The Annals of Probability
1994, Vol. 22, No. 3, 1252–1272

EQUILIBRIUM FLUCTUATIONS OF A ONE-DIMENSIONAL NONGRADIENT GINZBURG–LANDAU MODEL

BY SHENGLIN LU

University of Michigan

We study equilibrium fluctuations for hydrodynamic limits of a nongradient Ginzburg–Landau model. We prove that the limit fluctuation process is governed by a generalized stationary Ornstein-Uhlenbeck process.

1. Notation and summary. In this article we study the equilibrium fluctuations of a one-dimensional nongradient Ginzburg–Landau model. The dynamics in this model is governed by a conservation law together with random noise which also conserves the total charge. The resulting process is nongradient and reversible with respect to a family of time-independent Ginzburg–Landau Gibbs measures.

Equilibrium fluctuations for the gradient Ginzburg–Landau model have been studied by Chang [1] and Zhu [10] and, for the interacting Brownian motion model, by Spohn [7]. The gradient condition on dynamics means the regular summation by parts can be performed. With this property for the dynamics, the original contributions from the oscillations of fluctuation fields can be largely reduced. For the nongradient system, we also need to control these large oscillations. The usual idea to replace local microscopic quantities by averaging with respect to the Gibbs state does not suffice. Varadhan [9] introduced a perturbation technique to estimate the contributions of large oscillations to lower-order perturbation terms. The hydrodynamic scaling limits for the nongradient Ginzburg–Landau model are then derived with the bulk diffusion coefficient as a thermodynamic quantity. Quastel [6] extended this approach to the diffusion of color in a symmetric exclusion process. In considering fluctuations, we will also use this perturbation method. We identify the correct diffusion and drift terms for the limit fluctuation process by using estimates on the central limit theorem variances.

We now introduce the setup. Let S be the unit circle viewed as the interval [0, 1] with 0 and 1 identified. For each integer N, let S_N denote the periodic lattice $\{j/N\}$ with $j = 1, 2, \ldots, N$. Let $x_j(t)$ represent the continuous charges at the site j at time t. The dynamics of the charge configuration are governed by the infinitesimal generator

$$\mathcal{L}_N = \frac{N^2}{2}\sum_{i=1}^{N} a(x_i, x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 - \frac{N^2}{2}\sum_{i=1}^{N} W(x_i, x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right),$$

Received April 1992.

AMS 1991 *subject classification*. 60K35.

Key words and phrases: Fluctuation, central limit theorems, Ginzburg–Landau model, Ornstein–Uhlenbeck process.

with

$$W(x,y) = -a_1(x,y) + a_2(x,y) + a(x,y)\big(\phi'(x) - \phi'(y)\big),$$

where $a(x,y)$ is a function of x, y with bounded continuous first derivatives satisfying

$$0 < C_1 \le a(x,y) \le C_2 < \infty. \tag{1}$$

The functions a_1 and a_2 are the partial derivatives

$$a_1 = \frac{\partial a}{\partial x} \quad \text{and} \quad a_2 = \frac{\partial a}{\partial y}$$

and ϕ is a continuously differentiable function from R into R satisfying the following properties:

$$\int_{-\infty}^{\infty} e^{-\phi(x)}\,dx = 1, \tag{2}$$

$$\int_{-\infty}^{\infty} e^{\lambda x - \phi(x)}\,dx = M(\lambda) < \infty \qquad \text{for all } \lambda \in R, \tag{3}$$

$$\int_{-\infty}^{\infty} e^{\sigma|\phi'(x)| - \phi(x)}\,dx < \infty \quad \text{for all } \sigma > 0 \tag{4}$$

and $\phi'(x) = d\phi(x)/dx$.

If we let $\Phi_N(x) = \exp(-\Sigma_{i=1}^N \phi(x_i))$ on R^N, then this will be the density relative to the Lebesgue measure of a probability measure on R^N. The generator $\mathcal{L}_N$ is formally symmetric with respect to the density Φ_N and defines a reversible process with invariant density $\Phi_N(x)\,dx$.

Equivalently, one can describe the dynamics by the stochastic differential equations

$$\begin{aligned} dx_i(t) = {} & \frac{N^2}{2}\Big[W\big(x_{i-1}(t), x_i(t)\big) - W\big(x_i(t), x_{i+1}(t)\big)\Big]\,dt \\ & + N\Big[\sigma\big(x_{i-1}(t), x_i(t)\big)\,d\beta_i(t) - \sigma\big(x_i(t), x_{i+1}(t)\big)\,d\beta_{i+1}(t)\Big], \end{aligned}$$

where $(\beta_1(t), \dots, \beta_N(t))$ are N-independent Brownian motions and $\sigma(x,y) = \sqrt{a(x,y)}$.

Assuming

$$\int_{-\infty}^{\infty} x e^{-\phi(x)}\,dx = \rho,$$

we define the empirical density fluctuation field by

$$\zeta_t^N(\cdot) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \big(x_i(t) - \rho\big)\delta_{i/N},$$

where $\zeta_{\cdot}^N$ is a measure valued process. We denote by $\mu_{N,\rho}$ the product measure on R^N with density $\Phi_N(x)$ and average ρ and denote by P_N the distribution of $\zeta_{\cdot}^N$ with initial distribution $\mu_{N,\rho}$. So P_N lives on the space $C([0,1], \mathcal{M}(S))$, with $\mathcal{M}(S)$ as the space of signed measures on S. We denote by $H_{-\alpha}(S), \alpha > 0$, the conjugate space of regular Sobolev space $H_\alpha(S)$. Our main result is the following theorem.

THEOREM 1. *Suppose conditions* (1) *to* (4) *are satisfied. Then P_N converges weakly in $C([0,1], H_{-2}(S))$ to a generalized stationary Ornstein–Uhlenbeck process characterized by the following SDE*:

$$d\zeta^\infty(t) = \tfrac{1}{2}\widehat{a}(\rho)h''(\rho)\,\partial_\theta\partial_\theta\zeta^\infty(t)\,dt + \sqrt{\widehat{a}(\rho)}\,d\partial_\theta\,\beta(\theta,t), \tag{5}$$

where $\partial_\theta\beta(\theta,t)$ is the Gaussian random field with covariance

$$E\Big[\partial_\theta\beta(\theta,t)\big(J_1(\theta)\big)\,\partial_\theta\beta(\theta,s)\big(J_2(\theta)\big)\Big] = \min(t,s)\int_S J_1'(\theta)J_2'(\theta)\,d\theta \tag{6}$$

for any smooth functions J_1, J_2 on S. Here the function h is the free energy defined by

$$h(y) = \sup\big[\lambda y - \rho(\lambda)\big],$$
$$\rho(\lambda) = \log M(\lambda)$$

and the function $\widehat{a}(\cdot)$ will be defined below.

