The Annals of Probability
1994, Vol. 22, No. 3, 1273–1294

ON THE FILTRATION OF HISTORICAL BROWNIAN MOTION[1]

BY MARTIN T. BARLOW AND EDWIN A. PERKINS

University of British Columbia

We show that the historical Brownian motion may be recovered from ordinary super-Brownian motion when the dimension d of the underlying Brownian motion is greater than 4. We outline a proof showing that this conclusion is false if $d \leq 3$. The state of affairs in the critical dimension $d = 4$ is left unresolved. Some extensions are given for $1+\beta$ stable branching mechanisms where $\beta \in (0, 1]$.

1. Introduction. An emerging theme in much of the recent work on branching measure-valued Markov processes or superprocesses is that these processes themselves are, in general, seriously deficient. In recording only the current locations of individuals in the population, one loses critical genealogical information, such as which particles at time t are descended from which ancestors at time $s < t$. This information is clearly essential from a modeling perspective and has also proved to be important in the mathematical study of the superprocess itself. Several authors have introduced enriched structures which encode additional genealogical information. We mention a few recent contributions but hasten to add that this theme is an old and familiar one in the branching process literature. Perkins (1988) used a nonstandard model (essentially the usual branching particle system with infinite intensity and infinitesimal mass on each particle) to analyze the Hausdorff measure of the closed supports. Dynkin's characterization of polar sets [Dynkin (1991), (1992)] starts with a superprocess indexed by a class of stopping times for the underlying Markov process which is much richer than the class of constant times. Le Gall (1993) introduces a symmetric path-valued process which runs up and down the tree of branching Markov processes. In this work we follow the approach of Dawson and Perkins (1991) which introduced a historical process to record the entire past history of each individual in the population and not just the current location. A discussion of these last two approaches may be found in Dawson (1993). The natural question which we address in this work is: do these enriched structures really contain additional information? The somewhat surprising (to us) answer for super-Brownian motion is: it depends on the dimension d of the underlying Brownian motion.

To give a description of the historical process, we recall the branching particle construction of a Y-superprocess when Y is a continuous Feller process

Received December 1992.
[1]Research supported in part by an NSERC Canada Operating Grant.
AMS 1991 *subject classifications.* Primary 60G57; secondary 60G07.
Key words and phrases. Superprocesses, historical processes, branching measure-valued diffusions.

in $\mathbb{R}^d$. $O(N)$ particles follow independent copies of Y on each time interval $[i/N, (i+1)/N)$. At $t = i/N$ each particle independently splits into two or dies with equal probability. Let X_t^N be the random measure on $\mathbb{R}^d$ assigning mass N^{-1} to each individual alive at time t. Assume X_0^N converges to m as $N \to \infty$ in the space $M_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$ with the weak topology. Then $\mathbb{P}(X^N \in \cdot)$ converges weakly on $D([0,\infty), M_F(\mathbb{R}^d))$ to $\mathbb{P}_m$, the law of the Y-superprocesses on $\Omega_X = C([0,\infty), M_F(\mathbb{R}^d))$. [See Ethier and Kurtz (1986), Section 9.4, for a slightly modified version of this result.]

This construction is valid for more general time-inhomogeneous Markov processes, including the path-valued Markov process $Y_{\cdot \wedge t}$. This leads us to replace X_t^N by H_t^N, the random measure on $C = C([0,\infty), \mathbb{R}^d)$ which assigns mass $1/N$ to each function $y \in C$ which is constant on $[t, \infty)$ and which on $[0, t]$ equals a Y trajectory which is alive at time t. Identifying constant $\mathbb{R}^d$-valued trajectories with $\mathbb{R}^d$, if $H_0^N = X_0^N \to m$ in $M_F(\mathbb{R}^d)$ then $\mathbb{P}(H^N \in \cdot)$ converges weakly on $D([0,\infty), M_F(C))$ to $\mathbb{Q}_m$, the law of the Y-*historical process* on $(\Omega, \mathcal{G})$, where $\Omega = C([0,\infty), M_F(C))$ and $\mathcal{G}$ is its Borel σ-field [see Dawson and Perkins (1991), Corollary 7.17]. Let $H_t(w) = w(t)$ denote the coordinate variables on Ω. For $y \in C$ let $y^s(t) = y(t \wedge s)$, $C^s = \{y \in C: y = y^s\}$ and $M_F(C)^s = \{m \in M_F(C): m(C^s) = m(C)\}$: then clearly $H_s \in M_F(C)^s$ for all $s \geq 0$ $\mathbb{Q}_m$-a.s. If $s \geq 0$ and $m \in M_F(C)^s$ it is easy to extend the above convergence result to construct the natural law $\mathbb{Q}_{s,m}$ on $(\Omega, \mathcal{G}([s,\infty)))$ [here $\mathcal{G}(I) = \sigma(H_s: s \in I)$], corresponding to the Y-historical process starting at m at time s. Then $(\Omega, \mathcal{G}, \mathcal{G}([s, t+]), H_t, \mathbb{Q}_{s,m})$ is a time-inhomogeneous Borel strong Markov process with continuous paths in $M_F(C)$ [see Dawson and Perkins (1991), Theorems 2.2.3 and 2.1.5].

For $t \geq 0$ define π_t: $M_F(C) \to M_F(\mathbb{R}^d)$ by $\pi_t(\nu)(A) = \nu(y: y_t \in A)$ and let $X_t = \pi_t(H_t)$. It should be clear from the above that $\mathbb{Q}_{0,m}(X \in \cdot) = \mathbb{P}_m(\cdot)$; that is, X is the Y-superprocess starting at m [see Dawson and Perkins (1991), Theorem 2.2.4]. Hence the superprocess can be recovered from the historical process by a trivial projection. We are interested in the converse: can the historical process H be recovered from the superprocess X?

If $(\mathcal{C}_s)_{s\geq 0}$ is the canonical filtration on C, then for $s < t$ the measure $H_t|_{\mathcal{C}_s}$ is purely atomic [Dawson and Perkins (1991), Proposition 3.5] and gives the family tree up to time s of all particles alive at time t. Hence we are asking whether or not these family trees can be recovered from X itself.

In this work we consider the case when Y is a Brownian motion on $\mathbb{R}^d$, H is the associated historical Brownian motion and $X_t = \pi(H_t)$ is super-Brownian motion. We work on $(\Omega, \mathcal{G})$ with respect to $\mathbb{Q}_m = \mathbb{Q}_{0,m}$ where $m \neq 0$. Let $\mathcal{N}_m$ be the class of $\mathbb{Q}_m$-null sets in $\mathcal{G}$, $\mathcal{F}_t^H = \mathcal{G}([0,t]) \vee \mathcal{N}_m$ and $\mathcal{F}_t^X = \sigma(X_s: s \leq t) \vee \mathcal{N}_m$. From the remarks above we have $\mathcal{F}_t^X \subseteq \mathcal{F}_t^H$. Here then is our main result.

THEOREM 1.1. (a) *If $d \geq 5$, then $\mathcal{F}_t^H = \mathcal{F}_t^X$ for all $t \geq 0$.*
(b) *If $d = 1$, then $\mathcal{F}_t^H \neq \mathcal{F}_t^X$.*

The situation for dimensions $d = 2, 3, 4$ is as follows. If $d = 2, 3$, then we can show that $\mathcal{F}_t^X \neq \mathcal{F}_t^H$. We give a heuristic description of the proof below and defer the details (which at present are somewhat involved and ugly) to a forthcoming work on singular interactions. The situation for $d = 4$ is left open but the discussion below suggests the following conjecture.

CONJECTURE 1.2. *If $d = 4$, then $\mathcal{F}_t^X = \mathcal{F}_t^H$ for all $t \geq 0$.*

The basic fact underlying Theorem 1.1(a) and our belief in Conjecture 1.2 is that a Brownian motion will collide with the support of an independent super-Brownian motion if and only if $d < 4$. Write $S(\mu)$ for the closed support of a measure μ, let

$G(X) = \{(t,x)\colon x \in S(X_t),\ t > 0\} \subset (0,\infty) \times \mathbb{R}^d$ be the graph of X,

$\overline{G}(X)$ denote the closure of $G(X)$ in $(0,\infty) \times \mathbb{R}^d$,

$G(y) = \{(t,y(t))\colon t \geq 0\}$ be the graph of $y\colon [0,\infty) \to \mathbb{R}^d$,

and let $B(x,r)$ and $\overline{B}(x,r)$ denote the open and closed balls in $\mathbb{R}^d$ centered at x and with radius r, respectively.

PROPOSITION 1.3. *Let X be a super-Brownian motion in $M_F(\mathbb{R}^d)$ starting at $m \neq 0$ and let B be an independent d-dimensional Brownian motion. Then*

$$G(B) \cap \overline{G}(X) = \emptyset \quad \text{a.s. if } d \geq 4, \tag{1.1}$$

$$G(B) \cap G(X) \neq \emptyset \quad \text{with positive probability if } d < 4. \tag{1.2}$$

The intuitive idea behind the proof of Theorem 1.1(a) is as follows. For $x \in S(X_1)$ we wish to construct the path(s) $y \in S(H_1)$ with $y(1) = x$ from the process $\{X_t,\ 0 \leq t \leq 1\}$. If a unique such $y(x)$ can be constructed for X_1-a.a. x, then it is easy to reconstruct H_1 from X as the image of X_1 under the map $x \to y(x)$. If $d \geq 6$ we claim that for X_1-a.a. x there is a unique continuous path $y(t)$, $t \in [0,1]$, with $y(1) = x$ and $G(y)$ contained in $\overline{G}(X)$. To see this, use the fact that the graphs of two independent super-Brownian motions do not intersect if $d \geq 6$ [Barlow, Evans and Perkins (1991), Theorem 3.6]. If $f(t)$ is a second continuous path on [0, 1] which ends at x and whose graph is contained in $\overline{G}(X)$, choose a rational $r \in (0,1)$ and a "rational ball" B such that $y(r) \in B$ but $f(r) \notin \overline{B}$. The descendants of those particles in B at time r and the descendants of those particles in B^c at time r form a pair of conditionally (given the past up to time r) independent super-Brownian motions which must collide by time 1, an event which has probability 0. The union of these events over (r,B) form a null set off which y is the unique continuous path with the above properties. It is quite easy to turn these heuristics into a rigorous proof of Theorem 1.1 (a) for $d \geq 6$. Note that we must work under the Campbell measure on $\Omega \times C$ (the path y above is the second coordinate), defined at the beginning of Section 2 below.

