The Annals of Probability
1994, Vol. 22, No. 3, 1195–1126

ON THE STABILITY OF A POPULATION GROWTH MODEL WITH SEXUAL REPRODUCTION ON $\mathbf{Z}^d$, $d \geq 2$

BY HWA-NIEN CHEN

Purdue University, Calumet

We continue our study on the stability properties of a population growth model with sexual reproduction on $\mathbf{Z}^d, d \geq 2$. In the author's previous work, it was proved that in the type IV process (the two-dimensional symmetric model on $\mathbf{Z}^2$), the vacant state $\emptyset$ is stable under perturbation of the initial state (the first kind of perturbation), and it is unstable under perturbation of the birth rate (the second kind of perturbation). In this paper we prove that in the type III process on $\mathbf{Z}^2$, the vacant state $\emptyset$ is stable under the second kind of perturbation, and in three or higher-dimensional symmetric models, the vacant state $\emptyset$ is unstable under the first kind of perturbation. These results, combined with the results obtained earlier, provide a fairly complete picture concerning the stability properties of these models.

0. Introduction. We consider a population growth model on $\mathbf{Z}^d$ which is a Markov process whose state space $\mathbf{S}$ is the set of all subsets of $\mathbf{Z}^d$. Let ξ_t denote the state of the system at time $t \geq 0$, which is the set of *sites* (points in $\mathbf{Z}^d$) that are *occupied* at time t. Sometimes we will also treat ξ_t as a function from $\mathbf{Z}^d$ to $\{0,1\}$, with

$$\xi_t(x) = \begin{cases} 1, & \text{if } x \in \xi_t \quad (x \text{ is occupied}), \\ 0, & \text{if } x \notin \xi_t \quad (x \text{ is vacant}). \end{cases}$$

Therefore, the state space can also be expressed as $\mathbf{S} = \{0,1\}^{\mathbf{Z}^d}$.

The system evolves according to the following rules:

(0.1) occupied sites are vacated at a constant rate $\delta > 0$, that is, if $x \in \xi_t$, then $P(x \notin \xi_{t+s} \mid \xi_t) = \delta s + o(s)$ as $s \to 0$;

(0.2) vacant sites become occupied at rate $b_x(\xi)$, that is, if $x \notin \xi_t$, then $P(x \in \xi_{t+s} \mid \xi_t) = b_x(\xi_t)s + o(s)$ as $s \to 0$,

where $o(s) \to 0$ as $s \to 0$.

As usual, $b_x(\xi)$ and δ are called the *birth* and the *death* rates at x, respectively. In particular, the death rate in the systems under our consideration is identically 1; that is, $\delta = 1$. Depending on the different definitions on the birth rates b_x, we will get different systems.

Received June 1992; revised December 1992.

AMS 1991 *subject classification*. 60K35.

Key words and phrases. Population growth model, sexual contact process, stability properties, block renormalization.

EXAMPLE 1. The threshold contact process on $\mathbf{Z}^2$ (an example of an asexual contact process). Let

$$b_x(\xi) = \begin{cases} \lambda, & \text{if any site in } \{x \pm e_1, x \pm e_2\} \text{ is occupied,} \\ 0, & \text{otherwise,} \end{cases}$$

where $e_1 = (1,0), e_2 = (0,1)$ denote the standard basis vectors in $\mathbf{Z}^2$.

In this example, in order to produce a child particle at a vacant site x, only one site in its neighbor needs to be occupied by a parent particle.

EXAMPLE 2. Five types of sexual contact processes on $\mathbf{Z}^2$.

For $x \in \mathbf{Z}^2$, we label its neighboring sites $\{x-e_1, x-e_2\}$ as *pair* 1, $\{x+e_1, x-e_2\}$ as *pair* 2, $\{x+e_1, x+e_2\}$ as *pair* 3 and $\{x-e_1, x+e_2\}$ as *pair* 4. The birth rates $b_x(\xi)$ for each type of process are defined as follows:

Type I:	$b_x(\xi) = \lambda$	if pair 1 is occupied;
Type II(a):	$b_x(\xi) = \lambda$	if pair 1 or pair 2 is occupied;
Type II(b):	$b_x(\xi) = \lambda$	if pair 1 or pair 3 is occupied;
Type III:	$b_x(\xi) = \lambda$	if any one of the pairs $i, i = 1, 2, 3$ is occupied;
Type IV:	$b_x(\xi) = \lambda$	if any one of the pairs $i, i = 1, 2, 3, 4$ is occupied;

and for all of the above types

$$b_x(\xi) = 0 \quad \text{otherwise.}$$

In all these five types of birth rates, in order to produce a child particle at a site x, a pair of neighboring sites needs to be occupied by parent particles. That is why they are said to have sexual reproduction. The type IV system is often called the symmetric model.

The above rules (0.1) and (0.2) specify a unique Markov process [see Liggett (1985), Chapter 1]. Furthermore, all processes can be constructed explicitly by using a graphical representation that goes back to Harris (1978). A detailed construction which is well suited for our purpose can be found in Durrett and Gray (1990). (We will give a brief description of this construction at the end of this section.) It is a consequence of this construction that there exists a single probability space (Ω, F, P) such that all the growth models under consideration in this paper can be defined jointly on (Ω, F, P). This fact enables us to make comparisons between processes with different rates and different initial states. For example, for any given set of rates described by the above statements (0.1) and (0.2), if we use ξ_t^A and ξ_t^B to denote the states of the system at time t when the initial states are A and B, respectively, we can define the processes ξ_t^A and ξ_t^B on (Ω, F, P) in such a way that if $A \subset B$, then $\xi_t^A \subset \xi_t^B$ for all $t \geq 0$. Also, if ξ_t is a process with birth rates $b_x(\xi)$ and death rate δ, and if ζ_t is another process with death rate $\delta^* \geq \delta$ and birth rate $b_x^*(\xi) \leq b_x(\xi)$ for all $x \in \mathbf{Z}^2$ and $\xi \in \mathbf{S}$, then ξ_t and ζ_t can be defined in such a way that $\zeta_t \subset \xi_t$ for all $t \geq 0$, provided both processes have the same initial state. In this case we often simply say that ξ_t *dominates* ζ_t.

A system is called *attractive* if the birth and death rates b_x and d_x satisfy the condition that $b_x(\xi) \geq b_x(\eta)$ and $d_x(\xi) \leq d_x(\eta)$, whenever $\eta \subset \xi \subset \mathbf{Z}^2$. In the systems described in Examples 1 and 2, the death rates are identically 1 and the birth rates b_x are nondecreasing functions of the number of occupied sites in the set $\{x \pm e_1, x \pm e_2\}$, so the above condition is satisfied. It was first shown by Holley (1972) that systems with attractive rates have certain useful monotonicity properties. Let ξ_t^0 and ξ_t^1 denote the state of the system at time t when the initial states are $\emptyset$ and $\mathbf{Z}^2$, respectively. Then, for all $A \subset \mathbf{Z}^2$ and $0 \leq s < t < \infty$, $P(\xi_t^0 \cap A \neq \emptyset) \geq P(\xi_s^0 \cap A \neq \emptyset)$ and $P(\xi_t^1 \cap A \neq \emptyset) \leq P(\xi_s^1 \cap A \neq \emptyset)$. Thus ξ_t^0 and ξ_t^1 converge weakly ($\Rightarrow$) as $t \to \infty$ to stationary distributions which we denote as ξ_∞^0 and ξ_∞^1, respectively. For the processes in Examples 1 and 2, we have $\xi_t^0 = \emptyset$ for all t, hence $\xi_\infty^0 = \delta_\emptyset$ (the point mass concentrated on the state $\emptyset$). Thus $\delta_\emptyset$ is a trivial equilibrium. Let $\rho(\lambda) = \lim_{t\to\infty} P(0 \in \xi_t^1) = P(0 \in \xi_\infty^1)$. If $\rho(\lambda) = 0$, then $\xi_\infty^1 = \xi_\infty^0 = \delta_\emptyset$, and, by attractiveness, it follows that, for all initial configurations, $\xi_t \Rightarrow \delta_\emptyset$ as $t \to \infty$. On the other hand, if $\rho(\lambda) > 0$, then $\xi_\infty^1 \neq \xi_\infty^0$. Let $\lambda_c = \inf\{\lambda\colon \rho(\lambda) > 0\}$. Then $\xi_\infty^1 = \xi_\infty^0 = \delta_\emptyset$ if $\lambda < \lambda_c$ and $\xi_\infty^1 \neq \xi_\infty^0$ if $\lambda > \lambda_c$. In both Examples 1 and 2, it was proved that $0 < \lambda_c < \infty$. [See Durrett and Gray (1990).] Therefore, we know that, for both Examples 1 and 2, $\xi_\infty^0 = \delta_\emptyset$ is a trivial equilibrium for the systems regardless of the value of λ, whereas there exists a critical value $\lambda_c \in (0, \infty)$ such that ξ_∞^1 is nontrivial and distinct from ξ_∞^0 when $\lambda > \lambda_c$. Besides the behaviors of ξ_t^0 and ξ_t^1 as $t \to \infty$, it is also interesting to investigate the behavior of ξ_t as $t \to \infty$, when $\lambda > \lambda_c$ and ξ_t starts from simple initial distributions other than the ones concentrated at $\emptyset$ or $\mathbf{Z}^2$. In particular, we may consider the process ξ_t^p whose initial distribution ξ_0^p satisfies the conditions that $\{x \in \xi_0^p\}$, $x \in \mathbf{Z}^2$, are independent, and $\forall\, x \in \mathbf{Z}^2$, $P(x \in \xi_0^p) = p$. This initial distribution can be considered as a perturbation of the absorbing state $\emptyset$. If $\xi_t^p \Rightarrow \delta_\emptyset$ as $t \to \infty$, we say $\delta_\emptyset$ (or the vacant state $\emptyset$) is *stable* under perturbation of the initial state, otherwise it is *unstable*. This kind of perturbation and stability will be called the first kind of perturbation and stability, respectively. Another kind of perturbation that interests us is to add a small quantity $\beta > 0$ to all birth rates ("spontaneous births at rate" β). Namely, for each process described in Examples 1 and 2, the birth rates for the corresponding new system are equal to $b_x(\xi) + \beta$. Let $\xi_t^{0,\beta}$ and $\xi_t^{1,\beta}$ denote the states at time t for the system with spontaneous births at rate β and initial states $\emptyset$ and $\mathbf{Z}^2$, respectively. It is clear that the new systems are still attractive. As we mentioned above, the monotonicity properties of systems with attractive rates imply that, as $t \to \infty$, $\xi_t^{0,\beta}$ and $\xi_t^{1,\beta}$ converge to stationary distributions (denoted as) $\xi_\infty^{0,\beta}$ and $\xi_\infty^{1,\beta}$, respectively. The objective is to study the behavior of $\xi_\infty^{0,\beta}$ as $\beta \to 0$. If, as $\beta \to 0$, $\xi_\infty^{0,\beta} \Rightarrow \xi_\infty^0 = \delta_\emptyset$, then we say $\delta_\emptyset$ is *stable* under perturbation of the birth rate, otherwise it is *unstable*. This kind of perturbation and stability will be called the second kind of perturbation and stability, respectively.

It is known that, in the asexual contact process, $\delta_\emptyset$ is unstable under either kind of perturbation. [See, e.g., Durrett and Gray (1990) or Durrett (1985).]

For the systems with sexual reproduction, the results concerning the above two kinds of stability of $\delta_{\emptyset}$ were first studied in Durrett and Gray (1990). They proved the following two results for the type I system.

1. There exists a $p^* \in (0,1)$ that is independent of λ, such that if $p < p^*$, then $\xi_t^p \Rightarrow \delta_{\emptyset}$ as $t \to \infty$.
2. For any $\lambda > 0$, $\xi_\infty^{0,\beta} \Rightarrow \delta_{\emptyset}$ as $\beta \to 0$.

These results mean that in the type I system, $\delta_{\emptyset}$ is stable under either kind of perturbation, which is contrary to what happens in the asexual contact process.

All sexual contact processes in Example 2 have an important feature that is quite different from the asexual contact process. A sexual contact process starting with any finite set will die out almost surely. That is, if we let $\lambda_f = \inf\{\lambda\colon P(\xi_t^A \neq \emptyset \text{ for all } t) > 0 \text{ for some finite set } A\}$, then $\lambda_f = \infty$ if ξ_t is a sexual contact process, and $\lambda_f < \infty$ if ξ_t is an asexual contact process. Based on this fact and the above results 1 and 2 obtained by Durrett and Gray, one may naturally raise a more general question: can we conclude that in a population growth model as described in the beginning of this paper, $\delta_{\emptyset}$ is stable under either kind of perturbation if and only if $\lambda_f = \infty$? As discussed above, as far as the asexual contact process and the type I sexual contact process are concerned, this speculation seems plausible. However, in Chen (1992) the two kinds of stability problems were further studied for the other types of sexual contact processes in Example 2, and we will see soon that the criteria speculated are not valid in general. By applying a new method called successive block renormalization, the following two results were proved in that paper.

THEOREM 1. *Let ξ_t^p denote the state of the type IV process at time t with initial distribution ξ_0^p described as follows: the events $\{x \in \xi_0^p\}, x \in \mathbf{Z}^2$, are independent, and $\forall\, x \in \mathbf{Z}^2$, $P(x \in \xi_0^p) = p$. For any $\lambda \in (1,\infty)$, if $p > 0$ is sufficiently small (p may depend on λ), then, for large t,*

$$P(0 \in \xi_t^p) \leq t^{-c \log_{2\lambda}(1/p)},$$

where c is a positive constant independent of λ and p.

