widetilde{h}(r) &= r^{-1}|-\varepsilon r G_-(\varepsilon r) + \varepsilon r M(\varepsilon r) + \varepsilon r G_+(\varepsilon r)| + \varepsilon^2 G(\varepsilon r) + \varepsilon^2 K(\varepsilon r) \\ &\leq 2\varepsilon\big(|M(\varepsilon r)| + G(\varepsilon r) + K(\varepsilon r)\big) \\ &= 2\varepsilon h(\varepsilon r). \end{aligned}$$

Hence

$$I \geq \frac{1 + cJ(\varepsilon r)/h(\varepsilon r)}{(2+\varepsilon)}.$$

Combining this with the estimate for II in (3.2) completes the proof. □

LEMMA 3.5. *Let Z_k be asymmetric random walk with*

$$P(Z_1 = 1) = 1 - P(Z_1 = -1) \geq \eta,$$

where $\eta > \frac{1}{2}$. Let $N \geq 1$ be an integer and

$$\sigma_N = \min\{J: |Z_j| = 2N\}.$$

Then, for any $\xi > 0$,

$$P(\sigma_N > \xi N) \le \frac{2}{\xi(2\eta - 1)} \tag{3.4}$$

and

$$P(Z_j \text{ hits } 2N \text{ before } -N) \ge 1 - a^{-N}, \tag{3.5}$$

where $a = \eta/(1-\eta)$.

PROOF. For $N \ge 1$, by Markov's inequality and Wald's identity,

$$\begin{aligned} P(\sigma_N > \xi N) &\le \frac{E\sigma_N}{\xi N} \\ &= \frac{EZ_{\sigma_N}}{EZ_1 \xi N} \\ &\le \frac{2}{\xi(2\eta - 1)}. \end{aligned}$$

Define $\varphi: \mathbb{Z} \to \mathbb{R}$ by

$$\varphi(j) = \left(\frac{1-\eta}{\eta}\right)^j = a^{-j}.$$

Then $\varphi(Z_j)$ is a supermartingale. Hence, by standard arguments (noting that φ is decreasing),

$$\begin{aligned} P\big(Z_j \text{ hits } 2N \text{ before } -N\big) &\ge \frac{\varphi(-N) - \varphi(0)}{\varphi(-N) - \varphi(2N)} \\ &\ge \frac{\varphi(-N) - \varphi(0)}{\varphi(-N)}, \end{aligned}$$

which yields (3.5). □

PROPOSITION 3.6. *Fix an integer* $N \ge 1$ *and let* $\delta = (2N)^{-1}$. *Fix* $\eta > \frac{1}{2}$. *Assume that*

$$P\big(S_{T_{\delta r}} > 0\big) \ge \eta.$$

Then, for every $\gamma > 0$ *and* $\xi > 0$, *if* $\gamma \xi \delta^{-1} < 1$,

$$P\big(S_{T_r} > 0\big) \ge 1 - a^{-N} - \frac{2}{\xi(2\eta - 1)} - c_{\gamma\delta^{-1}\xi N}\left(\frac{G_-(\gamma r)}{h(\gamma r)}\right), \tag{3.6}$$

where $a = \eta/(1-\eta)$.

PROOF. Let

$$\begin{aligned} \tau_1 &= \min\big\{n: |S_n - S_0| > \delta r\big\}, \\ \tau_{j+1} &= \tau_j + \tau_1 \circ \theta_{\tau_j}, \\ Z_{j+1} &= Z_j + \Big((-\delta r) \vee \big(S_{\tau_{j+1}} - S_{\tau_j}\big) \wedge \delta r\Big)/(\delta r), \end{aligned}$$

where $Z_0 = 0$. Thus Z_j is asymmetric random walk with

$$P(Z_1 = 1) = 1 - P(Z_1 = -1) \geq \eta.$$

Let

$$\sigma_N = \inf\{j\colon |Z_j| = 2N\}.$$

Let $\gamma > 0$ and $\xi > 0$ be such that $\gamma\xi\delta^{-1} < 1$. Put

$$\begin{aligned} A_N &= \{Z_{\sigma_N} = 2N,\ Z_j > -N \text{ all } j \leq \sigma_N\},\\ B_N &= \{\sigma_N \leq \xi N\},\\ C_N &= \{S_{\tau_{j+1}} - S_{\tau_j} \geq -(\gamma+\delta)r \text{ all } j \leq \xi N\}. \end{aligned}$$

Then

$$\{S_{T_r} > 0\} \supseteq A_N B_N C_N.$$

Thus

$$P(S_{T_r} > 0) \geq P(A_N) - P(B_N^c) - P(C_N^c).$$

The estimates for the first two terms follow immediately from (3.4) and (3.5). For the third term we have

$$\begin{aligned} P(C_N^c) &\leq \xi N P(S_{T_{\delta r}} < -(\gamma+\delta)r)\\ &\leq \xi N P(X_i < -\gamma r \text{ for some } i \leq T_{\delta r})\\ &\leq \xi N c_{\gamma\delta^{-1}} \frac{G_-(\gamma r)}{h(\gamma r)} \end{aligned}$$

by (3.2). □

THEOREM 3.7. *Fix a sequence $\{r_k\}$. Then*

$$P^k(S_{T_{r_k}} > 0) \to 1 \tag{3.7}$$

if and only if the following two conditions hold:

$$\sup_{\lambda>0} \limsup_{k\to\infty} \frac{G_-^k(\lambda r_k)}{h^k(\lambda r_k)} = 0, \tag{3.8}$$

$$\inf_{\lambda>0} \liminf_{k\to\infty} \frac{J^k(\lambda r_k)}{h^k(\lambda r_k)} > 0. \tag{3.9}$$

PROOF. Assume (3.7). Then (cf. Remark 2.1), for every $\lambda > 0$,

$$P^k(S_{T_{\lambda r_k}} > 0) \to 1.$$

Hence (3.8) follows from (2.7) and the one-dimensional case of Proposition 3.1, while (3.9) follows from Proposition 3.2.

Now assume (3.8) and (3.9). Let $\sigma \in (0,1)$ be a strict lower bound for the left-hand side of (3.9). Then, for any $\delta > 0$, by setting $\varepsilon = \delta\sigma c$ in (3.3), we have that, for all sufficiently large k,

$$P^k(S_{T_{\delta r_k}} > 0) \geq \eta,$$

where $\eta = (1 + c\sigma)/(2 + c\sigma)$ is a constant greater than $\frac{1}{2}$. Thus in (3.6), by first choosing $\delta = (2N)^{-1}$ sufficiently small, then ξ sufficiently large and γ sufficiently small such that $\gamma\xi\delta^{-1} < 1$ and then finally k sufficiently large, we can make $P^k(S_{T_{r_k}} > 0)$ as close to 1 as we please. Hence (3.7) holds. □

An astute reader might feel uncomfortable with this proof. That is because if the infimum in (3.9) happens to fall in the interval $(0, 1/4c)$, where c is the universal constant appearing in (3.1), then Theorem 3.7 asserts that $P^k(S_{T_{r_k}} > 0) \to 1$, while Proposition 3.2 and Remark 2.1 imply that $\liminf P^k(S_{T_{r_k}} < 0) > 0$. After trying to discover an error in the proof, we concluded that the answer to our dilemma must be that the infimum cannot lie in this range. In fact, the following remarkable result is true:

PROPOSITION 3.8. *Assume*

$$\sup_{\lambda > 0} \limsup_{k \to \infty} \frac{G_-^k(\lambda r_k)}{h^k(\lambda r_k)} = 0.$$

Then the following are equivalent:

$$\limsup_{\lambda \downarrow 0} \liminf_{k \to \infty} \frac{G_+^k(\lambda r_k) + M^k(\lambda r_k)}{h^k(\lambda r_k)} > 0,$$

$$\inf_{\lambda > 0} \liminf_{k \to \infty} \frac{G_+^k(\lambda r_k) + M^k(\lambda r_k)}{h^k(\lambda r_k)} \geq \frac{1}{2}.$$

The proof of Theorem 3.7 shows that under (3.8), if

$$\limsup_{\delta \downarrow 0} \liminf_{k \to \infty} P^k(S_{T_{\delta r_k}} > 0) > \frac{1}{2},$$

then $\liminf_{k \to \infty} P^k(S_{T_{r_k}} > 0) = 1$. Proposition 3.8 is the analytical counterpart of this discontinuity. In order to avoid a lengthy detour at this stage, we defer its rather technical proof to the Appendix. However, we will take this opportunity to reformulate Theorem 3.7 as follows:

THEOREM 3.9. *Fix a sequence* $\{r_k\}$. *Then*

$$P^k(S_{T_{r_k}} > 0) \to 1 \tag{3.10}$$

if and only if the following two conditions hold:

$$\sup_{\lambda>0} \limsup_{k\to\infty} \frac{G_-^k(\lambda r_k)}{h^k(\lambda r_k)} = 0, \tag{3.11}$$

$$\inf_{\lambda>0} \liminf_{k\to\infty} \frac{J^k(\lambda r_k)}{h^k(\lambda r_k)} \geq \frac{1}{2}. \tag{3.12}$$

One might wonder whether (3.11) and (3.12) can be simplified to $G_-^k(r_k)/h^k(r_k) \to 0$ and $\liminf J^k(r_k)/h^k(r_k) \geq 1/2$. Easy examples show this is not the case. However, if we consider the case of a single distribution and let $r \to \infty$ through all real numbers, then we easily obtain the following result.

COROLLARY 3.10. $P(S_{T_r} > 0) \to 1$ *if and only if the following two conditions hold*:

$$\lim_{r\to\infty} \frac{G_-(r)}{h(r)} = 0, \tag{3.13}$$

$$\liminf_{r\to\infty} \frac{J(r)}{h(r)} \geq \frac{1}{2}. \tag{3.14}$$

At the time this work was being carried out, we learned that Harry Kesten and Ross Maller were working on some related problems. In particular, they were interested in finding necessary and sufficient conditions for $S_n \to \infty$ in probability. The condition they found is

$$\frac{J(r)}{G_-(r)} \to +\infty \quad \text{as } r \to \infty \tag{3.15}$$

(under the assumption $EX^2 = \infty$). It turns out that under this assumption (3.15) is equivalent to (3.13) and (3.14). For a proof of this and further remarks on the connection between the two results, see [7].

We conclude this section with several interesting examples. The first is taken from [4], Example 7.1.

EXAMPLE 3.11. Let X be bounded above, have mean 0 and satisfy $G_-(r) = 1/(r(\log r)^2)$ for large r. Then a simple computation (see [4]) shows that M dominates, that is,

$$\lim_{r\to\infty} \frac{M(r)}{h(r)} = 1.$$

Hence, by Corollary 3.10, $P(S_{T_r} > 0) \to 1$.

EXAMPLE 3.12. Let X have mean 0 and be in the domain of attraction of a stable law of index α where $1 < \alpha < 2$. Then, for some $p, q \geq 0$ with $p + q = 1$,

$$\lim_{r\to\infty} \frac{G_+(r)}{G(r)} = p, \qquad \lim_{r\to\infty} \frac{G_-(r)}{G(r)} = q. \tag{3.16}$$

Furthermore,

$$\lim_{r\to\infty}\frac{K(r)}{G(r)}=\frac{\alpha}{2-\alpha},$$
$$\lim_{r\to\infty}\frac{M(r)}{G(r)}=(q-p)\frac{\alpha}{\alpha-1}.$$

Thus if $p, q > 0$, then

$$\liminf_{r\to\infty}\frac{G_{\pm}(r)}{h(r)}>0.$$

Therefore (E) holds by applying Theorem 3.9 to X and $-X$. For the remaining case $p \wedge q = 0$, we may assume $p = 0$ and $q = 1$. (For example, X could be bounded above as in the previous example.) Then $\liminf_{r\to\infty} P(S_{T_r} < 0) > 0$ by Theorem 3.9, since $\liminf_{r\to\infty} G_-(r)/h(r) > 0$. To see that $\liminf_{r\to\infty} P(S_{T_r} > 0) > 0$, one can check from the above that

$$\frac{J(r)}{h(r)}\to\frac{2-\alpha}{4\alpha-\alpha^2-2},$$

which is greater than 0 for $1 < \alpha < 2$. Thus (E) holds.

The case $\alpha = 2$ can be handled similarly. In this case K dominates, that is, $K(r)/h(r) \to 1$. Hence again (E) holds by Theorem 3.9. The case $\alpha = 1$ is much more delicate; Example 3.11 is of this type. In this situation (3.16) still holds, and if $q > p$, then $M(r)/h(r) \to 1$. Hence $P(S_{T_r} > 0) \to 1$. If $p = q$, M may or may not dominate, and a more detailed study of the specific example is required to determine the exit behavior.

The previous example may lead one to hope that (E) holds whenever $EX = 0$ and $X \in L^p$ for some $p > 1$. This, however, is not the case.

EXAMPLE 3.13. Let $1 < \alpha < 2$ and set $u_k = 2^{2^k}$ and $p_k = u_k^{-\alpha}$. Assume X is bounded above, has mean 0 and the negative part of X has distribution given by $P(X = -u_k) = p_k$ for $k \geq 1$. Clearly, $X \in L^p$ for every $p < \alpha$. The key to computing with this example is that, for every $\beta > 0$,

$$\sum_{j=k}^{\infty}(p_j)^{\beta}\sim(p_k)^{\beta},\qquad\sum_{j=1}^{k}(p_j)^{-\beta}\sim(p_k)^{-\beta}. \tag{3.17}$$

Using this, one obtains that, for $u_k \leq r < u_{k+1}$,

$$G(r)\sim u_{k+1}^{-\alpha},$$
$$M(r)\sim\frac{u_{k+1}^{1-\alpha}}{r},$$
$$K(r)\sim\frac{u_k^{2-\alpha}}{r^2}.$$

Let $v_k = u_k^{2-\alpha} u_{k+1}^{\alpha-1}$. Then $u_k v_k^{-1} \to 0$ and $v_k u_{k+1}^{-1} \to 0$. Fix a sequence $\{r_j\}$. Then for some k_j we have $u_{k_j} \le r_j < u_{k_j+1}$. By choosing a subsequence of $\{k_j\}$ if necessary, we may assume $r_j/v_{k_j} \to a \in [0, \infty]$. If $a = \infty$ we may further assume $r_j/u_{k_j+1} \to b \in [0, 1]$. We claim that along this further subsequence (E) holds, unless both $a = \infty$ and $b = 0$, in which case (E_+) holds. To prove this, first observe that in the latter case

$$\frac{M(\lambda r_k)}{h(\lambda r_k)} \to 1$$

for every $\lambda > 0$. Hence (E_+) holds by Theorem 3.9. Next, if $a \in (0, \infty)$,

$$\frac{M(\lambda r_k)}{K(\lambda r_k)} \to \lambda a \quad \text{and} \quad \frac{G(\lambda r_k)}{K(\lambda r_k)} \to 0$$

for every $\lambda > 0$. Hence (E) holds, this time by applying Theorem 3.9 to X and $-X$. If $a = 0$, then

$$\frac{K(r_k)}{h(r_k)} \to 1.$$

Hence (E) holds as above by Theorem 3.9. Finally, if $a = \infty$ and $b \in (0, 1]$, then

$$\frac{K(2r_k/b)}{h(2r_k/b)} \to 1,$$

and so again (E) holds by Theorem 3.9.