For given $y \in R$, if we let $\lambda = h'(y)$, then we have a product measure μ_y on $\Pi_{-\infty}^{\infty}R = \Omega$ with each coordinate having the distribution $[1/M(\lambda)]e^{\lambda x - \phi(x)}\,dx$. Let $F(x_{-l},\dots,x_l)$ be a smooth function of $(2l+1)$ variables. We will view this as a function $F(\omega)$ defined on Ω. If we denote by T the unit shift operator $(T\omega)(i) = x_{i+1}$ if $\omega(i) = x_i$, we can form the formal infinite sum

$$\Psi = \sum_{k=-\infty}^{\infty} F(T^k\omega).$$

Although Ψ does not really make sense, the partial derivatives $\partial\Psi/\partial x_i$ are all well defined. Then we define

$$\widehat{a}(y) = \inf_{\Psi} E^{\mu_y} a(x_0,x_1)\left(1 - \left(\frac{\partial\Psi}{\partial x_0} - \frac{\partial\Psi}{\partial x_1}\right)\right)^2,$$

where the infimum is taken over all functions F, varying l as well as the function of $(2l+1)$ variables.

To prove Theorem 1, we have to control local fluctuations and replace the local average by proper macroscopic quantities at the fluctuation level. This effect for fluctuations is usually called the Boltzmann–Gibbs principle which will be given in the following form.

THEOREM 2. *For any smooth function J on S,*

$$\lim_{N\to\infty} E^{P_N}\left[\frac{1}{\sqrt{N}}\int_0^t \sum_{i=1}^N J\left(\frac{i}{N}\right)\left\{\phi'(x_i(s)) - h'(\rho) - h''(\rho)(x_i(s)-\rho)\right\} ds\right]^2 = 0. \tag{7}$$

REMARK 1. It is not hard to get the asymptotic process for equilibrium fluctuations in higher-dimensional space. In fact, a similar approach works for all other nongradient dynamics.

REMARK 2. There is no efficient estimation to control the oscillations for the fluctuations of nongradient systems in nonequilibrium. It does not suffice to use just the estimates developed in [2].

This paper is organized as follows: Section 1 presents the notation and a summary of the main results. In Section 2 we prove the tightness for the fluctuation fields. In Section 3 estimations are derived for the central limit theorem variances. Section 4 contains the proof of the Boltzmann–Gibbs principle for our dynamics. The proof of our main result is covered in Section 5 by using the techniques developed in Sections 3 and 4.

2. Tightness. In this section we shall prove several estimates which will lead to the tightness for P_N. These estimates are essential to using Garsia's lemma. We get the tightness on the measure valued space $C([0,1], H_{-2}(S))$.

For any test function $J(\cdot)$ on S, we have

$$\begin{aligned} d\langle J, \zeta_t^N\rangle &= d\frac{1}{\sqrt{N}}\sum_{i=1}^N J\left(\frac{i}{N}\right)(x_i(t)-\rho) \\ &= \frac{\sqrt{N}}{2}\sum_{i=1}^N \nabla J\left(\frac{i}{N}\right) W_{(x_i, x_{i+1})}\, dt \\ &\quad + \frac{1}{\sqrt{N}}\sum_{i=1}^N \nabla J\left(\frac{i}{N}\right)\sigma(x_i, x_{i+1})\, d\beta_i(t) \\ &= d\langle \nabla J, \gamma_N(t)\rangle + d\langle \nabla J, \nu_N(t)\rangle, \end{aligned} \tag{8}$$

where

$$\nabla J\left(\frac{i}{N}\right) = N\left(J\left(\frac{i+1}{N}\right) - J\left(\frac{i}{N}\right)\right),$$

the derivative of J in a lattice sense, and

$$\gamma_N(t)(dx) = \frac{\sqrt{N}}{2}\sum_{i=1}^N \delta_{i/N}(dx) W_{(x_i, x_{i+1})}\, dt, \tag{9}$$

$$\nu_N(t)(dx) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \delta_{i/N}(dx)\sigma_{(x_i, x_{i+1})}\, d\beta_i(t). \tag{10}$$

Let Q_N denote the induced measure of (γ_N, ν_N) on $C([0,1], (\mathcal{M}(S))^2)$.

THEOREM 2.1. *P_N is tight as a sequence of probability measures on $C([0,1], H_{-2}(S))$.*

PROOF. From the relation given above, we only need to prove the tightness for Q_N.

The following estimates will be used to prove the tightness. Methods are standard; see [6], [9] and [8].

LEMMA 2.1. *For γ_N, we have, for all $\alpha > 0$,*

$$E^{Q_N}\Big[\exp\Big(N\alpha\big|\langle J, \gamma_N(t)\rangle - \langle J, \gamma_N(s)\rangle\big|\Big)\Big] \le 2\exp\left(C_4|t-s|N\alpha^2\sum_{i=1}^{N} J^2\left(\frac{i}{N}\right)\right)$$

for some uniform constant $C_4 > 0$ depending only on the bound of $a(x,y)$.

PROOF. By stationarity, if $t > s$,

$$\begin{aligned} E^{Q_N}&\Big[\exp\Big(N\alpha\big(\langle J, \gamma_N(t)\rangle - \langle J, \gamma_N(s)\rangle\big)\Big)\Big] \\ &= E^{Q_N}\Big[\exp\big(N\alpha\langle J, \gamma_N(t-s)\rangle\big)\Big] \\ &\le \exp\big[(t-s)\Lambda(J)\big], \end{aligned}$$

by the spectral theorem and the Feynman–Kac formula, where

$$\begin{aligned} \Lambda(J) = \sup_{f\in D^{N,\rho}} &\left\{\frac{1}{2}N^{3/2}\alpha E^{\Phi_N}\sum_{i=1}^{N} J\left(\frac{i}{N}\right)W(x_i, x_{i+1})f(x)\right. \\ &\left. -\frac{N^2}{8}E^{\Phi_N}\sum_{i=1}^{N} a(x_i, x_{i+1})\frac{1}{f}\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}}\right)^2\right\} \\ = \sup_{f\in D^{N,\rho}} &\left\{N^{3/2}\alpha\sum_{i=1}^{N} J\left(\frac{i}{n}\right)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial_{x_{i+1}}}\right)\Phi_N\,dx\right. \\ &\left. -\frac{N^2}{8}\sum_{i=1}^{N}\int a(x_i, x_{i+1})\frac{1}{f}\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}}\right)^2\Phi_N\,dx\right\} \\ \le 2C_3\alpha^2 N &\sum_{i=1}^{N} J^2\left(\frac{i}{N}\right). \end{aligned}$$

Here C_3 is the bound of $a(x,y)$ and $D^{N,\rho} = \{f : f \text{ is smooth density wrt } \mu_{N,\rho}\}$. Lemma 2.1 is then proved by using $e^{|x|} \le e^x + e^{-x}$. □

LEMMA 2.2. *For ν_N, we have*

$$E^{Q_N}\big[\langle J, \nu_N(t)\rangle\big]^6 \le C_5\left(\frac{1}{N}\sum_{i=1}^{N} J^2\left(\frac{i}{N}\right)\right)^3 t^3,$$

where C_5 is constant depending only on the bound of $a(x,y)$.

This lemma is proved simply by applying Itô's formula.
We will use Garsia's lemma in the following form.