THEOREM 2. *Let $\xi_t^{0,\beta}$ denote the state of the type IV process at time t with spontaneous birth at rate $\beta > 0$ and initial state $\emptyset$. Let $\xi_\infty^{0,\beta}$ denote its limiting stationary distribution as $t \to \infty$. Suppose that λ is sufficiently large. Then*

$$\lim_{\beta \to 0} P(0 \in \xi_\infty^{0,\beta}) > 0.$$

Theorems 1 and 2 show that in the type IV system, $\delta_{\emptyset}$ is stable under the first kind of perturbation, but it is unstable under the second kind of perturbation. Theorem 2 proves that the "if part" of the criteria speculated previously is false concerning the second kind of stability. Since the type IV system dominates all other four types of systems, it follows that in all other four types of systems, $\delta_{\emptyset}$ is also stable under the first kind of perturbation. Thus, to offer a complete picture regarding the two kinds of stability properties for all five types of systems described in Example 2, we only need to determine the second kind of stability of $\delta_{\emptyset}$ in the type II and type III systems. This is the first objective of this paper. The second objective of this paper is to study the first kind of stability properties for a system with symmetric sexual reproduction on $\mathbf{Z}^d$, for $d \geq 3$. This system is analogous to the type IV system on $\mathbf{Z}^2$. Its birth rates $b_x(\xi)$ are defined as follows:

$$b_x(\xi) = \begin{cases} \lambda, & \text{if } x \text{ has two occupied neighbors of the form} \\ & \{x \pm e_i, x \pm e_j\},\ i, j \in \{1, 2, \dots, d\},\ i \neq j, \\ 0, & \text{otherwise,} \end{cases}$$

where $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0), \dots, e_d = (0, 0, \dots, 1)$ denote the standard basis in $\mathbf{Z}^d$. Notice that, when $d \geq 3$, λ_f is still equal to ∞. We intend to prove that when $d \geq 3$, in the system with symmetric sexual reproduction on $\mathbf{Z}^d$, $\delta_{\emptyset}$ is unstable under the first kind of perturbation and thus the "if part" of the criteria speculated previously is also false in regard to the first kind of stability.

The paper will consist of two sections. In Section 1 we will prove that in the type III system $\delta_{\emptyset}$ is stable under the second kind of perturbation. The result is formulated as the following theorem.

THEOREM 3. *Let $\xi_t^{0,\beta}$ be the state of the type III process on $\mathbf{Z}^2$ at time t with spontaneous births at rate β and initial state $\emptyset$. Let $\xi_\infty^{0,\beta}$ denote its limiting stationary distribution as $t \to \infty$. Then $\forall\, \lambda \in (0, \infty)$, $\xi_\infty^{0,\beta}$ converges weakly ($\Rightarrow$) to $\delta_{\emptyset}$ as $\beta \to 0$.*

In Section 2 we will prove that when $d \geq 3$, in the symmetric model on $\mathbf{Z}^d$, $\delta_{\emptyset}$ is no longer stable under the first kind of perturbation. The result is formulated as the following theorem.

THEOREM 4. *Let ξ_t^p, $t \geq 0$, denote the system with symmetric sexual reproduction on $\mathbf{Z}^d, d \geq 3$, described in the preceding paragraph, with initial distribution ξ_0^p defined as follows: for all $x \in \mathbf{Z}^d$, the events $\{x \in \xi_0^p\}$ are independent and $P(x \in \xi_0^p) = p$. Suppose λ is sufficiently large. Then $\forall\, p > 0$, $\lim_{t\to\infty} P(x \in \xi_t^p) > 0$.*

A crucial difference between the type III and the type IV systems on $\mathbf{Z}^2$ can be illustrated heuristically as follows. When λ is sufficiently large, in the type III system, an isolated finite region occupied by individuals will die out much more quickly than that in the type IV system. For instance, consider that an $N \times N$ square region is entirely occupied but all remaining sites in $\mathbf{Z}^2$ are vacant. Then, as we have already shown in the proof of Theorem 2 in Chen (1992), in the type IV system, with probability greater than $1 - \exp\{-\alpha(\lambda)N\}$, the region can sustain a high density for $\exp\{CN\}$ time units, where $\alpha(\lambda) > 0$ when λ is sufficiently large. However, as we will prove in subsection 1.1. of this paper, in the type III process, the region will be wiped out by death at a linear rate even though λ is large. Therefore, when spontaneous births $\beta > 0$ are added to the birth mechanism for each type of system, the consequences are drastically different. In the type IV system, the isolated islands of occupied sites can sustain long enough so that the spontaneous births occurring at their boundaries will have plenty of time to accumulate and produce children to make them continue to grow. In contrast, in the type III system, those isolated islands of individuals are wiped out too rapidly so that the spontaneous births will not have a chance to help them to survive.

For the symmetric system ξ_t^p on $\mathbf{Z}^d$, $d \geq 2$, described in Theorems 1 and 4, a necessary condition for a particular occupied island to continue to grow is that there are always individuals attached at the boundaries of the growing island in question. Loosely speaking, when λ is sufficiently large, the average amount of time for a typical growing island to fill a d-dimensional box with edge length n is at most n^d/λ, and an isolated occupied d-dimensional box with edge length n can survive about $\exp\{Cn^{d-1}\}$ time units with large probability. Let us use a specific scenario to visualize the way that the population evolves in the system ξ_t^p on $\mathbf{Z}^d$. Suppose that a population island has already filled a box with edge length N centered at the origin (we call it the "main island"). By the argument given above, the "main island" will have a good chance to continue to grow, if each of its two-dimensional faces of boundaries (it contains N^{d-1} sites) contains a box with edge length at least $C^{-1}(\log(N^d\lambda^{-1}))^{1/(d-1)}$, which is "well structured" initially (we will call such a box a "minor island"). When the dimension d increases, the number of sites contained in the main island's two-dimensional faces of boundaries also increase, and the minimum size required for the minor islands decreases. Based on an intuitive observation, a lower-dimensional well-structured subset in a d-dimensional box will have a good chance to produce offspring to fill the entire box. Thus the probability that a particular d-dimensional box is initially well-structured is nondecreasing with respect to the dimension d. Also, this probability increases when the edge length of the box decreases. This crude analysis suggests that the probability the main island will continue to grow is expected to be an increasing function of the dimension d. Hence it is not unreasonable to predict that there is an integer $d_0 > 2$, such that, when $d \geq d_0$, the behavior of the symmetric system ξ_t^p on $\mathbf{Z}^d$ will change drastically. In other words, when $d \geq d_0$, in the symmetric system on $\mathbf{Z}^d$, $\delta_\emptyset$ will become unstable under the first kind of perturbation. Theorem 4 proves this prediction is indeed correct and actually $d_0 = 3$.

The method employed in the proof of Theorem 3 is a successive block renormalization procedure. By applying the block renormalization scheme inductively, we will obtain for each $k = 1, 2, \ldots$, a new level k discrete-time process that has much simpler birth and death mechanisms. The procedure allows us to study the asymptotic behavior of the original process by studying the level k processes. The method of block renormalization is a powerful tool for investigating the asymptotic behavior of a large class of processes. For a survey of this method, see Bramson and Gray (1992). The method applied in the proof of Theorem 4 is quite similar to what was used in the proof of Theorem 2 in Chen (1992). We will first define a specific kind of configuration structure and prove that the probability that the initial configuration ξ_0^p is such a structure is very large. Then we will use a coupling argument to prove that, when λ is sufficiently large, if the system starts with one such configuration, then, with large probability, it will be quickly filled by occupied individuals. Combining these two steps, we know that as the system ξ_t^p evolves, with large probability, it will quickly posses a high density. An inductive procedure will prove that, with large probability, the system ξ_t^p will sustain a high density and thus survive forever.

Combining the results obtained from Theorems 1 to 4, the stability properties concerning the population growth model with sexual reproduction on $\mathbf{Z}^d$, $d \geq 2$, can be summarized as follows.

1. In the systems of type I, II or III on $\mathbf{Z}^2$, the vacant state $\emptyset$ is stable under either the first or the second kind of perturbation.
2. In the type IV system on $\mathbf{Z}^2$, the vacant state $\emptyset$ is stable under the first kind of perturbation but is unstable under the second kind of perturbation.
3. When $d \geq 3$, in the system with symmetric sexual reproduction on $\mathbf{Z}^d$, the vacant state $\emptyset$ is unstable under either the first or the second kind of perturbation.

The rest of the paper will be devoted to the proofs of Theorems 3 and 4. Throughout, we will construct the process under consideration ξ_t, $t \geq 0$, and all other processes on the same probability space (Ω, F, P) by using the same construction of the graphical representation as introduced in Section 0 of Chen (1992). Details can be found, for instance, in Durrett and Gray (1990). To make reference easier, we recall the construction briefly as follows.

For each $x \in \mathbf{Z}^d$, let $S_n(x)$ and $T_n(x), n \geq 1$, be independent Poisson processes with rate 1 and λ, respectively. We label certain points in the space–time graph $\mathbf{Z}^d \times [0, \infty)$ using the Poisson processes:

1. Mark the points $D_x = \{(x, S_n(x)) \colon n \geq 1\}$ with δ's (for death), and interpret the δ as to vacate site x at time $S_n(x)$, if x is occupied.
2. Mark the points $B_x\{(x, T_n(x)) : n \geq 1\}$ with λ's (for life), and interpret the λ as a birth at site x at time $T_n(x)$, provided the necessary conditions are met. That is, $x \notin \xi_{T_n(x)-}$ and $b_x(\xi_{T_n(x)}) = \lambda$, where $\xi_{T_n(x)-}$ denotes the limit of ξ_t as $t \uparrow T_n(x)$.

Having marked the space–time graph, we can compute the evolution of the process according to the rules for interpreting the δ's and λ's given in rules 1 and 2. Notice that this construction takes care of all the processes without spontaneous births under our consideration. For the processes with spontaneous births at rate $\beta > 0$, we need to augment the construction to allow for the spontaneous births. For each $x \in \mathbf{Z}^d$, we let $U_n(x)$, $n \geq 1$, be a Poisson process with rate β, independent of the processes $S_n(x)$ and $T_n(x)$. There is now a third rule (rule 3) in the description of the process, corresponding to spontaneous births at rate $\beta > 0$:

3. Mark the points $B_x^* = \{(x, U_n(x)) : n \geq 1\}$ with β's (for birth), and interpret the β as a (spontaneous) birth at site x at time $U_n(x)$ if x is vacant.

It is also guaranteed that the process constructed by using graphical representation is unique [for details, see Durrett and Gray (1990)].

1. Proof of Theorem 3. To prove Theorem 3, it is necessary and sufficient to prove that, $\forall\, x \in \mathbf{Z}^2$ and $\lambda \in (0, \infty)$,

$$\lim_{\beta \to 0} P\big(x \in \xi_\infty^{0,\beta}\big) = 0. \tag{1.1}$$

By translation invariance, to prove (1.1), it suffices to prove that

$$\lim_{\beta \to 0} P\big(0 \in \xi_\infty^{0,\beta}\big) = 0. \tag{1.2}$$

Since ξ_t, $t \geq 0$, is attractive, for each $\beta > 0$ and $\lambda \in (0, \infty)$, $P(0 \in \xi_s^{0,\beta}) \leq P(0 \in \xi_t^{0,\beta})$ if $s \leq t$. Hence, to prove (1.2), it suffices to prove that there is a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k \to \infty$ as $k \to \infty$, such that

$$\lim_{k \to \infty} P\big(0 \in \xi_{t_k}^{0,\beta}\big) = \rho(\beta) \quad \text{and} \quad \lim_{\beta \to 0} \rho(\beta) = 0. \tag{1.3}$$

The main weapon to prove (1.3) is the procedure of block renormalization. The proof consists of three parts: subsections 1.1 to 1.3. Subsection 1.1 concerns some preliminaries. We will prove that an occupied isolated $N \times N$ square will die out at a linear rate. In subsection 1.2 we will apply the procedure of block renormalization. We will first partition the $\mathbf{Z}^2$ lattice into square regions (blocks) and rescale the $\mathbf{Z}^2$ lattice by regarding each block as a "site." We will call the original lattice the level 0 lattice and the rescaled lattice the level 1 lattice. The time units will be rescaled accordingly. We will then define a discrete-time process that has much simpler birth and death mechanisms. The new process will be called the level 1 process. Roughly speaking, the birth and death mechanisms of level 1 process will be defined based on the following guidelines. For a particular partition block described previously, we will regard its corresponding site at level 1 as occupied at level 1 time $t[1] = m$, provided that any of the following three things happens in that block in the mth time interval: (b.1) at least two spontaneous births occur at two different sites; (b.2) although only a single spontaneous birth occurs at a certain site, the resulting new particle survives too long; and (b.3) significant amounts of particles spread from the neighboring blocks. After the meaning of birth is defined at level 1, we will regard a level 1 site that is occupied at $t[1] = m$ as vacated (by death) at $t[1] = m + 1$, provided both of the following, (d.1) and (d.2), happen in its corresponding block at level 0 in the $(m+1)$st time interval: (d.1) none of the above (b.1) to (b.3), occurs; and (d.2) the corresponding block at level 0 is wiped out by level 0 death. We will repeat this same procedure inductively and obtain for each $k = 1, 2, \dots$ a new level k process. In subsection 1.3 we will conclude our proof by examining the relationship between the level 0 process and the level k process and obtain the desired estimate formulated in (1.3) from the level k process. To implement our scheme, we first introduce the following preliminaries.

1.1. *Preliminaries. Some useful facts concerning the type III process.* Let $\Lambda(N)$ denote the square region $(-N/2, N/2]^2$ and let $\xi_t^{\Lambda(N)}$ denote the type III process with initial state $\Lambda(N)$. (For simplicity, in what follows we will often omit "$\cap \mathbf{Z}^2$" in the description of a $\mathbf{Z}^2$ region, unless such a distinction is necessary.) By the nature of the birth mechanism, $\xi_t^{\Lambda(N)}$ will always be confined to $\Lambda(N)$. That is, $\{\xi_t^{\Lambda(N)} = \emptyset\} = \{\xi_t^{\Lambda(N)} \cap \Lambda(N) = \emptyset\}$ for all $t \geq 0$. Let $\zeta_t = \mathbf{Z}^2 \backslash \xi_t$, $t \geq 0$. Then for any set $A \subset \mathbf{Z}^2$, $x \notin \xi_t^A$ if and only if $x \in \zeta_t^{\mathbf{Z}^2 \backslash A}$, for all $x \in \mathbf{Z}^2$ and $t \geq 0$.