For $d = 4, 5$ Proposition 1.3 ensures that y does not hit any cluster of unrelated particles and hence, using the Markov property at each rational time t, that y never hits any cluster of relations. However, (1.1) does not rule out the possibility that clusters of particles accumulate at y without any one cluster actually hitting y. We therefore used the following argument to prove Theorem 1.1(a).

Let $n \geq 1$, $\delta = \delta(n)$ and estimate $y(t)$ by a $\mathcal{F}_1^X$-measurable path Y_t^n defined as follows. Set $Y_1^n = y(1) = x$, and, given $Y_{(i+1)2^{-n}}^n \in S(X_{(i+1)2^{-n}}^n)$, let $Y_{i2^{-n}}^n \in S(X_{i2^{-n}}^n) \cap B(Y_{(i+1)2^{-n}}^n, \delta)$ for $0 \leq i < 2^n$. We then interpolate to define Y_t^n, $t \in [0,1]$. For $d \geq 5$ the estimate [Dawson and Perkins (1991), Theorem 2.2.4] shows that there exists $\varepsilon_n \downarrow 0$ such that, with high probability, all the "X-particles" in $B(y(t), \delta)$ are descendants of $y(t-\varepsilon_n)$ for all $t \in \{i2^{-n}: i \leq 2^n\}$. By a Borel–Cantelli argument, it then follows that, for all large n, the estimated ancestor $Y_{i2^{-n}}^n$ is a close relation of the true ancestor $y(i2^{-n})$, and from this we deduce that $\lim_n |Y_t^n - y(t)| = 0$. For $d = 4$ our estimates give $\varepsilon_n = O(1)$, so that this argument fails.

When $d \leq 3$ it follows from Proposition 1.3 that $y(t)$ (which is a Brownian path when chosen according to H_1) collides with unrelated clusters with positive probability. Thus, for $d \leq 3$, y is not the unique continuous path satisfying $y(1) = x$ and with $G(y) \subset \overline{G}(X)$. This strongly suggests that it is not possible to reconstruct H_1 from $\{X_t: 0 \leq t \leq 1\}$.

We now outline a proof of $\mathcal{F}_t^X \not\subseteq \mathcal{F}_t^H$ for $d \leq 3$. We argue with approximating branching particle systems and suppress many of the complications which arise when taking weak limits. Divide the initial population into a group of red particles and the complementary group of white particles. We say a red and a white particle collide if they come within ε_N of each other (N^{-1} is the mass of each particle). We do not know what ε_N should be, but do know there is a sequence $\{\varepsilon_N\}$ decreasing to 0 sufficiently slowly for this argument to work. When a red and a white particle collide, each for the first time, they switch offspring. That is, the white particle's descendants become red and are considered to be the descendants of the corresponding red particle and vice versa. (The incompetent nurse manages to switch all the babies in the first generation after each "admissible" collision.) This gives rise to two historical processes:

H_t = the original historical process of the entire population without any baby swapping;

$\widetilde{H}_t$ = the historical process of the entire population which incorporates the infant-trading interaction.

One must of course prove that in the limit $\widetilde{H}$ is a historical Brownian motion but this is not difficult as each particle feels a single additional jump of size $\varepsilon_N \to 0$ if it is "swapped." The fact that we only swap particles if it is the first interspecies collision for each helps here. Clearly, H and $\widetilde{H}$ both project down to the same super-Brownian motion X because the way the population is divided into red and white particles does not affect the empirical distribution of the entire population. We claim that

$$H \neq \widetilde{H} \quad \text{with positive probability.} \tag{1.3}$$

This shows H cannot be a function of X a.s. and hence $\mathcal{F}_t^X \neq \mathcal{F}_t^H$.

Claim (1.3) would follow from:

(1.4) Some of the colliding red particles have descendants which contribute to the population at a time strictly in the future.

To see that (1.4) implies (1.3), note that the contributions of the descendants described in (1.4) to H and $\tilde{H}$ are quite different. The former is supported by the trajectory of the colliding red particle prior to the collision, while the latter is supported by that of the colliding white particle prior to the collision.

To prove (1.4), we analyze a related model in which colliding red and white particles annihilate each other. When $d \leq 3$ these approximating particle systems are tight, and each limit point in $(\Omega_X)^2$ satisfies a natural martingale problem. Any solution to this martingale problem is a pair of measure-valued processes with zero collision local time (this is the most delicate part of the omitted proof) and therefore the pair of noninteracting (i.e., independent) red and white populations is distinct from the interacting pair of red and white populations [use Proposition 5.11(b) of Barlow, Evans and Perkins (1991)]. The difference between the interacting (i.e., annihilating) red and white populations and the dominating pair of independent super-Brownian motions shows that the descendants of the colliding particles do contribute to the total population at future times; that is, (1.4) holds.

The details of the above argument are deferred to a future work where we plan to study this and related martingale problems for singular interactions.

In Section 2 we prove Theorem 1.1(a). Section 3 gives a proof of an extended Markov property for the Campbell measure associated with H. This result, which is intuitively clear from the branching particle picture, plays an essential role in Section 2. In Section 4 we give a version of Theorem 1.1(a) for $1+\beta$-stable branching mechanisms ($\beta \in (0,1)$), and in Section 5 we prove Theorem 1.1(b) and Proposition 1.3.

NOTATION. If $(E, \mathcal{E})$ is a measurable space, then $b\mathcal{E}$ and $p\mathcal{E}$ denote the spaces of bounded and nonnegative $\mathcal{E}$-measurable functions from E to $\mathbb{R}$, respectively, and $bp\mathcal{E} = b\mathcal{E} \cap p\mathcal{E}$.

$\mathcal{B}_d$ and $\mathcal{C}$ denote the σ-fields of Borel sets on $\mathbb{R}^d$ and C, respectively.

2. Proof of Theorem 1.1(a).

DEFINITION. For $t \geq \tau \geq 0$ and $m \in M_F(C)^\tau - \{0\}$ the *normalized Campbell measure* associated with $\mathbb{Q}_{\tau,m}$ at time t is the probability $\widehat{\mathbb{Q}}_{\tau,m,t}$ on $(\widehat{\Omega}, \widehat{G}([\tau,\infty))) = (\Omega \times C, \mathcal{G}([\tau,\infty)) \times \mathcal{C})$ given by

$$\widehat{\mathbb{Q}}_{\tau,m,t}(A \times B) = \mathbb{Q}_{\tau,m}\big(1_A(H)H_t(B)\big)/m(C).$$

We work in the setting of Theorem 1.1(a) and so set $\tau = 0$, fix $m \in M_F(\mathbb{R}^d)$ and write $\widehat{\mathbb{Q}}_t$ for $\widehat{\mathbb{Q}}_{0,m,t}$. Let (H, y) denote a sample point in $\widehat{\Omega}$ and set $X_t = \overline{\pi}_t(H_t)$ on Ω. Under $\widehat{\mathbb{Q}}_t$, y denotes a typical path chosen according to H_t and clearly

$$\widehat{\mathbb{Q}}_t(H \in \cdot) \ll \mathbb{Q}_m. \tag{2.1}$$

For $s \geq 0$ define $M_F(C)$-valued processes on $\widehat{\Omega}$ by

$$H_u^{(s)}(H,y)(A) = H_u(\{w \in A\colon w^s \neq y^s\}), \qquad u \geq s,$$
$$\widetilde{H}_u^{(s)}(H,y)(A) = H_u(\{w \in A\colon w^s = y^s\}), \qquad u \geq s.$$

Hence $\widetilde{H}_u^{(s)}$ measures the contribution to H_u from those cousins of y which have branched off from y after time s and $H_u^{(s)}$ is the rest of the mass. Let $X_u^{(s)} = \pi_u(H_u^{(s)})$ and $\widetilde{X}_u^{(s)} = \pi_u(\widetilde{H}_u^{(s)})$ give the corresponding partition of X_u. Let $\widehat{\mathcal{G}}_s = \mathcal{G}([0,s+]) \times \mathcal{C}$. The extended Markov property we need is the following.

THEOREM 2.1. *For $t \geq s$ the processes $H_\cdot^{(s)}$ and $\widetilde{H}_\cdot^{(s)}$ are $\widehat{\mathbb{Q}}_t$-a.s. continuous on $[s,\infty)$ and satisfy*

$$\widehat{\mathbb{Q}}_t\Big(H_\cdot^{(s)} \in A \mid \widehat{\mathcal{G}}_s \vee \sigma\big(\widetilde{H}_\cdot^{(s)}\big)\Big)(y,H) = \mathbb{Q}_{s,H_s}(A) \quad \widehat{\mathbb{Q}}_t\text{-a.s. for all } A \in \mathcal{G}\big([s,\infty)\big).$$

It is easy to use the approximating branching particle system to see why this result is true. It states that if we condition on the trajectory y of a typical particle alive at t, the past of the entire process up to s and the evolution of all descendants of y_s, then the branching particle system starting at time s with the particles distinct from y_s evolves like an ordinary super-Brownian motion. It is possible to turn these heuristics into a proof but in the next section we give another argument based on a representation for the associated Palm measure from Evans (1993) and Dawson and Perkins (1991).

NOTATION. Let $\Omega_X = C([0,\infty), M_F(\mathbb{R}^d))$ and its canonical filtration be $\mathcal{G}_t^X = \sigma(w_s\colon s \leq t)$. Denote by $S(\nu)$ the closed support of a measure ν, and let $h(t) = (t\log^+(1/t))^{1/2}$.

LEMMA 2.2. (a) $H_t(\{w\colon w^s = y^s\}) > 0$ *for all* $s < t$, $y \in S(H_t)$ $\mathbb{Q}_m$*-a.s., for all* $t \geq 0$.

(b) $w^s \in S(H_s)$ *whenever* $w \in S(H_t)$ *for some* $t \geq s$ $\mathbb{Q}_m$*-a.s., for all* $s \geq 0$.