LEMMA 2.3 (Garsia, Rodemich and Ramsey). *For any continuous function $f \in C([0,1])$, if $\Psi(x)$ is a strictly increasing function with $\Psi(0) = 0$ and $\lim_{x\to\infty}\Psi(x) = \infty$ and*

$$B = \int_0^1\int_0^1 \Psi\left(\frac{|f(t) - f(s)|}{\sqrt{|t-s|}}\right) ds\, dt, \tag{11}$$

then, for $0 \le s < t \le 1$,

$$|f(t) - f(s)| \le 4\int_0^{(t-s)} \Psi^{-1}\left(\frac{4B}{u^2}\right)\frac{du}{\sqrt{u}}. \tag{12}$$

THEOREM 2.2. *Q_N is tight as a sequence of measures on $C([0,1]; (H_{-1}(S))^2)$.*

PROOF. It suffices to prove

$$\mathrm{I}: \quad \limsup_{N\to\infty} Q_N\left(\sup_{0\le t\le 1}\|\mu(t)\|_{L^2} \ge l\right) = 0$$

and, for every $\varepsilon > 0$,

$$\mathrm{II}: \quad \limsup_{N\to\infty} Q_N\left(\sup_{0\le s<t\le 1,\, |t-s|<\delta}\|\mu(t) - \mu(s)\|_{H_{-1}} > \varepsilon\right) = 0$$

for $\mu = \nu_N, \gamma_N$.
Since the proofs of I and II are basically the same, we only prove II for ν_N, γ_N. □

PROOF OF II FOR γ_N. We will use estimations from Lemmas 2.1 and 2.3. In Lemma 2.3 we take $\Psi(x) = \exp(N\alpha x) - 1$ and $\alpha > 0$. It is not hard to show that, for $0 \le t - s \le 1$,

$$\begin{aligned} &4\int_0^{(t-s)} \Psi^{-1}\left(\frac{4B}{u^2}\right)\frac{du}{\sqrt{u}} \\ &\quad \le \frac{8\sqrt{(t-s)}}{N\alpha}\Big\{\log\big[4B + (t-s)^2\big] + \log 4 - 2\log(t-s)\Big\} \end{aligned}$$

and, by Lemma 2.1,

$$E^{Q_N}B \leq 2\ \exp\left(C_4 N\alpha^2 \sum_{i=1}^{N} J^2\left(\frac{i}{N}\right)\right) - 1.$$

Then, using Garsia's inequality, we have, for $0 \leq \delta < 1$,

$$\begin{aligned}
&E^{Q_N}\left[\exp\frac{N\alpha}{8\delta^{1/4}} \sup_{0 \leq s < t \leq 1,\, |t-s| < \delta} \left|\langle J, \gamma_N(t)\rangle - \langle J, \gamma_N(s)\rangle\right|\right] \\
&\qquad \leq 10E^{Q_N}[4B + \delta^2] \\
&\qquad \leq C_6\ \exp\left(N\alpha^2 \sum_{i=1}^{N} J^2\left(\frac{i}{N}\right)\right),
\end{aligned}$$

with $C_6 > 0$. Integrating the inequality above over $\alpha > 0$, we get

$$E^{Q_N}\left[\exp\left(\frac{1}{C_7\delta^{1/2}} \sup_{0 \leq s < t \leq 1,\, |t-s| < \delta} \frac{\left|\langle J, \gamma_N(t)\rangle - \langle J, \gamma_N(s)\rangle\right|^2}{(1/N)\sum_{i=1}^{N} J^2(i/N)}\right)\right]$$

and

$$\begin{aligned}
&E^{Q_N}\left[\sup_{0 \leq s < t \leq 1,\, |t-s| < \delta} \left|\langle J, \gamma_N(t)\rangle - \langle J, \gamma_N(s)\rangle\right|^2\right] \\
&\qquad \leq C_9\delta^{1/2}\frac{1}{N}\sum_{i=1}^{N} J^2\left(\frac{i}{N}\right),
\end{aligned}$$

with C_7, C_8 and C_9 uniform positive constants. Then II follows for γ_N. □

PROOF OF II FOR ν_N. In Lemma 2.3 take $\Psi(x) = x^6$. Then

$$E^{Q_N}B \leq C_{10}\left[\frac{1}{N}\sum_{i=1}^{N} J^2\left(\frac{i}{N}\right)\right]^3$$

and

$$\begin{aligned}
&4\int_0^{(t-s)} \Psi^{-1}\left(\frac{4B}{u^2}\right)\frac{du}{\sqrt{u}} \\
&\qquad \leq 4\int_0^{(t-s)} \left(\frac{4B}{u^2}\right)^{1/6} \frac{du}{\sqrt{u}} \\
&\qquad \leq C_{11}B^{1/6}|t-s|^{1/6}.
\end{aligned}$$

Then

$$E^{Q_N}\left[\sup_{0\le s<t\le 1,\,|t-s|<\delta}\left|\langle J,\nu_N(t)\rangle-\langle J,\nu_N(s)\rangle\right|^2\right]$$
$$\le C_{12}E^{Q_N}\big[B^{1/3}\big]\delta^{1/3}$$
$$\le C_{13}\delta^{1/3}\left(\frac{1}{N}\sum_{i=1}^{N}J^2\left(\frac{i}{N}\right)\right),$$

with C_{10}, C_{11}, C_{12} and C_{13} uniform positive constants. Then II follows for ν_N.

This completes the proof of Lemma 2.4. □

3. Estimation of central limit theorem variances. In this section we shall estimate central limit theorem variances. Since we are in an equilibrium environment, fluctuation fields can be estimated by the central limit theorem for scaling limits. These ideas are from [4] and [9].

Let us suppose that L is the infinitesimal generator of a Markov process which is reversible and ergodic with respect to an invariant probability measure μ on some state space X. We denote by P_μ the stationary Markov process with marginal distribution μ. If $V(x)$ is square integrable with respect to μ and has mean 0, by the ergodic theorem,

$$\lim_{t\to\infty}E^{P_\mu}\left[\frac{1}{t}\int_0^t V\big(x(s)\big)\,ds\right]^2=0.$$

We are interested in calculating

$$\lim_{t\to\infty}E^{P_\mu}\left\{\frac{1}{\sqrt{t}}\int_0^t V\big(x(s)\big)\,ds\right\}^2=\Delta(V,V) \tag{13}$$

if it exists. There is a symmetric bilinear form

$$\lim_{t\to\infty}E^{P_\mu}\left\{\frac{1}{\sqrt{t}}\int_0^t V\big(x(s)\big)\,ds\frac{1}{\sqrt{t}}\int_0^t U\big(x(s)\big)\,ds\right\}=\Delta(V,U).$$

It is known that the limit (13) always exists. It is finite if and only if V is in the range of $(-L)^{1/2}$ and in that case

$$\Delta(V,V)=2\langle(-L)^{-1/2}V,(-L)^{-1/2}V\rangle.$$

By standard arguments, if U and V are in the range of $(-L)^{1/2}$, then

$$\Delta(V,U)=2\langle(-L)^{-1/2}V,(-L)^{-1/2}V\rangle.$$

Here $\langle,\rangle$ is the standard inner product in $L_2(\mu)$. By duality $\Delta(V,V)=2c^2$, where c is the smallest constant such that

$$\langle U,V\rangle\le c\big[D(U)\big]^{1/2}\quad\text{for all } U\in\mathcal{D}(L).$$