Let $T(\Lambda(N)) = \inf\{t\colon \xi_t^{\Lambda(N)} = \emptyset\} = \inf\{t\colon \xi_t^{\Lambda(N)} \cap \Lambda(N) = \emptyset\} = \inf\{t\colon \Lambda(N) \subset \zeta_t^{\mathbf{Z}^2 \backslash \Lambda(N)}\}$. We want to study some important properties concerning $T(\Lambda(N))$. In words, $T(\Lambda(N))$ is the first time that the initially occupied region $\Lambda(N)$ is "wiped out" by death of the individuals. Notice that, although $\xi_t^{\Lambda(N)} \subset \Lambda(N)$ for all $t \geq 0$, it will eventually die out; how long it can survive is really vital in this study. For instance, in the type IV system, the relation $\xi_t^{\Lambda(N)} \subset \Lambda(N)$, $t \geq 0$, still holds, but, as mentioned in Section 0, $P(T(\Lambda(N)) > \exp\{CN\}) \geq 1 - \exp\{-\alpha(\lambda)N\}$, where $\alpha(\lambda) > 0$ when λ is sufficiently large. In the type III process $\xi_t^{\Lambda(N)}$, $t \geq 0$, the nature of the birth mechanism is quite different. In this case, if a death occurs at the site $(N/2, -N/2)$, it will be permanently vacant. In general, suppose that $x \in \Lambda(N)$ and $x + e_1, x - e_2$ have both been permanently vacated. Then x will be permanently vacated if a death occurs at it. Thus, if we let $\eta_t, t \geq 0$, denote the type I process with death rate identically 0 and birth rate $b_x(\xi)$ as follows:

$$b_x(\xi) = \begin{cases} 1, & \text{if } x + e_1 \text{ and } x - e_2 \text{ are occupied,} \\ 0, & \text{otherwise,} \end{cases}$$

then $\zeta_t^{\mathbf{Z}^2 \backslash \Lambda(N)}$ dominates $\eta_t^{\mathbf{Z}^2 \backslash \Lambda(N)}$ for all $t \geq 0$ in the sense described in Section 0. The type I process has been studied in Durrett and Gray (1990). From a result included in their proof of Theorem 1 [which is quoted as Lemma 6 in Chen (1992)], it follows that we can find a constant $\mu \in (1, \infty)$, μ independent of λ, such that

$$P\Big(T\big(\Lambda(N)\big) > \mu N\Big) \leq \exp\big\{-C(\mu)N\big\}, \tag{1.4}$$

where $C(\mu) > 0$. From (1.4) we know that the type III process has drastically different behavior from the type IV process.

1.2. *The procedure of block renormalization.* In terms of the process $\xi_t^{0,\beta}$ ($\beta > 0$, sufficiently small), we will define for each $k = 1, 2, \dots$ a new discrete-time process $\xi[k]_{t[k]}^{0,\beta_k}$, $t[k] = 0, 1, \dots$, using the following procedure of block renormalization.

STEP 1. For $k = 1$, we consider the $\mathbf{Z}^2$ lattice as a subset of $\mathbf{R}^2$ and partition $\mathbf{R}^2$ into a lattice of $L_0 \times L_0$ squares, where $L_0 = 2\lfloor 1/(2\beta^a) \rfloor - 1, a$ is a fixed constant such that $0 < a < 1/8$ and $\lfloor x \rfloor$ indicates the greatest integer less than or equal to x. The squares are so situated that one of them is centered at the origin. That is, the level 1 origin $O[1]$ corresponds to the level 0 region

$(-L_0/2, L_0/2]^2$, which is denoted by $\Lambda(O[1])$. We obtain the level 1 lattice $\mathbf{Z}^2[1]$ by regarding each square as a site. We will define a new level 1 process $\xi[1]_{t[1]}$ on $\mathbf{Z}^2[1]$. The time parameter $t[1]$ of the level 1 process is discrete, $t[1] \in \{0, 1, \ldots\}$. We scale time so that one time unit at level 1 equals τ_0 level 0 time units, where $\tau_0 = \mu L_0$ and μ is the same constant introduced in (1.4). Since the original level 0 process starts with $\emptyset$, it is natural to choose $\emptyset$ as the initial state of $\xi[1]_{t[1]}$ also. That is, $\xi[1]_{O[1]} = \emptyset$. Now let us define the birth and death mechanisms for $\xi[1]_{t[1]}$. We want them to be translation invariant, so we will specify a set of rules to define the meaning of birth and death for the level 1 origin $O[1]$ and then apply them to other sites in $\mathbf{Z}^2[1]$ in a translation-invariant manner. To begin with, we introduce the following definitions.

DEFINITION 1. The event $\Delta^{(m)}[1], m = 0, 1, \ldots$.
For $m = 0, 1, \ldots$, let $\xi_{t(m)+u}^{(\Lambda, t(m))}$ denote the state of the type III system at $t(m)+u$, assuming the state of the system at $t(m)$ is Λ, where $t(m) = m\tau_0$, $\Lambda = \Lambda(O[1])$. Let $T^{(m)} = \inf\{u > 0 : \xi_{t(m)+u}^{(\Lambda, t(m))} = \emptyset\}$. Define $\Delta^{(m)}[1] = \{T^{(m)} \le \tau_0\}$.

DEFINITION 2. The event $G^{(m)}[1], m = 0, 1, \ldots$, is defined as follows.
At level 0, there are (at least) two sites x' and x'' in $\Lambda(O[1])$, such that $(x', t') \in B^*_{x'}$ and $(x'', t'') \in B^*_{x''}$ for some $t', t'' \in (m\tau_0, (m+1)\tau_0]$. Here the space–time set B^*_x is defined in the description of graphical representation at the end of Section 0.

DEFINITION 3. The event $G^{*(m)}[1]$, $m = 0, 1, 2, \ldots$, is defined as follows.
There is (at least) on site $x \in \Lambda(O[1])$ such that $(x, s) \in B^*_x$ for some $s \in (m\tau_0, (m+1)\tau_0]$, and x remains occupied for all $t \in (s, s+\beta^{-b}]$, where b is a fixed constant and $0 < b < a/2$.

DEFINITION 4. The event $H^{(m)}[1], m = 0, 1, \ldots$.
To define the event $H^{(m)}[1]$, we let R_i, $i = 1, 2, 3, 4$, denote the four border strips with width 2 of $\Lambda(O[1])$ described as follows:

$$R_1 = (L_0/2 - 2, L_0/2] \times (-L_0/2, L_0/2],$$
$$R_2 = (-L_0/2, L_0/2] \times (L_0/2 - 2, L_0/2],$$
$$R_3 = (-L_0/2, -L_0/2 + 2] \times (-L_0/2, L_0/2],$$
$$R_4 = (-L_0/2, L_0/2] \times (-L_0/2, -L_0/2 + 2].$$

For $m = 0, 1, \ldots$, we define the events $H_i^{(m)}[1], i = 1, 2, 3, 4$, as follows:

$H_1^{(m)}[1]$: R_1 is crossed by the spreading of individuals from the region$(L_0/2, 3L_0/2, 3L_0/2] \times (-L_0/2, L_0/2]$ during the time interval $(m\tau_0, (m+1)\tau_0]$.

The rest of the events $H_i^{(m)}[1]$, $i = 2, 3, 4$, are defined accordingly. Finally, we let $H^{(m)}[1] = (\bigcup_{i=1}^4 H_i^{(m)}[1])$.

We are now ready to define the meaning of birth an death for the level 1 origin $O[1]$. For each $m = 0, 1, \ldots$, we first suppose that $O[1]$ is vacant at time $t[1] = m$. We say that $O[1]$ is *occupied* at time $t[1] = m + 1$ if the event $B^{(m)}[1] = G^{(m)}[1] \cup G^{*(m)}[1] \cup (\Delta^{(m)}[1])^c \cup H^{(m)}[1]$ occurs.

Next, we suppose that $O[1]$ is occupied at $t[1] = m$. We say that $O[1]$ is *vacated* at time $t[1] = m + 1$ if the complement of $B^{(m)}[1]$, that is, $\Delta^{(m)}[1] \cap (G^{(m)}[1] \cup G^{*(m)}[1])^c \cap (H^{(m)}[1])^c$ occurs.

After the birth and death mechanisms of the level 1 process are defined, we want to evaluate, for each site $x[1] \in \mathbf{Z}^2[1]$, the birth probability β_1 and the death probability $\delta_1 = 1 - \beta_1$ for the process $\xi[1]_{t[1]}^{0,\beta_1}$. Actually, we wish to show that when β is sufficiently small, $\beta_1 < \beta^{1+\gamma}$ for some constant $\gamma > 0$. But we will first apply our procedure of block renormalization successively to obtain a level k process for each $k = 2, 3, \ldots$, which is illustrated as follows.

STEP 2. After the level 1 process is defined, we repeat the same procedure to define the level k processes ($k = 2, 3, \ldots$). The strategies are essentially the same except for some minor adjustments. Suppose that $\xi[k]_{t[k]}^{0,\beta_k}$ has been well defined on the level k lattice with $\beta_k < \beta_{k-1}^{1+\gamma}$, where γ is the same real number as in Step 1 and $\delta_k = 1 - \beta_k$. To define $\xi[k+1]_{t[k+1]}^{0,\beta_{k+1}}$, we rescale the level k lattice and level k time once more. That is, we partition the level k lattice $\mathbf{Z}^2[k]$ into $L_k \times L_k$ square regions with $L_k = 2\lfloor 1/(2\beta_k^a) \rfloor - 1$ (level k units). Each square region is regarded as a site of the level $k + 1$ lattice. The square regions are so situated that the level $k + 1$ origin $O[k + 1]$ corresponds to the level k square region $(-L_k/2, L_k/2]^2$, which is denoted by $\Lambda^{(k)}(O[k + 1])$. Moreover, we let one time unit at level $k + 1$ equal τ_k level k time units, where $\tau_k = L_k$. The constant a is the same as in Step 1. The initial state of the level $k + 1$ process is chosen to be $\xi[k+1]_{O[k+1]}^{0,\beta_{k+1}} = \emptyset$, just as the initial state of the level k process.

To define the birth and death mechanisms of the level $k + 1$ process, we first introduce the following definitions, which are parallel to Definitions 1 to 4.

DEFINITION 5. The event $\Delta^{(m)}[k + 1], m = 0, 1, \ldots.$

The event $\Delta^{(m)}[k + 1], m = 0, 1, \ldots$, is defined in much the same manner as the event $\Delta^{(m)}[1]$. With a little abuse of notation, let $\xi_{t(m)+u}^{(\Lambda, t(m))}$ denote the state of the type III system at $t(m) + u$, assuming the state of the system at $t(m)$ is Λ, where $t(m) = m\tau_k\tau_{k-1}\cdots\tau_0$, $\Lambda = \Lambda(O[k + 1])$. Let $T^{(m)} = \inf\{u > 0 : \xi_{t(m)+u}^{(\Lambda, t(m))} = \emptyset\}$. Define $\Delta^{(m)}[k + 1] = \{T^{(m)} \le \tau_k\tau_{k-1}\cdots\tau_0\}$.

DEFINITION 6. The event $G^{(m)}[k + 1], m = 0, 1, \ldots$, is defined as follows.

At level k there are (at least) two sites $x'[k]$ and $x''[k]$ in $\Lambda^{(k)}(O[k+1])$ at which the level k births occur at time $t'[k]$ and $t''[k]$, respectively, where $t'[k]$ and $t''[k]$ are some integers in the interval $(m\tau_k, (m + 1)\tau_k]$.

DEFINITION 7. The event $G^{*(m)}[k + 1], m = 0, 1, \ldots$, is defined as follows.

At some site $x[k] \in \Lambda^{(k)}(O[k + 1])$, a birth occurs at $t[k] \in (m\tau_k, (m + 1)\tau_k]$, and the resulting occupied site remains occupied for longer than one level k unit of time.

DEFINITION 8. The event $H^{(m)}[k+1], m = 0, 1, \ldots$.

To define the event $H^{(m)}[k+1]$, we consider the region $\Lambda(O[k+1]) = (-M_k/2, M_k/2]^2$, where $M_k = \prod_{i=0}^{k} L_i$, which is the corresponding region of the level $k+1$ origin $O[k+1]$ at level 0. Let $R_{k,i}, i = 1, 2, 3, 4$, denote the subset of $\Lambda(O[k+1])$ described as follows.

$$R_{k,1} = \Big((1-\beta^{a/2})M_k/2, M_k/2\Big] \times \big(-M_k/2, M_k/2\big],$$

$$R_{k,2} = \big(-M_k/2, M_k/2\big] \times \Big((1-\beta^{a/2})M_k/2, M_k/2\Big],$$

$$R_{k,3} = \Big(-M_k/2, -(1-\beta^{a/2})M_k/2\Big] \times \big(-M_k/2, M_k/2\big],$$

$$R_{k,4} = \big(-M_k/2, M_k/2\big] \times \Big(-M_k/2, -(1-\beta^{a/2})M_k/2\Big].$$

For $m = 0, 1, \ldots$, we define the event $H_i^{(m)}[k+1], i = 1, 2, 3, 4$, as follows.

$H_1^{(m)}[k+1]$: At level $0, R_{k,1}$ is crossed by the spreading of individuals from the region $(M_k/2, 3M_k/2] \times (-M_k/2, M_k/2]$ during the time interval $(m\tau_k\tau_{k-1}\cdots\tau_0, (m+1)\tau_k\tau_{k-1}\cdots\tau_0]$.

The rest of the events $H_i^{(m)}[k+1]$, $i = 2, 3, 4$, are defined accordingly. Finally, we let

$$H^{(m)}[k+1] = \left(\bigcup_{i=1}^{4} H_i^{(m)}[k+1]\right) \cap \left(G^{(m)}[k+1] \cup G^{*(m)}[k+1]\right)^c.$$

We are now ready to define for each $k = 1, 2, \ldots$, the meaning of birth and death for the level $k+1$ origin $O[k+1]$ in the process $\xi[k+1]_{t[k+1]}^{0,\beta_{k+1}}$.

For $m = 0, 1, \ldots$, we first suppose that $O[k+1]$ is vacant at time $t[k+1] = m$. We say $O[k+1]$ is *occupied* at $t[k+1] = m+1$ if the event $B^{(m)}[k+1] = G^{(m)}[k+1] \cup G^{*(m)}[k+1] \cup (\Delta^{(m)}[k+1])^c \cup H^{(m)}[k+1]$ occurs.

Next, we suppose $O[k+1]$ is occupied at time $t[k+1] = m$. We say it is *vacated* at time $t[k+1] = m+1$ if the complemented of $B^{(m)}[k+1]$ occurs.

From the above inductive procedure, we obtain for each $k = 1, 2, \ldots$, the level k process $\xi[k]_{t[k]}^{0,\beta_k}, t[k] = 1, 2, \ldots$, with birth probability β_k, death probability δ_k and initial state $\emptyset$. This completes our procedure of successive block renormalization.

REMARK. The fact that, for each $k = 1, 2, \ldots$, the level k process is a discrete-time, simple birth–death process is the reason why in Definition 8 we consider the spreading of individuals at level 0 only.