In concluding this example, let us point out that if we took $\alpha = 2$, then (E) would hold along the entire sequence. This is because X would be in the domain of attraction of a stable law of index 2, a case discussed briefly at the end of the previous example.

Our final example shows that the constant $\frac{1}{2}$ in Proposition 3.8 is sharp.

EXAMPLE 3.14. Let u_k and p_k be as in the previous example, except now $0 < \alpha < 1$. Assume $X \ge 0$ with $P(X = u_k) = p_k$ for $k \ge 1$ and the remaining mass all at 0. Trivially, (E_+) holds and so, by Corollary 3.10,

$$\liminf_{r\to\infty} \frac{J(r)}{h(r)} \ge \frac{1}{2}. \tag{3.18}$$

Let $r_k = u_k$. Then, using (3.17) as above, one can readily verify that

$$\lim_{k\to\infty} \frac{M(r_k)}{K(r_k)} = 1 \quad \text{and} \quad \lim_{k\to\infty} \frac{G(r_k)}{K(r_k)} = 0.$$

Thus the lim inf in (3.18) equals $\frac{1}{2}$.

4. Condition (*E*) in multidimensions. Our aim now is to extend the results of the previous section to multidimensions and also obtain a characterization of when $S_{T_r}\|S_{T_r}\|^{-1}$ has only nondegenerate subsequential limits. We begin by introducing, for each $\theta \in S^{d-1}$, the conditions

$$(D_\theta) \qquad P^k\Big(\big|\langle S_{T_{r_k}}\|S_{T_{r_k}}\|^{-1}, \theta\rangle\big| > \varepsilon\Big) \to 0 \quad \text{for all } \varepsilon > 0,$$

$$(\widehat{D}_\theta) \qquad \widehat{P}^k\Big(\big|\langle S_{T_{r_k}}\|S_{T_{r_k}}\|^{-1}, \theta\rangle\big| > \varepsilon\Big) \to 0 \quad \text{for all } \varepsilon > 0,$$

$$(H_\theta) \qquad P^k\Big(\langle S_{T_{r_k}}\|S_{T_{r_k}}\|^{-1}, \theta\rangle < -\varepsilon\Big) \to 0 \quad \text{for all } \varepsilon > 0,$$

$$(\widehat{H}_\theta) \qquad \widehat{P}^k\Big(\langle S_{T_{r_k}}\|S_{T_{r_k}}\|^{-1}, \theta\rangle < -\varepsilon\Big) \to 0 \quad \text{for all } \varepsilon > 0.$$

We will find necessary and sufficient conditions for (D_θ) and (H_θ) to hold for each fixed θ. The first step is to reduce this to the problem of $(\widehat{D}_\theta)$ and $(\widehat{H}_\theta)$. This is the purpose of the first three lemmas.

It will be convenient to use the following notation. For $\theta \in S^{d-1}$ and $\beta > 0$, let $\Gamma(\theta, \beta) = \{x\colon \langle x, \theta\rangle > \beta\|x\|\}$. Thus, in the Euclidean case, $\Gamma(\theta, \beta)$ denotes the open cone with vertex at the origin and having β as the cosine of the angle between θ and any generator of the cone. Also let $\mathrm{Ann}(0, a, b) = \{x \in \mathbb{R}^d\colon\ a < \|x\| < b\}$.

LEMMA 4.1. *Suppose, for a given $\theta \in S^{d-1}$, $\beta > 0$, $0 < \varepsilon < 1$, $r > 0$ and $p > 0$, we have*

$$P(S_i \in \Gamma(\theta, \beta) \cap \mathrm{Ann}(0, \varepsilon r, r) \text{ for some } 0 < i < T_r) \ge p.$$

Then

$$P\left(S_{T_r} \in \Gamma\left(\theta, \frac{\beta\varepsilon}{8n}\right)\right) \ge p\big[1 - (1-p)^{1/n}\big]^n,$$

where $n = [2\rho/\beta\varepsilon] + 1$. Here ρ is the constant in (2.1).

PROOF. Let $\Sigma = \Gamma(\theta, \beta) \cap \mathrm{Ann}(0, \varepsilon r, r)$, $L_1 = \{x\colon \langle x, \theta\rangle = \frac{1}{2}\beta\varepsilon r\}$ and $L_2 = \{x\colon \langle x, \theta\rangle = \rho r\}$. Note that if $x \in \Sigma$, then $\langle x, \theta\rangle > \beta\|x\| > \beta\varepsilon r$, while if $\langle x, \theta\rangle > \rho r$, then $\|x\| > \rho^{-1}\|x\|_E > r$. Hence Σ is contained between hyperplanes L_1 and L_2.

Let $\mu = \min\{n > 0\colon |\langle S_n - S_0, \theta\rangle| \ge \frac{1}{2}\beta\varepsilon r\}$ and define a sequence of stopping times $\mu_0, \mu_1, \dots$ as follows:

$$\mu_0 = \min\{n\colon S_n \in \Sigma\},$$
$$\mu_j = \mu \circ \theta_{\mu_{j-1}} + \mu_{j-1}, \qquad j \ge 1.$$

Let $n = [2\rho/\beta\varepsilon] + 1$ and

$$E_0 = \Big\{\mu_0 < T_r,\ \langle S_{\mu_1} - S_{\mu_0}, \theta\rangle > 0, \dots, \langle S_{\mu_n} - S_{\mu_{n-1}}, \theta\rangle > 0,\ \|S_{\mu_n}\|^* \le 4nr\Big\},$$

where $\|S_\ell\|^* = \max_{0 \le i \le \ell}\|S_i\|$. Note that, on the event E_0, the random walk must hit Σ before time T_r and then cross L_2 before crossing L_1. In particular, on E_0, $\langle S_{T_r}, \theta \rangle > \frac{1}{2}\beta\varepsilon r$ and $\|S_{T_r}\|^* \leq 4nr$, so that $E_0 \subseteq \{S_{T_r} \in \Gamma(\theta, \beta\varepsilon/8n)\}$. By the strong Markov property,

$$\begin{aligned} P(E_0) &\geq pP\Big(\big\langle S_{\mu_1} - S_{\mu_0}, \theta\big\rangle > 0,\ \max_{\mu_0 \leq j \leq \mu_1} \|S_j - S_{\mu_0}\| \leq 3r\Big)^n \\ &= pP_0\Big(\big\langle S_\mu, \theta\big\rangle > 0,\ \|S_\mu\|^* \leq 3r\Big)^n. \end{aligned}$$

The idea now is that $P_0(\langle S_\mu, \theta\rangle > 0,\ \|S_\mu\|^* \leq 3r)$ cannot be too small; otherwise the random walk would either fail to cross L_1 or else jump far outside the ball of radius r, missing Σ in either case. To make this precise, introduce stopping times $\nu_1, \nu_2, \dots$ by defining $\nu_1 = \mu$ and $\nu_j = \nu_1 \circ \theta_{\nu_{j-1}} + \nu_{j-1}$ for $j \geq 2$. Observe that, on the event

$$\begin{aligned} &\Big(\Big\{\big\langle S_{\nu_1}, \theta\big\rangle < 0\Big\} \cup \Big\{\|S_{\nu_1}\|^* > 3r\Big\}\Big) \\ &\quad \cap \bigcap_{j=2}^{n} \left(\left\{\Big\langle\big(S_{\nu_j} - S_{\nu_{j-1}}\big), \theta\Big\rangle < 0\right\} \cup \left\{\max_{\nu_{j-1} \leq k \leq \nu_j} \|S_k - S_{\nu_{j-1}}\| > 3r\right\}\right), \end{aligned}$$

the random walk cannot hit Σ before time T_r. Thus, by the strong Markov property again,

$$P_0\Big(\big\{\langle S_{\nu_1}, \theta\rangle < 0\big\} \cup \big\{\|S_{\nu_1}\|^* > 3r\big\}\Big)^n \leq 1 - p,$$

so that

$$P_0\Big(\big\langle S_{\nu_1}, \theta\big\rangle > 0,\ \|S_{\nu_1}\|^* \leq 3r\Big) \geq 1 - (1-p)^{1/n}.$$

This completes the proof of Lemma 4.1. □

LEMMA 4.2. *Let $r > 0$, $0 < \beta < 1$ and $\theta \in S^{d-1}$. Then, for any η such that $0 < \eta < P(\langle \widehat{S}_{T_r}, \theta\rangle \geq \beta\|\widehat{S}_{T_r}\|)$, we have*

$$P\Big(\big\langle S_{T_r}, \theta\big\rangle > \frac{\beta^2}{128n}\|S_{T_r}\|\Big) \geq \frac{\eta}{2}\left[1 - \left(1 - \frac{\eta}{2}\right)^{1/n}\right]^n,$$

where $n = [32\rho/\beta^2] + 1$.

PROOF. Fix $1 > \beta > 0$ and $\eta > 0$ such that $P\big(\widehat{S}_{T_r} \in \Gamma(\theta, \beta)\big) > \eta$. Choose β_1 satisfying $0 < \beta_1 < \frac{2}{3}\beta$. Let $0 < \varepsilon < (2\beta - 3\beta_1)$. We will show that, for a given $\theta \in S^{d-1}$ and $r > 0$, at least one of (4.1) and (4.2) below must hold:

$$P(S_n \in \Gamma(\theta, \varepsilon) \cap \operatorname{Ann}(0, \varepsilon r, r) \text{ for some } 0 < n < T_r) \geq \frac{\eta}{2}, \tag{4.1}$$

$$P\big(S_{T_r} \in \Gamma(\theta, \beta_1)\big) \geq \frac{\eta}{2}. \tag{4.2}$$

We begin with some geometry.

CLAIM. Let $x, y \in \mathbb{R}^d$ satisfy $\|x\| < r$, $\|y\| = 3r$ and $x + y \in \Gamma(\theta, \beta)$. Then either $\langle y, \theta \rangle > 3r\beta_1$ or $x \in \Gamma(\theta, \varepsilon) \cap \mathrm{Ann}(0, \varepsilon r, r)$.

We show this by contraposition, *assuming throughout that* $\|x\| < r$ and $\|y\| = 3r$.

Suppose $\langle y, \theta \rangle \le 3r\beta_1$ and $x \notin \Gamma(\theta, \varepsilon) \cap \mathrm{Ann}(0, \varepsilon r, r)$. Then $\langle x, \theta \rangle \le \varepsilon r$. Now $\langle y + x, \theta \rangle \le 3r\beta_1 + \varepsilon r < 3r\beta_1 + 2\beta r - 3\beta_1 r = 2\beta r \le \beta\|y + x\|$. Thus $x + y \notin \Gamma(\theta, \beta)$, and the claim is proved.

We will apply the claim below with $y = \widehat{X}_{T_r}$ and $x = S_{T_r - 1} (= \widehat{S}_{T_r - 1})$. Let

$$\begin{aligned} A &= \{S_n \in \Gamma(\theta, \varepsilon) \cap \mathrm{Ann}(0, \varepsilon r, r) \text{ for some } 0 < n < T_r\}, \\ B &= \{S_{T_r} \in \Gamma(\theta, \beta_1)\}, \\ C &= \{\widehat{X}_{T_r} \in \Gamma(\theta, \beta_1),\ \widehat{S}_{T_r} \in \Gamma(\theta, \beta),\ \|\widehat{X}_{T_r}\| = 3r\}. \end{aligned}$$

Observe that on C, since X_{T_r} is a multiple of $\widehat{X}_{T_r}$, we have

$$\langle S_{T_r}, \theta \rangle = \langle \widehat{S}_{T_r}, \theta \rangle + \langle (S_{T_r} - \widehat{S}_{T_r}), \theta \rangle \ge \beta\|\widehat{S}_{T_r}\| + \beta_1\|S_{T_r} - \widehat{S}_{T_r}\| \ge \beta_1\|S_{T_r}\|.$$

Thus $C \subseteq B$. Next on $\{S_{T_r} \ne \widehat{S}_{T_r}\}$ we have $\|\widehat{X}_{T_r}\| = 3r$. Thus, by the claim above,

$$\begin{aligned} \{\widehat{S}_{T_r} \in \Gamma(\theta, \beta),\ \widehat{S}_{T_r} \ne S_{T_r}\} &\subseteq C \cup \{S_{T_r - 1} \in \Gamma(\theta, \varepsilon) \cap \mathrm{Ann}(0, \varepsilon r, r)\} \\ &\subseteq B \cup A. \end{aligned}$$

On the other hand, clearly $\{\widehat{S}_{T_r} \in \Gamma(\theta, \beta),\ \widehat{S}_{T_r} = S_{T_r}\} \subseteq B$. Thus $\{\widehat{S}_{T_r} \in \Gamma(\theta, \beta)\} \subseteq A \cup B$. So $P(A \cup B) \ge \eta$ implies either $P(A) \ge \eta/2$ or $P(B) \ge \eta/2$. Finally, if we take, for example $\beta_1 = \beta/2$, $\varepsilon = \beta/4$, then the desired result follows by combining (4.1), (4.2) and Lemma 4.1. □

LEMMA 4.3. *Let* $0 < r < \infty$, $0 < \beta < 1$ *and* $\theta \in S^{d-1}$. *Let* $\eta > 0$ *be any number satisfying* $P(\langle S_{T_r}, \theta \rangle > \beta\|S_{T_r}\|) > \eta$. *Then we have*

$$P\left(\langle \widehat{S}_{T_r}, \theta \rangle > 0.3\beta\|\widehat{S}_{T_r}\|\right) \ge \left(\frac{\eta}{3}\right) \wedge \left(1 - \left(1 - \frac{\eta}{6}\right)^{1/N}\right), \tag{4.3}$$

where $N = [600c\rho^2/9\eta\beta^2] + 1$, *and* c *is the constant in* (2.5).

Before giving the proof we require three elementary geometric lemmas. Throughout these we assume $\|x\| < r$, $\|y\| > 3r$ and we let $\widehat{y} = (3r/\|y\|)y$.