Here $D(U)$ is the Dirichlet form

$$D(U) = \langle -LU, U\rangle.$$

If $U = Lf$ for some f in $\mathcal{D}(L)$ and $\Delta(V, U) < \infty$, then

$$\Delta(V, U) = -2\langle f, v\rangle.$$

In particular, $\Delta(U, U) < \infty$ and

$$\Delta(U, U) = -2\langle f, Lf\rangle = 2D(f).$$

We want to explore these ideas in a specific context. We fix N and consider $R^N - N$ copies of R. On R^N we have the product measure $\exp[-\sum_{i=1}^N \phi(x_i)]\,dx$ and the conditioned measures $\mu_{N,y}(dx)$ on the hyperplane $(1/N)\sum_{i=1}^N x_i = y$. We look at the Dirichlet form

$$D_{N,y}(f) = \frac{1}{2}\int_{R^N} \sum_{i=1}^N a(x_i, x_{i+1})\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}}\right)^2 d\mu_{N,y}(dx),$$

and the corresponding operator is L_N as given in Section 1 but without scaling, that is,

$$L_N = \frac{1}{2}\sum_{i=1}^N a(x_i, x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 - \frac{1}{2}\sum_{i=1}^N W(x_i, x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right).$$

We will be looking at three classes of functions,

$$A_N = \frac{1}{2}\sum_{i=1}^N \nabla J\left(\frac{i}{N}\right) W(x_i, x_{i+1}),$$

$$B_N = \sum_{i=1}^N \nabla J\left(\frac{i}{N}\right)\big(\phi'(x_i) - \phi'(x_{i+1})\big),$$

$$C_N = \sum_{i=1}^N \nabla J\left(\frac{i}{N}\right) L_N F^i(x),$$

where $F^i(x) = F(x_{i+1}, \dots, x_{i+k})$ with $F(x_1, \dots, x_k)$ a smooth function of k variables for some fixed k, and

$$\nabla J\left(\frac{i}{N}\right) = N\left(J\left(\frac{i+1}{N}\right) - J\left(\frac{i}{N}\right)\right)$$

An elementary calculation yields

$$A_N = NL_N\left(\sum_{i=1}^N \nabla J\left(\frac{i}{N}\right)x_i\right),$$

$$C_N = L_N\left(\sum_{i=1}^N \nabla J\left(\frac{i}{N}\right)F^i(x)\right).$$

On the other hand, B_N is also in the range of $(-L_N)^{1/2}$, because for any smooth test function u,

$$\int_{R^N} uB_N \, d\mu_{N,y}(dx) = \int_{R^N} \sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right)\left(\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_{i+1}}\right) d\mu_{N,y}(dx)$$
$$\leq C\sqrt{N}\big(D_{N,y}(u)\big)^{1/2}.$$

It is clear then that A_N, B_N and H_N are all in the range of $(-L_N)^{1/2}$. So central limit theorem variances and covariances exist. We will estimate these variances and covariances in the following several lemmas.

For a given $F(x_1, \ldots, x_k)$ on R^k, we formally define

$$\Psi_F = \sum_{k=-\infty}^{\infty} F(T^k \omega)$$

and

$$\widehat{a}_F = E^{\mu_y} a(x_0, x_1)\left(1 - \left(\frac{\partial \Psi_F}{\partial x_0} - \frac{\partial \Psi_F}{\partial x_1}\right)\right)^2,$$

where μ_y is the product measure defined in Section 1, and T, which is defined in Section 1, is the unit shift operator. It is not hard to see that $\widehat{a}_F$ is meaningful, although Ψ_F is just formally defined.

LEMMA 3.1.

$$\lim_{N\to\infty} \frac{1}{N} \Delta_{N,y}(A_N, A_N) = \|J'\|^2_{L^2(S)} E^{\mu_y} a(x_0, x_1).$$

PROOF. Since

$$\Delta_{N,y}(A_N, A_N) = E^{\mu_{N,y}} \sum_{j=1}^{N} a(x_j, x_{j+1}) \left[\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}}\right)\left(\sum_{i=1}^{N} NJ\left(\frac{i}{N}\right) x_i\right)\right]^2,$$

Lemma 3.1 follows from the translation invariance of μ_y^N. $\square$

LEMMA 3.2.

$$\lim_{N\to\infty} \frac{1}{N} \Delta_{N,y}(A_N, B_N) = \|J'\|^2_{L^2(S)}.$$

PROOF. Since

$$\Delta_{N,y}(A_N, B_N) = E^{\mu_{N,y}}\left[\left(\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right) x_i\right)\left(\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right) \phi'(x_i)\right)\right]$$
$$= E^{\mu_{N,y}}\left[\left(\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right)(x_i - y)\right)\left(\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right) \phi'(x_i)\right)\right]$$

and

$$\lim_{N\to\infty} E^{\mu_{N,y}}\big[(x_i - y)\phi'(x_i)\big] = 1,$$

Lemma 3.2 is then proved. □

LEMMA 3.3.

$$\lim_{N\to\infty}\frac{1}{N}\Delta_{N,y}(A_N, C_N) = \|J'\|^2_{L^2(S)} E^{\mu_y} a(x_0, x_1)\left(\frac{\partial\Psi_F}{\partial x_0} - \frac{\partial\Psi_F}{\partial x_1}\right).$$

PROOF.

$$\Delta_{N,y}(A_N, C_N) = E^{\mu_{N,y}}\sum_{j=1}^{N} a(x_j, x_{j+1})\Big(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}}\Big)\left(\sum_{i=1}^{N}\nabla J\Big(\frac{i}{N}\Big)x_i\right)$$
$$\times\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}}\right)\left(\sum_{i=1}^{N}\nabla J\Big(\frac{i}{N}\Big)F^i(x)\right).$$

Since F depends only on finitely many variables and by the translation invariance of the measure, we have Lemma 3.3. □

LEMMA 3.4.

$$\lim_{N\to\infty}\frac{1}{N}\Delta_{N,y}(B_N, B_N) \le \|J'\|^2_{L^2(S)}\frac{1}{\widehat{a}(y)}.$$

PROOF. For a given integer l, we define

$$B_N^l = \sum \nabla J\Big(\frac{il}{N}\Big)\Big[\phi'\big(x_{(i+1)l}\big) - \phi'(x_{il})\Big].$$

By the smoothness of J and finite action for different bonds, we have

$$\lim_{N\to\infty}\frac{1}{N}\Delta_{N,y}(B_N, B_N) \le \lim_{l\to\infty}\lim_{N\to\infty}\frac{1}{N}\Delta\big(B_N^l, B_N^l\big)$$
$$\le \|J'\|^2 \lim_{l\to\infty}\frac{1}{l}E^{\mu_y}\big[\Delta_{l,(\Sigma_{i=1}^l x_i)/l}(D_l, D_l)\big],$$

where

$$D_l = \big(\phi'(x_l) - \phi'(x_0)\big).$$

By the theorem in [8], we know

$$\lim_{N\to\infty}\sup_{y'\to y}\frac{1}{l}\Delta_{l,y'}(D_l, D_l) \le \frac{1}{\widehat{a}(y)}.$$

By the law of large numbers, we then have

$$\lim_{N\to\infty}\frac{1}{N}\Delta_{N,y}(B_N,B_N)\le\frac{1}{\widehat{a}(y)}.$$