PROOF. (a) It suffices to prove the result for a fixed $s \in (0,t)$ by monotonicity in s. By Dawson and Perkins (1991), Proposition 3.5(a), $H_t(\{w\colon w^s \in \cdot\})$ has finite support $\mathbb{Q}_m$-a.s. and this implies

$$\Big\{y\colon H_t\big(\{w\colon w^s = y^s\}\big) > 0\Big\}$$

is $\mathbb{Q}_m$-a.s. closed. Therefore it suffices to prove

$$H_t\big(\{w\colon w^s = y^s\}\big) > 0 \quad \text{for } H_t\text{-a.a. } y \ \mathbb{Q}_m\text{-a.s., for all } t > s > 0.$$

For $y \in C^s$ let $R_{s,t}(y, d\nu)$ denote the canonical measure for H_t starting at (s,δ_y)—see Dawson and Perkins (1991), pages 62 and 63. Proposition 4.1.8 of Dawson and Perkins (1991) implies

$$\begin{aligned}
&\mathbb{Q}_m\left(\int 1\big(H_t(\{w\colon w^s = y^s\}) = 0\big)H_t(dy)\right)\\
&= \lim_{\theta\to\infty} \mathbb{Q}_m\left(\int \exp\big(-\theta H_t(\{w\colon w^s = y^s\})\big)H_t(dy)\right)\\
&= \lim_{\theta\to\infty} \exp\left(-\int_s^t \int (1-e^{-\theta\nu(C)})R_{r,t}(y^r, d\nu)\,dr\right)\\
&= \exp\left(-\int_s^t (t-r)^{-1}\,dr\right) \quad \text{[by Dawson and Perkins (1991), (4.1.5)]}\\
&= 0.
\end{aligned}$$

(b) First fix $t \geq s \geq 0$. Then

$$\begin{aligned}
&\mathbb{Q}_m\left(\int 1(w^s \notin S(H_s))H_t(dw)\right)\\
&= \mathbb{Q}_m\left(\mathbb{Q}_{s,H_s(w)}\left(\int 1\big(w^s \notin S(H_s(w))H_t(dw)\big)\right)\right)\\
&= \mathbb{Q}_m\left(P_{s,H_{s(w)}}\left(\big\{y\colon y^s \notin S\big(H_s(w)\big)\big\}\right)\right) \quad \text{(superprocess property)}\\
&= 0.
\end{aligned}$$

The weak continuity of $\{H_t\colon t \geq s\}$ now implies

$$\int 1(w^s \notin S(H_s))H_t(dw) = 0 \quad \forall\, t \geq s\ \mathbb{Q}_m\text{-a.s. } \forall\, s \geq 0,$$

and from this we get

$$\forall\, t \geq s \text{ and } w \in S(H_t) \text{ one has } w^s \in S(H_s)\ \mathbb{Q}_m\text{-a.s. } \forall\, s \geq 0. \qquad \square$$

REMARK 2.3. Proposition 8.11 of Dawson and Perkins (1991) gives (b) for all $s \geq 0$ simultaneously w.p.1. The above result suffices for our needs, and the simple proof immediately extends to discontinuous branching mechanisms by appealing to the right-continuity of H instead of continuity (see Section 4).

Here is the key result we need to prove Theorem 1.

PROPOSITION 2.4. *If $d \geq 5$ and $t > 0$, there is a $\mathcal{G}_t^X \times \mathcal{B}_d$-measurable map $\varphi\colon \Omega_X \times \mathbb{R}^d \to C$ such that $\varphi(X, y(t)) = y$ $\widehat{\mathbb{Q}}_t$-a.s.*

PROOF. We may take $t = 1$ to simplify the notation. Let $\mathcal{K}$ denote the space of nonempty compact subsets of $\mathbb{R}^d$ with the Hausdorff metric. By Stroock and Varadhan (1979), Theorem 12.10.1 (take $E = \mathcal{K}$ and $K_q = q$), there exists a $\mathcal{B}(\mathcal{K})$ measurable map $\psi\colon \mathcal{K} \to \mathbb{R}^d$ such that $\psi(K) \in K$ for all $K \in \mathcal{K}$.

Since $d > 4$ we may choose $\alpha \in (0,1)$ such that

$$2 + \alpha d/2 - d/2 < 0. \tag{2.2}$$

Let $\varepsilon_n = 2^{-n\alpha}$ and let m_n be the smallest integer for which $m_n 2^{-n} \geq \varepsilon_n$. Write $h_n = h(2^{-n})$. Define random variables $Y^n_{i2^{-n}}$ and $\widetilde{Y}^n_{i2^{-n}}$ on $\widehat{\Omega}$ by backwards induction on $i \in \{m_n, m_n+1, \dots, 2^n\}$ as follows:

$$Y^n(1) = \widetilde{Y}^n(1) = y(1),$$

$$K_{i,n} = S(X_{i2^{-n}}) \cap \overline{B}(Y^n_{(i+1)2^{-n}}, 3h_n) \in \mathcal{K},$$

$$\widetilde{K}_{i,n} = S(\widetilde{X}^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}) \cap \overline{B}(\widetilde{Y}^n_{(i+1)2^{-n}}, 3h_n) \in \mathcal{K},$$

$$Y^n_{i2^{-n}} = \begin{cases} \psi(K_{i,n}), & \text{if } K_{i,n} \neq \emptyset, \\ 0, & \text{if } K_{i,n} = \emptyset. \end{cases}$$

$$\widetilde{Y}^n_{i2^{-n}} = \begin{cases} \psi(\widetilde{K}_{i,n}), & \text{if } \widetilde{K}_{i,n} \neq \emptyset, \\ y(i2^{-n}), & \text{if } \widetilde{K}_{i,n} = \emptyset. \end{cases}$$

The following properties are immediate:

(2.3) If $\widetilde{Y}^n_{(i+1)2^{-n}} = Y^n_{(i+1)2^{-n}}$ and both $\widetilde{X}^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\left(B(\widetilde{Y}^n_{(i+1)2^{-n}}, 3h_n)\right) > 0$ and $X^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\left(\overline{B}(\widetilde{Y}^n_{(i+1)2^{-n}}, 3h_n)\right) = 0$, then $K_{i,n} = \widetilde{K}_{i,n}$ and so $\widetilde{Y}^n_{i2^{-n}} = Y^n_{i2^{-n}}$.

(2.4) $Y^n_{i2^{-n}} = f_{i,n}(X, y(1))$ for some $\mathcal{G}^X_t \times \mathcal{B}_d$-measurable map $f_{i,n}\colon \Omega_X \times \mathbb{R}^d \to \mathbb{R}^d$.

(2.5) $\widetilde{Y}^n_{i2^{-n}} = \widetilde{f}_{i,n}(\widetilde{H}^{(i2^{-n}-\varepsilon_n)}, y)$ for some $\widehat{\mathcal{G}}_1$-measurable $\widetilde{f}_{i,n}\colon \widehat{\Omega} \to \mathbb{R}^d$.

For (2.5) note that if $s \leq r \leq 1$ and $u \geq r$, then $\widetilde{H}^{(r)}_u$ is a measurable function of $\widetilde{H}^{(s)}_u$ and y^r.

We now define the paths $Y^n(t)$, $\widetilde{Y}^n(t)$ by interpolation:

$$Y^n(t) = Y^n(m_n 2^{-n}) 1(t \leq m_n 2^{-n}) + \sum_{i=m_n+1}^{2^n} Y^n_{i2^{-n}} 1_{((i-1)2^{-n}, 2^{-n}]}(t),$$

$$\widetilde{Y}^n(t) = \widetilde{Y}^n(m_n 2^{-n}) 1(t \leq m_n 2^{-n}) + \sum_{i=m_n+1}^{2^n} \widetilde{Y}^n_{i2^{-n}} 1_{((i-1)2^{-n}, i2^{-n}]}(t).$$

Theorem 8.7 of Dawson and Perkins (1991) and the trivial observation $y \in S(H_1)$ $\widehat{\mathbb{Q}}_1$-a.s. imply

(2.6) For $\widehat{\mathbb{Q}}_1$-a.a. (H, y) there exist $\delta(H) > 0$ such that for all $s \leq 1$, $w \in S(H_s)$ and $u \leq v \leq s$ satisfying $v - u < \delta(H)$ one has $|w(u) - w(v)| < 3h(v-u)$. In particular, this conclusion holds for $s = 1$ and $w = y$.

LEMMA 2.5. *For $\widehat{\mathbb{Q}}_1$-a.a. (H, y) satisfying $\delta(H) > 2\varepsilon_n$,*

$$\sup_{s \le 1} \left|\widetilde{Y}^n(s) - y(s)\right| \le 11h(\varepsilon_n). \tag{2.7}$$

PROOF. Consider first $s = i2^{-n} \ge m_n 2^{-n}$. If $\widetilde{Y}^n = y(s)$ there is nothing to prove so assume otherwise. Then $s < 1$ and $\widetilde{Y}^n_{i2^{-n}} \in S(\widetilde{X}^{(i2^{-n}-\varepsilon_n)}(i2^{-n}))$. As in the proof of Dawson and Perkins (1991), Theorem 8.10(b) [recall (2.1)], one easily sees that

$$S\big(\widetilde{X}^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\big) = \pi_{i2^{-n}}\Big(S\big(\widetilde{H}^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\big)\Big) \quad \forall\, i \ge m_n \text{ and } n \in \mathbb{N}\ \widehat{\mathbb{Q}}_1\text{- a.s.}, \tag{2.8}$$

where π_t: $C \to \mathbb{R}^d$ are the projection maps. Therefore off a $\widehat{\mathbb{Q}}_1$-null set there is a $w \in S(\widetilde{H}^{(i2^{-n}-\varepsilon_n)}(i2^{-n})) \subset S(H(i2^{-n}))$ such that $\widetilde{Y}^n_{i2^{-n}} = w(i2^{-n})$ and $w^{i2^{-n}-\varepsilon_n} = y^{i2^{-n}-\varepsilon_n}$. This together with (2.6) shows that, $\widehat{\mathbb{Q}}_1$-a.s. on $\{\delta(H) > \varepsilon_n\}$,

$$\begin{aligned}\left|\widetilde{Y}^n_{i2^{-n}} - y(i2^{-n})\right| &\le \left|w(i2^{-n}) - w(i2^{-n} - \varepsilon_n)\right| \\ &\quad + \left|y(i2^{-n}) - y(i2^{-n} - \varepsilon_n)\right| < 6h(\varepsilon_n).\end{aligned} \tag{2.9}$$