LEMMA 4.4. *If* $x \in \Gamma(\theta, 0.3\beta)$ *and* $x + y \in \Gamma(\theta, \beta)$, *then* $x + \widehat{y} \in \Gamma(\theta, 0.3\beta)$.

This follows at once from the convexity of $\Gamma(\theta, 0.3\beta)$ and $\Gamma(\theta, \beta) \subseteq \Gamma(\theta, 0.3\beta)$.

LEMMA 4.5. *If* $x + y \in \Gamma(\theta, \beta)$ *and* $x \notin \Gamma(\theta, 0.3\beta)$, *then* y *and* $\widehat{y}$ *belong to* $\Gamma(\theta, \beta/2)$.

PROOF. It suffices to prove this for y. Now $\langle x+y,\theta\rangle \geq \beta\|x+y\| \geq \beta\|y\| - \frac{6}{5}\beta\|x\|$. Thus $\langle y,\theta\rangle \geq \beta\|y\| - \frac{6}{5}\beta\|x\| - \langle x,\theta\rangle \geq \beta\|y\| - \frac{3}{2}\beta\|x\| \geq (\beta/2)\|y\|$. □

LEMMA 4.6. *If $y \in \Gamma(\theta,\beta/2)$ and $\langle x,\theta\rangle \geq -0.3\beta r$, then $x+\widehat{y} \in \Gamma(\theta, 0.3\beta)$.*

PROOF. We have $\langle x+\widehat{y},\theta\rangle \geq \frac{1}{2}\beta\|\widehat{y}\| - 0.3\beta r = 0.4\beta\|\widehat{y}\|$. The result follows, since clearly $\|x+\widehat{y}\| \leq \frac{4}{3}\|y\|$. □

PROOF OF LEMMA 4.3. Since $a+b+c \geq \eta$, where

$$a = P\big(S_{T_r} \in \Gamma(\theta,\beta),\ S_{T_r} = \widehat{S}_{T_r}\big),$$
$$b = P\big(S_{T_r} \in \Gamma(\theta,\beta),\ S_{T_r} \neq \widehat{S}_{T_r},\ S_{T_r-1} \in \Gamma(\theta, 0.3\beta)\big)$$

and

$$c = P\big(S_{T_r} \in \Gamma(\theta,\beta),\ S_{T_r} \neq \widehat{S}_{T_r},\ S_{T_r-1} \notin \Gamma(\theta, 0.3\beta)\big),$$

at least one of the following must hold: $a \geq \eta/3$, $b \geq \eta/3$ or $c \geq \eta/3$. If either of the first two possibilities holds, then (4.3) follows easily; if $a \geq \eta/3$, (4.3) is immediate, while if $b \geq \eta/3$, apply Lemma 4.4 with $x = S_{T_r-1}$ and $y = X_{T_r}$.

For the rest of the proof we assume $c \geq \eta/3$. Define a stopping time σ by

$$\sigma = \min\big\{n\colon \|X_n\| > 3r \text{ and } \langle X_n,\theta\rangle > \tfrac{1}{2}\beta\|X_n\|\big\}.$$

By Lemma 4.5, $P(\sigma = T_r) \geq \eta/3$. Set $\varepsilon = 0.3\beta\rho^{-1}$. Then $\|x\| < \varepsilon r \Rightarrow \langle x,\theta\rangle \geq -0.3\beta r$.

Now define $\mu_1 = \min\{n\colon \|S_n - S_0\| > \varepsilon r\}$ and $\mu_k = \mu_1 \circ \theta_{\mu_{k-1}} + \mu_{k-1}$. Let $W_1 = S_{\mu_1}$, $W_2 = S_{\mu_2} - S_{\mu_1}, \dots$ and $\tau = \min\{n\colon \|\sum_{i=1}^n W_i\| > r\}$. Then the W_i are i.i.d., and since $\|W_i\| \geq \varepsilon r$ we have $K_{W_1}(r) \geq \varepsilon^2$. Thus, by (2.5), taking $N = [6c/\eta\varepsilon^2]$, we have $P(\tau > N) \leq \eta/6$. Since $T_r \leq \mu_r$, this means $P(\mu_N < T_r) \leq P(N < \tau) \leq \eta/6$. Hence, setting $q = P(\sigma = T_{\varepsilon r})$, we have, by the strong Markov property,

$$1 - \frac{\eta}{3} \geq P(\sigma > T_r) \geq P(\sigma > \mu_N,\ \mu_N \geq T_r) \geq P(\sigma > \mu_N) - \frac{\eta}{6} = (1-q)^N - \frac{\eta}{6}.$$

Thus $q \geq 1 - (1-\eta/6)^{1/N}$. Finally, by Lemma 4.6,

$$P\big(\widehat{S}_{T_r} \in \Gamma(\theta, 0.3\beta)\big) \geq P(\sigma = T_{\varepsilon r}) \geq \left[1 - \left(1 - \frac{\eta}{6}\right)^{1/N}\right].$$

The proof of Lemma 4.3 is complete. □

As an immediate consequence of Lemmas 4.2 and 4.3, we have the following result.

PROPOSITION 4.7. *Fix a sequence $\{r_k\}$ and $\theta \in S^{d-1}$. Then*

(4.4) *(D_θ) holds along $\{r_k\}$ if and only if $(\widehat{D}_\theta)$ holds along $\{r_k\}$,*

(4.5) *(H_θ) holds along $\{r_k\}$ if and only if $(\widehat{H}_\theta)$ holds along $\{r_k\}$.*

In the next result, we denote by T_r^θ the first exit time from $[-r, r]$ of the one-dimensional random walk with steps given by $\langle X, \theta\rangle$.

LEMMA 4.8. *Fix $\{r_k\}$ and assume that $\|S_{T_{r_k}}\| r_k^{-1}$ is tight. Then (D_θ) is equivalent to*

$$P^k\big(T^\theta_{\varepsilon r_k} \le T_{r_k}\big) \to 0 \quad \text{for all } \varepsilon > 0. \tag{4.6}$$

PROOF. That (4.6) implies (D_θ) follows from (2.1) and

$$\left\{\left|\left\langle S_{T_{r_k}} \|S_{T_{r_k}}\|^{-1}, \theta\right\rangle\right| \le \varepsilon\rho\right\} \supseteq \left\{T^\theta_{\varepsilon r_k} > T_{r_k}\right\}.$$

Now assume (4.6) fails. Thus, along some subsequence $\{k(1)\}$,

$$P^{k(1)}\big(T^\theta_{\varepsilon r_{k(1)}} \le T_{r_{k(1)}}\big) \to \sigma > 0$$

for some $\varepsilon > 0$. By tightness, for some L and all $k(1)$,

$$P^{k(1)}\Big(\|S_{T_{r_{k(1)}}}\| > L r_{k(1)}\Big) \le \sigma/2.$$

Hence, for each large $k(1)$, either

$$P^{k(1)}\Big(T^\theta_{\varepsilon r_{k(1)}} = T_{r_{k(1)}},\ \|S_{T_{r_{k(1)}}}\| \le L r_{k(1)}\Big) \ge \sigma/4 \tag{4.7}$$

or

$$P^{k(1)}\big(T^\theta_{\varepsilon r_{k(1)}} < T_{r_{k(1)}}\big) \ge \sigma/4. \tag{4.8}$$

If (4.7) holds, then a little geometry shows that

$$P^{k(1)}\left(\left|\left\langle S_{T_{r_{k(1)}}} \|S_{T_{r_{k(1)}}}\|^{-1}, \theta\right\rangle\right| > \varepsilon L^{-1}\right) \ge \sigma/4.$$

Hence (D_θ) fails. If (4.8) holds, then either

$$P^{k(1)}\Big(S_n \in \Gamma(\theta, \varepsilon) \cap \mathrm{Ann}\big(0, \varepsilon\rho^{-1} r_{k(1)}, r_{k(1)}\big) \text{ for some } n < T_{r_{k(1)}}\Big) \ge \sigma/8$$

holds or the analogous statement with θ replaced by $-\theta$ holds. In either case we can apply Lemma 4.1 to conclude that (D_θ) fails. □

LEMMA 4.9. *If*

$$\sup_{\lambda > 0} \limsup_{k \to \infty} \frac{G^k(\lambda r_k)}{h^k(\lambda r_k)} = 0,$$

then, for every $\varepsilon \in (0, 1)$,

$$\limsup_{k \to \infty} \frac{h^k(r_k)}{h^k(\varepsilon r_k)} \le \varepsilon.$$

PROOF. For any $r > 0$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned} h(r) &= |M(r)| + K(r) + G(r) \\ &\le \varepsilon|M(\varepsilon r)| + \varepsilon^2 K(\varepsilon r) + G(r) + 2G(\varepsilon r) \\ &\le \varepsilon h(\varepsilon r) + 3G(\varepsilon r), \end{aligned}$$

and the result follows. □

THEOREM 4.10. *Assume that*

$$\|S_{T_{r_k}}\| r_k^{-1} \text{ is tight.} \tag{4.9}$$

Then (D_θ) *is equivalent to*

$$\limsup_{k\to\infty} \frac{h_\theta^k(r_k)}{h^k(r_k)} = 0. \tag{4.10}$$

PROOF. Assume that (4.10) holds. Then, by the doubling property, for any $\varepsilon > 0$,

$$\limsup_{k\to\infty} \frac{h_\theta^k(\varepsilon r_k)}{h^k(r_k)} = 0. \tag{4.11}$$

Now, for any ξ, by (2.5) and (2.6),

$$\begin{aligned} P^k\left(T_{\varepsilon r_k}^\theta \le \frac{\xi}{h_\theta^k(\varepsilon r_k)}\right) &\le c\xi, \\ P^k\left(T_{r_k} \le \frac{\xi}{h_\theta^k(\varepsilon r_k)}\right) &\ge 1 - \frac{c h_\theta^k(\varepsilon r_k)}{\xi h^k(r_k)}. \end{aligned}$$

Thus, by (4.11),

$$P^k\left(T_{\varepsilon r_k}^\theta \le T_{r_k}\right) \to 0. \tag{4.12}$$

Hence (D_θ) holds by Lemma 4.8. (Note that this part of the proof does not require the tightness assumption.)

Now assume (4.10) fails. We consider two cases:

$$\sup_{\lambda>0} \limsup_{k\to\infty} \frac{G_\theta^k(\lambda r_k)}{h^k(\lambda r_k)} > 0. \tag{A}$$

It then follows easily by doubling and the monotonicity of G that, for some subsequence $\{k(1)\}$ and some $\alpha \in (0, 1)$, either

$$\frac{G_{\theta+}^{k(1)}\left(2\rho\alpha r_{k(1)}\right)}{h^{k(1)}(\alpha r_{k(1)})} \to \sigma > 0 \tag{4.13}$$

or

$$\frac{G_{\theta-}^{k(1)}\big(2\rho\alpha r_{k(1)}\big)}{h^{k(1)}(\alpha r_{k(1)})} \to \sigma > 0. \tag{4.14}$$

Without loss of generality, assume (4.13). Then by Proposition 3.1, for some universal constant $c > 0$,

$$\liminf_{k(1)\to\infty} P^{k(1)}\Big(\big\langle S_{T_{\alpha r_{k(1)}}}, \theta\big\rangle > \rho\alpha r_{k(1)}\Big) \geq c\sigma.$$

In particular, since $T_{\alpha r} \leq T_r$ and $\rho \geq 1$,

$$\liminf_{k(1)\to\infty} P^{k(1)}\big(T^{\theta}_{\alpha r_{k(1)}} \leq T_{r_{k(1)}}\big) \geq c\sigma.$$

By Lemma 4.8 this means (D_θ) fails along $\{r_{k(1)}\}$.

$$\sup_{\lambda>0}\limsup_{k\to\infty} \frac{G_\theta^k(\lambda r_k)}{h^k(\lambda r_k)} = 0. \tag{B}$$

Since (4.10) fails, along some subsequence $\{k(1)\}$,

$$\frac{h_\theta^{k(1)}(r_{k(1)})}{h^{k(1)}(r_{k(1)})} \to \zeta > 0. \tag{4.15}$$

Then by (B) and (2.7)

$$\frac{G_\theta^{k(1)}(\lambda r_{k(1)})}{h_\theta^{k(1)}(\lambda r_{k(1)})} \to 0 \quad \text{for all } \lambda > 0.$$

Thus, by Lemma 4.9, for any $\varepsilon > 0$,

$$\limsup_{k(1)\to\infty} \frac{h_\theta^{k(1)}(r_{k(1)})}{h_\theta^{k(1)}(\varepsilon r_{k(1)})} \leq \varepsilon. \tag{4.16}$$

Now, for any $\delta > 0$, by (2.5) and (2.6),

$$\begin{aligned} P^k\big(T_{r_k} < T^\theta_{\varepsilon r_k}\big) &\leq P^k\left(T^\theta_{\varepsilon r_k} > \frac{\delta}{h_\theta^k(r_k)}\right) + P^k\left(T_{r_k} \leq \frac{\delta}{h_\theta^k(r_k)}\right) \\ &\leq \frac{c h_\theta^k(r_k)}{\delta h_\theta^k(\varepsilon r_k)} + c\delta\frac{h^k(r_k)}{h_\theta^k(r_k)}. \end{aligned}$$

Letting $k(1) \to \infty$ and using (4.15) and (4.16),

$$\limsup_{k(1)\to\infty} P^{k(1)}\big(T_{r_{k(1)}} < T^\theta_{\varepsilon r_{k(1)}}\big) \leq c\varepsilon\delta^{-1} + c\delta\zeta^{-1}.$$

This can be made less than 1 by appropriate choice of δ and ε. Hence, by Lemma 4.8, (D_θ) fails. □

Since $\|\widehat{S}_{T_r}\|r^{-1}$ is always tight, as an immediate consequence of Proposition 4.7 and Theorem 4.10, we have the following result.

THEOREM 4.11. *Fix $\{r_k\}$. Then (D_θ) holding along $\{r_k\}$ is equivalent to*

$$\limsup_{k\to\infty} \frac{\widehat{h}_\theta^k(r_k)}{\widehat{h}^k(r_k)} = 0. \tag{4.17}$$

To illustrate the need for considering $\widehat{h}$ rather than h, we now give an example showing that the previous result is false if we replace $\widehat{h}$ with h. In addition, it shows that Lemma 4.8 and Theorem 4.10 are false without the tightness assumption.