Lemma 3.4 then follows. □

LEMMA 3.5.

$$\lim_{N\to\infty}\frac{1}{N}\Delta_{N,y}(B_N,C_N)=0.$$

PROOF.

$$\begin{aligned}\Delta_{N,y}(B_N,C_N)&=E^{\mu_y^N}\left(\sum_{i=1}^N\nabla J\left(\frac{i}{N}\right)(\phi'(x_{i+1})-\phi'(x_i))\right)\left(\sum_{i=1}^N\nabla J\left(\frac{i}{N}\right)F^i(x)\right)\\&=E^{\mu_y^N}\left(\sum_{i=1}^N\nabla J\left(\frac{i}{N}\right)\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_{j+1}}\right)\right)\left(\sum_{i=1}^N\nabla J\left(\frac{i}{N}\right)F^i(x)\right).\end{aligned}$$

Since $\int J'(x)\,dx=0$ and F only depends on finite variables, Lemma 3.5 follows by the translation invariance of μ_y^N. □

LEMMA 3.6.

$$\lim_{N\to\infty}\frac{1}{N}\Delta_{N,y}(C_N,C_N)=\|J'\|^2_{L^2(S)}E^{\mu_y}a(x_0,x_1)\left(\frac{\partial\Psi_F}{\partial x_0}-\frac{\partial\Psi_F}{\partial x_1}\right)^2.$$

PROOF.

$$\Delta_{N,y}(C_N,C_N)=E^{\mu_y^N}\sum_{j=1}^N a(x_j,x_{j+1})\left[\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_{j+1}}\right)\left(\sum_{i=1}^N\nabla J\left(\frac{i}{N}\right)F^i(x)\right)\right]^2.$$

Since F depends only on finite variables and J is smooth, we have Lemma 3.6. □

These lemmas can help us to estimate macroscopic limits for the microscopic quantities. We will use these estimations to compute the contributions from the fluctuations in Section 5.

4. Boltzmann–Gibbs principle. In this section we shall establish the Boltzmann–Gibbs principle for our dynamics. The Boltzmann–Gibbs principle was discussed by Chang [1], Spohn [7] and Zhu [10] for equilibrium fluctuations for different gradient dynamics, and by Chang and Yau [2] for nonequilibrium gradient model. Our method is similar to Chang [1] but different from Spohn [7] and Zhu [10].

First, we want to truncate the unbounded function ϕ'. For $l \geq 0$, the truncation ψ_l of ϕ' is defined by

$$\psi_l = \begin{cases} \phi', & \text{for } |\phi'| \leq l, \\ l, & \text{for } \phi' > l, \\ -l, & \text{for } \phi' < -l. \end{cases}$$

Let

$$\widetilde{\psi}_l(x) = M(\lambda)^{-1} \int e^{\lambda y - \phi(y)} \psi_l(y)\, dy,$$

with $\lambda = h'(x)$. From Lemma 6.4 in [3], we know for each x, $\widetilde{\psi}_l(x) \to h'(x)$ and $\widetilde{\psi}_l'(x) \to h''(x)$ as $l \to \infty$.

LEMMA 4.1.

$$\lim_{l \to \infty} \limsup_{N \to \infty} E^{P_N} |F(N, l, t)|^2 = 0,$$

with

$$F(N, l, t) = \int_0^t \frac{1}{\sqrt{N}} \sum_{i=1}^N J\left(\frac{i}{N}\right) \left[\phi'\big(x_i(s)\big) - h'(\rho) - \Big(\psi_l\big(x_i(s)\big) - \widetilde{\psi}_l(\rho)\Big)\right] ds.$$

PROOF. By the Schwarz inequality and stationarity,

$$\limsup_{N \to \infty} E^{P_N} |F(N, l, t)|^2 \leq t^2 \|J\|_\infty^2 E\Big[\phi'(x) - h'(\rho) - \big(\psi_l(x) - \widetilde{\psi}_l(\rho)\big)\Big]^2,$$

where E is the expectation with respect to the probability measure $e^{\lambda x - \phi(x)}\, dx$ and $\lambda = h'(\rho)$.

Since

$$|\psi_l(x)| \leq |\phi'(x)| \quad \text{for } \forall x$$

and

$$|\widetilde{\psi}_l(\rho)| \leq \int |\phi'(x)| e^{-\phi(x)}\, dx$$

using Lebesgue's dominated convergence theorem, we get Lemma 4.1. □

From Lemma 4.1, we can analyze local microscopic quantities by considering only the truncated function. The next step is to localize our dynamics.

Let K be a positive integer and let $N = mK + r$, $0 \leq r < K$. Let B_q be the block containing sites at $(q-1)K + 1/N, \dots, qK/N$, $q = 1, \dots, m$, and let B_{m+1} contain the remaining sites at $mK + 1/N, \dots, N/N$.

Define

$$L_{i,i+1} = \begin{cases} 0, & \text{if } i = 0 \bmod k), \\ \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^* a(x_i, x_{i+1})\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right), & \text{otherwise,} \end{cases}$$

where * means the adjoint with respect to $\Phi_N\,dx$,

$$L_{B_q} = \sum_{i \in B_q} J\left(\frac{i}{N}\right) L_{i,i+1}$$

and

$$f(x_{B_q}(s)) = \begin{cases} 0, & \text{if } q = m+1, \\ f(x_{(q-1)K+1}(s), \dots, x_{qK}(s)), & \text{otherwise,} \end{cases}$$

where f is any function depending on K variables.

LEMMA 4.2. *For any nice f, we have*

$$\lim_{N\to\infty} E^{P_N}\left| \int_0^t \frac{1}{\sqrt{N}} \sum_{i=1}^{m+1} L_{B_q} f(x_{B_q}(s))\, ds \right| = 0.$$

PROOF. By the ergodic theorem on each hyperplane, we have

$$\begin{aligned} E^{P_N}&\left[\int_0^t \frac{1}{\sqrt{N}} \sum_{i=1}^{m+1} L_{B_q} f(x_{B_q}(s))\, ds \right]^2 \\ &\le \frac{t}{N^2} \langle V, L^{-1} V\rangle \\ &= \frac{t}{N^2} \sup_{u \in H^1, D_N(u)=1} \left|\langle u, V\rangle\right|^2 \\ &\le \frac{t}{N^2} \int \frac{1}{N} \sum_{q=1}^{m+1} \sum_{i,i+1 \in B_q} \left[\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) J\left(\frac{i}{N}\right) f(x_{B_q}) \right]^2 \Phi_N\, dx \\ &\to 0, \end{aligned}$$

where

$$D_N(u) = \int \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_{i+1}} \right)^2 \Phi_N\, dx.$$

Based on the observations in Lemmas 4.1 and 4.2, we can reduce the proof of Theorem 2 to showing the following relation:

$$\lim_{K\to\infty} \inf_{f} \lim_{N\to\infty} E^{P_N}\left[\frac{1}{\sqrt{N}} \int_0^t \sum_{q=1}^{m+1} \sum_{i \in B_q} J\left(\frac{i}{N}\right) F(i, K, f)\, ds \right]^2 = 0, \tag{14}$$

where

$$F(i,K,f) = \psi_l\big(x_i(s)\big) - \widetilde{\psi}_l(\rho) - \widetilde{\psi}_l'(\rho)\big(x_i(s) - \rho\big) - L_{i,i+1}f\big(x_{B_q}(s)\big).$$