We are now going to evaluate the birth probability β_k and death probability $\delta_k = 1 - \beta_k$ for each $k = 1, 2, \ldots$. We will first obtain the evaluation in the case $k =$

1 and then carry out the results in general by induction. Notice the probability that a birth occurs at a certain site in $\Lambda(O[1])$ for some $t \in (m\tau_0, (m+1)\tau_0]$ is equal to $1 - \exp(-\beta\tau_0)$, and the total number of sites in the region $\Lambda(O[1])$ is equal to $L_0^2 < (\beta^{-a})^2 = \beta^{-2a}$. Hence, by virtue of $1 - e^{-x} \le x$, we have

$$\begin{aligned} P\big(G^{(m)}[1]\big) &\le \Big(\big(1 - \exp(-\beta\tau_0)\big)\beta^{-2a}\Big)^2 \\ &\le (\beta\tau_0\beta^{-2a})^2 \le (\mu\beta^{-a}\beta^{1-2a})^2 \\ &= \mu^2\beta^{2-6a}. \end{aligned} \tag{1.5}$$

Furthermore, the probability that the occupied site resulting from a single spontaneous birth remains occupied for longer than β^{-b} time units is equal to $\exp(-\beta^{-b})$. It follows that

$$\begin{aligned} P\big(G^{*(m)}[1]\big) &\le \big(1 - \exp(-\beta\tau_0)\big)\beta^{-2a}\exp(-\beta^{-b}) \\ &\le \mu\beta^{-a}\beta^{1-2a}\exp(-\beta^{-b}) \\ &= \mu\beta^{1-3a}\exp(-\beta^{-b}). \end{aligned} \tag{1.6}$$

Therefore,

$$P\big(G^{(m)}[1] \cup G^{*(m)}[1]\big) \le \mu^2\beta^{2-6a} + \mu\beta^{1-3a}\exp(-\beta^{-b}). \tag{1.7}$$

To evaluate $P(\Delta^{(m)}[1])$, we notice that, by the Markov property, the distribution of $T^{(m)}$ is the same as the distribution of $T(\Lambda(O[1]))$ defined in subsection 1.1. By (1.4) (replacing N by L_0), we obtain

$$\begin{aligned} 1 - P\big(\Delta^{(m)}[1]\big) &= P\big(T^{(m)} > \tau_0\big) \\ &= P\Big(T\Big(\Lambda\big(O[1]\big)\Big) > \mu L_0\Big) \\ &\le \exp(-CL_0) = \exp(-C\beta^{-a}). \end{aligned} \tag{1.8}$$

We emphasize that, for convenience, here and in what follows, C always denotes a positive constant whose value may change from line to line.

To deal with $H^{(m)}[1]$, we first make some observations about the level 0 process in the region $\Lambda(O[2])$, the corresponding region of the level 2 origin $O[2]$ at level 0, for $t \in (0, \tau_0]$. Since the process starts with $\emptyset$, it is easy to see that for each square region $\Lambda(x[1])$ in $\Lambda(O[2])$, if it becomes a source of spreading, then $x[1]$ must be occupied at level 1 at time $t[1] = 1$. In general for $t \in (m\tau_0, (m+1)\tau_0], m = 1, 2, \ldots$, if $\Lambda(x[1])$ is a source of spreading, then $x[1]$ must be occupied at level 1 at time $t[1] = m + 1$. Therefore, in the consideration of $H^{(m)}[1]$, we do not have to consider the situation in which individuals spread into $\Lambda(O[1])$ in $t \in (m\tau_0, (m+1)\tau_0]$ due to the combined effect of any two neighboring regions (because this implies there are at least two occupied level 1 sites at time $t[1] = m + 1$, in the corresponding region of $O[2]$ at level 1, and thus implies that $O[2]$ is occupied at level 2 at the respective time). We may modify the definition of $H^{(m)}[1]$ by interpreting it as "exactly one of the events $H_i^{(m)}[1], i = 1, 2, 3, 4$, occurs."

After this modification it is easy to observe that, according to the nature of the level 0 process $\xi_t^{0,\beta}$, $H^{(m)}[1] \subset G^{(m)}[1]$ for each m. It follows from (1.5) to (1.8) that

$$\begin{aligned}\beta_1 &= P\Big(G^{(m)}[1] \cup G^{*(m)}[1] \cup \big(\Delta^{(m)}[1]\big)^c \cup H^{(m)}[1]\Big)\\ &\le P\big(G^{(m)}[1]\big) + P\big(G^{*(m)}[1]\big) + P\Big(\big(\Delta^{(m)}[1]\big)^c\Big)\\ &\le \mu^2\beta^{2-6a} + \mu\beta^{1-3a}\exp(-\beta^{-b}) + \exp(-C\beta^{-a}).\end{aligned}$$

Since $a \in (0, 1/8)$ and $b \in (0, a/2)$, when β is sufficiently small, $\beta_1 < \beta^{1+\gamma}$ for some $\gamma > 0$.

The death probability for each level site is $\delta_1 = 1 - \beta_1$.

When $k \ge 1$, for each level $k+1$ site $x[k+1] \in \mathbf{Z}^2[k+1]$, the birth probability β_{k+1} and the death probability δ_{k+1} in the process $\xi[k+1]_{t[k+1]}^{0,\beta_{k+1}}$ can be evaluated as follows.

To evaluate $P(\Delta^{(m)}[k+1])$, we apply once again the result of (1.4) (replacing N by M_k this time). Notice that $M_k = \prod_{i=0}^k L_i$ and $\tau_k\tau_{k-1}\cdots\tau_0 = \mu\prod_{i=0}^k L_i = \mu M_k$. It follows then that

$$\begin{aligned}1 - P\big(\Delta^{(m)}[k+1]\big) &= P\big(T^{(m)} > \tau_k\tau_{k-1}\cdots\tau_0\big)\\ &= P\big(T^{(m)} > \mu M_k\big) \le \exp(-CM_k)\\ &= \exp\big\{-C(\beta_k\beta_{k-1}\cdots\beta_1\beta)^{-a}\big\}.\end{aligned} \tag{1.9}$$

A very similar argument to that applied in the evaluation of $P(G^{(m)}[1])$ yields

$$P\big(G^{(m)}[k+1]\big) \le \beta_k^{2-6a}. \tag{1.10}$$

Next, $G^{*(m)}[k+1]$ occurs means a single level k birth occurs at some $x[k] \in \Lambda^{(k)}(O[k+1])$ during the time $t[k] \in (m\tau_k, (m+1)\tau_k)$, and the corresponding region of $x[k]$ at level 0 survives longer than $\tau_{k-1}\tau_{k-2}\cdots\tau_0$ level 0 time units. Therefore,

$$\begin{aligned}P\big(G^{*(m)}[k+1]\big) &\le \beta_k^{1-3a}P\Big(\big(\Delta^{(m)}[k]\big)^c\Big)\\ &\le \beta_k^{1-3a}\exp\big\{-C(\beta_{k-1}\cdots\beta_1\beta)^{-a}\big\}.\end{aligned} \tag{1.11}$$

The evaluation of $P(H^{(m)}[k+1]), m = 0, 1, \ldots,$ is given by the following lemma.

LEMMA 1. *There exist constants $C_1, C_2 \in (0, \infty)$, such that, for each $k = 1, 2\cdots$ and $m = 0, 1, \ldots, P(H^{(m)}[k+1]) \le C_1\exp\{-C_2(\beta_k\beta_{k-1}\cdots\beta_1\beta)^{-a}\}$.*

To prove Lemma 1, we need first to introduce the following definition.

DEFINITION 9. Let $\mathbf{Z}_f^2$ denote the finite subsets of $\mathbf{Z}^2$ and let $A, B \in \mathbf{Z}_f^2$. A function π from $[s,t]$ to $\mathbf{Z}_f^2$ is said to be an *occupancy path* from (A,s) to (B,t) if π is right continuous, has a finite number of discontinuities and satisfies:

(1.12) $\pi(s) = A$ and $\pi(t) \supset B$;

(1.13) if $s \le u \le v \le t$, then $\xi(v; u, \pi(u)) \supset \pi(v)$;

(1.14) π is minimal $\big[$i.e., if $\pi'(u) \subset \pi(u)$ for all $u \in [s,t]$ and if π' satisfies both (1.12) and (1.13), then $\pi' = \pi\big]$;

where $\xi(v; u, \pi(u))$ denotes the state of the process at time v when its state at time u is $\pi(u)$.

PROOF OF LEMMA 1. It follows from Definition 8 that

$$\begin{aligned} P\big(H^{(m)}[k+1]\big) &= P\left(\bigcup_{i=1}^{4} H_i^{(m)}[k+1] \cap \big(G^{(m)}[k+1] \cup G^{*(m)}[k+1]\big)^c\right) \\ &= P\left(\bigcup_{i=1}^{4} H_i^{(m)}[k+1] \,\middle|\, \big(G^{(m)}[k+1] \cup G^{*(m)}[k+1]\big)^c\right) \\ &\quad \times P\left(\big(G^{(m)}[k+1] \cup G^{*(m)}[k+1]\big)^c\right) \\ &\le P\left(\bigcup_{i=1}^{4} H_i^{(m)}[k+1] \,\middle|\, \big(G^{(m)}[k+1] \cup G^{*(m)}[k+1]\big)^c\right) \\ &\le 4P\left(H_1^{(m)}[k+1] \,\middle|\, \big(G^{(m)}[k+1] \cup G^{*(m)}[k+1]\big)^c\right). \end{aligned}$$

To bound the probability involved in the last inequality, we label the vertical strips with unit width (measured in level 0 units) that partition the region $R_{k,1}$ from right to left as $S_1, S_2, \dots, S_{n(k)}$, where $n(k) = \beta^{\alpha/2} M_k/2$. Consider the space–time region $\Lambda^*(O[k+2]) = \Lambda(O[k+2]) \times (mT_k, (m+1)T_k]$, where $\Lambda(O[k+2]) = (-M_{k+1}/2, M_{k+1}/2]^2 = (-L_{k+1}M_k/2, L_{k+1}M_k/2]^2$, which is the corresponding region of the level $k+2$ origin at level 0, and $T_k = \tau_k \tau_{k-1} \cdots \tau_0$. Let $R_{k,1}^* = R_{k,1} \times (mT_k, (m+1)T_k]$. For $j = 1, 2, \dots k+1$, if a level j site $x[j]$ is occupied at level j time $t[j] = n$, we will regard the region $\Lambda(x[j]) \times (n\tau_{j-1}\tau_{j-2} \cdots \tau_0, (n+1)\tau_{j-1}\tau_{j-2} \cdots \tau_0]$ entirely occupied at level 0. (As usual, $\Lambda(x[j])$ denotes the corresponding region of $x[j]$ at level 0.) For convenience, a region with the form $\Lambda(x[j]) \times (n\tau_{j-1}\tau_{j-2} \cdots \tau_0, (n+1)\tau_{j-1}\tau_{j-2} \cdots \tau_0]$ contained in $\Lambda^*(O[k+2])$ will be called a *basic j-box*. We may rule out the situation in which $\Lambda(O[k+2])$ contains two or more level $k+1$ sites that are occupied at time $t[k+1] = m$. For if this happens, it will imply $O[k+2]$ is occupied at a certain level $k+2$ time corresponding to $t[k+1] = m$. Without loss of generality, we may assume that the site (1, 0) at level $k+1$ is occupied at $t[k+1] = m$, and all other level $k+1$ sites are vacant at $t[k+1] = m$. Notice that when $k \ge 1$, even under this assumption, the relation $H^{(m)}[k+1] \subset G^{(m)}[k+1]$ is no longer true. That is because the level

0 spontaneous births that result in the spreading may not necessarily be abundant enough to trigger the higher level births. For example, the spreading may be initiated by a single short-lived level 0 birth in S_1, and it can invade into S_2 by another single short-lived level 0 birth in S_2. These two level 0 births will not result in any level 1 births if they belong to two different basic 1-boxes.

Let us first consider a special case in which $H_1^{(m)}[k+1]$ occurs but $R^*_{k,1}$ contains no occupied basic j-boxes for $j = 1, 2, \dots, k$. In this case the spreading of individuals can only be initiated from the sites in S_1 by the occurrence of the level 0 spontaneous births. The spreading will take place in a strip-by-strip fashion. After part of S_i is invaded, the individuals can spread into S_{i+1} if a spontaneous birth occurs at a site in S_{i+1} adjacent to the occupied area of S_i. Let π be an occupancy path from (A_k, u) to $(\{x\}, v)$, where $A_k = (M_k/2, 3M_k/2] \times (-M_k/2, M_k/2]$, the corresponding region of the level $k+1$ site $(1,0)$ at level $0, x \in S_{n(k)}$ and $mT_k \le u < v \le (m+1)T_k$. For each $i = 1, \dots, n(k)$, let $(x_i, t_i) \in \cup_{t \in (mT_k, (m+1)T_k]}((\pi(t) \cap S_i) \times \{t\})$ be the first point at which a spontaneous birth occurs. Then $t_1 < \cdots < t_{n(k)}$, and the sequence $(x_1, t_1), \dots, (x_{n(k)}, t_{n(k)})$ describes the route and timetable of the spreading path as it crosses the region $R_{k,1}$. Since $R^*_{k,1}$ contains no occupied basic j-boxes, the points $(x_1, t_1), \dots, (x_{n(k)}, t_{n(k)})$ must belong to different basic 1-boxes. Let d_k denote the distance traveled by individuals spreading from S_1 to $S_{n(k)}$. Then $d_k \ge |x_2 - x_1| + \cdots + |x_{n(k)} - x_{n(k)-1}|$. Since the heights of $R^*_{k,1}$ and each of the basic 1-boxes are T_k and τ_0 level 0 time units, respectively, it follows that, for each region $\Lambda(x[1]) \subset R_{k,1}$, the rectangular cylinder $\Lambda(x[1]) \times (mT_k, (m+1)T_k]$ contains T_k/τ_0 basic 1-boxes. Let j be an index in $\{1, 2, \dots, n(k)\}$ such that (x_j, t_j) and (x_{j+1}, t_{j+1}) belong to two different basic 1-boxes that are located in the same cylinder, say, $\Lambda(x[1]) \times (mT_k, (m+1)T_k]$. Then, since $t_1 < \cdots < t_{n(k)}$, this can only happen for at most T_k/τ_0 many j's. Hence there are at least $n(k) - T_k/\tau_0$ many sites among $\{x_i, \dots, x_{n(k)}\}$ which belong to different regions of the form $\Lambda(x[1])$ contained in $R_{k,1}$. Note that $n(k) = \beta^{a/2} M_k/2 = \beta^{a/2} T_k/2\mu$ and $T_k/\tau_0 \le \beta^a(1 + o(\beta^a))T_k/\mu < 2(1 + o(\beta^a))\beta^{a/2} n(k)$; we get $n(k) - T_k/\tau_0 \ge (1 - 2(1 + o(\beta^a))\beta^{a/2} n(k)$. This means that

$$\begin{aligned} d_k &\ge L_0\bigl(1 - 2(1 + o(\beta^a)\bigr)n(k) \\ &\ge \frac{1}{2\mu\beta^{a/2}}\bigl(1 - 3\beta^{a/2}\bigr)T_k \\ &\ge \frac{1}{2\mu\beta^{a/2}}\bigl(1 - 3\beta^{a/2}\bigr)(t_{n(k)} - t_1). \end{aligned}$$