EXAMPLE 4.12. Let $d = 2$ and $\|\ \| = \|\ \|_E$. Set $\varepsilon_k = k^{-1}$ and $u_k = p_k^{-1} = 2^{2^k}$. Define a random variable X by

$$P\big(X = u_k e^{i\varphi_k}\big) = \frac{1}{4}p_k \quad \text{for } k \ge 2,$$

where $\varphi_k = \pm(\pi/2 \pm \varepsilon_k)$ (i.e., put equal mass at each of the four points), with the remaining mass at 0. Let $r_k = u_k|\cos\varphi_k|$. Thus $r_k \sim u_k/k$. Since $u_{k-1} < r_k < u_k$ if k is sufficiently large, a simple computation shows that

$$\begin{aligned} M(r_k) &= 0,\\ K(r_k) &\sim \frac{u_{k-1}^2 p_{k-1}}{r_k^2} = o(p_k),\\ G(r_k) &\ge p_k. \end{aligned}$$

Hence

$$\lim_{k\to\infty} \frac{G(r_k)}{h(r_k)} = 1.$$

On the other hand, if we let $\theta = (1, 0)$, then $G_\theta(r_k) = G(r_k)$ for all k sufficiently large. Since $h_\theta \le h$ in this example, this means

$$1 \le \frac{h(r_k)}{h_\theta(r_k)} \le \frac{h(r_k)}{G_\theta(r_k)} \le \frac{h(r_k)}{G(r_k)} \to 1.$$

Thus (4.17) fails with h replacing $\widehat{h}$.

On the other hand, (4.17) does hold. To see this, observe that, for large k,

$$\begin{aligned} \widehat{h}_\theta(r_k) &= \widehat{K}_\theta(r_k)\\ &\le \widehat{K}(r_k)\\ &= o(p_k). \end{aligned}$$

Since $\widehat{h}(r_k) = h(r_k) \ge G(r_k) \ge p_k$, this means

$$\frac{\widehat{h}_\theta(r_k)}{\widehat{h}(r_k)} \to 0.$$

To state the next result, we need to introduce the following condition:

$$(E_{\theta+}) \qquad P^k\Big(\langle S_{T^0_{r_k}}, \theta\rangle > 0\Big) \to 1.$$

PROPOSITION 4.13. *Fix* $\{r_k\}$ *and assume* $\|S_{T_{r_k}}\|r_k^{-1}$ *is tight. Then, for each* $\theta \in S^{d-1}$, (H_θ) *is equivalent to*:

(4.18) *Every subsequence* $\{k(1)\}$ *contains a further subsequence* $\{k(2)\}$ *along which either* (D_θ) *holds or* $(E_{\theta+})$ *holds.*

PROOF. Assume (4.18) holds. Then, given any subsequence $\{k(1)\}$, there is a further subsequence $\{k(2)\}$ along which either (D_θ) holds or $(E_{\theta+})$ holds. If (D_θ) holds, then trivially (H_θ) holds along $\{k(2)\}$. If $(E_{\theta+})$ holds, then, by Remark 2.1 for any $\varepsilon > 0$,

$$P^{k(2)}\Big(\langle S_n, \theta\rangle \text{ hits } (\rho r_{k(2)}, \infty) \text{ before } \big(-\infty, -\varepsilon r_{k(2)}\big)\Big) \to 1.$$

Clearly, this forces (H_θ) to hold along $\{k(2)\}$ also. Hence (H_θ) holds along the entire sequence.

Now assume (H_θ). Given any subsequence $\{k(1)\}$, if there is a further subsequence $\{k(2)\}$ along which (D_θ) holds, we are done. Hence we may assume (D_θ) fails along every further subsequence of $\{k(1)\}$. Thus we can find a $\delta \in (0, 1)$ and $\sigma < 1$ such that

$$P^{k(1)}\left(\left|\left\langle S_{T_{r_k}} \|S_{T_{r_k}}\|^{-1}, \theta\right\rangle\right| \le \delta\right) < \sigma$$

for all large $k(1)$. Fix L large and $0 < \varepsilon < \delta$ (ε will be chosen very small later). Define stopping times τ_j, $j = 0, 1, 2, \dots$, by $\tau_0 = 0$ and

$$\tau_{j+1} = \tau_j + \tau \circ \theta_{\tau_j},$$

where $\tau = \min\{n\colon \|S_n - S_0\| > r_{k(1)}\}$. Let

$$A_j = \left\{\left\langle (S_{\tau_j} - S_{\tau_{j-1}})\|S_{\tau_j} - S_{\tau_{j-1}}\|^{-1}, \theta\right\rangle \in (-\varepsilon, \delta);\ \|S_{\tau_j} - S_{\tau_{j-1}}\| \le Lr_{k(1)}\right\}$$

and

$$C_{r_{k(1)}} = \left\{\langle S_n, \theta\rangle \text{ hits } \big(-\infty, -2r_{k(1)}\big) \text{ before } \left(\frac{\delta r_{k(1)}}{2}, \infty\right)\right\}.$$

We claim that

$$\begin{aligned} C_{r_{k(1)}} \subset &\left(\bigcap_{j=1}^{[\delta/2\varepsilon L]} A_j\right) \cup \left\{\|S_{\tau_j} - S_{\tau_{j-1}}\| > Lr_{k(1)} \text{ for some } j \le \frac{\delta}{2\varepsilon L}\right\} \\ &\cup \left\{\left\langle (S_{\tau_j} - S_{\tau_{j-1}})\|S_{\tau_j} - S_{\tau_{j-1}}\|^{-1}, \theta\right\rangle \le -\varepsilon \text{ for some } j \le \frac{\delta}{2\varepsilon L}\right\} \\ &= B_1 \cup B_2 \cup B_3. \end{aligned}$$

To see this, first observe that

$$\langle S_{\tau_j} - S_{\tau_{j-1}}, \theta \rangle > -\varepsilon L r_{k(1)}$$

on A_j. Thus, on $\bigcap_{j=1}^{m} A_j$, $m \geq 1$,

$$\langle S_{\tau_m}, \theta \rangle \geq -\varepsilon L m r_{k(1)}$$

and

$$\langle S_n, \theta \rangle \geq -r_{k(1)} - \varepsilon L(m-1) r_{k(1)} \quad \text{for all } n < \tau_m.$$

Now if none of B_1, B_2 or B_3 occur, there is a first $j \leq [\delta/2\varepsilon L]$ for which A_j fails, say $j = m+1$. In that case

$$\begin{aligned} \langle S_{\tau_{m+1}}, \theta \rangle &\geq \langle S_{\tau_m}, \theta \rangle + \delta r_{k(1)} \\ &\geq -\varepsilon L m r_{k(1)} + \delta r_{k(1)} \\ &\geq \frac{\delta r_{k(1)}}{2} \end{aligned}$$

and

$$\langle S_n, \theta \rangle \geq -r_{k(1)} - \varepsilon L m r_{k(1)} \quad \text{for all } n < \tau_{m+1}.$$

Thus $C_{r_{k(1)}}$ fails, proving the claim.

By the strong Markov property,

$$\begin{aligned} P^{k(1)}(B_1) &= P^{k(1)}\Big(\Big\langle S_{T_{r_{k(1)}}} \big\|S_{T_{r_{k(1)}}}\big\|^{-1}, \theta \Big\rangle \in (-\varepsilon, \delta);\ \big\|S_{T_{r_{k(1)}}}\big\| \leq L r_{k(1)}\Big)^{[\delta/2\varepsilon L]} \\ &\leq P^{k(1)}\Big(\Big\langle S_{T_{r_{k(1)}}} \big\|S_{T_{r_{k(1)}}}\big\|^{-1}, \theta \Big\rangle \in (-\varepsilon, \delta)\Big)^{[\delta/2\varepsilon L]} \\ &\leq \sigma^{[\delta/2\varepsilon L]} \end{aligned}$$

for large $k(1)$. Again, by the strong Markov property,

$$P^{k(1)}(B_2) \leq \frac{\delta}{2\varepsilon L} P^{k(1)}\Big(\big\|S_{T_{r_{k(1)}}}\big\| > L r_{k(1)}\Big)$$

and

$$P^{k(1)}(B_3) \leq \frac{\delta}{2\varepsilon L} P^{k(1)}\Big(\Big\langle S_{T_{r_{k(1)}}} \big\|S_{T_{r_{k(1)}}}\big\|^{-1}, \theta \Big\rangle \leq -\varepsilon\Big).$$

Given any M (large), choose L so large that

$$P^{k(1)}\Big(\big\|S_{T_{r_k}}\big\| > L r_{k(1)}\Big) \leq \frac{1}{M^2}$$

for all $k(1)$. This is possible by tightness. Now let $\varepsilon = \delta/2LM$. Then, for large $k(1)$,

$$P^{k(1)}\big(C_{r_{k(1)}}\big) \leq \sigma^{[M]} + \frac{1}{M} + M P^{k(1)}\Big(\Big\langle S_{T_{r_{k(1)}}} \big\|S_{T_{r_{k(1)}}}\big\|^{-1}, \theta \Big\rangle \leq -\varepsilon\Big).$$

Letting $k(1) \to \infty$ and using (H_θ), then letting $M \to \infty$, we see that

$$P^{k(1)}(C_{r_{k(1)}}) \to 0.$$

By Remark 2.1 this forces $(E_{\theta+})$ to hold along $\{k(1)\}$. □

As an immediate consequence of Propositions 4.7 and 4.13 we have the following result.

THEOREM 4.14. *Fix a sequence $\{r_k\}$. Then (H_θ) holds along $\{r_k\}$ if and only if every subsequence $\{k(1)\}$ contains a further subsequence $\{k(2)\}$ along which either $(\widehat{D}_\theta)$ holds or $(\widehat{E}_{\theta+})$ holds, that is, either*

$$\limsup_{k(2)\to\infty} \frac{\widehat{h}_\theta^{k(2)}(r_{k(2)})}{\widehat{h}^{k(2)}(r_{k(2)})} = 0$$

or both

$$\sup_{\lambda>0} \limsup_{k(2)\to\infty} \frac{\widehat{G}_{\theta-}^{k(2)}(\lambda r_{k(2)})}{\widehat{h}_\theta^{k(2)}(\lambda r_{k(2)})} = 0$$

and

$$\inf_{\lambda>0} \liminf_{k(2)\to\infty} \frac{\widehat{J}_\theta^{k(2)}(\lambda r_{k(2)})}{\widehat{h}_\theta^{k(2)}(\lambda r_{k(2)})} \ge \frac{1}{2}.$$

We are now ready to give necessary and sufficient conditions for (E) to hold.

THEOREM 4.15. *Fix a sequence $\{r_k\}$. Then (E) holds along $\{r_k\}$ if and only if, for every $\theta \in S^{d-1}$,*

$$\liminf_{k\to\infty} \frac{\widehat{h}_\theta^{k}(r_k)}{\widehat{h}^{k}(r_k)} > 0$$

and along every subsequence $\{k(1)\}$ either

$$\limsup_{k(1)\to\infty} \frac{\widehat{G}_{\theta-}^{k(1)}(\lambda r_{k(1)})}{\widehat{h}_\theta^{k}(\lambda r_{k(1)})} > 0 \quad \textit{for some } \lambda > 0$$

or

$$\liminf_{k(1)\to\infty} \frac{\widehat{J}_\theta^{k(1)}(\lambda r_{k(1)})}{\widehat{h}_\theta^{k(1)}(\lambda r_{k(1)})} < \frac{1}{2} \quad \textit{for some } \lambda > 0.$$

5. Condition (E) and Laplace transforms. As mentioned in Section 1, a more concise necessary and sufficient condition for (E) to hold, at least in one dimension, can be stated in terms of Laplace transforms. In higher dimensions a slight variant of the Laplace transform is needed. The key to this approach is the pair of probability estimates obtained in Propositions 5.1 and 5.2 below.

Fix $r > 0$. For $x \in \mathbb{R}^d$ let

$$\widehat{x} = xI(\|x\| \le 3r) + 3rx\|x\|^{-1}I(\|x\| > 3r).$$

For $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $r > 0$, define

$$\xi_r(x, y) = -(d-1)\rho^2 \frac{\|x\|_E^2}{r^2} + \langle x, y\rangle. \tag{5.1}$$

In the results of this section, the quantity $Ee^{\xi_r(\hat{X}, y)}$ will play an important role. Observe that in one dimension it is simply the Laplace transform of $\widehat{X}$.

PROPOSITION 5.1. *Fix $\theta \in S^{d-1}$, $r > 1$ and $\beta > 0$. Assume that $t > 0$ and*

$$E \exp\left(\xi_r\left(\widehat{X}, t\theta - \frac{2(d-1)\rho^2 y}{r^2}\right)\right) \ge 1 \tag{5.2}$$

for all $y \in \mathbb{R}^d$ with $\|y\| \le r$. Then

$$P(\widehat{S}_{T_r} \in \Gamma(\theta, \beta)) \ge (1 - e^{-1+4t\beta r})e^{-4t\rho r}.$$

PROOF. We begin by observing that

$$x \notin \Gamma(\theta, \beta),\ r \le \|x\| \le 4r \quad \Rightarrow \quad \xi_r(x, t\theta) \le -1 + 4t\beta r. \tag{5.3}$$

If $d = 1$ this follows since $r > 1$ and the assumptions in (5.3) force $x\theta$ to be negative. If $d \ge 2$ we have $\xi_r(x, t\theta) \le -1 + t\beta\|x\| \le -1 + 4t\beta r$. Also note that, by the Cauchy–Schwarz inequality,

$$r \le \|x\| \le 4r \quad \Rightarrow \quad \xi_r(x, t\theta) \le 4t\rho r. \tag{5.4}$$

Next, observe that

$$\begin{aligned}
\xi_r(\widehat{S}_n, t\theta) &= \xi_r(\widehat{S}_{n-1} + \widehat{X}_n, t\theta) \\
&= \xi_r(\widehat{S}_{n-1}, t\theta) + \xi_r(\widehat{X}_n, t\theta) - \frac{2(d-1)\rho^2\langle\widehat{S}_{n-1}, \widehat{X}_n\rangle}{r^2} \\
&= \xi_r(\widehat{S}_{n-1}, t\theta) + \xi_r\left(\widehat{X}_n, t\theta - \frac{2(d-1)\rho^2\widehat{S}_{n-1}}{r^2}\right).
\end{aligned}$$

Since $\|\widehat{S}_{n-1}\| \le r$ on $\{T_r \ge n\}$, we have, by (5.2),

$$E\left[\exp\left(\xi_r\left(\widehat{X}_n, t\theta - \frac{2(d-1)\rho^2\widehat{S}_{n-1}}{r^2}\right)\right)\middle|\mathcal{F}_{n-1}\right]1_{\{T_r \ge n\}} \ge 1_{\{T_r \ge n\}},$$

where $\mathcal{F}_{n-1} = \sigma(\widehat{X}_1, \dots, \widehat{X}_{n-1})$. It follows that $M_n = e^{\xi_r(\hat{S}_{n\wedge T_r}, t\theta)}$ is a bounded submartingale. Hence

$$\begin{aligned}1 \le E_0(M_\infty) &= E\big[e^{\xi_r(\hat S_{T_r}, t\theta)};\ \widehat{S}_{T_r} \in \Gamma(\theta,\beta)\big] + E\big[e^{\xi_r(\hat S_{T_r}, t\theta)};\ \widehat{S}_{T_r} \notin \Gamma(\theta,\beta)\big]\\ &\le e^{4t\rho r} P\big(\widehat{S}_{T_r} \in \Gamma(\theta,\beta)\big) + e^{-1+4t\beta r} P\big(\widehat{S}_{T_r} \notin \Gamma(\theta,\beta)\big),\end{aligned}$$

where we used (5.4) to estimate the first term and (5.3) to estimate the second. Thus $1 \le e^{4t\rho r} P(\widehat{S}_{T_r} \in \Gamma(\theta,\beta)) + e^{-1+4t\beta r}$, which gives the desired result. □

PROPOSITION 5.2. *Fix $\theta \in S^{d-1}$, $r > 0$ and $\beta > 0$. Assume that $t > 0$ and $Ee^{\xi_r(\hat X, t\theta)} \le 1$. Then*

$$P\big(\widehat{S}_{T_r} \in \Gamma(\theta,\beta)\big) \le \frac{(1+c)}{\beta r t},$$

where c is a universal constant depending only on dimension.