Since our measure is translation invariant, by the smoothness of J and the lack of interactions between blocks,

$$\lim_{N\to\infty} E^{P_N}\left[\frac{1}{\sqrt{N}}\int_0^t \sum_{i=1}^N J\left(\frac{i}{N}\right)F(i,K,f)\,ds\right]^2$$
$$\leq \|J\|_2^2 t^2 \frac{1}{K}E^{K\rho}\left|\sum_{i=1}^K \big(\psi_l(x_i) - \widetilde{\psi}_l(\rho) - \widetilde{\psi}_l'(\rho)(x_i - \rho) - L_K f(x_1,\dots,x_K)\right|^2,$$

where $L_K = L_{1,2} + \cdots + L_{K-1,K}$ and $E^{K,\rho}$ is the expectation wrt $\mu_{K,\rho}$ on R^K.

Let us denote by ν_y^K the conditional probability and also conditional expectation of $x_1,\dots,x_K$ on the hyperplane $\sum_{i=1}^K x_i = s_K = Ky$. Since, when restricted on the hyperplane, L_K is elliptic and ergodic,

$$\inf_f E^{K,\rho}\left|\sum_{i=1}^K \big(\psi_l(x_i) - \widetilde{\psi}_l(\rho) - \widetilde{\psi}_l'(\rho)(x_i - \rho) - L_K f(x_1,\dots,x_K)\right|^2$$
$$= K^2 E^{K,\rho}\left|\nu_{s_K/K}^K\Big(\psi_l(x_1) - \widetilde{\psi}_l(\rho) - \widetilde{\psi}_l'(\rho)\Big(\frac{s_K}{K} - \rho\Big)\Big)\right|^2,$$

where the inf is taken over all functions f depending only on K variables.

PROOF OF THEOREM 2. We shall prove Theorem 2 by the following lemmas.

LEMMA 4.3.

$$\lim_{K\to\infty} KE^{K,\rho}\left[\nu_{s_K/K}^K\big(\psi_l(x_1)\big) - \widetilde{\psi}_l\left(\frac{s^K}{K}\right)\right]^2 = 0.$$

PROOF. We will use Cramér's trick. Assume $\rho = 0$. Let us define

$$f(z,y) = \frac{1}{M(\lambda)}e^{\lambda(z+y) - \phi(z+y)},$$

where $y \in R, \lambda = h'(y)$ and $y = \rho'(\lambda)$. Let $f_K(z,y)$ be the probability density of $Z_1 + \cdots + Z_K/\sqrt{\rho''(\lambda)K}$, where $Z_1,\dots,Z_K$ are independent, identically distributed random variables with a common density $f(z,y)$. Note that $EZ_i = 0, \operatorname{Var} Z_i = \rho''(\lambda)$. We denote

$$y(x) = \frac{y - x}{\sqrt{\rho''(\lambda)(K-1)}},$$

$$\Phi_K(dx_1\dots dx_K) \doteq \exp\left[-\sum_{i=1}^K \phi(x_i)\right]dx_1\dots dx_K,$$

$$\nu_y^K\big(\psi_l(x_1)\big) = \frac{I_1}{I_2},$$

where

$$I_1 = \int \psi_l(x_1) K e^{-\phi(Ky - x_1 - \cdots - x_{K-1})} \Phi_{K-1}(dx_1 \ldots ds_{K-1})$$
$$= \int \psi_l(x_1) K e^{-\phi(x)} \frac{M(\lambda)^{K-1} e^{\lambda(x-Ky)}}{\sqrt{\rho''(\lambda)(K-1)}} f_{K-1}(y(x), y)\, dx,$$
$$I_2 = \int K e^{-\phi(Ky - x_1 - \cdots - x_{K-1})} \Phi_{K-1}(dx_1 \ldots dx_{K-1})$$
$$= K \frac{M(\lambda)^K e^{-\lambda K y}}{\sqrt{\rho''(\lambda) K}} f_K(0, y).$$

Then we obtain

$$\nu_y^K(\psi_l(x_1)) - \widetilde{\psi}_l(y) = \int \psi_l(x) \left\{ \sqrt{\frac{K}{K-1}} \frac{f_{K-1}\big(y(x), y\big)}{f_K(0, y)} - 1 \right\} \frac{e^{\lambda x - \phi(x)}}{M(\lambda)}\, dx.$$

Now we use a proposition which can be found in Petrov [5].

PROPOSITION 4.4. *For any y_0 in R, there exists a $\delta > 0$ such that*

$$f_K(z, y) = (2\pi)^{-1/2} G(z) \left\{ 1 + \frac{\rho'''(\lambda)}{6\sqrt{K\rho''(\lambda)^3}} H_3(z) \right\} + o\left(\frac{1}{\sqrt{K}}\right),$$

uniformly in z and $y \in [y_0 - \delta, y_0 + \delta]$, where

$$G(Z) = e^{-z^2/2},$$
$$H_3(z) = z^3 - 3z, \quad \textit{the Chebyshev-Hermite polynomial of degree 3.}$$

Take $y_0 = 0$. By Proposition 4.4, there exists a $\delta > 0$ such that for $y \in [-\delta, \delta]$, we have

$$\sqrt{\frac{K}{K-1}} \{f_K(0, y)\}^{-1} = 1 + o\left(\frac{1}{\sqrt{K}}\right),$$
$$f_{K-1}\big(y(x), y\big) = 1 + Q\big(K-1, y(x)\big) + o\left(\frac{1}{\sqrt{K}}\right),$$

uniformly in $x \in R$, where

$$Q\big(K-1, y(x)\big) = G\big(y(x)\big) - 1 + \frac{\rho'''(\lambda)}{6\sqrt{(K-1)\rho''(\lambda)^3}} (GH_3)\big(y(x)\big).$$

By the continuity of ρ'', ρ''' in y and positivity of $\rho''(0) = \sigma^2$, we may choose a

δ_0, $0 < \delta_0 < \delta$, such that $\rho''(\lambda)^{-1} \geq c > 0$ for $y \in [-\delta_0, \delta_0]$. Then

$$|Q(K-1, y(x))| \leq |G(y(x)) - 1| + \frac{c}{\sqrt{K-1}}|y(x)^3 - y(x)|$$
$$\leq \frac{C'}{K-1}\left\{\sum_{i=1}^{3}|y-x|^i\right\}.$$