Another application of (2.6) gives

$$\sup_{m_n 2^{-n} \le s \le 1} \left|\widetilde{Y}^n(s) - y(s)\right| < 9h(\varepsilon_n), \quad \widehat{\mathbb{Q}}_1\text{-a.s. on } \big\{\delta(H) > \varepsilon_n\big\}.$$

Note that $m_n 2^{-n} < 2\varepsilon_n$. Therefore if $s < m_n 2^{-n}$, then $\widehat{\mathbb{Q}}_1$-a.s. on $\{\delta(H) > 2\varepsilon_n\}$ one has

$$\begin{aligned}\left|\widetilde{Y}^n(s) - y(s)\right| &\le \left|\widetilde{Y}^n(m_n 2^{-n}) - y(m_n 2^{-n})\right| + \left|y(m_n 2^{-n}) - y(s)\right| \\ &< 6h(\varepsilon_n) + 3h(2\varepsilon_n) \quad \big[\text{by (2.9) and (2.6)}\big] \\ &< 11h(\varepsilon_n),\end{aligned}$$

proving (2.7). □

Dawson and Perkins (1991), Proposition 8.10(b), Lemma 2.2(b) and (2.1) imply that $\widehat{\mathbb{Q}}_1$-a.s.:

(2.10) If $0 \le s \le t \le 1$ with s rational, and $w \in S(H_t)$, then $w^s \in S(H_s)$ and $w_s \in S(X_s)$. In particular, $y^s \in S(H_s)$ and $y_s \in S(H_s)$ for all $s \in [0, 1] \cap \mathbb{Q}$.

Lemma 2.2(a) and (2.1) imply

$$\forall\, t \in \mathbb{Q} \cap [0, 1]\ \forall\, s < t\ H_t\big(\{w' \colon (w')^s = w^s\}\big) > 0 \quad \forall\, w \in S(H_t)\ \widehat{\mathbb{Q}}_1\text{-a.s.} \tag{2.11}$$

Fix H, y outside a $\widehat{\mathbb{Q}}_1$-null set so that (2.6), (2.8), (2.10) and (2.11) hold and $y \in S(H_1)$. Choose n so that $2^{-n} < \delta(H)$. We claim that

(2.12) if $t \in \mathbb{Q} \cap [0, 1]$, $0 \le s \le t$, and $w \in S(H_t)$ satisfies $w^s = y^s$, then $w \in S\big(\widetilde{H}^{(s)}_t\big)$.

If s, t and w satisfy the hypotheses of (2.12), then (2.11) shows that, for all $u \in (s,t)$, $\widetilde{H}_t^{(s)}(\{\widetilde{w} \in C: \widetilde{w}^u = w^u\}) > 0$. Let $u \uparrow t$ and use (2.6) to conclude that $w \in S(\widetilde{H}_t^{(s)})$. This proves the claim.

Statement (2.10) allows us to apply (2.12) to $w = y^t$ and conclude

$$y^t \in S\big(\widetilde{H}_t^{(s)}\big) \quad \forall\, s < t\ \forall\, t \in \mathbb{Q} \cap [0,1]. \tag{2.13}$$

The definition of $\widetilde{Y}^n$, (2.8) and (2.13) show that

$$\widetilde{Y}^n_{(i+1)2^{-n}} \in \pi_{(i+1)2^{-n}}\Big(S\big(\widetilde{H}^{((i+1)2^{-n}-\varepsilon_n)}_{((i+1)2^{-n})}\big)\Big) \quad \text{for } m_n \le i < 2^n.$$

For i as above we may choose $w \in S(H((i+1)2^{-n}))$ such that $w^{(i+1)2^{-n}-\varepsilon_n} = y^{(i+1)2^{-n}-\varepsilon_n}$ and $w((i+1)2^{-n}) = \widetilde{Y}^n_{(i+1)2^{-n}}$. Next, (2.10) shows that $w^{i2^{-n}} \in S(H(i2^{-n}))$, and then (2.12) and (2.8) imply that $w(i2^{-n}) \in S(\widetilde{X}^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}})$. Statement (2.6) shows that $|w(i2^{-n}) - \widetilde{Y}^n_{(i+1)2^{-n}}| < 3h_n$ and therefore we have

$$\widetilde{X}^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\Big(B\big(\widetilde{Y}^n_{(i+1)2^{-n}}, 3h_n\big)\Big) > 0 \quad \forall\, m_n \le i < 2^n \text{whenever } 2^{-n} < \delta(H)\ \widehat{\mathbb{Q}}_1\text{-a.s.}$$

This, (2.3) and a backwards induction on i show that up to $\widehat{\mathbb{Q}}_1$-null sets

$$\begin{aligned}&\bigcap_{i=m_n}^{2^n-1}\Big\{(H,y): X^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\Big(\overline{B}\big(\widetilde{Y}^n_{(i+1)2^{-n}}, 3h_n\big)\Big) = 0\Big\} \cap \big\{\delta(H) > 2^{-n}\big\}\\ &\qquad \subset \big\{(H,y): \widetilde{Y}^n(s) = Y^n(s) \quad \forall\, s \in [0,1]\big\} \cap \big\{\delta(H) > 2^{-n}\big\}.\end{aligned} \tag{2.14}$$

We are now ready for the key estimate. If $m_n \le i < 2^n$, then

$$\begin{aligned}&\widehat{\mathbb{Q}}_1\Big(X^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\Big(\overline{B}\big(\widetilde{Y}^n_{(i+1)2^{-n}}, 3h_n\big)\Big) > 0\Big)\\ &\quad = \int \widehat{\mathbb{Q}}_1\Big(X^{(i2^{-n}-\varepsilon_n)}_{i2^{-n}}\Big(\overline{B}\big(\widetilde{Y}^n_{(i+1)2^{-n}}(H,y), 3h_n\big)\Big) > 0\\ &\qquad\qquad \mid \widehat{\mathcal{G}}_{i2^{-n}-\varepsilon_n} \vee \sigma\big(\widetilde{H}^{(i2^{-n}-\varepsilon_n)}\big)\Big)(H,y)\, d\widehat{\mathbb{Q}}_1(H,y).\end{aligned} \tag{2.15}$$

Here we have used the fact that $\widetilde{Y}^n_{(i+1)2^{-n}}$ is $\sigma(\widetilde{H}^{((i+1)2^{-n}-\varepsilon_n)}) \vee \sigma(y)$-measurable [see (2.5)] and hence also $\sigma(\widetilde{H}^{(i2^{-n}-\varepsilon_n)}) \vee \sigma(y)$-measurable. By Theorem 2.1 the conditional probability in the above integral equals, using Dawson and Perkins (1991), Theorem 2.2.4

$$\begin{aligned}&\widehat{\mathbb{Q}}_{i2^{-n}-\varepsilon_n, H(i2^{-n}-\varepsilon_n)}\Big(X_{i2^{-n}}\Big(\overline{B}\big(\widetilde{Y}^n_{(i+1)2^{-n}}(H,y), 3h_n\big)\Big) > 0\Big)\\ &\quad = \mathbb{P}_{X(i2^{-n}-\varepsilon_n)}\Big(X_{\varepsilon_n}\Big(\overline{B}\big(\widetilde{Y}^n_{(i+1)2^{-n}}(H,y), 3h_n\big)\Big) > 0\Big)\\ &\quad \le c_{2.1}(3h_n)^{d-2}\varepsilon_n^{-d/2} X_{i2^{-n}-\varepsilon_n}(\mathbb{R}^d).\end{aligned} \tag{2.16}$$

In the last inequality we have used the hitting estimate Theorem 3.1(a) in Dawson, Iscoe and Perkins (1989). Returning to (2.15), we have

$$\begin{aligned}&\widehat{\mathbb{Q}}_1\left(\bigcup_{i=m_n}^{2^n-1}\left\{X_{i2^{-n}}^{(i2^{-n}-\varepsilon_n)}\left(\overline{B}\left(\widetilde{Y}^n_{(i+1)2^{-n}}, 3h_n\right)\right) > 0\right\}\right)\\&\quad\le c_{2.2}2^{-nd/2+n+nd\alpha/2}n^{d/2-1}\sum_{i=m_n}^{2^n}\mathbb{P}_m\left(X_1(1)X_{i2^{-n}-\varepsilon_n}(1)\right)\\&\quad\le c_{2.3}n^{d/2-1}2^{(2+d\alpha/2-d/2)n},\end{aligned}$$

where $c_{2.3}$ may depend on $m(\mathbb{R}^d)$. This bound is summable in n by (2.2). A Borel–Cantelli argument and (2.14) therefore show that

$$\widetilde{Y}^n(s) = Y^n(s) \quad \text{for all } s \in [0,1], \quad \text{for a large } n\ \widehat{\mathbb{Q}}_1\text{-a.s.} \tag{2.17}$$

The property (2.4) shows that $Y^n = \varphi_n(X, y(1))$ for a $\mathcal{G}_1^X \times \mathcal{B}_d$-measurable $\varphi_n\colon \Omega_X \times \mathbb{R}^d \to D([0,1], \mathbb{R}^d)$. Define

$$\varphi(X,x) = \begin{cases}\lim_{n\to\infty}\varphi_n(X,x), & \text{if it converges uniformly on } [0,1] \text{ to a limit in } C([0,1],\mathbb{R}^d),\\ 0, & \text{otherwise.}\end{cases}$$

Then (2.7) and (2.17) imply $y = \varphi(X, y(1))$ $\widehat{\mathbb{Q}}_1$ -a.s. and the proof is complete. □

PROOF OF THEOREM 1.1(a). Fix $t > 0$ and let φ be as in Proposition 2.4. If $A \in \mathcal{C}$, then

$$\begin{aligned}H_t(A) &= \int 1_A\left(\varphi(X, y(t))\right)H_t(dy) \quad \mathbb{Q}_m\text{-a.s.}\\&= \int 1_A(\varphi(X,x))X_t(dx).\end{aligned}$$

Therefore $H_t(A)$ is $\mathcal{F}_t^X$-measurable. The continuity of H now shows that $\mathcal{F}_t^H \subseteq \mathcal{F}_t^X$. □

3. An extended Markov property for the Campbell measure. Our goal in this section is the proof of Theorem 2.1. While the key construction is essentially that of Evans (1993), Theorem 2.2, there are enough differences to warrant a separate argument.