PROOF. Observe that there exists a nonnegative constant C, depending only on dimension and norm, such that

$$E\left[\sum_{j=1}^{T_r} (d-1)\rho^2 \frac{\|\widehat{X}_j\|_E^2}{r^2}\right] \le C\frac{Q(r)}{h(r)} \le C. \tag{5.5}$$

[Use Wald's lemma, (2.4) and (2.7).]

Now, since $E\big[\exp\big(\xi_r(\hat X, t\theta)\big)\big] \le 1$,

$$M_n = \exp\left\{-\sum_{j=1}^{n\wedge T_r} (d-1)\rho^2 \frac{\|\widehat{X}_j\|_E^2}{r^2} + \big\langle t\theta, \widehat{S}_{n\wedge T_r}\big\rangle\right\}$$

is a bounded supermartingale. Thus, letting $\eta = P(\widehat{S}_{T_r} \in \Gamma(\theta,\beta))$, we have

$$\begin{aligned}1 \ge E_0 M_\infty &\ge E\big[M_{T_r};\ \widehat{S}_{T_r} \in \Gamma(\theta,\beta)\big]\\ &\ge e^{t\beta r} E\left\{\exp\left[-\sum_{j=1}^{T_r}(d-1)\rho^2\frac{\|\widehat{X}_j\|_E^2}{r^2}\right];\ \widehat{S}_{T_r} \in \Gamma(\theta,\beta)\right\}\\ &= e^{t\beta r}\eta E\left\{\exp\left[-\sum_{j=1}^{T_r}(d-1)\rho^2\frac{\|\widehat{X}_j\|_E^2}{r^2}\right]\middle|\ \widehat{S}_{T_r} \in \Gamma(\theta,\beta)\right\}\\ &\ge e^{t\beta r}\eta \exp\left\{E\left[-\sum_{j=1}^{T_r}(d-1)\rho^2\frac{\|\widehat{X}_j\|_E^2}{r^2}\middle|\ \widehat{S}_{T_r} \in \Gamma(\theta,\beta)\right]\right\}\\ &= e^{t\beta r}\eta \exp\left\{\frac{-1}{\eta}E\left[\sum_{j=1}^{T_r}(d-1)\rho^2\frac{\|\widehat{X}_j\|_E^2}{r^2};\ \widehat{S}_{T_r} \in \Gamma(\theta,\beta)\right]\right\}\\ &\ge e^{t\beta r}\eta \exp\left\{\frac{-1}{\eta}E\left[-\sum_{j=1}^{T_r}\rho^2(d-1)\rho^2\frac{\|\widehat{X}_j\|_E^2}{r^2}\right]\right\}\\ &\ge e^{t\beta r}\eta e^{-C/\eta},\end{aligned}$$

where we used (5.5) in the last inequality. Since $\eta \ge e^{-1/\eta}$, the result follows. □

THEOREM 5.3. *Fix a sequence* r_k. *Let* $\alpha(r_k) \geq 0$ *be the smallest number such that, for all* $\theta \in S^{d-1}$,

$$t \geq \alpha(r_k) \quad \Rightarrow \quad E^k\big[\exp\big(\xi_{r_k}(\hat{X},\, t\theta)\big)\big] \geq 1. \tag{5.6}$$

[*Set* $\alpha(r_k) = +\infty$ *if there is no such number.*] *Then*

$$(E) \text{ holds along the sequence } r_k \quad \Longleftrightarrow \quad \sup_k r_k \alpha(r_k) < \infty.$$

PROOF. By Lemmas 4.2 and 4.3, it is equivalent to prove the result with (E) replaced by $(\widehat{E})$.

($\Leftarrow$) Let $t = 2\alpha(r_k)\rho + 2(d-1)\rho^3/r_k$. Then, for all $\|y\| \leq r_k$,

$$\left\| t\theta - \frac{2(d-1)\rho^2 y}{r_k^2} \right\| \geq \rho^{-1} t - \frac{2(d-1)\rho^2}{r_k} = 2\alpha(r_k).$$

Hence (5.2) holds and so, by Proposition 5.1,

$$P^k\big(\widehat{S}_{T_{r_k}} \in \Gamma(\theta, \beta)\big) \geq \big(1 - e^{-1 + 8\beta r_k \rho\alpha(r_k) + 8(d-1)\rho^3\beta}\big) e^{-8\rho^2 r_k \alpha(r_k) - 8(d-1)\rho^4}.$$

Since this is uniformly bounded in θ away from 0 if β is sufficiently small, this proves that $(\widehat{E})$ holds.

($\Rightarrow$) Let $t = \alpha(r_k)/2$ and choose $\theta_k \in S^{d-1}$ such that $E^k\big[\exp\big(\xi_r(\hat{X},\, t\theta_k)\big)\big] \leq 1$. Then, by Proposition 5.2, for any $\beta > 0$,

$$P^k\big(\widehat{S}_{T_{r_k}} \in \Gamma(\theta, \beta)\big) \leq \frac{2(1+c)}{\beta r_k \alpha(r_k)}.$$

By a simple compactness argument, the left-hand side is bounded away from 0 if (E) holds, provided β is sufficiently small (see Lemma 4.1 of [4]). Hence the result follows. □

REMARK 5.4. The $\|\widehat{X}\|_E^2$ term in (5.1) cannot be omitted in higher dimensions. Indeed, any random vector whose distribution is symmetric with respect to the origin satisfies $Ee^{\langle y, \hat{X}\rangle} \geq 1$ for every y but it is easy to construct such distributions for which (E) fails. See, for example, [4], Example 7.2.

REMARK 5.5. Let $\alpha_\theta(r_k) \geq 0$ be the smallest number for which

$$t \geq \alpha_\theta(r_k) \quad \Rightarrow \quad E^k E^{\langle \hat{X}, t\theta\rangle} \geq 1.$$

Clearly, $\alpha_\theta(r_k) \leq \alpha(r_k)$. Hence, if (E) holds, then

$$\sup_k \sup_\theta r_k \alpha_\theta(r_k) < \infty.$$

Thus the projections satisfy the one-dimensional condition for (E) uniformly. However, the converse is false by the above remark. Hypotheses requiring uniformity in the behavior of one-dimensional projections have appeared in several recent studies on sums of i.i.d. random variables in higher dimensions. See, for example, the work of Hahn and Klass [5] on operator norming.

6. The limiting exit position. Assume that X_i are i.i.d. genuinely d-dimensional mean 0 random vectors with finite variance and that $\omega(\theta\colon \ell_\theta^*(X)$ is lattice) = 0. We will first prove the sufficiency part of Theorem 1.2, that is, give the limiting joint distribution of $(\|S_{T_r}\| - r, S_{T_r}/\|S_{T_r}\|)$ as $r \to \infty$.

Before giving the proof we discuss some preliminaries related to the convex geometry of the $B_r = \{x : \|x\| \le r\}$. Let $x \in \partial B_1$. Recall that a *support hyperplane at* x is a set of the form $\{\xi\colon \ell^*(\xi) = 1\}$, where the linear functional ℓ^* satisfies $\|\ell^*\| = \ell^*(x) = 1$. Denote the set of such functionals by SH_x. Let

$$\rho'(x; z) = \lim_{t \downarrow 0} \frac{\|x + tz\| - 1}{t}.$$

The limit exists because the function $t \to \|x + tz\|$ is convex. Clearly, $|\rho'(x; z)| \le \|z\|$. For fixed x, the function $z \to \rho'(x; z)$ is sublinear, that is, $\rho'(x; z_1 + z_2) \le \rho'(x; z_1) + \rho'(x; z_2)$, and positively homogeneous. See [6], page 28. We define the *modulus of smoothness at* x, Λ_x, by

$$\Lambda_x = \lim_{t \to 0} \sup_{\|z\| = 1} \frac{\|x + tz\| + \|x - tz\| - 2}{t}.$$

(We should remark that this modulus of smoothness is defined somewhat differently than the modulus of smoothness of a uniformly smooth Banach space.) Clearly, $\Lambda_x \ge 0$ (take $z = x$). It is easy to show, for example, by arguing as in Lemma 6.1 below, that

$$\Lambda_x = \sup_{\|z\| = 1} \left[\rho'(x; z) + \rho'(x; -z)\right], \tag{6.1}$$

so that Λ_x may be viewed as a measure of the departure of the function $z \to \rho'(x; z)$ from linearity. We also have that (see [6], pages 30 and 31)

$$\Lambda_x = 0 \quad \Leftrightarrow \quad SH_x \text{ is a singleton, } \{\ell^*\},$$

in which case $\ell^*(z)$ is equal to $\rho'(x; z)$. Such an x is called a *smooth point*. It is easy to see that Λ_x is an upper-semicontinuous function of x. The set of nonsmooth points is of first category and has surface measure 0. The latter fact follows from standard differentiation theory, since the norm is Lipschitz on ∂B_1. For the former, see, for example, [6] page 171.

Suppose $x \in \partial B_1$ is fixed. Then by convexity, for each z the function ε_z defined on $\mathbb{R}_+$ by

$$\varepsilon_z(t) = \frac{\|x + tz\| - 1}{t} - \rho'(x; z)$$

is nonnegative, continuous and $\lim_{t \downarrow 0} \varepsilon_z(t) = 0$.

LEMMA 6.1. *Let* $\varepsilon(t) = \max_{\|z\| = 1} \varepsilon_z(t)$. *Then* $\lim_{t \downarrow 0} \varepsilon(t) = 0$.

PROOF. If $\|z\| = \|w\| = 1$, then

$$t|\varepsilon_z(t) - \varepsilon_w(t)| \le t\|z - w\| + t|\rho'(x; z) - \rho'(x; w)|.$$

By sublinearity,

$$\begin{aligned}\rho'(x;z) &= \rho'(x;w+z-w) \le \rho'(x;w) + \rho'(x;z-w)\\ &= \rho'(x;w) + \rho'\left(x;\frac{z-w}{\|z-w\|}\right)\|z-w\|.\end{aligned}$$

Hence $|\rho'(x;z)-\rho'(x;w)| \le \|z-w\|$, and so $|\varepsilon_z(t)-\varepsilon_w(t)| \le 2\|z-w\|$. The desired result now follows by an easy compactness argument. □

LEMMA 6.2. *For any $\eta > 0$ there is a $\Delta = \Delta(x,\eta) > 0$ such that, for $0 < \delta < \Delta(x,\eta)$, we have the following implication*:

$$\|y\| \ge 1 \text{ and } \ell^*(y) \le 1-\delta \text{ for some } \ell^* \in SH_x \quad \Rightarrow \quad \|y-(1-\delta)x\| \ge \frac{\delta}{\Lambda_x+\eta}.$$

PROOF. Let $z' = y-(1-\delta)x$ so that $\ell^*(z') \le 0$. We have

$$\begin{aligned}1 \le \|y\| = \|(1-\delta)x+z'\| &= (1-\delta)\left\|x+\frac{z'}{1-\delta}\right\|\\ &\le (1-\delta)+\rho'(x;z')+\varepsilon\left(\frac{\|z'\|}{1-\delta}\right)\|z'\|.\end{aligned}$$

Now, for any $\ell^* \in SH_x$, we have $\ell^*(\cdot) \le \rho'(x;\,\cdot)$ ([6], Theorem 1.7.F). Hence $\rho'(x;\,-z') \ge -\ell^*(z') \ge 0$. Thus, by (6.1), we have $\rho'(x;\,z') \le \Lambda_x\|z'\|$. Hence

$$\|z'\| \ge \frac{\delta}{\Lambda_x+\varepsilon\|z'\|/(1-\delta)}. \tag{6.2}$$

Now let t_0 be such that $\varepsilon(t) \le \eta$ for $t < t_0$, and take Δ so small that

$$\delta < \Delta \Rightarrow \frac{\delta}{(\Lambda_x+\eta)(1-\delta)} < t_0.$$

We claim that $\|z'\| \ge \delta/(\Lambda_x+\eta)$. If not, then $\|z'\| < \delta/(\Lambda_x+\eta)$. Hence $\|z'\|/(1-\delta) < t_0$. But this leads to a contradiction by (6.2). □

REMARK 6.3. The following useful fact follows from [6], Theorem 1.7.F, as in the proof of Lemma 6.2. If $\ell_1, \ell_2 \in SH_x$, then $\|\ell_1-\ell_2\| \le \Lambda_x$.

LEMMA 6.4. *For any y we have*

$$\lim_{r\to\infty}\big(\|rx+y\|-r\big) = \sup_{\ell^*\in SH_x}\ell^*(y), \tag{6.3}$$

and the convergence is uniform on compact sets of y.