Let $D = \{(x_1, \ldots, x_K) \mid |s_K| \leq \delta_0 K\}$. Then we can see from above that

$$KE^{K,\rho}\left\{\left|\nu^K_{s_K/K}(\psi_l(x_1)) - \tilde{\psi}_l\left(\frac{s_K}{K}\right)\right|^2 1_D\right\}$$
$$\leq KE^{K,\rho}\left\{\left|\tilde{\psi}_l(y)o\left(\frac{1}{\sqrt{K}}\right)\right.\right.$$
$$\left.\left. + \int \psi_l(x)\left[Q(K-1, y(x))\left(1 + o\left(\frac{1}{\sqrt{K}}\right)\right)\right]\frac{e^{\lambda x - \phi(x)}}{M(\lambda)}\,dx\right|^2_{s_K = Ky} 1_D\right\}$$
$$\leq l^2 o(1) + 2Kl^2 E^{K,\rho}\left\{\int Q(K-1, y(x))^2\left(1 + o\left(\frac{1}{\sqrt{K}}\right)\right)\frac{e^{\lambda x - \phi(x)}}{M(\lambda)}\,dx\right\}_{s_K = Ky}$$
$$= o(1)l^2 \quad \text{as } K \to \infty.$$

Here we use the facts that

$$|Q(K-1, y(x))|^2 \leq \frac{3C'^2}{(K-1)^2}\left\{\sum_{i=1}^{3}|y-x|^{2i}\right\}$$

and

$$\int \sum_{i=1}^{3}|y-x|^{2i}\frac{e^{\lambda x - \phi(x)}}{M(\lambda)}\,dx \text{ is uniformly bounded for } |y| \leq \delta_0.$$

On the other hand, we also have

$$\lim_{K\to\infty} KE^{K,\rho}\left\{\left|\nu^K_{s_K/K}(\psi_l(x_1)) - \tilde{\psi}_l\left(\frac{s_K}{K}\right)\right|^2 1_{D^c}\right\}$$
$$\leq \lim_{K\to\infty} KE^{K,\rho}\left\{2\left|\nu^K_{s_K/K}(\psi_l(x_1))\right|^2 + 2\left|\tilde{\psi}_l\left(\frac{s_K}{K}\right)\right|^2\left(\delta_0^{-1}\frac{s_K}{K}\right)^4\right\}$$
$$\leq \lim_{K\to\infty}\frac{4l^2}{\delta_0^4 K^3}E^{K,\rho}|s_K|^4 = 0.$$

This completes the proof of Lemma 4.3. □

LEMMA 4.5.

$$\lim_{K\to\infty} KE^{K,\rho}\left[\tilde{\psi}_l\left(\frac{s_K}{K}\right) - \tilde{\psi}_l(\rho) - \tilde{\psi}'_l(\rho)\left(\frac{s_K}{K} - \rho\right)\right]^2 = 0.$$

PROOF. Let $D = \{(x_1, \dots, x_K) | \, |s_K| \le K\}$. Outside of this set, we have

$$\begin{aligned}
&\lim_{K\to\infty} KE^{K,\rho}\left[\left|\widetilde{\psi}_l\left(\frac{s_K}{K}\right) - \widetilde{\psi}_l(\rho) - \widetilde{\psi}_l(\rho)\frac{s_K}{K}\right|^2 1_{D^c}\right] \\
&\quad \le \lim_{K\to\infty} KE^{K,\rho}\left[6l^2K^{-4}|s_K|^4 + 3\sigma^{-2}l^2K^{-4}|s_K|^4\right] \qquad (15)\\
&\quad = \lim_{K\to\infty} 3l^2K^{-3}(2+\sigma^{-2})E^{K,\rho}|s_K|^4 = 0.
\end{aligned}$$

On the other hand, since $\left|\widetilde{\psi}_l(x) - \widetilde{\psi}_l(\rho) - \widetilde{\psi}_l'(\rho)(x-\rho)\right| \le C(x-\rho)^2$ for $|x| \le 1$, where C is a finite constant,

$$\begin{aligned}
&\lim_{K\to\infty} KE^{K,\rho}\left[\left|\widetilde{\psi}_l\left(\frac{s_K}{K}\right) - \widetilde{\psi}_l(\rho) - \widetilde{\psi}_l(\rho)\left(\frac{s_K}{K} - \rho\right)\right|^2 1_D\right] \qquad (16)\\
&\quad \le C^2 \lim_{K\to\infty} K^{-3}E^{K,\rho}|s_K|^4 = 0.
\end{aligned}$$

Combining (15) and (16), we prove Lemma 4.5. □

5. Proof of Theorem 1. In this section we shall prove Theorem 1 by using the estimations given in Sections 3 and 4.

For any test function $J(\cdot)$ on S, we have

$$\begin{aligned}
d\langle \zeta_N(t), J\rangle &= \frac{\sqrt{N}}{2}\sum_{i=1}^{N} N\left(J\left(\frac{i}{N}\right) - J\left(\frac{i+1}{N}\right)\right)W(x_i, x_{i+1})\,dt \\
&\quad + \frac{1}{\sqrt{N}}\sum_{i=1}^{N} N\left(J\left(\frac{i}{N}\right) - J\left(\frac{i+1}{N}\right)\right)\sigma(x_i, x_{i+1})\,d\beta_i(t) \qquad (17)\\
&= \mathrm{I}(J) + \mathrm{II}(J) + \mathrm{III}(J) + \mathrm{IV}(J) + \mathrm{V}(J).
\end{aligned}$$

Here

$$\mathrm{I}(J) = \frac{\sqrt{N}}{2}\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right)\Big(W(x_i, x_{i+1}) - \widehat{a}(\rho)\big(\phi'(x_i) - \phi'(x_{i+1})\big) - LF^i(x)\Big)dt,$$

$$\mathrm{II}(J) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \nabla J\left(\frac{1}{N}\right)\sigma(x_i, x_{i+1})\left(1 - \left(\frac{\partial\sum_{j=1}^N F^j(x)}{\partial x_i} - \frac{\partial\sum_{j=1}^N F^j(x)}{\partial x_{i+1}}\right)\right)d\beta_i,$$

$$\mathrm{III}(J) = \frac{\sqrt{N}}{2}\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right)\widehat{a}(\rho)\big(\phi'(x_i) - \phi'(x_{i+1})\big)\,dt,$$

$$\mathrm{IV}(J) = \frac{\sqrt{N}}{2}\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right)LF^i(x)\,dt,$$

$$\mathrm{V}(J) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \nabla J\left(\frac{i}{N}\right)\sigma(x_i, x_{i+1})\left(\frac{\partial\sum_{j=1}^N F^j(x)}{\partial x_i} - \frac{\partial\sum_{j=1}^N F^j(x)}{\partial x_{i+1}}\right)d\beta_i,$$

where $F = F(x_1, \dots, x_k)$ is a smooth function depending only on finitely many variables, $F^i(x) = F(x_{i+1}, \dots, x_{k+1})$ is the ith translation of F and

$$\nabla J\left(\frac{i}{N}\right) = N\left(J\left(\frac{i+1}{N}\right) - J\left(\frac{i}{N}\right)\right).$$

Choose a sequence $\{F_k\}_{k=1}^{\infty}$ such that $\widehat{a}_{F_k}(y)$ goes to $\widehat{a}(y)$ for $|y-\rho| \le M$ for some constant M. This is because of the uniform continuity of $\widehat{a}(y)$ on a bounded set.

PROOF OF THEOREM 1. We shall estimate the terms in (17). We prove Theorem 1 by establishing the following lemmas.