DEFINITION. Let $(E_s, \mathcal{E}_s)_{s\ge 0}$ be a collection of topological spaces with their Borel σ-fields. A collection of maps $P_{s,t}\colon E_s \times \mathcal{E}_t \to [0,1]$, $0 \le s \le t < \infty$, is a Markov semigroup on $\{E_s\}$ if and only if for all $s \le t \le u$,

(i) $P_{s,t}(x, \cdot)$ is a probability on $(E_t, \mathcal{E}_t)$ for all $x \in E_s$;
(ii) $P_{s,t}(\cdot, A)$ is $\mathcal{E}_s$-measurable for all $A \in \mathcal{E}_t$;

(iii) $\int P_{t,u}(y,A)P_{s,t}(x,dy) = P_{s,u}(x,A)$ for all $(x,A) \in E_s \times \mathcal{E}_u$.

We also use $P_{s,t}$ to denote the induced semigroup of operators $P_{s,t}: b\mathcal{E}_t \to b\mathcal{E}_s$.

Let $\{Q_{s,t}: 0 \le s \le t < \infty\}$ denote the Markov semigroup on $\{M_F(C)^s: s \ge 0\}$ associated with the laws of the historical Brownian motion $\{\mathbb{Q}_{s,m}\}$ and let $V_{s,t}: bp\mathcal{C} \to bp\mathcal{C}$ denote the nonlinear semigroup satisfying

$$\mathbb{Q}_{s,m}\Big(\exp\big(-H_t(\varphi)\big)\Big) = \exp\big(-m(V_{s,t}\varphi)\big) \quad \forall\, \varphi \in bp\mathcal{C},\ m \in M_F(C)^s. \tag{3.1}$$

See Dawson and Perkins (1991), Theorem 2.2.3, and note we are setting $V_{s,t}\,\varphi(y) = V_{s,t}\varphi(y^s)$ for all $y \in C$. If $\tau \ge 0$ and $y \in C^\tau$, $P_{\tau,y}$ is the probability on $(C, \mathcal{C})$ given by

$$P_{\tau,y}(A) = P_{y(\tau)}\Big(\big\{w \in C: y/\tau/w \in A\big\}\Big),$$

where P_x is a d-dimensional Wiener measure starting at x and $y/s/w$ is the $P_{y(s)}$-a.s. continuous path given by

$$\big(y/s/w\big)(t) = \begin{cases} y(t), & \text{if } t < s, \\ w(t-s), & \text{if } t \ge s. \end{cases}$$

If $m \in M_F(C)^\tau$, let $P_{\tau,m} = \int P_{\tau,y} m(dy)$. Let $T_{r,t}$ denote the Markov semigroup on $\{C^s: s \ge 0\}$ given by $T_{r,t}\varphi(y) = \int \varphi(w^t)dP_{r,y}(w)$. For $0 \le s \le t$ and $y \in C^t$, let $(R_{s,t})_y$ be the unique probability on $M_F(C)$, with its Borel sets $\mathcal{M}$, such that

$$\int e^{-\mu(\varphi)}(R_{s,t})_y(d\mu) = \exp\Big(-\int_s^t V_{r,t}\varphi(y^r)\,dr\Big) \quad \text{for all } \varphi \in bp\mathcal{C}. \tag{3.2}$$

The explicit construction in Dawson and Perkins (1991), Proposition 4.1.5, shows the right-hand side of (3.2) is the Laplace functional of a random measure on C.

LEMMA 3.1. *For $(R_{s,t})_y$-a.a. μ:*

(a) $w^s = y^s$ *for* μ-a.a. w;
(b) $w = w^t$ *for* μ-a.a. w.

PROOF. (a) Let $y \in C$ be fixed and set $\varphi(w) = 1(w^s \ne y^s)$. In view of (3.2) it suffices to show $V_{r,t}\varphi(y^r) = 0$ for all $r \in [s,t]$, and (3.1) shows this is equivalent to

$$\mathbb{Q}_{r,\delta_{y^r}}\big(H_t(\varphi)\big) = 0 \quad \forall\, s \le r \le t.$$

The left-hand side of the above equals $T_{r,t}\varphi(y^r) = 0$ by the superprocess property [Dawson and Perkins (1991), Theorem 2.1.5(d)].

(b) This may be proved in a similar way with $\varphi(w) = 1(w \ne w^t)$. □

DEFINITION. If $y \in C$, $0 \le r \le s < \infty$ and $m \in M_F(C)^r$, let $U^y_{r,s}(m, \cdot)$ be the probability on $(M_F(C), \mathcal{M})$ given by

$$U^y_{r,s}(m, \cdot) = Q_{r,s}(m, \cdot) * (R_{r,s})_y.$$

(Here $*$ denotes convolution of measures.)

Lemma 3.1(b) shows that $U^y_{r,s}(m, \cdot)$ is supported by $M_F(C)^s$.

LEMMA 3.2. *For each $y \in C$, $\{U^y_{r,s}: 0 \le r \le s < \infty\}$ is a Markov semigroup on $\{M_F(C)^s: s \ge 0\}$. For any bounded Borel function ψ on $M_F(C)^s$ and $0 \le r \le s$, $U^y_{r,s}(m, \psi)$ is jointly Borel measurable in $(y, m) \in C \times M_F(C)^r$.*

PROOF. The last assertion is clear. To check the semigroup property, let $0 \le r < s < t$, $\varphi \in p\mathcal{C}$ and set $e_\varphi(\mu) = \exp\{-\mu(\varphi)\}$ for $\mu \in M_F(C)$. Then

$$\begin{aligned} U^y_{r,s}\big(U^y_{s,t}e_\varphi\big)(m) &= U^y_{r,s}\left(e_{V_{s,t}\varphi} \exp\left(-\int_s^t V_{u,t}\varphi(y^u)du\right)\right)(m) \\ &= e_{V_{r,s} \circ V_{s,t}\varphi}(m) \exp\left(-\int_r^s V_{u,s} \circ V_{s,t}\varphi(y^u)\,du - \int_s^t V_{u,t}\varphi(y^u)\,du\right) \\ &= U^y_{r,t}(e_\varphi)(m). \end{aligned}$$

□

DEFINITION. Given $t \in [0, \infty)$ and $y \in C^t$, define a Markov semigroup $\{U^{y,t}_{r,s}: 0 \le r \le s < \infty\}$ on $\{M_F(C)^s: s \ge 0\}$ by

$$U^{y,t}_{r,s} = \begin{cases} U^y_{r,s}, & \text{if } r \le s \le t, \\ Q_{r,s}, & \text{if } t \le r \le s, \\ U^y_{r,t} \circ Q_{t,s}, & \text{if } r \le t \le s. \end{cases}$$

NOTATION. Let $\overline{K}_t(w) = w(t)$, $t \ge 0$, denote the coordinate mappings on $\overline{\Omega} = M_F(C)^{[0,\infty)}$. If I is a subinterval of $[0, \infty)$, $\mathcal{G}(I) = \sigma(\overline{K}_t: t \in I)$. Clearly any probability $\mathbb{Q}$ on $(\Omega, \mathcal{G}(I))$ induces a probability $\overline{\mathbb{Q}}$ on $(\overline{\Omega}, \overline{\mathcal{G}}(I))$ via

$$\overline{\mathbb{Q}}\big(\overline{K}_{t_i} \in A_i,\ i = 1, \dots, n\big) = \mathbb{Q}\big(H_{t_i} \in A_i,\ i = 1, \dots, n\big). \tag{3.3}$$

If $\overline{\mathbb{Q}}$ is a given law on $(\overline{\Omega}, \overline{\mathcal{G}}(I))$, a law $\mathbb{Q}$ on $(\Omega, \mathcal{G}(I))$ satisfying (3.3) is unique if it exists (it need not) and is called the extension of $\overline{\mathbb{Q}}$ to $(\Omega, \mathcal{G}(I))$.

DEFINITION. If $\tau \ge 0$, $m \in M_F(C)^\tau$, $t \ge 0$ and $y \in C^t$, let $\overline{\mathbb{P}}_{\tau,m,y,t}$ be the unique law on $(\overline{\Omega}, \overline{\mathcal{G}}([\tau, \infty)))$ under which $(\overline{K}_t: t \ge \tau)$ is a Markov process with semigroup $\{U^{y,t}_{r,s}: r \le s\}$ and $\overline{K}_\tau = m$ a.s.

LEMMA 3.3. *If* $\overline{K}^i(t)$, $i = 1, 2$, $t \geq 0$, *denote the coordinate variables on* $\overline{\Omega}^2$, *then*

$$\overline{\mathbb{P}}_{\tau, m_1, y, t} \times \overline{\mathbb{Q}}_{\tau, m_2}\big(\overline{K}^1 + \overline{K}^2 \in \cdot\big) = \overline{\mathbb{P}}_{\tau, m_1+m_2, y, t}(\cdot)$$

for all $m_1, m_2 \in M_F(C)^\tau$, $y \in C$ *and* $\tau, t \geq 0$.

PROOF. Let $\tau \leq u \leq s \leq t$ and $\varphi \in p\mathcal{C}$. Then

$$\begin{aligned}\overline{\mathbb{P}}_{\tau, m_1, y, t} \times \overline{\mathbb{Q}}_{\tau, m_2} &\Big(\exp\Big(-(\overline{K}^1_s + \overline{K}^2_s)(\varphi)\Big)\Big|\overline{K}^1_r, \overline{K}^2_r,\ r \leq u\Big)\\ &= U^y_{u,s}(e_\varphi)\big(\overline{K}^1_u\big)Q_{u,s}(e_\varphi)\big(\overline{K}^2_u\big)\\ &= Q_{u,s}(e_\varphi)\big(\overline{K}^1_u\big)(R_{u,s})_{y^s}(e_\varphi)Q_{u,s}(e_\varphi)\big(\overline{K}^2_u\big)\\ &= U^{y,t}_{u,s}(e_\varphi)\big(\overline{K}^1_u + \overline{K}^2_u\big)\end{aligned}$$

by the multiplicative property of $Q_{u,s}$ [see (3.1)]. The corresponding identity for $s \geq u \geq t$ is easier and left for the reader. The result follows. □

DEFINITION. The existence of $\widehat{\mathbb{Q}}_{r,m,t}$ regular conditional probabilities shows there is a collection of probabilities $\{Q_{\tau,m,t}(y): y \in C^t\}$ on $(\Omega, \mathcal{G}([\tau, \infty)))$ such that $Q_{\tau,m,t}(y)(A)$ is Borel measurable in y for each $A \in \mathcal{G}([\tau, \infty))$ and

$$\int_B Q_{\tau,m,t}(y^t)(A)P_{\tau,m}(dy) = \mathbb{Q}_{\tau,m}\big(1_A(H)H_t(B)\big) \quad \forall A \in \mathcal{G}\big([\tau, \infty)\big),\ B \in \mathcal{C}.$$

The collection $\{Q_{\tau,m,t}(y): y \in C^t\}$ are the Palm measures associated with $\mathbb{Q}_{\tau,m}$ at time t. The collection is unique up to $P_{\tau,m}(y^t \in \cdot)$-null sets.