PROOF. For any $\ell^* \in SH_x$ and $r > 0$, we have $\|rx+y\|-r \ge \ell^*(rx+y)-r = \ell^*(y)$. First suppose $\ell^*(y) \ge 0$ for at least one $\ell^* \in SH_x$. Let $c > \sup_{\ell^*\in SH_x}$. The vector $y - cx$ then satisfies $\ell^*(y - cx) < 0$ for every $\ell^* \in SH_x$, and it follows from the Hahn–Banach theorem that $x + t(y - cx) \in B_1$ for all sufficiently small t. Thus, for large enough r,

$$\left\| x + \frac{1}{r} y \right\| \le \frac{c}{r} + \left\| x + \frac{1}{r}(y - cx) \right\| \le \frac{c}{r} + 1.$$

Hence $\limsup_{r\to\infty} \|rx + y\| - r \le c$. If $\ell^*(y) < 0$ for every $\ell^* \in SH_x$, then choose a large enough γ so that $\ell^*(y + \gamma x) = \ell^*(y) + \gamma > 0$. Then, by the first part of the proof,

$$\begin{aligned} \lim_{r\to\infty} \|rx + y\| - r &= \lim_{r\to\infty} \|(r - \gamma)x + y + \gamma x\| - (r - \gamma) - \gamma \\ &= \sup_{\ell^* \in SH_x} \ell^*(y + \gamma x) - \gamma = \sup_{\ell^* \in SH_x} \ell^*(y). \end{aligned}$$

This shows that (6.3) holds for each y. It follows from the triangle inequality that the expression on the left-hand side of (6.3) is nonincreasing in r. Thus the uniformity in y follows from Dini's theorem. □

PROOF OF THEOREM 1.2 Since we work with an arbitrary norm, there is no loss of generality in assuming that $EX_1^T X_1$, the covariance matrix of X_1, is the identity. Thus $\omega(\cdot) = \lim_{r\to\infty} P(S_{T_r}/\|S_{T_r}\| \in (\cdot))$ is ordinary harmonic measure on ∂B_1. It is not difficult to show that B_1 is a Lipschitz domain and hence admits a surface area measure σ. By Dahlberg's theorem [2], $\omega \ll \sigma$.

For $a > 0$, let $G_a = \{\theta \in \partial B_1 : \Lambda_\theta < a\}$ and let S be the set of smooth points. Then each G_a is open, $S \subset G_a$ for all $a > 0$ and $\omega(G_a^c) = 0$ for every $a > 0$. To each $\theta \in \partial B_1$ assign a functional $\ell_\theta^* \in \mathrm{SH}_\theta$. [Recall that $\|\ell_\theta^*\| = 1$ and $\ell_\theta^*(\theta) = 1$.] Make the arbitrary choice at nonsmooth points in a measurable way. If $\theta_0 \in S$, then it is easy to check from the definitions that the map $\theta \to \ell_\theta^*$ is continuous at θ_0 no matter how ℓ_θ^* is defined off S. Finally, let L_θ^* be a random variable satisfying (1.4) for the choice $Z_j = \ell_\theta^*(X_j)$ as described in Section 1 and denote the set of θ for which Z_j is nonlattice by D.

By the basic theory of weak convergence, it is enough to prove

$$\begin{aligned} \liminf_{r\to\infty} P\left(\|S_{T_r}\| - r \in I,\ \frac{S_{T_r}}{\|S_{T_r}\|} \in \Omega \right) &\ge \int_\Omega \omega(d\theta) P(L_\theta^* \in I) \\ &= \int_{\Omega\cap S\cap D} \omega(d\theta) P(L_\theta^* \in I) \end{aligned} \tag{6.4}$$

for each open interval $I \subset (0, \infty)$ and open set $\Omega \subset \partial B_1$ such that $\omega(\partial\Omega) = 0$. [Note that $P(L_\theta^* = 0) = 0$ for every $\theta \in D$ since each L_θ^* has an absolutely continuous distribution.]

For $0 < N < r$ let ω_r^N be the distribution of $S_{T_{r-N}}/\|S_{T_{r-N}}\|$. By the strong

Markov property at time T_{r-N},

$$
\begin{aligned}
&P_0\left(\|S_{T_r}\| - r \in I, \frac{S_{T_r}}{\|S_{T_r}\|} \in \Omega\right) \\
(6.5)\qquad &\geq \int_\Omega \omega_r^N(d\theta) \inf_{v\in[N/2,N]} P_{(r-v)\theta}(\|S_{T_r}\| - r \in I) - P\left(\|S_{T_{r-N}}\| > r - \frac{N}{2}\right) \\
&\quad - P\left(\left\{\frac{S_{T_r}}{\|S_{T_r}\|} \in \Omega\right\} \Delta \left\{\frac{S_{T_{r-N}}}{\|S_{T_{r-N}}\|} \in \Omega\right\}\right).
\end{aligned}
$$

The last two terms here may be handled by using the tightness of the overshoot distributions (Theorem 1.3 of [4]) and Lemma 6.5 below, respectively.

LEMMA 6.5. *For any $\epsilon > 0$ and $\Omega \subset \partial B_1$ open such that $\omega(\partial\Omega) = 0$, we have*

$$\limsup_{r\to\infty} P\left(\left\{\frac{S_{T_{r-N}}}{\|S_{T_{r-N}}\|} \in \Omega\right\} \Delta \left\{\frac{S_{T_r}}{\|S_{T_r}\|} \in \Omega\right\}\right) < \epsilon$$

for N large enough.

PROOF. Let $K \subset \Omega$ be compact. Fix $v \in [N/2, N]$ and $r > N$. Since Ω is open, a little geometry shows that there is a constant $\sigma > 0$ so that $\|S_{T_r} - S_0\| \geq \sigma r$ on $\{S_0 = (r-v)\theta,\ S_{T_r}/\|S_{T_r}\| \notin \Omega\}$ for all $\theta \in K$. Thus, on the same event, we have $S_\tau^* > \sigma r$, where τ is the first passage time of S_n to the half-space $\{z\colon \ell_\theta^*(z) > r\}$, and S_τ^* is the maximal function defined by $S_\tau^* = \sup_{n\leq\tau} \|S_n - S_0\|$. By Theorem 3.1 of [9], we have that, for any $0 < p < 1$,

$$E_w(S_\tau^*)^p \leq C \operatorname{dist}(w, H)^p + C, \tag{6.6}$$

where C is a constant depending only on p, ρ and the distribution of X_1. Thus, using (6.6) and the strong Markov property at time T_{r-N}, we obtain

$$\limsup_{r\to\infty} P\left(\|S_{T_{r-N}}\| \in \left[r - N, r - \frac{N}{2}\right], \frac{S_{T_{r-N}}}{\|S_{T_{r-N}}\|} \in K, \frac{S_{T_r}}{\|S_{T_r}\|} \notin \Omega\right) = 0.$$

The same then holds with K replaced by Ω since K was arbitrary and $\omega(\partial\Omega) = 0$. Since the overshoot distribution is tight, we may fix N so that $P\|S_{T_{r-N}}\| > r - N/2) < \epsilon/2$ for all r sufficiently large. Thus

$$\limsup_{r\to\infty} P\left(\left\{\frac{S_{T_{r-N}}}{\|S_{T_{r-N}}\|} \in \Omega\right\} \Big\backslash \left\{\frac{S_{T_r}}{\|S_{T_r}\|} \in \Omega\right\}\right) < \frac{\epsilon}{2}.$$

Applying this to $\overline{\Omega}^c$ gives

$$\limsup_{r\to\infty} P\left(\left\{\frac{S_{T_{r-N}}}{\|S_{T_{r-N}}\|} \in \overline{\Omega}^c\right\} \Big\backslash \left\{\frac{S_{T_r}}{\|S_{T_r}\|} \in \overline{\Omega}^c\right\}\right) < \frac{\epsilon}{2}.$$

The desired result now follows, since

$$\limsup_{r\to\infty} P\left(\frac{S_{T_{r-N}}}{\|S_{T_{r-N}}\|} \in \partial\Omega\right) \le \omega(\partial\Omega) = 0. \qquad \square$$

The idea now is that, for most θ (i.e., the smooth points) and large r, we may approximate the curved boundary of B_r by a hyperplane $\{z\colon \ell_\theta^*(z) = r - \delta\}$. The overshoot of the first passage time to this hyperplane may then be analyzed using classical renewal theory.

Let $L_\gamma(\theta) = \min\{m\colon \sum_{j=1}^m \ell_\theta^*(X_j) > \gamma\}$ and let $0 < \delta < \inf\{t\colon t \in I\} \wedge N/2$. Then, for any $v \in [N/2, N]$,

$$\begin{aligned} &P_{(r-v)\theta}\big(\|S_{T_r}\| - r \in I\big) \\ (6.7)\qquad &\ge P\left(\left\|\sum_{j=1}^{L_{v-\delta}(\theta)} X_j + (r-v)\theta\right\| \in I + r\right) - P_{(r-v)\theta}(L_{v-\delta}(\theta) > T_r). \end{aligned}$$

The following technical result is needed to handle the second term in (6.7).

LEMMA 6.6. *For all $\epsilon > 0$ and N and δ as above, there exists a constant $\lambda = \lambda(\epsilon, N, \delta) > 0$ such that, for θ satisfying $\Lambda_\theta < \lambda$, we have*

$$\limsup_{r\to\infty} \sup_{v\in[N/2,N]} P_{(r-v)\theta}(L_{v-\delta}(\theta) > T_r) < \epsilon.$$

PROOF. On the event $\{L_{v-\delta}(\theta) > T_r, S_0 = (r-v)\theta\}$, we have $\ell_\theta^*(S_{T_r} - (r-v)\theta) \le v - \delta$ or $\ell_\theta^*(S_{T_r}/r) \le 1 - \delta/r$. Applying Lemma 6.2 with $y = S_{T_r}/r$ and δ replaced by δ/r, we have that, for any $\eta > 0$,

$$\big\|S_{T_r} - (r-v)\theta\big\| \ge \big\|S_{T_r} - (r-\delta)\theta\big\| - (N-\delta) \ge \frac{\delta}{\Lambda_\theta + \eta} - (N-\delta),$$

as soon as $\delta/r < \Delta(\theta, \eta)$. For such r and any $v \in [N/2, N]$, we thus have

$$(6.8)\quad P_{(r-v)\theta}\big(L_{v-\delta}(\theta) > T_r\big) \le P_{(r-v)\theta}\left(\big\|S_{T_r} - (r-v)\theta\big\| > \frac{\delta}{\Lambda_\theta + \eta} - (N-\delta)\right).$$

Let τ denote the first passage time of S_n to the half-space $H = \{z\colon \ell_\theta^*(z) > r - \delta\}$. Taking, say, $p = \frac{1}{2}$ in (6.6) and applying this inequality on the right-hand side of (6.8), we obtain

$$P_{(r-v)\theta}\big(L_{v-\delta}(\theta) > T_r\big) \le \frac{CN^{1/2} + C}{\big(\delta/(\Lambda_\theta + \eta) - N + \delta\big)^{1/2}}$$

for any $v \in [N/2, N]$ and any $\eta > 0$ if r is sufficiently large. Hence, letting $r \to \infty$ and $\eta \downarrow 0$,

$$\limsup_{r\to\infty} \sup_{v\in[N/2,N]} P_{(r-v)\theta}(L_{v-\delta}(\theta) > T_r) \le \frac{(CN^{1/2} + C)\Lambda_\theta^{1/2}}{[\delta - N\Lambda_\theta + \delta\Lambda_\theta]^{1/2}},$$

which clearly yields the desired result. □

For fixed N and $\epsilon > 0$, let λ be as in Lemma 6.6. Then, by (6.7) and Lemma 6.6,

$$\int_{\Omega} \omega_r^N(d\theta) \inf_{v \in [N/2, N]} P_{(r-v)\theta}(\|S_{T_r}\| - r \in I)$$
$$\geq \int_{G_\lambda \cap \Omega} \omega_r^N(d\theta) \inf_{v \in [N/2, N]} P\left(\left\|\sum_{j=1}^{L_{v-\delta}(\theta)} X_j + (r-v)\theta\right\| - r \in I\right) - \epsilon,$$

once r is sufficiently large.

Fix $L > 0$ large and $\lambda > 0$ small. Then, by Lemma 6.4 and Remark 6.3, if r is sufficiently large, for any $\|y\| \leq L$ and $\theta \in G_\lambda$,

$$\left|(\|y + r\theta\| - r) - l_\theta^*(y)\right| \leq 2\Lambda_\theta \|y\| < 2\lambda L.$$

We apply this with $y = Y = \sum_{j=1}^{L_{v-\delta}(\theta)} X_j - v\theta$. Observe that Y is finite a.s. Then, for sufficiently large r,

$$P\left(\left\|\sum_{j=1}^{L_{v-\delta}(\theta)} X_j + (r-v)\theta\right\| - r \in I\right)$$
$$\geq P\left(\sum_{j=1}^{L_{v-\delta}(\theta)} \ell_\theta^*(X_j) - (v-\delta) \in I' + \delta;\ \|Y\| < L\right),$$

where $I' = \{t \in I: \operatorname{dist}(t, I^c) > 2\lambda L\}$.

For fixed δ, λ and L, let $g(v, \theta)$ denote the term on the right-hand side. Also let $f(\theta) = \inf_{v \in [N/2, N]} g(v, \theta)$. By the classical limit theorems for spent and residual lifetimes, it is easy to see that, for each $\theta \in S \cap D$,

$$\lim_{\eta \to 0} \lim_{\eta \to \infty} P(A(v, \theta, \eta)) = 1,$$

where

$$A(v, \theta, \eta) = \left\{\sum_{j=1}^{L_v(\theta)-1} \ell_\theta^*(X_j) < v - \eta < v + \eta < \sum_{j=1}^{L_v(\theta)} \ell_\theta^*(X_j)\right\}.$$

Now fix $\varepsilon > 0$. Then by a simple measure-theoretic argument we can find $N = N(\varepsilon)$ and $\eta = \eta(\varepsilon)$ such that

$$E_\varepsilon \equiv \left\{\theta \in S \cap D : \inf_{v \geq N/2} P(A(v - \delta, \theta, \eta)) > 1 - \frac{\varepsilon}{2}\right\}$$

satisfies $\omega(E_\varepsilon) \geq 1 - \varepsilon$. Next by (6.6) with $p = 1/2$, if ε is sufficiently small [depending only on the constants in (6.6)], then

$$\inf_{v \in [N/2, N]} P_{(r-v)\theta}(S^*_{L_{v-\delta}(\theta)} \leq N\varepsilon^{-3}) \geq 1 - \frac{\varepsilon}{2}.$$