The speed of the spreading from (x_1, t_1) to $(x_{n(k)}, t_{n(k)})$ is dominated by a one-dimensional asexual contact process with death rate 1 and birth rate λ. We have just shown that when β is sufficiently small, there is a sufficiently small positive constant c such that $\lambda(t_{n(k)} - t_1) < cd_k$. It follows from Lemma 9 of Durrett (1988) that there exist constants $C_1, C_2 \in (0, \infty)$ such that

$$\begin{aligned} &P(H_1^{(m)}[k+1] \mid R^*_{k,1} \text{ contains on occupied basic } j\text{-boxes for } j = \\ &1, 2, \dots, k) \le C_1 \exp(-C_2 d_k) \le C_1 \exp\bigl\{-C_2(\beta_k \beta_{k-1} \dots \beta_1 \beta)^{-a}\bigr\}. \end{aligned} \tag{1.15}$$

In general, under the condition $(G^{(m)}[k+1] \cup G^{*(m)}[k+1])^c$, $R^*_{k,1}$ contains at most one occupied basic k-box. Consequently, for each $j = 2, 3, \dots, k-1$, every basic j-box in $R^*_{k,1}$, excluding possibly one exception, can contain at most one occupied basic $(j-1)$-box. Therefore, if we denote $V = \mathrm{vol}(R^*_{k,1})$, V_j = total volume of all occupied basic j-boxes in $R^*_{k,1}$ that are not contained in an occupied basic $(j+1)$-box, then $V = \frac{1}{2}\beta^{a/2}M_k^2 T_k = \frac{1}{2}\beta^{a/2}\prod_{i=0}^k L_i^2\tau_i$ (level 0 units). The total number of basic $(j+1)$-boxes in $R^*_{k,1}$ is $V(\prod_{i=0}^j L_i^2\tau_i)^{-1}$, which is $\frac{1}{2}\beta^{a/2}\prod_{i=j+1}^k L_i^2\tau_i$. One of them is (possibly) entirely occupied; each of the others can contain at most one occupied basic j-box. Thus the total number of all occupied basic j-boxes contained in $R^*_{k,1}$ which are not contained in an occupied basic $(j+1)$-box, is $\frac{1}{2}\beta^{a/2}\prod_{i=j+1}^k L_i^2\tau_i - 1$. Since each basic j-box has volume $\prod_{i=0}^{j-1} L_i^2\tau_i$, it follows that

$$\begin{aligned} V_j &= \frac{1}{2}\beta^{a/2}\left(\prod_{i=j+1}^{k} L_i^2\tau_i - 1\right)\prod_{i=0}^{j-1} L_i^2\tau_i \\ &\le \frac{1}{2}\beta^{a/2}\prod_{\substack{i=0\\ i\neq j}}^{k} L_i^2\tau_i = V/L_j^2\tau_j \le 2\beta_j^{3a}V. \end{aligned}$$

By the induction hypothesis, for $j = 1, 2, \dots, k$, $\beta_j < \beta_{j-1}^{1+\gamma}$, where β_0 is interpreted as β and $\gamma > 0$ is the same as in Step 1. It follows that $\beta_j^{3a} < \beta_{j-1}^{3a}/2$, when β is sufficiently small. Hence

$$\begin{aligned} V_1 + V_2 + \cdots + V_k &\le 2V\sum_{j=1}^{k}\beta_j^{3a} \\ &< 2V\beta^{3a}\sum_{j=1}^{k}\frac{1}{2^j} < 2V\beta^{3a} = o(\beta^a)V. \end{aligned} \tag{1.16}$$

Now we may apply the same argument as in the special case discussed previously to the region $R^*_{k,1}\backslash Q(k)$, where $Q(k)$ denotes the union of all occupied basic j-boxes in $R^*_{k,1}$ for $j = 1, 2, \dots, k$. By the estimate we just obtained in (1.16), we know that in this case the distance traveled by spreading is equal to $(1 - o(\beta^a))d_k$. Therefore the arguments that carried out (1.15) still apply, and we obtain

$$\begin{aligned} &P\Big(H_1^{(m)}[k+1] \,\Big|\, \big(G^{(m)}[k+1] \cup G^{*(m)}[k+1]\big)^c\Big) \\ &\qquad \le C_1\exp\big\{-C_2(\beta_k\beta_{k-1}\cdots\beta_1\beta)^{-a}\big\}. \end{aligned}$$

Hence

$$P\big(H^{(m)}[k+1]\big) \le C_1\exp\big\{-C_2(\beta_k\beta_{k-1}\cdots\beta_1\beta)^{-a}\big\}. \qquad \square$$

It follows from (1.9) to (1.11) and Lemma 1, that, when β is sufficiently small,

$$\beta_{k+1} = P\Big(G^{(m)}[k+1] \cup G^{*(m)}[k+1] \cup \big(\Delta^{(m)}[k+1]\big)^c \cup H^{(m)}[k+1]\Big)$$
$$\le \beta_k^{2-6a} + \beta_k^{1-3a}\exp\{-C(\beta_{k-1}\dots\beta_1\beta)^{-a}\} + \exp\{-C(\beta_k\beta_{k-1}\dots\beta_1\beta)^{-a}\}$$
$$+ C_1\exp\{-C_2(\beta_k\beta_{k-1}\dots\beta_1\beta)^{-a}\}$$
$$< \beta_k^{1+\gamma},$$

where $\gamma > 0$ is the same constant as in Step 1.

The death probability for each level $k+1$ site is $\delta_{k+1} = 1 - \beta_{k+1}$.

1.3. *The conclusion of the Proof of Theorem* 3. We are now in a position to complete our proof of (1.3). We will consider the level 0 time sequence $\{t_k\}_{k=1}^{\infty}$, where $t_1 = \tau_0$ and $t_k = \tau_{k-1}\tau_{k-2}\dots\tau_0, k = 2,3,\dots$. The sequence is chosen such that $t = t_k$ corresponds to $t[k] = 1$ at level k. We wish to establish an upper bound of $P(0 \in \xi_{t_k}^{0,\beta})$ by using the results that can be obtained readily from the level j processes, $j \ge 1$. To illustrate the idea, we use $t_1 = \tau_0$ as our example. Suppose that the origin O is occupied at time t_1. Then the cause of the occupation can be divided into four cases as follows: (i) a spontaneous birth occurs at O at time $s \in (t_1 - \beta^{-b}, t_1]$, and O remains occupied up to time t_1; (ii) a spontaneous birth occurs at O at time $s \in (0, t_1 - \beta^{-b})$, and O remains occupied up to time t_1; (iii) O is invaded by individuals inside the region $\Lambda(O[1])$ by time t_1, and it remains occupied up to t_1; (iv) O is invaded by individuals outside the region $\Lambda(O[1])$ by time t_1, and it remains occupied up to t_1. The probability involved in case (i) is easy to evaluate directly. We will show that either (ii) or (iii) implies $O[1] \in \xi[1]_{t[1]}^{0,\beta_1}$ for $t[1] = 1$, and (iv) implies that $O[j] \in \xi[j]_1^{0,\beta_j}$ for some $j \ge 2$. Consequently, the probability involved in cases (ii), (iii) and (iv) can be assessed easily as well. To apply this idea to the general case and make the argument more rigorous, we introduce the following notation. Denote

$$E(k,0) = \{\text{at level } 0, (0,s) \in B_x^* \text{ for some } s \in (t_k - \beta^{-b}, t_k]\},$$
$$E(k,1) = \{O[1] \in \xi[1]_{t[1]}^{0,\beta_1} \text{ for } t[1] = t_k/\tau_0\}.$$

For $j = 2,3,\dots,k-1$, let

$$E(k,j) = \{O[j] \in \xi[j]_{t[j]}^{0,\beta_j} \text{ for } t[j] = t_k/(\tau_{j-1}\tau_{j-2}\dots\tau_0) = \tau_{k-1}\tau_{k-2}\dots\tau_j\}.$$

For $j = 1,2,\dots,$ denote

$$A(j) = \{O[j] \in \xi[j]_1^{0,\beta_j}\}.$$

Let $E^*(k) = (\bigcup_{j=0}^{k-1}E(k,j)) \cup (\bigcup_{j=k}^{\infty}A(j))$.

We are now going to prove the following lemma.

LEMMA 2. *For each* $k = 1,2,\dots$, *let* $E(k,j), j = 1,2,\dots,k-1$, *and* $E^*(k)$ *be defined as above. Then* $\{0 \in \xi_{t_k}^{0,\beta}\} \subset E^*(k)$, *and therefore*

$$P(0 \in \xi_{t_k}^{0,\beta}) \le P(E^*(k)) \le \beta^{1-b} + \sum_{j=1}^{\infty}\beta_j \le \beta^{1-b} + \beta, \tag{1.17}$$

where $0 < b < a/2 < 1$ *as defined in Definition* 3.

Since the right-hand side of the last inequality in (1.17) is independent of k, (1.3) follows from Lemma 2 immediately.

PROOF OF LEMMA 2. Suppose $0 \in \xi_{t_k}^{0,\beta}$. Then there is an occupancy path π from $(\{x\}, t)$ to $(\{0\}, t_k)$ for some $x \in \mathbf{Z}^2$ and $t < t_k$. We can find the smallest j such that π is contained in $\Lambda(O[j]) \times [t, t_k]$.

Case I. Suppose $j = 1$.

I(a). If $x = 0$, then the occupation of the origin at time t_k is due to a spontaneous birth that occurs at the origin at some time $t \le t_k$ and the origin remains occupied during the time period $(t, t_k]$. If $t_k - t < \beta^{-b}$, then $E(k, 0)$ occurs. If $t_k - t \ge \beta^{-b}$, then, by Definition 3, $G^{*(m)}[1]$ occurs for $m = \tau_{k-1}\tau_{k-2}\cdots\tau_1 - 1$. Therefore, $O[1]$ is occupied by $\xi[1]_{t[1]}^{0,\beta_1}$ at $t[1] = \tau_{k-1}\tau_{k-2}\cdots\tau_1 = t_k/\tau_0$; thus $E(k, 1)$ occurs.

I(b). If $x \ne 0$ and $t_k - t \le \tau_0$, then, by Definition 2, $G^{(m)}[1]$ occurs for $m = \tau_{k-1}\tau_{k-2}\cdots\tau_1 - 1$; hence $E(k, 1)$ occurs. If $x \ne 0$ and $t_k - t > \tau_0$, then, by Definition 1, $(\Delta^{(m)}[1])^c$ occurs for $m = \tau_{k-1}\tau_{k-2}\cdots\tau_j - 1$. This again implies that $E(k, 1)$ occurs.

Case II. Suppose $j \in \{2, 3, \ldots\}$.

II(a). If $j \le k$ and $t_k - t \le \tau_{j-1}\tau_{j-2}\cdots\tau_0$, then $\Delta^{(m)}[j] \cap (G^{(m)}[j] \cup G^{*(m)}[j])^c$ occurs for $m = \tau_{k-1}\tau_{k-2}\cdots\tau_j - 1$. Then the only possibility that there exists such an occupancy path as π is that the individuals spread from one of the neighboring regions of $\Lambda(O[j])$, after $\Lambda(O[j])$ is wiped out by deaths (due to the occurrence of $\Delta^{(m)}[j]$). By Definition 8, this implies that $H^{(m)}[j]$ occurs for $m = \tau_{k-1}\tau_{k-2}\cdots\tau_j - 1$, which in turn implies that $O[j]$ is occupied by the level j process at time $t[j] = \tau_{k-1}\tau_{k-2}\cdots\tau_j$, and thus $E(k, j)$ occurs.

II(b). If $j \le k$ and $t_k - t > \tau_{j-1}\tau_{j-2}\cdots\tau_0$. Then, by Definition 5, this implies that $(\Delta^{(m)}[j])^c$ occurs for $m = \tau_{k-1}\tau_{k-2}\cdots\tau_j - 1$. Hence it again implies that $O[j]$ is occupied by the level j process at time $t[j] = \tau_{k-1}\tau_{k-2}\cdots\tau_j$, and thus $E(k, j)$ occurs.

II(c). If $j > k$, then $t_k - t \le \tau_{k-1}\tau_{k-2}\cdots\tau_0 < \tau_{j-1}\tau_{j-2}\cdots\tau_0$. The arguments are essentially the same as II(a). Suppose $\Delta^{(0)}[j] \cap (G^{(0)}[j] \cup G^{*(0)}[j])^c$ occurs, then, by Definition 8, the existence of such as occupancy path as π must imply that $H^{(0)}[j]$ occurs. Hence $O[j] \in \xi[j]_1^{0,\beta_j}$, and thus $A(j)$ occurs.

From the above analysis, we conclude that $\{0 \in \xi_{t_k}^{0,\beta}\} \subset E^*(k)$. It is fairly straightforward that

$$\begin{aligned} P(E^*(k)) &= P\left(\bigcup_{j=0}^{k-1} E(k,j) \cup \bigcup_{j=k}^{\infty} A(j)\right) \\ &\le \beta^{1-b} + \sum_{j=1}^{\infty} \beta_j \le \beta^{1-b} + \beta \sum_{j=1}^{k} \frac{1}{2^j} \\ &\le \beta^{1-b} + \beta. \end{aligned}$$

This completes the proof of Lemma 2 and thus the proof of Theorem 3. □

2. Proof of Theorem 4. In this section we will prove Theorem 4 in the case $d = 3$. The conclusion holds also for all $d > 3$. Throughout this section ξ_t^p, $t \geq 0$, represents the three-dimensional symmetric system with initial distribution ξ_0^p, which is described in Section 0. In several places of the proof, the birth rate λ of the system is required to be sufficiently large. We assume that λ is fixed but large enough to make the proof work. Since $P(x \in \xi_t^p)$ is increasing with respect to p, we need only to prove Theorem 4 in the case that p is sufficiently small. We will always assume that $\lambda < (\log(1/p))^{-1}$.