Here then is our modified version of Evans (1993), Theorem 2.7(ii).

THEOREM 3.4. *Let* $t \geq \tau \geq 0$ *and* $m \in M_F(C)^\tau$. *For* $P_{\tau,m}$-*a.a.* y, $\overline{\mathbb{P}}_{\tau,m,y^t,t}$ *has an extension* $\mathbb{P}_{\tau,m,y^t,t}$ *to* $(\Omega, \mathcal{G}([\tau, \infty)))$. *These extensions satisfy* $\mathbb{P}_{\tau,m,y^t,t} = Q_{\tau,m,t}(y^t)$ *for* $P_{\tau,m}$-*a.a.* y, *and if* $m_1, m_2 \in M_F(C)^\tau$, *then*

$$\mathbb{P}_{\tau,m_1,y^t,t} \times \mathbb{Q}_{\tau,m_2}(H^1 + H^2 \in \cdot) = \mathbb{P}_{\tau,m_1+m_2,y^t,t}(\cdot) \quad \textit{for } P_{\tau,m_1}\text{-}a.a.\ y, \tag{3.4}$$

where (H^1, H^2) *denote the coordinate variables on* Ω^2.

PROOF. Theorem 4.1.1 and 4.1.3 of Dawson and Perkins (1991) imply

$$\mathbb{Q}_{r,m}\big(H_s(\psi)\Phi(H_s)\big) = \int U^{w^s}_{r,s}(m, \Phi)\psi(w^s)P_{r,m}(dw) \tag{3.5}$$

for all $r \leq s$, $m \in M_F(C)^r$, $\Phi \in b\mathcal{M}$, $\psi \in b\mathcal{C}$.

Let $\psi \in b\mathcal{C}$, $A_i \in \mathcal{M}$ for $i = 1, \ldots, n$ and $\tau \le s_1 < \cdots < s_n \le t$. Then

$$\begin{aligned}
&\int Q_{\tau,m,t}(y^t)\big(H_{s_i} \in A_i,\ i \le n\big)\psi(y^t)P_{\tau,m}(dy)\\
&= \mathbb{Q}_{\tau,m}\Big(1\big(H_{s_i} \in A_i,\ i < n\big)\mathbb{Q}_{s_{n-1},H_{s_{n-1}}}\Big(1\big(H_{s_n} \in A_n\big)H_{s_n}\big(T_{s_n,t}\psi\big)\Big)\Big)\\
&\qquad\qquad\qquad\qquad\text{(Markov and superprocess properties)}\\
&= \mathbb{Q}_{\tau,m}\Big(1\big(H_{s_i} \in A_i,\ i < n\big)\\
&\quad\times \int\int U^{w^{s_n}}_{s_{n-1},s_n}\big(H_{s_{n-1}}, A_n\big)T_{s_n,t}\psi(w^{s_n})P_{s_{n-1},y}(dw)H_{s_{n-1}}(dy)\Big) \quad [\text{by } (3.5)]\\
&= \int\int\int 1\big(H_{s_i} \in A_i,\ i < n\big)U^{w^{s_n}}_{s_{n-1},s_n}\big(H_{s_{n-1}}, A_n\big)\,dQ_{\tau,m,s_{n-1}}(y^{s_{n-1}})\\
&\quad\times T_{s_n,t}\psi(w^{s_n})P_{s_{n-1},y}(dw)P_{\tau,m}(dy)\\
&= \int\int 1\big(H_{s_i} \in A_i,\ i < n\big)U^{y^{s_n}}_{s_{n-1},s_n}\big(H_{s_{n-1}}, A_n\big)\,dQ_{\tau,m,s_{n-1}}\big(y^{s_{n-1}}\big)\\
&\quad\times \psi(y^t)P_{\tau,m}(dy).
\end{aligned}$$

The above implies

$$\begin{aligned}
&Q_{\tau,m,t}(y^t)\big(H_{s_i} \in A_i,\ i \le n\big)\\
&\quad= Q_{\tau,m,s_{n-1}}(y^{s_{n-1}})\Big(1\big(H_{s_i} \in A_i,\ i < n\big)U^{y^{s_n}}_{s_{n-1},s_n}\big(H_{s_{n-1}}, A_n\big)\Big)\\
&\qquad\qquad\text{for } P_{\tau,m}\text{-a.a. } y.
\end{aligned} \tag{3.6}$$

Let $A_n = C$ and each A_i, $i < n$, range over a countable determining class in $\mathcal{M}$ to conclude that $(H_{s_1}, \ldots, H_{s_{n-1}})$ has the same distribution under $Q_{\tau,m,t}(y^t)$ and $Q_{\tau,m,s_{n-1}}(y^{s_{n-1}})$ for $P_{\tau,m}$-a.a. y. Use this in (3.6) and let A_i, $i < n$, range over a countable determining class to see that, for $P_{\tau,m}$-a.a. y,

$$\begin{aligned}
&Q_{\tau,m,t}(y^t)\big(H_{s_n} \in A_n \,|\, H_{s_1}, \ldots, H_{s_{n-1}}\big)\\
&\quad= U^{y^{s_n}}_{s_{n-1},s_n}\big(H_{s_{n-1}}, A_n\big) \quad Q_{\tau,m,t}(y^t)\text{-a.s.}\\
&\quad= U^{y^t,t}_{s_{n-1},s_n}\big(H_{s_{n-1}}, A_n\big).
\end{aligned} \tag{3.7}$$

Let $\tau \le s_1 < \cdots < s_n = t = u_1 < \cdots < u_k$, where $k > 1$, $A_i, B_i \in \mathcal{M}$ and $\psi \in b\mathcal{C}$. Then, writing $F = \{H_{s_i} \in A_i,\ 1 \le i \le n\}$, we have

$$\begin{aligned}
&\int Q_{\tau,m,t}(y^t)\big(F \cap \{H_{u_j} \in B_j,\ 1 \le j \le k\}\big)\psi(y^t)P_{\tau,m}(dy)\\
&\quad= \mathbb{Q}_{\tau,m}\Big(\big(F \cap \{H_{u_j} \in B_j,\ 1 \le j \le k-1\}\big)H_t(\psi)Q_{u_{k-1},u_k}\big(H_{u_{k-1}}, B_k\big)\Big)\\
&\quad= \int Q_{\tau,m,t}(y^t)\Big(\big(F \cap \{H_{u_j} \in B_j,\ 1 \le j \le k-1\}\big)Q_{u_{k-1},u_k}\big(H_{u_{k-1}}, B_k\big)\Big)\\
&\qquad\times\ \psi(y^t)P_{\tau,m}(dy).
\end{aligned}$$

As above, this easily gives

$$Q_{\tau,m,t}(y^t)\big(H_{u_k} \in B_k \mid H_{s_1},\dots,H_{s_n}, H_{u_1},\dots,H_{u_{k-1}}\big) = Q_{u_{k-1},u_k}(H_{u_{k-1}}, B_k),\ Q_{\tau,m,t}(y^t)\text{-a.s. for } P_{\tau,m}\text{-a.a. } y. \tag{3.8}$$

(3.7) and (3.8) show that $\overline{Q_{\tau,m,t}(y^t)} = \overline{\mathbb{P}}_{\tau,m,y^t,t}$ on $\overline{\mathcal{G}}([\tau,\infty))$ for $P_{\tau,m}$-a.a. y. Hence off this $P_{\tau,m}$-null set of y's, $Q_{\tau,m,t}(y^t)$ is the required extension of $\overline{\mathbb{P}}_{\tau,m,y^t,t}$ to $\mathcal{G}([\tau,\infty))$. The last assertion is now immediate from Lemma 3.3 and the fact that $P_{\tau,m_1} \le P_{\tau,m_1+m_2}$. □

Recall the definitions of $H^{(s)}$ and $\widetilde{H}^{(s)}$ on $\widehat{\Omega} = \Omega \times C$ given at the beginning of Section 2.