Now since $\theta \to \ell_\theta^*$ is continuous at each point of $S \cap D$, we have by some elementary geometry that, for every $v \in [N/2, N]$ and $\theta \in S \cap D$,

$$A(v-\delta, \theta, \eta) \cap \left\{ \sum_{j=1}^{L_{v-\delta}(\theta)} \ell_\theta^*(X_j) - (v-\delta) \in I' + \delta,\ \|Y\| < L \right\} \cap \left\{ S^*_{L_{v-\delta}(\theta)} \le N\varepsilon^{-3} \right\}$$
$$\subset \left\{ \sum_{j=1}^{L_{v'-\delta}(\theta')} \ell_{\theta'}^*(X_j) - (v'-\delta) \in I' + \delta, \right\},$$

once $\|\theta' - \theta\|$ and $|v' - v|$ are sufficiently small. Thus, for each $v \in [N/2, N]$ and $\theta \in E_\varepsilon$, there is an $\mathcal{L} = \mathcal{L}(v, \theta) > 0$ such that, for $\|\theta' - \theta\| < \mathcal{L}$ and $|v' - v| < \mathcal{L}$, we have

$$g(v', \theta') > g(v, \theta) - \varepsilon \ge f(\theta) - \varepsilon.$$

By a simple argument involving compactness of $[N/2, N]$, it then follows that we can choose $\mathcal{L} = \mathcal{L}(\varepsilon) > 0$ such that

$$\|\theta' - \theta\| < \mathcal{L}(\theta) \quad \Rightarrow \quad f(\theta') \ge f(\theta) - \varepsilon$$

for all $\theta \in E_\varepsilon$. Let

$$\underline{f}(\theta) = \lim_{\mathcal{L} \downarrow 0} \inf_{\|\theta' - \theta\| < \mathcal{L}} f(\theta')$$

be the l.s.c. regularization of f. Then we have shown that $\underline{f}(\theta) \ge f(\theta) - \varepsilon$ for $\theta \in E_\varepsilon$. Thus

$$\begin{aligned} \liminf_{r\to\infty} \int_{G_\lambda \cap \Omega} f(\theta)\omega_r^N(d\theta) &\ge \liminf_{r\to\infty} \int_{G_\lambda \cap \Omega} \underline{f}(\theta)\omega_r^N(d\theta) \\ &\ge \int_{G_\lambda \cap \Omega} \underline{f}(\theta)\omega(d\theta) \\ &\ge \int_{G_\lambda \cap \Omega \cap E_\varepsilon} (f(\theta) - \varepsilon)\omega_r^N(d\theta) \\ &\ge \int_{G_\lambda \cap \Omega} f(\theta)\omega_r^N(d\theta) - 2\varepsilon. \end{aligned}$$

Hence if we let $r \to \infty$, followed by $\lambda \to 0$ and then $L \to \infty$, we obtain

$$\begin{aligned} &\liminf_{r\to\infty} \int_{G_\lambda \cap \Omega} \omega_r^N(d\theta) \inf_{v \in [N/2, N]} P_{(v-\delta)\theta}(\|S_{T_r}\| - r \in I) \\ &\quad \ge \int_{S \cap \Omega} \omega(d\theta) \inf_{v \in [N/2, N]} \left(\sum_{j=1}^{L_{v-\delta}(\theta)} \ell_\theta^*(X_j) - (v-\delta) \in I + \delta \right) - 3\varepsilon. \end{aligned}$$

Letting $N \to \infty$ and using the results of classical renewal theory, we obtain

$$\liminf_{r\to\infty} P\left(\|S_{T_r}\| - r \in I,\ \frac{S_{T_r}}{\|S_{T_r}\|} \in \Omega \right) \ge \int_{S \cap \Omega} \omega(d\theta) P(L_\theta^* \in I + \delta) - 3\varepsilon.$$

The desired result follows, since $\omega(S \cap \Omega) = \omega(\Omega)$, δ and ε were arbitrary and each L_θ^* has an absolutely continuous distribution function. □

We conclude this section by proving the necessity part of Theorem 1.2. The necessity of the conditions $EX = 0$ and $E\|X\|^2 < \infty$ is immediate from [4], Theorem 1.3. For $\theta \in \partial B_1$, set

$$A_\theta = \{\psi \in \partial B_1 : \ell_\psi^* = \ell_\theta^*\}.$$

Define an equivalence relation on ∂B_1 by $\theta \sim \psi$ if $A_\theta = A_\psi$ and denote the equivalence class of A_θ by $[A_\theta]$. Observe that there are at most countably many equivalence classes $[A_\theta]$ for which $\ell_\psi^*(X)$ is lattice, say $[A_i]$ for $i \geq 1$. Note also that by convexity, the surface measure of ∂A_i is 0 and hence so is its harmonic measure. Let $A_0 = \partial B_1 \setminus \cup_1^\infty A_i$. Then any Borel set $B \subset \partial B_1$ can be written as a disjoint union $\cup_0^\infty B_i$, where $B_i = B \cap A_i$. If $\omega(\partial B) = 0$, then the same is true of $\omega(\partial B_i)$. Thus, to compute

$$\lim_{r\to\infty} P\left(\|S_{T_r}\| - r \in I,\ \frac{S_{T_r}}{\|S_{T_r}\|} \in B\right), \tag{6.9}$$

it suffices to do so with B replaced by B_i. Now the proof of the sufficiency part of Theorem 1.2 gives the answer for $i = 0$. For $i \geq 1$ we may assume $\omega(A_{\theta_i}) > 0$, where $A_{\theta_i} = A_i$, else the limit in (6.9) is 0. Let $\delta_i > 0$ be such that the minimal lattice which supports $\ell_{\theta_i}^*(X)$ is $\{n\delta_i\, n \in \mathbb{Z}\}$. Then the proof of sufficiency above is easily modified to show that, for $0 \leq \gamma < \delta_i$,

$$\begin{aligned}\lim_{r\to\infty} P\left(\left\|S_{T_{n\delta_i+\gamma}}\right\| - (n\delta_i + \gamma) = k\delta_i - \gamma,\ \frac{S_{T_{n\delta_i}}}{\left\|S_{T_{n\delta_i}}\right\|} \in B_i\right) &= \int_{B_i} \omega(d\theta) P(L_\theta^* = k\delta_i)\\ &= P(L_\theta^* = k\delta_i)\omega(B_i).\end{aligned}$$

From this we see that the limit in (1.5) exists if and only if $\omega(A_0) = 1$. This completes the proof of Theorem 1.2. □

REMARK 6.7. If $\omega(A_0) < 1$, then the limit along subsequences may exist if $\cap\{\delta_i\mathbb{Z} : \omega(A_i) > 0, i \geq 1\} = \delta\mathbb{Z}$ for some $\delta > 0$ (the only other possibility is that the intersection is $\{0\}$). For example, if $r_n = n\delta + \gamma$ where $0 \leq \gamma < \delta$, then the limit along r_n can be computed using the above.

APPENDIX

The purpose of this Appendix is to prove Proposition 3.8. We begin by making a few simple observations. First, for any $0 < \lambda < \alpha$ we have

$$\alpha r M(\alpha r) = \lambda r M(\lambda r) + E(X;\ \lambda r < |X| \leq \alpha r), \tag{A1}$$

$$(\alpha r)^2 K(\alpha r) = (\lambda r)^2 K(\lambda r) + E(X^2;\ \lambda r < |X| \leq \alpha r). \tag{A2}$$

Since $E(X;\ \lambda r < X \le \alpha r) \ge (\alpha r)^{-1} E(X^2;\ \lambda r < X \le \alpha r)$ these imply

$$K(\alpha r) \le (\lambda\alpha^{-1})^2 K(\lambda r) + M(\alpha r) - (\lambda\alpha^{-1}) M(\lambda r) + O(G_-(\lambda r)). \tag{A3}$$

Next observe that if $|M(\lambda r)| \le \frac{1}{2} h(\lambda r)$, then

$$\begin{aligned} h(\lambda r) &\le 2Q(\lambda r) \\ &\le 2\big(\alpha\lambda^{-1}\big)^2 Q(\alpha r) \\ &\le 2\big(\alpha\lambda^{-1}\big)^2 h(\alpha r), \end{aligned} \tag{A4}$$

since $x^2 Q(x) \uparrow$.

LEMMA A.1. *Assume*

$$\sup_{\lambda > 0} \lim_{k \to \infty} \frac{G_-^k(\lambda r_k)}{h^k(\lambda r_k)} = 0. \tag{A5}$$

If

$$\limsup_{\lambda \downarrow 0} \liminf_{k \to \infty} \frac{M^k(\lambda r_k) + G_+^k(\lambda r_k)}{h^k(\lambda r_k)} > 0, \tag{A6}$$

then

$$\inf_{\lambda > 0} \liminf_{k \to \infty} \frac{M^k(\lambda r_k)}{h^k(\lambda r_k)} \ge 0. \tag{A7}$$

PROOF. If not, then along some subsequence $\{k(1)\}$,

$$\frac{M^{k(1)}(\alpha r_{k(1)})}{h^{k(1)}(\alpha r_{k(1)})} \to \mu < 0 \tag{A8}$$

for some α.

Let $\nu > 0$ be strictly smaller than the left-hand side of (A6). We consider two cases, the first of which is

$$\limsup_{k(1) \to \infty} \frac{G_+^{k(1)}(\lambda r_{k(1)})}{G_+^{k(1)}(\alpha r_{k(1)})} > \frac{1}{\nu} \quad \text{for some } \lambda \in (0, \alpha). \tag{A}$$

By (A6) and the fact that the left-hand side of (*A*) is monotone in λ, along a further subsequence $\{k(2)\}$ we can find a $\lambda \in (0, \alpha)$ to satisfy

$$G_+^{k(2)}(\lambda r_{k(2)}) > \nu^{-1} G_+^{k(2)}(\alpha r_{k(2)}), \tag{A9}$$

$$G_+^{k(2)}(\lambda r_{k(2)}) + M^{k(2)}(\lambda r_{k(2)}) > \nu h^{k(2)}(\lambda r_{k(2)}) \tag{A10}$$

for all $k(2)$. Hence, by (A1) and (A9),

$$\begin{aligned} M^{k(2)}(\alpha r_{k(2)}) &= \lambda\alpha^{-1} M^{k(2)}(\lambda r_{k(2)}) + (\alpha r_{k(2)})^{-1} E^{k(2)}(X;\ \lambda r_{k(2)} < X \le \alpha r_{k(2)}) \\ &\quad + O\big(G_-^{k(2)}(\lambda r_{k(2)})\big) \\ &\ge \lambda\alpha^{-1}\big(M^{k(2)}(\lambda r_{k(2)}) + (1-\nu) G_+^{k(2)}(\lambda r_{k(1)})\big) + O\big(G_-^{k(2)}(\lambda r_{k(2)})\big) \\ &\ge \lambda\alpha^{-1}\big(G_+^{k(2)}(\lambda r_{k(2)}) + M^{k(2)}(\lambda r_{k(2)}) - \nu h^{k(2)}(\lambda r_{k(2)})\big) \\ &\quad + O\big(G_-^{k(2)}(\lambda r_{k(2)})\big). \end{aligned}$$

Now divide by $h^{k(2)}(\lambda r_{k(2)})$ and use (A5) and (A10) to obtain

$$\liminf_{k(2)\to\infty} \frac{M^{k(2)}(\alpha r_{k(2)})}{h^{k(2)}(\lambda r_{k(2)})} \geq 0.$$

Since h^k satisfies the doubling property, this contradicts (A8). Next consider the case

$$\text{(B)} \qquad \limsup_{k(1)\to\infty} \frac{G_+^{k(1)}(\lambda r_{k(1)})}{G_+^{k(2)}(\alpha r_{k(1)})} \leq \frac{1}{\nu} \quad \text{for all } \lambda \in (0, \alpha).$$

By (A1), if $\lambda \in (0, \alpha)$,

$$\text{(A11)} \qquad M^{k(1)}(\alpha r_{k(1)}) = \lambda\alpha^{-1} M^{k(1)}(\lambda r_{k(1)}) + O\big(G_-^{k(1)}(\lambda r_{k(1)})\big).$$

Thus if

$$\text{(A12)} \qquad \limsup_{k(1)\to\infty} \frac{M^{k(1)}(\lambda r_{k(1)})}{h^{k(1)}(\lambda r_{k(1)})} \geq 0$$

for some $\lambda \in (0, \alpha)$, then, by the doubling property of h^k,

$$\limsup_{k(1)\to\infty} \frac{M^{k(1)}(\alpha r_{k(1)})}{h^k(\alpha r_{k(1)})} \geq 0,$$

which contradicts (A8). Thus we may assume (A12) fails for all $\lambda \in (0, \alpha)$. Hence, by (A6), for arbitrarily small λ,

$$\liminf_{k(1)\to\infty} \frac{G_+^{k(1)}(\lambda r_{k(1)})}{h^{k(1)}(\lambda r_{k(1)})} > \nu.$$

Since $h^k \geq G_+^k + |M^k|$, this means that, for arbitrarily small λ,

$$\text{(A13)} \qquad \limsup_{k(1)\to\infty} \frac{|M^{k(1)}(\lambda r_{k(1)})|}{G_+^{k(1)}(\lambda r_{k(1)})} < \frac{1-\nu}{\nu}.$$

Thus, by (B) and (A13), for arbitrarily small λ,

$$\text{(A14)} \qquad \begin{aligned} \limsup_{k(1)\to\infty} \frac{|M^{k(1)}(\lambda r_{k(1)})|}{h^{k(1)}(\alpha r_{k(1)})} &\leq \limsup_{k(1)\to\infty} \frac{|M^{k(1)}(\lambda r_{k(1)})|}{G_+^{k(1)}(\alpha r_{k(1)})} \\ &\leq \frac{1}{\nu} \limsup_{k(1)\to\infty} \frac{|M^{k(1)}(\lambda r_{k(1)})|}{G_+^{k(1)}(\lambda r_{k(1)})} \\ &\leq (1-\nu)\nu^{-2}. \end{aligned}$$

Since we are assuming (A12) fails for all $\lambda \in (0, \alpha)$, it follows from (A5), (A11), (A14) and the doubling property of h^k that, for arbitrarily small λ,

$$\liminf_{k(1)\to\infty} \frac{M^{k(1)}(\alpha r_{k(1)})}{h^{k(1)}(\alpha r_{k(1)})} \geq \frac{-\lambda(1-\nu)}{\alpha\nu^2}.$$

Letting $\lambda \downarrow 0$ then contradicts (A8). □

COROLLARY A.2. *Assume* (A5). *Then, for any* $\alpha > 0$, *we have the implication*

$$\liminf_{k\to\infty} \frac{M^k(\alpha r_k) - K^k(\alpha r_k)}{h^k(\alpha r_k)} \geq 0 \qquad \Longrightarrow \qquad \liminf_{k\to\infty} \frac{G_+^k(\alpha r_k) + M^k(\alpha r_k)}{h^k(\alpha r_k)} \geq \frac{1}{2}.$$

PROOF. We have

$$h^k(\alpha r_k) \leq 2\big(G_+^k(\alpha r_k) + |M^k(\alpha r_k)|\big) + K^k(\alpha r_k) - |M^k(\alpha r_k)| + G_-^k(\alpha r_k).$$