Briefly, the main ideas of the proof can be illustrated as follows. First of all, we will define a class of configurations with special structures on the cube $\Lambda_L = (-L/2, L/2]^3$, where L will be specified later. For simplicity, we will call a cube with such a configuration a Ψ-cube. The definitions can be applied to other cubes centered at $x \in \mathbf{Z}^3$ in a translation-invariant manner. Second, we will prove that the probability that Λ_L is a Ψ-cube initially is close to 1. Namely, the probability that $\Lambda_L \cap \xi_0^p$ is a Ψ-cube is close to 1. Third, we will prove that if the initial state of the process is η such that $\Lambda_L \cap \eta$ is a Ψ-cube, then, as the process evolves, Λ_L will quickly become completely filled by the system, except for some isolated, short-lived holes resulting from the constant deaths. Combined with the second and third steps described above, we know that if we partition $\mathbf{Z}^3$ into cubic regions with edge length L, then, as the system ξ_t^p evolves, each of them will have large probability to become completely filled. The last step of the proof is to apply an inductive procedure that will conclude that, with large probability, the system ξ_t^p will survive for all t, as desired.

The rest of the paper will be devoted to implementing the ideas described in the preceding paragraph. In subsection 2.1 we will define the Ψ-cubes, in subsection 2.2 we will evaluate the probability that $\Lambda_L \cap \xi_0^p$ is a Ψ-cube, in subsection 2.3 we will study the behavior of the process when its initial state is a Ψ-cube and, finally, in subsection 2.4 we will conclude the proof by an inductive procedure.

2.1. *Defining the Ψ-cube.* We will introduce a set of definitions the will eventually give us Ψ-cubes. For simplicity, in what follows when we say "Γ is an $l \times w \times h$ box region," we always mean that it is of the form $(m - 1/2, m - 1/2 + l] \times (n - 1/2, n - 1/2 + w] \times (q - 1/2, q - 1/2 + h]$, where m, n, and q are integers. Moreover, Γ could be either an $\mathbf{R}^3$ or a $\mathbf{Z}^3$ region. In the latter case it should be interpreted as $\Gamma \cap \mathbf{Z}^3$.

As indicated in Section 0 of Chen (1992), for $d \geq 2$, the d-dimensional symmetric system $\xi_t^p, t \geq 0$, has a close relationship with the bootstrap percolation models on $\mathbf{Z}^d$. One can find certain similarities between the structures being defined in the following definitions and the structures of the "critical droplets" described in Aizenman and Lebowitz (1988).

DEFINITION 10. Let D be the principal diagonal of Λ_L and let S be the right circular cylinder whose central axis is D and cross-sectional area is L^α, where

$\alpha \in (0, 1/4)$ is a fixed constant. For a given configuration η, we say that $\Lambda_L \cap \eta$ is a Δ-*cube* if $\Lambda_L \cap \eta \cap S$ is entirely occupied.

DEFINITION 11. Suppose that $\exp(1/p) < L \le \exp\{\exp(2/p)\}$. Let $L_1 = \exp(1/p)$. For a given configuration η, we say that $\Lambda_L \cap \eta$ is a Θ-*cube* if Λ_{L_1} is a Δ-cube and, for $k = L_1 + 2, L_1 + 4, \dots, L$, each of the six planes of $\Lambda_k \backslash \Lambda_{k-2}$ intersects at least one Δ-cube with edge length $l \ge \lambda(\log k)^{1/2}$.

DEFINITION 12. Let $L = \exp\{\exp(2/p)\}$ and $M = \exp\{\exp((2-\alpha)/p)\}$. Partition Λ_L into $M \times M \times M$ blocks. Let Σ denote the collection of all $M/2 \times M/2$ planes contained in Λ_L that are parallel to one of the coordinate planes. For a given configuration η, we say that $\Lambda_L \cap \eta$ is a Ψ-cube, if the following conditions are met:

(2.1) For every plane in Σ, it intersects at least one Δ -cube with edge length $l \ge \lambda \exp(1/p)$.

(2.2) Λ_L contains a Θ-cube that is a translate of Λ_M, located on one of the $M \times M \times M$ blocks.

2.2. *The evaluation of the probability that $\Lambda_L \cap \xi_0^p$ is a Ψ-cube.* To evaluate the probability that $\Lambda_L \cap \xi_0^P$, $L = \exp\{\exp(2/p)\}$, is a Ψ-cube, we need first to evaluate the probabilities that $\Lambda_L \cap \xi_0^p, L \in \{1, 3, \dots\}$, is a Δ-cube and $\Lambda_L \cap \xi_0^P, L = \exp\{\exp((2-\alpha)/p)\}$, is a Θ-cube. It follows directly from Definition 10 that

$$P\big(\lambda_L \cap \xi_0^p \text{ is a } \Delta\text{-cube}\big) \ge p^{L^{1+\alpha}} = \exp\big\{-L^{1+\alpha}\log(1/p)\big\}. \tag{2.3}$$

In particular, for $L = \exp(1/p)$, this yields

$$P\big(\Lambda_L \cap \xi_0^p \text{ is a } \Delta\text{-cube}\big) \ge \exp\Big\{-\exp\big((1+\alpha)/p\big)\log(1/p)\Big\}.$$

Let A be the event that $\Lambda_L \cap \xi_0^P$ is a Θ-cube, A_0 be the event that $\Lambda_{L_1} \cap \xi_0^p$ is a Δ-cube and A_k be the event that each of the six planes of $\Lambda_k \backslash \Lambda_{k-2}$ intersects at least one Δ-cube with edge length $l = \lambda(\log k)^{1/2}$, for $k = L_1 + 2, L_1 + 4, \dots, L$. Then $A \supset A_0 \cap A_1 \cap \cdots \cap A_L$. Moreover, since the events $A_0, A_1, \dots, A_L$ are positively correlated, we have

$$P(A) \ge P(A_0 \cap A_1 \cap \cdots \cap A_L) \ge P(A_0) \prod_{k=L_1}^{L} P(A_k).$$

Applying (2.3) and using $1 - x \le e^{-x}$, we obtain

$$\begin{aligned} P(A_k) &\ge \Big(1 - \big(1 - p^{\lambda^{1+\alpha}(\log k)^{1/2+\alpha/2}}\big)^{k^2/(\lambda^2 \log k)}\Big)^6 \\ &\ge \Big(1 - \exp\Big\{-(k^2/(\lambda^2 \log k))p^{\lambda^{1+\alpha}(\log k)^{1/2+\alpha/2}}\Big\}\Big)^6. \end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4),

$$\begin{aligned} &P(\Lambda_L \cap \xi_0^p \text{ is a } \Theta\text{-cube}) \\ &\quad \geq P(A_0) \prod_{k=L_1}^{L} P(A_k) \\ &\quad \geq \exp\Big\{-\exp\big((1+\alpha)/p\big)\log(1/p)\Big\} \\ &\quad\quad \times \prod_{k=\exp(1/p)}^{\exp\{\exp(2/p)\}} \Big(1-\exp\Big\{-(k^2/(\lambda^2\log k))p^{\lambda^{1+\alpha}(\log k)^{1/2+\alpha/2}}\Big\}\Big)^6. \end{aligned}$$

As assumed in the beginning of this section, $\lambda < (\log(1/p))^{-1}$ when $k > \exp(1/p)$ and $\alpha \in (0, 1/4)$,

$$\begin{aligned} &(k^2/(\lambda^2\log k))p^{\lambda^{1+\alpha}(\log k)^{1/2+\alpha/2}} \\ &\quad = (k^2/(\lambda^2\log k))\exp\Big\{-\lambda^{1+\alpha}(\log k)^{1/2+\alpha/2}\log(1/p)\Big\} \\ &\quad \geq k^{2-\alpha^*} \quad \text{for some } \alpha^* \in (0, 1/2). \end{aligned}$$

Hence

$$\begin{aligned} &P(\Lambda_L \cap \xi_0^p \text{ is a } \Theta\text{-cube}) \\ &\quad \geq \exp\Big\{-\exp\big((1+\alpha)/p\big)\log(1/p)\Big\} \prod_{k=\exp(1/p)}^{\exp\{\exp(2/p)\}} \big(1-\exp(-k^{2-\alpha^*})\big)^6 \\ &\quad = \exp\Big\{-\exp\big((1+\alpha)/p\big)\log(1/p)\Big\} \\ &\quad\quad \times \exp\Bigg\{6 \prod_{k=\exp(1/p)}^{\exp\{\exp(2/p)\}} \log\big(1-\exp(-k^{2-\alpha^*})\big)\Bigg\}. \end{aligned}$$

Notice that, for $x < 1/2, \log(1-x) = -x - x^2/2 - x^3/3 - \cdots > -2x$. Thus $\log(1-\exp(-k^{2-\alpha^*})) > -2\exp(-k^{2-\alpha^*})$, when p is sufficiently small. So we obtain

$$\begin{aligned} &P\big(\Lambda_L \cap \xi_0^p \text{ is a } \Theta\text{-cube in } \xi_0^p\big) \\ &\quad \geq \exp\Big\{-\exp\big((1+\alpha)/p\big)\log(1/p)\Big\}\exp\Bigg\{-12 \sum_{k=\exp(1/p)}^{\exp\{\exp(2/p)\}} \exp(-k^{2-\alpha^*})\Bigg\} \qquad (2.5) \\ &\quad \geq \big(1-o(p)\big)\exp\Big\{-\exp\big((1+\alpha)/p\big)\log(1/p)\Big\} \end{aligned}$$

for all $L \in (\exp(1/p), \exp\{\exp(2/p)\}]$. Here and in what follows $o(p)$ represents a positive constant whose value may vary from line to line, and $o(p)/p \to 0$ as $p \to 0$.

Let A_1, A_2 be the events that the conditions (2.1) and (2.2), respectively, are initially satisfied. Then, by the bound given in (2.3), we know that for each plane in Σ, the probability that it does not intersect any Δ-cubes with edge length $l \geq \lambda \exp(1/p)$ is bounded above by the quantity $(1-\exp\{-\lambda^{1+\alpha} \exp((1+\alpha)/p) \log(1/p)\})^{(M/(2\lambda \exp(1/p)))^2}$, and the total number of the planes in Σ is at most $6L^3$. Therefore,

$$P(A_1^c) \leq 6L^3\Big(1-\exp\Big\{-\lambda^{1+\alpha}\exp\big((1+\alpha)/p\big)\big(\log 1/p\big)\Big\}\Big)^{(M/(2\lambda \exp(1/p)))^2}$$

Note that $M = \exp\{\exp((2-\alpha)/p)\}, 0 < \alpha < 1/4$. Thus $(M/(2\lambda \exp(1/p)))^2 = M^2/(4\lambda^2 \exp(2/p)) \gg M$, where $\gg$ means "much greater than." It follows from the inequality $1-x \leq e^{-x}$ that

$$\begin{aligned} P(A_1^c) &\leq 6L^3\Big(1-\exp\Big\{-\lambda^{1+\alpha}\exp\big((1+\alpha)/p\big)\big(\log 1/p\big)\Big\}\Big)^{M} \\ &\leq 6L^3\exp\Big\{-M\exp\Big\{-\lambda^{1+\alpha}\exp\big((1+\alpha)/p\big)\big(\log 1/p\big)\Big\}\Big\} \\ &= 6L^3\exp\Big\{-\exp\Big\{\big(1-o(p)\big)\exp\big((2-\alpha)/p\big)\Big\}\Big\} \\ &= \exp\Big\{-\exp\Big\{\big(1-o(p)\big)\exp\big((2-\alpha)/p\big)\Big\}\Big\}. \end{aligned} \tag{2.6}$$

To bound $P(A_2^c)$, we observe

$$P(A_2^c) = \Big(1-P\big(\Lambda_M \cap \xi_0^p \text{ is a } \Theta\text{-cube}\big)\Big)^{(L/M)^3}.$$

Applying (2.5), and again by the fact $1-x \leq e^{-x}$, we obtain

$$\begin{aligned} P(A_2^c) &\leq \Big(1-\big(1-o(p)\big)\exp\Big\{-\exp\big((1+\alpha)/p\big)\log\big(1/p\big)\Big\}\Big)^{(L/M)^3} \\ &\leq \exp\Big\{-(L/M)^3\big(1-o(p)\big)\exp\Big\{-\exp\big((1+\alpha)/p\big)\log\big(1/p\big)\Big\}\Big\}. \end{aligned}$$

Note $(L/M)^3 = (\exp\{\exp(2/p)\}/\exp\{\exp((2-\alpha)/p\})^3 = \exp\{(3-o(p))\exp(2/p)\}$. Hence

$$P(A_2^c) \leq \exp\Big\{-\exp\Big\{\big(3-o(p)\big)\exp\big(2/p\big)\Big\}\Big\}. \tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain that, for $L = \exp\{\exp(2/p)\}$,

$$\begin{aligned} p\big(\Lambda_L \cap \xi_0^p \text{ is a } \Psi\text{-cube}\big) &= 1-P\big(A_1^c \cup A_2^c\big) \geq 1-P\big(A_1^c\big)-P\big(A_2^c\big) \\ &\geq 1-\exp\Big\{-\exp\Big\{\big(1-o(p)\big)\exp\big((2-\alpha)/p\big)\Big\}\Big\}. \end{aligned} \tag{2.8}$$

2.3. *The behavior of the process starting with a Ψ-cube.* We are now going to study the behavior of the three-dimensional symmetric system in which the initial state η is such that $\Lambda \cap \eta$ is a Ψ-cube, where Λ is a cube with edge length $L = \exp\{\exp(2/p)\}$. In what follows, for a cubic region Λ that is described in Definitions 10 to 12, we will denote $\Delta(\Lambda) = \{\eta\colon \Lambda \cap \eta \text{ is a } \Delta\text{-cube}\}$, $\Theta(\Lambda) = \{\eta\colon \Lambda \cap \eta \text{ is a } \Theta\text{-cube}\}$ and $\Psi(\Lambda) = \{\eta\colon \Lambda \cap \eta \text{ is a } \Psi\text{-cube}\}$.