PROPOSITION 3.5. *Assume $m \in M_F(C)^s$ is atomless. For $P_{s,m}$-a.a. y, $H^{(s)}_\cdot(H,y)$ and $\widetilde{H}^{(s)}_\cdot(H,y)$ are continuous on $[s,\infty)$ for $Q_{s,m,t}(y^t)$-a.a. H. If $\varphi_i \in b\mathcal{G}([s,\infty))$ for $i = 1, 2$, then*

$$\int \varphi_1(H^{(s)})\varphi_2(\widetilde{H}^{(s)})\,dQ_{s,m,t}(y^t) = \mathbb{Q}_{s,m}(\varphi_1)\int \varphi_2(\widetilde{H}^{(s)})\,dQ_{s,m,t}(y^t) \quad \text{for } P_{s,m}\text{-a.a. } y. \tag{3.9}$$

PROOF. We work on $(\Omega^2, \mathcal{G}([s,\infty))^2)$ with respect to $\mathbb{P}_y = \mathbb{Q}_{s,m} \times \mathbb{P}_{s,0,y,t}$. Here y has been chosen outside a $P_{s,m}$-null set so that the conclusions of Theorem 3.4 hold with $m_1 = 0$ and $m_2 = m$ in (3.4). Let (H^1_u, H^2_u) denote the coordinate variables on Ω^2 and, abusing notation slightly, set $H = H^1 + H^2$. Lemma 3.1(a) and the continuity of H^2 imply

$$w^s = y^s \quad \forall w \in S(H^2_u)\ \forall u \in [s,t]\ \mathbb{P}_y\text{-a.s.} \tag{3.10}$$

Lemma 2.2 (b) implies

$$w^t \in S(H^2_t) \quad \forall w \in S(H^2_u)\ \forall u \ge t\ \mathbb{P}_y\text{-a.s.}$$

Using this in (3.10), we see that (3.10) holds for all $u \ge s$ $\mathbb{P}_y$-a.a. and therefore

$$H^2_u \le \widetilde{H}^{(s)}_u(H,y) \quad \text{for all } u \ge s\ \mathbb{P}_y\text{-a.s.} \tag{3.11}$$

Results of Fitzsimmons (1988) show that $H^1_u(\{w: w^s = y^s\})$ is a continuous martingale for $u \ge s$ [see Mueller and Perkins (1992), Theorem 2.3(b) or Theorem 2.7]. Since its initial value is $m(\{y^s\}) = 0$ (by the atomless hypothesis), $H^1_u(\{w: w^s = y^s\}) = 0$ for all $u \ge s$ $\mathbb{P}_y$-a.s. and hence

$$H^1_u \le H^{(s)}_u(H,y) \quad \text{for all } u \ge s\ \mathbb{P}_y\text{-a.s.} \tag{3.12}$$

The sum of the left-hand sides of (3.11) and (3.12) equal the sum of the right-hand sides (both equal H_u), and so we must have

$$H^{(s)}_u = H^1_u \quad \text{and} \quad \widetilde{H}^{(s)}_u = H^2_u \quad \text{for all } u \ge s\ \mathbb{P}_y\text{-a.s.} \tag{3.13}$$

Since $\mathbb{P}_y(H \in \cdot\,) = \boldsymbol{Q}_{s,m,t}(y^t)(\,\cdot\,)$ by Theorem 3.4 and the choice of y, this proves the a.s. continuity of $H^{(s)}$ and $\widetilde{H}^{(s)}$. Also, for $\varphi_i \in b\mathcal{G}([s, \infty))$ and y as above,

$$\begin{aligned}\int \varphi_1(H^{(s)})\varphi_2(\widetilde{H}^{(s)})d\boldsymbol{Q}_{s,m,t}(y^t) &= \int \varphi_1(H^1)\varphi_2(H^2)\,d\mathbb{P}_y \quad [\text{by } (3.13)]\\ &= \mathbb{Q}_{s,m}(\varphi_1)\int \varphi_2\,d\mathbb{P}_{s,0,y,t}\\ &= \mathbb{Q}_{s,m}(\varphi_1)\int \varphi_2(\widetilde{H}^{(s)})d\boldsymbol{Q}_{s,m,t}(y^t),\end{aligned}$$

where in the last line we used the previous equality with $\varphi_1 = 1$. □

PROOF OF THEOREM 2.1. The $\widehat{\mathbb{Q}}_t$-a.s. continuity of $H^{(s)}$ and $\widetilde{H}^{(s)}$ is immediate from the continuity established in Proposition 3.5. Let $B \in \mathcal{G}([0, s+])$, $\psi \in b\mathcal{C}$ and $\varphi_i \in b\mathcal{G}([s, \infty))$ for $i = 1, 2$. Then

$$\begin{aligned}&\widehat{\mathbb{Q}}_t\Big(\varphi_1(H^{(s)})\varphi_2(\widetilde{H}^{(s)})\psi(y)1_B\Big)m(C)\\ &= \mathbb{Q}_m\left(1_B\mathbb{Q}_{s,H_s}\left(\int \varphi_1(H^{(s)})\varphi_2(\widetilde{H}^{(s)})\psi(y)H_t(dy)\right)\right)\\ &= \mathbb{Q}_m\left(1_B\int \psi(y^t)\boldsymbol{Q}_{s,H_s,t}(y^t)\Big(\varphi_1(H^{(s)})\varphi_2(\widetilde{H}^{(s)})\Big)P_{s,H_s}(dy)\right)\\ &= \mathbb{Q}_m\left(1_B\int \psi(y^t)\mathbb{Q}_{s,H_s}(\varphi_1)\boldsymbol{Q}_{s,H_s,t}(y^t)\Big(\varphi_2(\widetilde{H}^{(s)})\Big)P_{s,H_s}(dy)\right)\end{aligned}$$

[Proposition 3.5 may be applied because H_s is a.s. atomless by Dawson and Perkins (1991), Proposition 4.1.8]

$$\begin{aligned}&= \mathbb{Q}_m\left(1_B\mathbb{Q}_{s,H_s}(\varphi_1)\int \psi(y)\varphi_2(\widetilde{H}^{(s)})H_t(dy)\right) \quad \text{(Markov property)}\\ &= \widehat{\mathbb{Q}}_t\Big(\mathbb{Q}_{s,H_s}(\varphi_1)\varphi_2(\widetilde{H}^{(s)})\psi(y)1_B\Big)m(C).\end{aligned}$$

The result follows. □

REMARK 3.6. The results and arguments of the section remain valid (with only trivial changes) in the general setting of Section 4.1 of Dawson and Perkins (1991). In particular, H may be a (Y, Φ)-historical process, where Y is a Borel right process with cadlag paths and a Lusin state space, and $\Phi(\lambda) = \gamma\lambda^{\beta+1}$ for some $\gamma > 0$ and $\beta \in (0, 1]$. (In the case of continuous branching considered above, $\gamma = \frac{1}{2}$ and $\beta = 1$.) Precise definitions may be found in Section 2.2 of Dawson and Perkins (1991). We must replace C by the Skorokhod space

$D = D([0, \infty), E)$, Ω by $D([0, \infty), M_F(D))$ and (3.2) by

$$\int e^{-(\mu)(\varphi)} (R_{s,t})_y(d\mu) = \exp\left(-\gamma(1+\beta)\int_s^t \left(V_{r,t}\varphi(y^r)\right)^\beta dr\right). \tag{3.2$'$}$$

Continuity is replaced by right-continuity with left limits in both the proof and conclusion of Proposition 3.5. The only other step which requires comment is the derivation of (3.12). The proof of Theorem 2.3 (b) in Mueller and Perkins (1992) remains valid for $1+\beta$-branching (the continuous branching assumption is not needed there). This shows $H_u^1(\{w: w^s = y^s\})$ is a.s. right-continuous in $u \geq s$ and since it is a.s. 0 for each u by the superprocess property, it is identically 0 for all $u \geq s$ a.s. and (3.12) holds.

4. Discontinuous branching. Consider now the analogue of Theorem 1.1(a) for the (Y, Φ)-historical process, where Y is a d-dimensional Brownian motion and $\Phi(\lambda) = \gamma\lambda^{1+\beta}$ for some $\gamma > 0$ and $\beta \in (0, 1)$. In this case H_t may have jumps and we must work on $D([0, \infty), M_F(C))$ rather than Ω. When the necessary notational changes are made, the following version of Theorem 1.1(a) holds.

THEOREM 4.1. *If $d > 2 + (2/\beta)$, then $\mathcal{F}_t^H = \mathcal{F}_t^X$ for all $t \geq 0$.*

The three key ingredients in the proof of Theorem 1.1(a) are:

(i) The uniform modulus of continuity for all paths in $S(H_t)$ [Dawson and Perkins (1991), Theorem 8.7]—see (2.6).

(ii) The fixed time estimates for X_t to charge a ball of radius r as $r \downarrow 0$ [Dawson, Iscoe and Perkins (1989), Theorem 3.1(a)]—see (2.16).

(iii) Theorem 2.1.

Once analogues of these three results are in place, our proof of Theorem 1.1(a) goes through quite generally. Note that the proof of Lemma 2.2 goes through without change in this setting and [Dawson and Perkins (1991), Theorem 8.10(b)] is valid once an analogue of (i) is established. (This result was used twice in the proof of Theorem 2.1.)

For the $1 + \beta$-branching case, it is straightforward to establish versions of (i) to (iii). We have already noted that Theorem 2.1 holds in this setting (see Remark 3.6). It is easy to modify the proof of Theorem 8.7 in [Dawson and Perkins (1991) to establish (2.6) with $[\varepsilon + (2(1+\beta)\beta^{-1})^{1/2}]h(v-u)$ in place of $3h(v-u)$ for any $\varepsilon > 0$. A slightly different result is proved in Theorem 1.2 of Dawson and Vinogradov (1992). Finally, the p.d.e. arguments of Dawson, Iscoe and Perkins (1988), Theorem 3.1(a), are easy to adapt to the $1+\beta$ setting to give the required analogue of (ii). Some of these arguments, but unfortunately not the particular one needed here, are carried out in Appendix 1 of Dawson and Vinogradov (1992). In our present setting the probability that X_t charges a ball of radius r goes to 0 like $r^{d-(2/\beta)}$ as $r \downarrow 0$ (assuming $d > 2/\beta$). Using this in (2.16), one easily obtains Theorem 4.1.

5. Intersections and the case $d = 1$.

PROOF OF PROPOSITION 1.3. If $h: [0, \varepsilon) \to [0, \infty)$ is nondecreasing for some $\varepsilon > 0$ and $h(0+) = 0$, then, following Taylor and Watson (1985), we define

$$C(t,x,r) = [t, t+r^2] \times \prod_{i=1}^{d} [x_i, x_i + r] \quad \text{for } t \geq 0,\ x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

$$q^h(A) = \liminf_{\delta \downarrow 0} \left\{ \sum_{i=1}^{\infty} h(r_i): A \subset \bigcup_{i=1}^{\infty} C(t_i, x_i, r_i),\ r_i < \delta \right\} \quad \text{for } A \subset [0,\infty) \times \mathbb{R}^d.$$

Consider first $d \geq 4$. Let $h(r) = r^4 \log^+ \log^+(1/r)$. Theorem 3.1 of Barlow, Evans and Perkins (1991) shows that $q^h(\overline{G}(X)) < \infty$. Actually, the above theorem gives a two-sided bound for $d \geq 5$, but the proof for the upper bound estimate on $q^h(\overline{G}(X))$ goes through unchanged for $d = 4$ as well. Therefore $q^{r^d}(\overline{G}(X)) = 0$, and (1.1) follows from Theorem 1 of Taylor and Watson (1985).