Thus

$$\frac{G_+^k(\alpha r_k) + |M^k(\alpha r_k)|}{h^k(\alpha r_k)} \geq \frac{1}{2} + \frac{1}{2}\frac{|M^k(\alpha r_k)| - K^k(\alpha r_k)}{h^k(\alpha r_k)} - \frac{G_-^k(\alpha r_k)}{h^k(\alpha r_k)}.$$

The result now follows, since the hypothesis implies

$$\liminf_{k\to\infty} M^k(\alpha r_k)/h^k(\alpha r_k) \geq 0.$$ □

LEMMA A.3. *Assume* (A5). *If*

$$\limsup_{\lambda\downarrow 0} \liminf_{k\to\infty} \frac{G_+^k(\lambda r_k)}{h^k(\lambda r_k)} = 1, \tag{A15}$$

then

$$\inf_{\lambda>0} \liminf_{k\to\infty} \frac{G_+^k(\lambda r_k) + M^k(\lambda r_k)}{h^k(\lambda r_k)} \geq \frac{1}{2}. \tag{A16}$$

PROOF. Fix $\alpha > 0$ and $\varepsilon \in (0, \frac{1}{2}]$. Then, by (A15), we can find a $\lambda \in (0, \alpha)$ such that for all k sufficiently large

$$G_+^k(\lambda r_k) \geq (1-\varepsilon) h^k(\lambda r_k). \tag{A17}$$

Note that this implies $|M^k(\lambda r_k)| < \frac{1}{2}|h^k(\lambda r_k)|$. Thus, by (A3), (A4) and (A5),

$$\frac{K^k(\alpha r_k)}{h^k(\alpha r_k)} \leq \frac{2K^k(\lambda r_k)}{h^k(\lambda r_k)} + \frac{M^k(\alpha r_k)}{h^k(\alpha r_k)} - \frac{\lambda}{\alpha}\frac{M^k(\lambda r_k)}{h^k(\alpha r_k)} + o(1). \tag{A18}$$

Next observe that (A15) implies (A6). Thus, by Lemma A1 and the doubling property of h_k^+,

$$\liminf_{k\to\infty} \frac{M^k(\lambda r_k)}{h^k(\alpha r_k)} \geq 0 \quad \text{for all } \lambda > 0. \tag{A19}$$

Hence for the λ chosen above

$$\text{(A20)}\qquad \begin{aligned}\liminf_{k\to\infty}\frac{M^k(\alpha r_k)-K^k(\alpha r_k)}{h^k(\alpha r_k)} &\geq -2\limsup_{k\to\infty}\frac{K^k(\lambda r_k)}{h^k(\lambda r_k)}\\ &\geq -2\varepsilon\end{aligned}$$

by (A17). Letting $\varepsilon \downarrow 0$ and using Corollary A2 then completes the proof. □

PROOF OF PROPOSITION 3.8. We must show, assuming (A5), that the following are equivalent:

$$\text{(A21)}\qquad \limsup_{\lambda\downarrow 0}\liminf_{k\to\infty}\frac{G_+^k(\lambda r_k)+M^k(\lambda r_k)}{h^k(\lambda r_k)} > 0,$$

$$\text{(A22)}\qquad \inf_{\lambda>0}\liminf_{k\to\infty}\frac{G_+^k(\lambda r_k)+M^k(\lambda r_k)}{h^k(\lambda r_k)} \geq \frac{1}{2}.$$

Clearly, (A22) implies (A21). For the converse we argue by contradiction. Thus assume that, for some α and some subsequence $\{k(1)\}$,

$$\text{(A23)}\qquad \lim_{k(1)\to\infty}\frac{G_+^{k(1)}(\alpha r_{k(1)})+M^{k(1)}(\alpha r_{k(1)})}{h^{k(1)}(\alpha r_{k(1)})} < \frac{1}{2}.$$

We consider two cases, the first of which is

$$\text{(A)}\qquad \limsup_{\lambda\downarrow 0}\limsup_{k(1)\to\infty}\frac{M^{k(1)}(\lambda r_{k(1)})}{h^{k(1)}(\lambda r_{k(1)})} > 0.$$

Let $\mu > 0$ be a strict lower bound for the left-hand side of (A). Then, for arbitrarily small λ, there is a subsequence $\{k(2)\}$ of $\{k(1)\}$ (depending on λ) along which

$$M^{k(2)}(\lambda r_{k(2)}) \geq \mu K^{k(2)}(\lambda r_{k(2)})$$

for all $k(2)$. Hence if λ is chosen small enough that $\lambda \leq \mu\alpha$, then

$$\lambda\alpha^{-1}M^{k(2)}(\lambda r_{k(2)}) \geq (\lambda\alpha^{-1})^2 K^{k(2)}(\lambda r_{k(2)}).$$

Thus, by (A3),

$$K^{k(2)}(\alpha r_{k(2)}) \leq M^{k(2)}(\alpha r_{k(2)}) + O\big(G_-^{k(2)}(\lambda r_{k(2)})\big).$$

Dividing by $h^{k(2)}(\alpha r_{k(2)})$ and letting $k(2) \to \infty$, we obtain

$$\liminf_{k(2)\to\infty}\frac{M^{k(2)}(\alpha r_{k(2)})-K^{k(2)}(\alpha r_{k(2)})}{h^{k(2)}(\alpha r_{k(2)})} \geq 0.$$

Then the conclusion of Corollary A2 contradicts (A23). Next consider the case

$$\text{(B)}\qquad \limsup_{\lambda\downarrow 0}\limsup_{k(1)\to\infty}\frac{M^{k(1)}(\lambda r_{k(1)})}{h^{k(1)}(\lambda r_{k(1)})} \leq 0,$$

and hence, by (A21) and Lemma A1,

$$\limsup_{\lambda\downarrow 0}\limsup_{k(1)\to\infty}\frac{|M^{k(1)}(\lambda r_{k(1)})|}{h^{k(1)}(\lambda r_{k(1)})}=0.$$

We first observe that by (A5), (A21) and (B) we can find arbitrarily small λ so that, for all large $k(1)$,

$$\frac{G_+^{k(1)}(\lambda r_{k(1)})}{K^{k(1)}(\lambda r_{k(1)})}\geq\frac{G^{k(1)}(\lambda r_{k(1)})}{h_+^{k(1)}(\lambda r_{k(1)})}\geq\nu,\tag{A24}$$

where $\nu>0$ is a strict lower bound for the left-hand side of (A21).

Subcase (B1): For every $\beta>0$ there is a $\gamma\in(0,\beta)$ such that

$$\liminf_{k(1)\to\infty}\frac{G_+^{k(1)}(\gamma r_{k(1)})}{G_+^{k(1)}(\beta r_{k(1)})}>2.\tag{A25}$$

Fix $\beta>0$ and choose $\gamma\in(0,\beta)$ so that (A25) holds. Then, for all $\lambda\in(0,\gamma]$ and all $k(1)$ sufficiently large,

$$G_+^{k(1)}(\lambda r_{k(1)})>2G_+^{k(1)}(\beta r_{k(1)}).$$

Thus, for large $k(1)$,

$$(\beta r_{k(1)})^{-2}E^{k(1)}(X^2;\ \lambda r_{k(1)}<X\leq\beta r_{k(1)})\geq(\lambda\beta^{-1})^2G_+^{k(1)}(\lambda r_{k(1)})/2.\tag{A26}$$

Next by (A24) we can find a $\lambda\in(0,\gamma)$ so that, for all large $k(1)$,

$$K^{k(1)}(\lambda r_{k(1)})\leq\nu^{-1}G_+^{k(1)}(\lambda r_{k(1)}).\tag{A27}$$

Fix a λ so that (A26) and (A27) hold for all large $k(1)$. Set $c_\nu=2\nu^{-1}+1$. Then by (A1), (A2) with α replaced by β, (A26) and (A27), for large $k(1)$,

$$\begin{aligned}K^{k(1)}(\beta r_{k(1)})&\leq(\lambda\beta^{-1})^2\nu^{-1}G_+^{k(1)}(\lambda r_{k(1)})+(\beta r_{k(1)})^{-2}E^{k(1)}(X^2;\ \lambda r_{k(1)}<X\leq\beta r_{k(1)})\\&\quad+O\big(G_-^{k(1)}(\lambda r_{k(1)})\big)\\&\leq c_\nu(\beta r_{k(1)})^{-2}E^{k(1)}(X^2;\ \lambda r_{k(1)}<X\leq\beta r_{k(1)})\\&\quad+O\big(G_-^{k(1)}(\lambda r_{k(1)})\big)\\&\leq c_\nu(\beta r_{k(1)})^{-1}E^{k(1)}\big(X;\ \lambda r_{k(1)}<X\leq\beta r_{k(1)}\big)\\&\quad+O\big(G_-^{k(1)}(\lambda r_{k(1)})\big)\\&\leq c_\nu\big(M^{k(1)}(\beta r_{k(1)})-(\lambda\beta^{-1})M^{k(1)}(\lambda r_{k(1)})+O(G_-^{k(1)}(\lambda r_{k(1)})\big).\end{aligned}$$

Dividing by $h^{k(1)}(\beta r_{k(1)})$ and using (A5), (A7) and the doubling property of h^k, we obtain, for every $\beta>0$,

$$\limsup_{k(1)\to\infty}\frac{K^{k(1)}(\beta r_{k(1)})}{h^{k(1)}(\beta r_{k(1)})}\leq c_\nu\limsup_{k(1)\to\infty}\frac{M^{k(1)}(\beta r_{k(1)})}{h^{k(1)}(\beta r_{k(1)})}.$$

Letting $\beta \downarrow 0$ and using (B), we conclude that

$$\liminf_{\beta \downarrow 0} \liminf_{k(1) \to \infty} \frac{G_+^{k(1)}(\beta r_{k(1)})}{h^{k(1)}(\beta r_{k(1)})} = 1.$$

By Lemma A3, this is more than enough to contradict (A23).

Subcase (B2): There is a $\beta > 0$ such that, for all $\gamma \in (0, \beta)$,

$$\liminf_{k(1) \to \infty} \frac{G_+^{k(1)}(\gamma r_{k(1)})}{G_+^{k(1)}(\beta r_{k(1)})} \leq 2. \tag{A28}$$

We first observe that there is a fixed subsequence $\{k(2)\}$ of $\{k(1)\}$ such that, for all $\gamma \in (0, \beta)$ and all $k(2)$,

$$G_+^{k(2)}(\gamma r_{k(2)}) \leq 3G_+^{k(2)}\big(\beta r_{k(2)}\big). \tag{A29}$$

To see this, let

$$A_\gamma = \big\{k(1) : G_+^{k(1)}(\gamma r_{k(1)}) \leq 3G_+^{k(1)}(\beta r_{k(1)})\big\}.$$

Then $A_{\gamma_1} \subset A_{\gamma_2}$ if $\gamma_1 < \gamma_2$ and $|A_\gamma| = \infty$ for each $\gamma \in (0, \beta)$. The required sequence is then obtained by letting

$$k(2) = k\text{th integer in } A_{\beta k^{-1}}.$$

Thus, for arbitrarily small $\gamma > 0$ and λ satisfying $\gamma < \lambda < \beta$, we have, by (A3), (A24), (A29) and the monotonicity of G_+^k,

$$\begin{aligned} K^{k(2)}(\lambda r_{k(2)}) \leq\ & \big(\gamma\lambda^{-1}\big)^2 K^{k(2)}(\gamma r_{k(2)}) + M^{k(2)}(\lambda r_{k(2)}) \\ & - \big(\gamma\lambda^{-1}\big) M^{k(2)}(\gamma r_{k(2)}) + O\big(G_-^{k(2)}(\gamma r_{k(2)})\big) \\ \leq\ & \big(\gamma\lambda^{-1}\big)^2 \nu^{-1} 3G_+^{k(2)}(\lambda r_{k(2)}) + M^{k(2)}(\lambda r_{k(2)}) \\ & - \big(\gamma\lambda^{-1}\big) M^{k(2)}(\gamma r_{k(2)}) + O\big(G_-^{k(2)}(\gamma r_{k(2)})\big). \end{aligned}$$

Dividing by $h^{k(2)}(\lambda r_{k(2)})$ and using the doubling property of h^k, (A5) and Lemma A1, we obtain

$$\limsup_{k(2) \to \infty} \frac{K^{k(2)}(\lambda r_{k(2)})}{h^{k(2)}(\lambda r_{k(2)})} \leq \frac{3\gamma^2}{\lambda^2 \nu} + \limsup_{k(2) \to \infty} \frac{M^{k(2)}(\lambda r_{k(2)})}{h^{k(2)}(\lambda r_{k(2)})}.$$

Now let $\gamma \downarrow 0$ and then $\lambda \downarrow 0$ and use (B) to obtain

$$\liminf_{\lambda \downarrow 0} \liminf_{k(2) \to \infty} \frac{G_+^{k(2)}(\lambda r_{k(2)})}{h^{k(2)}(\lambda r_{k(2)})} = 1.$$

By Lemma A3, this contradicts (A23). □

Acknowledgment. The authors would like to thank the referee for a careful reading of the original manuscript.

REFERENCES

[1] CHOW, Y. S. and TEICHER, H. (1978). *Probability Theory*. Springer, New York.
[2] DAHLBERG, B. E. J. (1977). Estimates of harmonic measure. *Arch. Rational Mech. Anal.* **65** 272–288.
[3] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, 2nd ed. Wiley, New York.
[4] GRIFFIN, P. S. and MCCONNELL, T. R. (1992). On the position of a random walk at the time of first exit from a sphere. *Ann. Probab.* **20** 825–854.
[5] HAHN, M. G. and KLASS, M. (1985). Affine normability of partial sums of i.i.d. random vectors: a characterization. *Z. Wahrsch. Verw. Gebiete* **69** 479–505.
[6] HOLMES, R. B. (1975). *Geometric Functional Analysis and Its Applications*. Springer, New York.
[7] KESTEN, H. and MALLER, R. A. (1994). Infinite limits and infinite limit points of random walks and trimmed sums. *Ann. Probab.* **22** 1473–1513.
[8] LAI, T. L. (1976). Asymptotic moments of random walks with applications to ladder variables and renewal theory. *Ann. Probab.* **4** 51–66.
[9] MCCONNELL, T. R. (1984). Exit times of N-dimensional random walks. *Z. Wahrsch. Verw. Gebiete* **67** 213–233.
[10] PRUITT, W. E. (1981). The growth of random walks and Lévy processes. *Ann. Probab.* **9** 948–956.
[11] PRUITT, W. E. (1981). General one-sided laws of the iterated logarithm. *Ann. Probab.* **9** 1–48.
[12] SPITZER, F. (1964). *Principles of Random Walk*. Van Nostrand, Princeton.

DEPARTMENT OF MATHEMATICS
SYRACUSE UNIVERSITY
SYRACUSE, NEW YORK 13244-1150