To accomplish our goal, we will apply a method that is very similar to the method employed in the proof of Theorem 2 in Chen (1992). First, we let $\eta_t^{(i)}, t \geq 0, i = 1, 2, \dots, 8$, be the processes with death rate identically 1 and birth rate $b_x^{(i)}(\xi)$, for $x \in \mathbf{Z}^3$, defined as follows:

$$
\begin{aligned}
b_x^{(1)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x-e_1, x-e_2\}, \{x-e_1, x-e_3\} \text{ or} \\
& \{x-e_2, x-e_3\} \text{ is occupied;} \\
b_x^{(2)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x+e_1, x-e_2\}, \{x+e_1, x-e_3\} \text{ or} \\
& \{x-e_2, x-e_3\} \text{ is occupied;} \\
b_x^{(3)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x+e_1, x+e_2\}, \{x+e_1, x-e_3\} \text{ or} \\
& \{x+e_2, x-e_3\} \text{ is occupied;} \\
b_x^{(4)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x-e_1, x+e_2\}, \{x-e_1, x-e_3\} \text{ or} \\
& \{x+e_2, x-e_3\} \text{ is occupied;} \\
b_x^{(5)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x-e_1, x-e_2\}, \{x-e_1, x+e_3\} \text{ or} \\
& \{x-e_2, x+e_3\} \text{ is occupied;} \\
b_x^{(6)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x+e_1, x-e_2\}, \{x+e_1, x+e_3\} \text{ or} \\
& \{x-e_2, x+e_3\} \text{ is occupied;} \\
b_x^{(7)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x+e_1, x+e_2\}, \{x+e_1, x+e_3\} \text{ or} \\
& \{x+e_2, x+e_3\} \text{ is occupied;} \\
b_x^{(8)}(\xi) = \lambda \quad & \text{if one of the pairs } \{x-e_1, x+e_2\}, \{x-e_1, x+e_3\} \text{ or} \\
& \{x+e_2, x+e_3\} \text{ is occupied.}
\end{aligned}
$$

Otherwise

$$b_x^{(i)}(\xi) = 0, \forall\, i = 1, 2, \dots, 8.$$

The initial state $\eta_0^{(i)}$ is $\mathbf{Z}^3, \forall\, i = 1, 2, \dots, 8$.

For $i = 1, 2, \dots, 8$, let $Q(i)$ denote the ith octant of $\mathbf{R}^3$ as in the conventional manner. Our strategy is to compare the process ξ_t^{ψ}, where $\psi \in \Psi(\Lambda_L)$ and $L = \exp\{\exp(2/p)\}$, with the process $\eta_t^{(i)}$ in the region $\Lambda_L \cap Q(i), i = 1, 2, \dots, 8$. We will prove that, $\forall\, i = 1, 2, \dots, 8$, with large probability, after a relatively short time period, ξ_t^{ψ} will start to dominate $\eta_t(i)$ in $\Lambda_L \cap Q(i)$. By a result proved in Durrett and Gray (1990), when λ is large, $\eta_t^{(i)}, t \geq 0$, is completely filled except for some isolated, short-lived holes. Thus the process $\xi_t^{\psi}, t \geq 0$, has the desired behavior.

To implement our strategy, we define, $\forall\, i = 1, 2, \dots 8$ and $t \geq 0$, the coupled process $\chi_t^{(i)} = (\xi_t^{\psi}, \eta_t^{(i)})$, where $\psi \in \Psi(\Lambda_L)$ and $L = \exp\{\exp(2/p)\}$. We regard a site

$x \in \mathbf{Z}^3$ as occupied by $\chi_t^{(i)}$, if $\chi_t^{(i)}(x) = (1,1), (1,0)$ or $(0,0)$ and x is not occupied by $\chi_t^{(i)}$, if $\chi_t^{(i)}(x) = (0,1)$. For convenience, we will also use the notation $x \in \chi_t^{(i)}$ and $x \notin \chi_t^{(i)}$ to describe these two cases, even though $\chi_t^{(i)}$ is not a set in $\mathbf{Z}^3$. Using this interpretation, the initial state of $\chi_t^{(i)}, i = 1,2,\dots,8$, is the same as Ψ. We are going to prove that, $\forall\ i = 1,2,\dots,8$, the probability that $\Lambda_L \cap Q(i)$ will be entirely occupied by $\chi_t^{(i)}$ for $t \in [L^3, L^6]$ is very large.

It follows from Definition 12 that Λ_L contains a Θ-cube which in turn contains a central Δ-cube. Since the arguments are essentially the same regardless of the location of the Θ-cube contained in Λ_L, we may assume, without loss of generality, that $\Lambda_M \cap \psi$ is a Θ-cube and thus $\Lambda_K \cap \psi$ is a Δ-cube, where $M = \exp\{\exp((2-\alpha)/p)\}, \alpha \in (0,1/4)$ and $K = \exp(1/p)$. Let $N = \max\{n : \Lambda_n \cap \psi$ is entirely occupied by $\xi_t^{\psi}\}$, then $N \geq bK^{\alpha/2} = b\exp(\alpha/2p)$ for some $b \in (1/2,1)$, due to the fact that $\Lambda_k \cap \psi$ is a Δ-cube. The desired results can be carried out through the following three steps:

1. We begin by investigating the behavior of the process $\chi_t^{(i)}$ in $\Lambda_N \cap Q(i), i = 1,2,\dots,8$. At time $t = 0, \Lambda_N$ is entirely occupied by $\chi_t^{(i)}, \forall i = 1,2,\dots,8$. We will prove that the probability that it will remain entirely occupied for all $t \in [0, N^6]$ is close to 1. This will be formulated as Lemma 3.
2. Now suppose Λ_N is entirely occupied by $\chi_t^{(i)}$ for all $t \in [0,N^6], \forall i = 1,2,\dots,8$. We will prove that, with large probability, Λ_N will continue to grow, and, by the time $t = (N+2)^3, \Lambda_{N+2} \cap Q(i)$ will be entirely occupied by $\chi_t^{(i)}, \forall i = 1,2,\dots,8$. Thus, by repeating the argument employed in step 1 above, we know that, with large probability, $\Lambda_{N+2} \cap Q(i)$ will continue to be occupied by $\chi_t^{(i)}$ for all $t \in [(N+2)^3, (N+2)^6]$. This will be formulated as Lemma 4.
3. We then apply mathematical induction and extend the above results to $n = N+4, N+6, \dots, K$ (Lemma 5), then to $n = K+2, K+4, \dots, M$ (Lemma 6) and finally to $n = M+2, M+4, \dots, L$ (Lemma 7). The reason for dividing the inductive procedure into three stages is due to the structure of the Ψ-cube under consideration.

It should be noted that in steps 2 and 3 above the cause of the continual growth of the process $\chi_t^{(i)} \cap \Lambda_n$ is different from the process studied in Chen (1992). In that case, the growth is due to the spontaneous births occurring on the boundary of $\Lambda_n \cap Q(i)$, but now it is due to the islands that are occupied initially and can survive long enough (such as Δ-cubes) to merge into the main island. This difference makes the technicalities in the proofs of the following lemmas a bit easier than those in the proofs in Proposition 6 (and the subsequent inductive procedure) in Chen (1992). We are now ready to present the Lemmas 3 to 7 as follows.

LEMMA 3. *Let $\chi_t^{(i)}$ and N be defined as in the preceding discussion. Then*

$$\begin{aligned} &P(\forall\ i = 1,2,\dots,8 \text{ and } t \in [0,N^6], \Lambda_N \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}) \\ &\quad \geq 1 - CN^6 \operatorname{vol}(\Lambda_N) \exp\{-a(\lambda)N\} = 1 - CN^9 \exp\{-a(\lambda)N\} \\ &\quad \geq 1 - \exp\{-\tfrac{3}{4}a(\lambda)N\}, \end{aligned}$$

where $a(\lambda)$ is an increasing function of λ independent of N, and, for sufficiently large $\lambda, a(\lambda) > 0$.

The method employed in obtaining the consequence (2.3) (Lemmas 5 and 6 and Corollaries 6 and 7) in Chen (1992) can be readily generalized to apply to this three-dimensional symmetric model case. We are not going to reproduce the proof here.

Based on Lemma 3 and a similar but simpler argument compared to that used in Chen (1992), we can obtain the following lemma which is parallel to Proposition 6 in the cited paper.

LEMMA 4.

$$\begin{aligned} P\Big(&\forall\, i = 1,2,\dots,8 \text{ and } t \in \big[(N+2)^3,(N+2)^6\big], \\ &\Lambda_{N+2} \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\Big) \\ &\geq \Big(1 - CN^9 \exp\{-a(\lambda)N\}\Big)\Big(1-(N+2)^6\exp\big(-a(\lambda)(N+2)^2\big)\Big) \\ &\quad\times\Big(1-\exp\big(-C(N+2)^2\big)\Big)\Big(1-C(N+2)^9\exp\{-a(\lambda)(N+2)\}\Big) \\ &\geq \left(1-\exp\left\{-\frac{3}{4}a(\lambda)N\right\}\right)\Big(1-\exp\{-\tfrac{3}{4}a(\lambda)(N+2)\}\Big). \end{aligned}$$

PROOF. Let

$$\begin{aligned} A(N,T) = \{&\forall\, i = 1,2,\dots,8 \text{ and } t\in[0,T], \Lambda_N \cap Q(i) \text{ is entirely} \\ &\text{occupied in } \chi_t^{(i)}\}, \end{aligned}$$

$$B(N+2,T) = \{\Lambda_{N+2}\backslash\Lambda_N \text{ is completely vacated in } \xi_t^{\psi} \text{ for some } t\in[0,T]\}$$

and

$$\tau(N+2) = \inf\{t\colon \forall\, i = 1,2,\dots,8, \Lambda_{N+2}\cap Q(i) \text{ is entirely occupied by } \xi_t^{(i)}\}.$$

We first claim that

$$\begin{aligned} P\Big(\tau(N+2) > (N+2)^3 \mid A(N,N^6)\cap B^c\big(N+2,(N+2)^3\big)\Big) \\ \leq \exp\{-C(N+2)^2\}. \end{aligned} \tag{2.9}$$

Under the condition $A(N,N^6)$, we only need to prove that

$$\begin{aligned} P\Big(\tau^*(N+2) > (N+2)^3 \mid A(N,N^6)\cap B^c\big(N+2,(N+2)^3\big)\Big) \\ \leq \exp\{-C(N+2)^2\}, \end{aligned} \tag{2.10}$$

where

$$\tau^* = \inf\Big\{t\colon \forall\, i = 1,2,\dots,8, \big(\Lambda_{N+2}\backslash\Lambda_N\big)\cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\Big\}.$$

$\Lambda_{N+2} \backslash \Lambda_N$ consists of six planes with edge length $N+2$. Under the condition $B^c(N+2,(N+2)^3)$, each of the six planes contains an island that is occupied not only by $\chi_t^{(i)}$ but also by ξ_t^{ψ} for all $t \in [0,(N+2)^3]$. We will focus our discussion on one of these six planes, say, the top face of Λ_{N+2}, which will be denoted by Σ. The same reasoning can be applied to the other five planes equally well. Let $x \in \xi_t^{\psi} \cap \Sigma, t \in [0, N^6], y_1 = x - e_3 + e_1, y_2 = x - e_3 + e_2, y_3 = x - e_3 - e_1$ and $y_4 = x - e_3 - e_2$. Then $y_j \in \Lambda_N, j = 1,2,3,4$, so $\chi_t^{(i)}(y_j) = (1,1), (1,0)$ or $(0,0)$ for $t \in [0, N^6]$. Hence, if we denote $P(\chi_t^{(i)}(y_j) = (1,1)$ or $(1,0)$ for all $j)$ by $\rho(\lambda)$, then, at rate $\lambda\rho(\lambda), x$ produces a ξ-birth at its vacant neighboring sites. The ξ-death rate is identically 1. Therefore, the growth of the occupied island $\xi_t^{\psi} \cap \Sigma, t \in [0, N^6]$, dominates an asexual contact process on $\mathbf{Z}^2$ with birth rate $\lambda\rho(\lambda)$ and death rate 1, described in Example 1 of Section 0. By Theorem 1 of Durrett and Gray (1990), $\forall x \in \mathbf{Z}^3$, when $\lambda \to \infty, P(\eta_t^{(i)}(x) = 1) \to 1$ for all t. Thus we may choose λ such that $\lambda\rho(\lambda)$ is sufficiently large. As a consequence of the shape theorem concerning the contact processes due to Durrett and Griffeath (1982), under the condition $B^c(N+2,(N+2)^3), \chi_t^{(i)} \cap \Sigma$, $t \in [0, N^6]$, will grow at a linear rate. This implies (2.10) and thus (2.9).

By the Markov property, we may apply the same argument as in Lemma 3 to $\chi_t^{(i)}$ in the region Λ_{N+2} and obtain

$$\begin{aligned} P\Big(&\forall\, i = 1,2,\dots,8 \text{ and } t \in \big[(N+2)^3,(N+2)^6\big],\ \Lambda_{N+2} \cap Q(i) \\ &\text{is entirely occupied by } \chi_t^{(i)} \mid A\big(N,N^6\big) \cap B^c\big(N+2,(N+2)^3\big)\Big) \\ &\ge \Big(1 - \exp\big\{-C(N+2)^2\big\}\Big)\Big(1 - C(N+2)^9 \exp\big\{-a(\lambda)(N+2)\big\}\Big). \end{aligned}$$

Since $S \cap \Lambda_K$ is entirely occupied by ξ_t^{ψ} at $t = 0$, each of the six planes of $\Lambda_{N+2} \backslash \Lambda_N$ contains a cross section of S that has area greater than N. It follows from Lemma 6 in Chen (1992) (with appropriate modification regarding dimensions) that

$$\begin{aligned} P\Big(B\big(N+2,(N+2)^3\big)\Big) &< \operatorname{vol}(\Lambda_{N+2})(N+2)^3 \exp\big\{-a(\lambda)(N+2)^2\big\} \\ &= (N+2)^6 \exp\big\{-a(\lambda)(N+2)^2\big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} P\Big(&\forall\, i = 1,2,\dots,8 \text{ and } t \in \big[(N+2)^3,(N+2)^6\big], \\ &\quad \Lambda_{N+2} \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\Big) \\ &\ge \Big(1 - CN^9 \exp\big\{-a(\lambda)N\big\}\Big)\Big(1 - (N+2)^6 \exp\big\{-a(\lambda)(N+2)^2\big\}\Big) \\ &\qquad \times \Big(1 - \exp\big(-C(N+2)^2\big)\Big)\Big(1 - C(N+2)^9 \exp\big\{-a(\lambda)(N+2)\big\}\Big) \\ &\ge \Big(1 - \exp\big\{-\tfrac{3}{4}a(\lambda)N\big\}\Big)\Big(1 - \exp\big\{-\tfrac{3}{4}a(\lambda)(N+2)\big\}\Big). \qquad \square \end{aligned}$$

By iterating the above arguments inductively, we can further obtain the following lemma.