LEMMA 5.1.

$$\lim_{k\to\infty}\lim_{N\to\infty} E^{P_N}\left[\int_0^T (\mathrm{IV}+\mathrm{V})(J)\right]^2 = 0.$$

PROOF. For a smooth function F, we know by Itô's formula that

$$\begin{aligned}
&\frac{1}{N^{3/2}}\sum_{i=1}^{N} J\left(\frac{i}{N}\right)\left[F^i(x(t)) - F^i(x(0))\right] \\
&= \int_0^t \frac{\sqrt{N}}{2}\sum_{i=1}^{N} J\left(\frac{i}{N}\right) LF^i(x(s))\,ds \\
&\quad + \int_0^t \frac{1}{\sqrt{N}}\sum_{i=1}^{N} J\left(\frac{i}{N}\right)\sum_{r=1}^{N} \sigma(x_r(s), x_{r+1}(s))\left(\frac{\partial F^i}{\partial x_r} - \frac{\partial F^i}{\partial x_{r+1}}\right) d\beta_r(s) \\
&= \int_0^t (\mathrm{IV} + V_1).
\end{aligned}$$

Since J is smooth and F is a function of finite variables, V_1 can be substituted by V since

$$E^P\left[\int_0^t (V_1 - \mathrm{V})\right]^2 = O\left(\frac{1}{N}\right) \quad \text{as } N \to \infty.$$

Because the left-hand side goes to 0 as N goes to ∞, we are done. □

LEMMA 5.2.

$$\lim_{k\to\infty}\lim_{N\to\infty} E^{P_N}\left[\int_0^T \mathrm{I}(J)\right]^2 = 0.$$

PROOF. Using the strong law of large numbers, we have

$$\lim_{k\to\infty}\lim_{N\to\infty} E^{P_N}\left[\int_0^N \mathrm{I}\right]^2$$
$$\leq \lim_{\varepsilon\to 0}\ \limsup_{k\to\infty,\,|y-\rho|\leq\varepsilon}\ \lim_{N\to\infty}\frac{1}{4N}\Delta_{N,y}\big(A_N-\widehat{a}(y)B_N-C_N\big)^2.$$

By the calculations given in Section 3, we then have

$$\lim_{\varepsilon\to 0}\ \limsup_{k\to\infty,\,|y-\rho|\leq\varepsilon}\ \lim_{N\to\infty}\frac{1}{N}\Delta_{N,y}\big(A_N-\widehat{a}(y)B_N-C_N\big)^2$$
$$\leq \lim_{\varepsilon\to 0}\ \limsup_{K\to\infty,\,|y-\rho|\leq\varepsilon} 2\big(\widehat{a}_K(y)-\widehat{a}(y)\big)$$
$$=0,$$

by our choice of sequence $\{F_K\}$. □

LEMMA 5.3.

$$\lim_{N\to\infty}\mathrm{III}(J)=\int_S J''(\theta)\widehat{a}(\rho)h''(\rho)\zeta^{\infty}(\theta)\,d\theta\,dt.$$

PROOF. By the tightness, we know

$$\zeta_N(t)\to\zeta^{\infty}(\theta,t)\,d\theta$$

and, by Theorem 2, we have

$$\lim_{N\to\infty}\int^t \mathrm{III}=\lim_{N\to\infty}\int_0^t\frac{1}{2\sqrt{N}}\sum_{i=1}^N\Delta J\left(\frac{1}{N}\right)\widehat{a}(\rho)\Big(\phi'\big(x_i(s)\big)\Big)ds$$
$$=\lim_{N\to\infty}\int_0^t\frac{1}{2\sqrt{N}}\sum_{i=1}^N\Delta J\left(\frac{i}{N}\right)\widehat{a}(\rho)h''(\rho)(x_i-\rho)\,ds$$
$$=\frac{1}{2}\int_0^t\int_S\widehat{a}(\rho)h''(\rho)J''(\theta)\zeta^{\infty}(\theta,s)\,d\theta\,ds,$$

where

$$\Delta J\left(\frac{i}{N}\right)=N^2\left(J\left(\frac{i+1}{N}\right)+J\left(\frac{i-1}{N}\right)-2J\left(\frac{i}{N}\right)\right)$$

is the second derivative of J in the lattice sense. □

LEMMA 5.4.

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathrm{II}(J)=\int_S J'(\theta)\sigma(\rho)\,d\beta(\theta,t)\,d\theta.$$

PROOF. Since II gives a Gaussian field, we only need to show, for J_1, J_2,

$$\lim_{k\to\infty}\lim_{N\to\infty} E^{P_N}\left[\int_0^t \mathrm{II}(J_1)\int_0^t \mathrm{II}(J_2)\right] = \int_0^t\int_S \widehat{a}(\rho)J_1'(\theta)J_2'(\theta)\,d\theta\,ds. \tag{18}$$

This is because

$$\begin{aligned} &E^{P_N}\left[\int_0^t \mathrm{II}(J_1)\int_0^t \mathrm{II}(J_2)\right] \\ &= \frac{1}{N}\int_0^t\sum_{i=1}^N \nabla J_1\left(\frac{i}{N}\right)\nabla J_2\left(\frac{i}{N}\right) \\ &\quad\times EE^{\mu_N,\, s_N/N}a(x_i,x_{i+1})\left[1-\left(\frac{\partial\sum_{j=1}^N F^j(x)}{\partial x_i}-\frac{\partial\sum_{j=1}^N F^j(x)}{\partial x_{i+1}}\right)\right]^2 ds, \end{aligned} \tag{19}$$

where $s_N = \sum_{i=1}^N x_i$.

By our choice of $\{F_K\}$, the translation invariance of the measure $\mu_{N,y}$ and the strong law of large numbers, (18) is just a standard transition of (19) from the canonical state to the grand canonical state. □

Acknowledgments. I wish to thank Professors S.R.S. Varadhan and H.T. Yau for many helpful discussions. I also thank the referee for pointing out mistakes in the first version of this work.

REFERENCES

[1] CHANG, C. C. (1992). Equilibrium fluctuations for gradient particle systems. Preprint.

[2] CHANG, C. C. and YAU, H. T. (1992). Fluctuation of one-dimensional Ginzburg–Landau models. *Comm. Math. Phys.* **145** 209–234.

[3] GUO, M. Z., PAPANICOLAOU, G. C. and VARADHAN, S. R. S. (1988). Nonlinear diffusion limit for a system with nearest neighbor interaction. *Comm. Math. Phys.* **118** 31–59.

[4] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple Exclusions. *Comm. Math. Phys.* **104** 1–19.

[5] PETROV, V. V. (1975). *Sums of Independent Random Variables*. Springer, New York.

[6] QUASTEL, J. (1990). Diffusion of color in the simple exclusion process. Thesis, New York Univ.

[7] SPOHN, H. (1986). Equilibrium fluctuations for interacting Brownian motions. *Comm. Math. Phys.* **103** 1–33.

[8] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multi-dimensional Diffusion Processes*. Springer, New York.

[9] VARADHAN, S. R. S. (1990). Nonlinear diffusion limit for a system with nearest neighbor interaction-II. *Proc. Taniguchi Symp.* Kyoto.

[10] ZHU, M. (1990). Equilibrium fluctuations for one-dimensional Ginsburg–Landau lattice model. *Nagoya Math. J.* **117** 63–92.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN 48109