The existence of a jointly continuous density $\{u(t,x): t > 0, x \in \mathbb{R}^d\}$ for X when $d = 1$ [Konno and Shiga (1988) or Reimers (1989)] makes (1.2) obvious, so let us assume $d = 2$ or 3. Let $f(r) = r^4(1 + \log^+(1/r))^{4-d}$, $G = G(X) \cap ([1,2] \times \mathbb{R}^d)$ and $Y(A) = \int_1^2 \int 1_A(s,x) X_s(dx)\, ds$. Corollary 4.8 of Barlow, Evans and Perkins (1991) implies that $\mathbb{Q}_m$-a.s. for sufficiently small $r > 0$,

$$\sup_{t \geq 0, x \in \mathbb{R}^d} Y\big(G \cap C(t,x,r)\big) \leq \sup_{t \geq 1, x \in \mathbb{R}^d} r^2 X_t\big(B(x,r)\big) \leq c_{5.1} f(r)$$

for some universal constant $c_{5.1}$. Lemma 2 in Taylor and Watson (1985) therefore shows that $q^f(G) \geq c_{5.2} Y(G)$, where $c_{5.2} > 0$, which is nonzero with positive probability. Theorem 2 of Taylor and Watson (1985) now gives (1.2). □

We now turn to the proof of Theorem 1.1(b). The next result should be compared with Proposition 2.4.

LEMMA 5.1. *Assume $d = 1$, $u \geq 0$ and $m \in M_F(C)^u$ is such that $\overline{\pi}_u(m)$ is not supported by a single point. There is no $\mathcal{B}([u,\infty)) \times \mathcal{F}_\infty^H \times \mathcal{B}_1$-measurable map $g: [u,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that*

(5.1) $\quad y(u) = g\big(s, H, y(s)\big)$ *for H_s-a.a. y and Lebesgue-a.a. $s \geq u$ $\mathbb{Q}_{u,m}$-a.s.*

PROOF. Suppose there is a measurable g satisfying (5.1). Let A_1, A_2 be disjoint Borel sets such that $\mathbb{R} = A_1 \cup A_2$ and $\overline{\pi}_u(m)(A_i) > 0$, $i = 1, 2$. Define $M_F(C)$-valued processes by

(5.2) $$H_t^i(A) = H_t\Big(\big\{y\colon y(u) \in A_i,\ y \in A\big\}\Big), \qquad t \geq u,\ i = 1, 2,$$

and let $m^i(A) = m(\{y \in A: y(u) \in A_i\})$. Then under $\mathbb{Q}_{u,m}$ the processes H^1 and H^2 are independent historical Brownian motions starting at m^1 and m^2, respectively, at time u. To see this, note that if we start with (H^1, H^2) with law $\mathbb{Q}_{u,m^1} \times \mathbb{Q}_{u,m^2}$ and set $H = H^1 + H^2$, then H has law $\mathbb{Q}_{u,m}$ by the multiplicative property and H^i may be recovered from H via (5.2). $X_t^i = \overline{\pi}_{u+t}(H_{u+t}^i)$, $t \geq 0$, $i = 1, 2$, are therefore independent super-Brownian motions starting at $\overline{\pi}_u(m^i)$, $i = 1, 2$ [Dawson and Perkins (1991), Theorem 2.2.4]. (5.1) and (5.2) imply

$$X_t^i(B) = X_{t+u}\Big(\big\{x: g(t+u, H, x) \in A_i, x \in B\big\}\Big)$$
$$\forall B \in \mathcal{B}_1 \text{ Lebesgue-a.a. } t \geq 0, \ \mathbb{Q}_{u,m}\text{-a.s.}$$

and therefore $\Lambda_i(t) = \{x: g(t+u, H, x) \in A_i\}$ is a support for X_t^i for Lebesgue-a.a. $t \geq 0$, $i = 1, 2$, $\mathbb{Q}_{u,m}$-a.s. If $\{u^i(t, x): t > 0, x \in \mathbb{R}\}$ is the jointly continuous density of X_t^i, then, since $\Lambda_1(t) \cap \Lambda_2(t) = \emptyset$, we see that $\int_0^\infty \int u^1(s,x) u^2(s,x)\,dx\,ds = 0$ $\mathbb{Q}_{u,m}$-a.s. On the other hand, the mean value of this integral is [write m_u^i for $\overline{\pi}_u(m^i)$]

$$\int_0^\infty \int \left(\int p(s, x-z) m_u^1(dz)\right)\left(\int p(s, x-z) m_u^2(dz)\right) dx\,ds > 0.$$

This contradiction completes the proof. □

NOTATION. $\mathcal{P}(\mathcal{F}_t)$ denotes the predictable σ-field on $[0, \infty) \times \Omega$ associated with a filtration $(\mathcal{F}_t)$ on Ω.

PROOF OF THEOREM 1.1(b). Suppose that $\mathcal{F}_t^X = \mathcal{F}_t^H$. Let $u > 0$, $\psi(x) = \arctan x$ and $M(t) = \int \psi(y(u)) H_t(dy)$ for $t \geq u$. Let Z^H (respectively, Z^X) be the orthogonal martingale measure on $(C, \mathcal{C})$ associated with H [respectively, on $(\mathbb{R}, \mathcal{B}_1)$ associated with X]. Z^H and its associated stochastic integral is constructed in Perkins (1992)—see the discussion there after Theorem 2.3. It follows from Proposition 2.4 of Perkins (1992) that $(M_t: t \geq u)$ is an $(\mathcal{F}_t^H)_{t \geq u}$-martingale under $\mathbb{Q}_m$ and satisfies

$$M(t) = M(u) + \int_u^t \int \psi\big(y(u)\big)\, dZ^H(s, y) \quad \forall\, t \geq u \ \mathbb{Q}_m\text{-a.s.} \tag{5.3}$$

Extend M_t to $t \in [0, \infty)$ so that it is a cadlag $(\mathcal{F}_t^H)_{t \geq 0}$-martingale. It is therefore also an $(\mathcal{F}_t^X)_{t \geq 0}$-martingale, and the predictable representation property [Evans and Perkins (1994), Theorem 1.2] shows there is a $P(\mathcal{F}_t^X) \times \mathcal{B}_1$-measurable map $f: [0, \infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$M(t) = M(u) + \int_u^t \int f(s, w, x)\, dZ^X(s, x) \quad \forall\, t \geq u, \ \mathbb{Q}_m\text{-a.s.}$$

The obvious connection between Z^X and Z^H [e.g., this is an easy consequence of the trivial case of Perkins (1992), Theorem 3.19] now gives

$$M(t) = M(u) + \int_u^t \int f\big(s, w, y(s)\big)\, dZ^H(s, y) \quad \forall\, t \geq u, \ \mathbb{Q}_m\text{-a.s.} \tag{5.4}$$

Take mean square differences of the right-hand sides of (5.3) and (5.4) to conclude

$$0 = \mathbb{Q}_m\left(\int_u^t \int \Big(\psi\big(y(u)\big) - f\big(s, w, y(s)\big)\Big)^2 H_s(dy)\, ds\right),$$

and therefore, by the Markov property of H,

$$y(u) = \tan\Big(f\big(s, w, y(s)\big)\Big)$$

for H_s-a.a. y and Lebesgue a.a. $s \geq u$ $\mathbb{Q}_{u,H_u}$-a.s. $\mathbb{Q}_m$-a.a. H_u. On the set where $H_u \neq 0$ [and hence $\overline{\pi}_u(H_u)$ is not supported by a single point since it has a nontrivial density], the above contradicts Lemma 5.1 and we are done. □

REFERENCES

BARLOW, M. T., EVANS, S. N. and PERKINS, E. A. (1991). Collision local times and measure-valued processes. *Canad. J. Math.* **43** 897–938.

DAWSON, D. A. (1993). Measure-valued processes. *Ecole d'Eté de Probabilités de Saint-Flour XXI. Lecture Notes in Math.* **1541**. Springer, Berlin.

DAWSON, D. A., ISCOE, I. and PERKINS, E. A. (1989). Super-Brownian motion: path properties and hitting probabilities. *Probab. Theory Related Fields* **83** 135–205.

DAWSON, D. A. and PERKINS, E. A. (1991). *Historical processes. Mem. Amer. Math Soc.* **93**. Amer. Math. Soc., Providence, RI.

DAWSON, D. A. and VINOGRADOV, V. (1992). Almost-sure properties of $(2, d, \beta)$-superprocesses. LRSP Technical Report 195.

DYNKIN, E. B. (1991). A probabilistic approach to one class of nonlinear differential equations. *Probab. Theory Related Fields* **89** 89–115.

DYNKIN, E. B. (1992). Superdiffusions and parabolic nonlinear differential equations. *Ann. Probab.* **20** 942–962.

ETHIER, S. and KURTZ, T. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.

EVANS, S. N. (1993). Two representations of a conditioned superprocess. *Journal of the Royal Society Edinburgh* **123A** 959–971.

EVANS, S. N. and PERKINS, E. A. (1994). Measure-valued branching diffusions with singular interactions. *Canad. J. Math.* **46** 120–168

FITZSIMMONS, P. J. (1988). Construction and regularity of measure-valued branching processes. *Israel J. Math.* **464** 337–361.

KONNO, N. and SHIGA, T. (1988). Stochastic differential equations for some measure-valued diffusions. *Probab. Theory Related Fields* **79** 201–225.

LE GALL, J.-F. (1993). A class of path-valued Markov processes and its applications to superprocesses. *Probab. Theory Related Fields* **95** 25–46.

MUELLER, C. and PERKINS, E. A. (1992). The compact support property for solutions to the heat equation with noise. *Probab. Theory Related Fields* **93** 325–358.

PERKINS, E. A. (1988). A space–time property of a class of measure-valued branching diffusions. *Trans. Amer. Math. Soc.* **305** 743–795.

PERKINS, E. A. (1992). Measure-valued branching diffusions with spatial interactions. *Probab. Theory Related Fields* **94** 189–246.

REIMERS, M. (1989). One dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Related Fields* **81** 319–340.

STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. Springer, Berlin.

TAYLOR, S. J. and WATSON, N. A. (1985). A Hausdorff measure classification of polar sets for the heat equation. *Math. Proc. Cambridge Philos. Soc.* **97** 325–344.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BRITISH COLUMBIA
CANADA V6T 1Z2