LEMMA 5. *For all* $n = N+4, N+6, \dots, K$,

$$P\big(\forall\, i = 1,2,\dots,8 \text{ and } t \in [n^3, n^6], \Lambda_n \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\big) \geq \prod_{j \in J(N,n)} \left(1 - \exp\left\{-\frac{3}{4}a(\lambda)j\right\}\right),$$

where $J(N,n)$ *is defined as the integer set* $\{N, N+2, \dots, n\}$.

PROOF. The arguments employed in Lemma 4 still apply without requiring any modification due to the fact that $S \cap \Lambda_K$ is entirely occupied by ξ_t^ψ at $t = 0$. □

Applying the inductive procedure repeatedly (with slight modification), we can further extend our results for $n = K+2, K+4, \dots, M$ and then for $n = M+2, M+4, \dots, L$. This is formulated as the following lemmas.

LEMMA 6. *For all* $n = K+2, K+4, \dots, M$,

$$P\big(\forall i = 1,2,\dots,8 \text{ and } t \in [n^3, n^6], \Lambda_n \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\big) \geq \prod_{j \in J(N,K)} \left(1 - \exp\left\{-\frac{3}{4}a(\lambda)j\right\}\right), \prod_{j \in J(K+2,n)} \big(1 - \exp\{-a(\lambda)\lambda \log j\}\big),$$

Here and in what follows, $J(\cdot,\cdot)$ *has the same meaning as in Lemma* 5.

PROOF. We are not going to present every detail of the induction. Rather, we will only illustrate the key ingredients that are different from those employed in the proof of Lemma 4. The modification needed here is that in this case we will use the fact that each of the six planes of $\Lambda_n \backslash \Lambda_{n-2}$ intersects at least one Δ-cube with edge length at least $\lambda(\log n)^{1/2}$. (See Definition 11.) Define the event $B(n,T)$ in much the same manner as $B(N,T)$; that is,

$$B(n,T) = \Big\{ \big(\Lambda_n \backslash \Lambda_{n-2}\big) \text{ is completely vacated in } \xi_t^\psi \text{ for some } t \in [0,T] \Big\}.$$

It is easy to check that

$$\begin{aligned} P\big(B(n,n^3)\big) &\leq \mathrm{vol}(\Lambda_n) n^3 \exp\Big\{-a(\lambda)\big(\lambda(\log n)^{1/2}\big)^2\Big\} \\ &\leq n^6 \exp\big\{-a(\lambda)\lambda^2 \log n\big\} \\ &= \exp\big\{-a(\lambda)\lambda^2 \log n + 6\log n\big\}. \end{aligned}$$

When λ is sufficiently large, $\frac{1}{2}a(\lambda)\lambda^2 \geq 6$. Therefore,

$$\begin{aligned} P\big(B(n,n^3)\big) &\leq \exp\big\{-\tfrac{1}{2}a(\lambda)\lambda^2 \log n\big\} \\ &\leq \exp\big\{-a(\lambda)\lambda \log n\big\}. \end{aligned}$$

The rest of the proof is essentially the same as in Lemma 4. □

LEMMA 7. *For all* $n = M+2, M+4, \dots, L$,

$$\begin{aligned}
&P\big(\forall i = 1,2,\dots,8 \text{ and } t \in [n^3, n^6], \Lambda_n \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\big)\\
&\quad\ge \prod_{j\in J(N,K)} \left(1-\exp\left\{-\frac{3}{4}a(\lambda)j\right\}\right) \prod_{j\in J(K+2,M)} \big(1-\exp\{-a(\lambda)\lambda\log j\}\big)\\
&\qquad\times \prod_{j\in J(M+2,n)} \big(1-\exp\{-a(\lambda)\lambda\log j\}\big)\\
&\quad= \prod_{j\in J(N,K)} \left(1-\exp\left\{-\frac{3}{4}a(\lambda)j\right\}\right) \prod_{j\in J(K+2,n)} \big(1-\exp\{-a(\lambda)\lambda\log j\}\big).
\end{aligned}$$

PROOF. Again, we are not going to present the complete inductive procedure. We will only illustrate the key ingredients to the success of the induction as follows. In this case we use the fact stated in (2.1) of Definition 12 that each of the six planes of $\Lambda_n\backslash\Lambda_{n-2}$ intersects at least one Δ-cube with edge length at least $\lambda\exp(1/p)$. Thus

$$\begin{aligned}
P\big(B(n,n^3)\big) &\le \mathrm{vol}(\Lambda_n) n^3 \exp\left\{-a(\lambda)\big(\lambda\exp(1/p)\big)^2\right\}\\
&\le n^6\exp\left\{-a(\lambda)\lambda^2\exp(2/p)\right\}.
\end{aligned}$$

Note that $n \le L = \exp\{\exp(2/p)\}$, so $\exp(2/p) \ge \log n$. Hence

$$P\big(B(n,n^3)\big) \le n^6\exp\{-a(\lambda)\lambda^2\log n\} \le \exp\{-a(\lambda)\lambda^2\log n + 6\log n\}.$$

When λ is sufficiently large, $\frac{1}{2}\,a(\lambda)\lambda^2 \ge 6$. Therefore,

$$\begin{aligned}
P\big(B(n,n^3)\big) &\le \exp\{-\tfrac{1}{2}a(\lambda)\lambda^2\log n\}\\
&\le \exp\{-a(\lambda)\lambda\log n\}.
\end{aligned}$$

The rest of the proof is essentially the same as in Lemma 4. □

2.4. *Conclusion of the Proof of Theorem* 4. It follows directly from Lemma 7 that, for $L = \exp\{\exp(2/p)\}$,

$$\begin{aligned}
&P\big(\forall i = 1,2,\dots,8 \text{ and } t \in [L^3, L^6], \Lambda_L \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\big)\\
&\quad\ge \prod_{j\in J(N,K)} \left(1-\exp\left\{-\frac{3}{4}a(\lambda)j\right\}\right) \prod_{j\in J(K+2,L)} \big(1-\exp\{-a(\lambda)\lambda\log j\}\big),
\end{aligned}$$

Recall that $K = \exp(1/p), N \geq bK^{\alpha/2} = b\exp(\alpha/2p)$, for some $b \in (1/2, 1)$. We have

$$\prod_{j \in J(N,K)} \left(1 - \exp\left\{-\frac{3}{4}a(\lambda)j\right\}\right) \geq 1 - \frac{4}{a(\lambda)} \exp\left\{-\frac{3}{4}a(\lambda)N\right\}$$
$$\geq 1 - \exp\left\{-\frac{1}{2}a(\lambda)N\right\} \geq 1 - \exp\left\{-\frac{b}{2}a(\lambda)\exp\left(\alpha/2p\right)\right\}$$

and

$$\prod_{j \in J(K+2,L)} \left(1 - \exp\{-a(\lambda)\lambda \log j\}\right) \geq \left(1 - \exp\left\{-\frac{3}{4}a(\lambda)\lambda \log K\right\}\right)$$
$$= 1 - \exp\left\{-\frac{3}{4}a(\lambda)\lambda\frac{1}{p}\right\}.$$

Therefore,

$$\begin{aligned} &P\left(\forall\, i = 1,2,\ldots,8 \text{ and } t \in [L^3,L^6], \Lambda_L \cap Q(i) \text{ is entirely occupied by } \chi_t^{(i)}\right) \\ &\quad \geq \left(1 - \exp\left\{-\frac{b}{2}a(\lambda)\exp\left(\alpha/2p\right)\right\}\right)\left(1 - \exp\left\{-\frac{3}{4}a(\lambda)\lambda\frac{1}{p}\right\}\right). \end{aligned} \tag{2.11}$$

Consider the coupled process $\chi_t^{(p,i)} = (\xi_t^p, \eta_t^{(i)}), t \geq 0, i = 1,2,\ldots,8$. The meaning of occupancy and vacancy for $\chi_t^{(p,i)}$ is defined in the same manner as for $\chi_t^{(i)}$. Combining (2.8) and (2.11), we obtain that

$$\begin{aligned} &P\left(\forall\, i = 1,2,\ldots,8 \text{ and } t \in [L^3,L^6], \Lambda_L \cap Q(i) \text{ is entirely occupied by } \chi_t^{(p,i)}\right) \\ &\quad \geq \left(1 - \exp\left\{-\exp\left\{(1 - o(p))\exp\left((2-\alpha)/p\right)\right\}\right\}\right) \\ &\quad\quad \times \left(1 - \exp\left\{-\frac{b}{2}a(\lambda)\exp\left(\alpha/2p\right)\right\}\right)\left(1 - \exp\left\{-\frac{3}{4}a(\lambda)\lambda\frac{1}{p}\right\}\right) \\ &\quad \geq 1 - \exp\left\{-C(\lambda)\frac{1}{p}\right\}, \end{aligned} \tag{2.12}$$

where $C(\lambda) = \frac{3}{4}(1 - o(p))a(\lambda)\lambda$. We may choose λ so that $C(\lambda) \geq 4$. Thus, if we denote the probability in the left-hand side of (2.12) by $P(L,p)$, then $P(L,p) \geq 1 - \exp(-4/p)$.

Now let us consider a sequence $\{p_n\}, n = 1,2,\ldots$, such that, for each n, p_n is defined as $p_n = p(1 + (p/2)\log(n+1))^{-1}$. Define the $\psi(p_n)$-cube by using the same definitions in subsection 2.1 (replacing p by p_n). For each n, we repeat the arguments applied in subsections 2.2 and 2.3 without any change except for replacing p by p_n. Therefore, we obtain that $P(L_n,p_n) \geq 1 - \exp(-4/p_n)$ for each n. Here L_n is the size of the $\psi(p_n)$-cube specified in Definition 12. That is,

$$L_n = \exp\Big\{\exp(2/p_n)\Big\} = \exp\Big\{\exp(2/p + \log(n+1))\Big\}$$
$$= \exp\Big\{(n+1)\exp(2/p)\Big\} = \Big(\exp\Big\{\exp(2/p)\Big\}\Big)^{(n+1)} = L^{n+1}.$$

Note that $p > p_n$, for all n, and, for fixed $L, P(L, p)$ is an increasing function of p. Hence

$$P(L_n, p) \geq P(L_n, p_n) \geq 1 - \exp\left(-4/p_n\right)$$
$$= 1 - \exp\Big\{-(4/p + 2\log(n+1))\Big\} = 1 - (n+1)^{-2}\exp\left(-4/p\right).$$

That is, for every $n = 1, 2, \dots,$

$$(2.13)\qquad \begin{aligned} &P\Big(\forall\, i = 1, 2, \dots, 8 \text{ and } t \in \big[L^{3(n+1)}, L^{6(n+1)}\big],\\ &\quad \Lambda\big(L^{n+1}\big) \cap Q(i) \text{ is entirely occupied by } \chi_t^{(p,i)}\Big)\\ &\quad \geq 1 - (n+1)^{-2}\exp\left(-4/p\right). \end{aligned}$$

Denote the events described in (2.13) by $E_n, n = 1, 2, \dots$, then $\{E_n\}$ are positively correlated. Note that, for each fixed n, $6(n+j) \geq 3(n+j+1)$, $j = 1, 2, \dots$. It follows from (2.12) and (2.13) that, for every $n = 1, 2, \dots$,

$$(2.14)\qquad \begin{aligned} &P\Big(\forall\, i = 1, 2, \dots, 8 \text{ and } t \in \big[L^{3(n+1)}, \infty\big),\\ &\quad \Lambda\big(L^{n+1}\big) \cap Q(i) \text{ is entirely occupied by } \chi_t^{(p,i)}\Big)\\ &\quad \geq \prod_{n=0}^{\infty}\Big(1 - (n+1)^{-2}\exp\big(-4/p\big)\Big) \geq 1 - 2\exp\big(-4/p\big). \end{aligned}$$

We are now ready to carry out the final step of our proof. For any given $x \in \mathbf{Z}^3$, there is an $L_0 \in \{L, L^2, \dots\}$ such that $x \in \Lambda(L_0) \cap Q(i)$ for some $i \in \{1, 2, \dots, 8\}$. Without loss of generality, we may assume $i = 1$. It follows from (2.14) that

$$P\Big(x \in \chi_t^{(p,1)} \text{ for all } t \in \big[L_0^3, \infty\big)\Big) \geq 1 - 2\exp\big(-4/p\big).$$

That is,

$$P\Big(\big(\xi_t^p(x), \eta_t^{(1)}(x)\big) = (1,1), (0,0) \text{ or } (1,0) \text{ for all } t \in \big[L_0^3, \infty\big)\Big)$$
$$\geq 1 - 2\exp\big(-4/p\big).$$

By Theorem 1 of Durrett and Gray (1990), for all $x \in \mathbf{Z}^3$ and $t \geq 0$, when λ is sufficiently large, $\lim_{t\to\infty} P(x \in \eta_t^{(1)}) > 0$. Hence $\lim_{t\to\infty} P(x \in \xi_1^p) > 0$. This completes the proof of Theorem 4. □

Acknowledgment. The methods of solving the problems studied in this paper were developed when the author worked on his dissertation [published earlier, cited as Chen (1992) in this paper] at the University of Minnesota, under the direction of Professor Lawrence F. Gray. The author would like to thank Professor Gray for his numerous valuable suggestions and strong support. The author would also like to thank the referee for his valuable comments concerning the revision of this paper.

REFERENCES

AIZENMAN, M. and LEBOWITZ, J. L. (1988). Metastability effects in bootstrap percolation. *J. Phys. A* **21** 3801–3813.

BRAMSON, M. and GRAY, L. (1992). A useful renormalization argument. In *Random Walks, Brownian Motion and Interacting Particle Systems: A Festschrift in Honor of Frank Spitzer* (R. Durrett and H. Kesten, eds.) 113–152. Birkhäuser, Boston.

CHEN, H.-N. (1992). On the stability of a population growth model with sexual reproduction on $\mathbf{Z}^2$. *Ann. Probab.* **20** 232–285.

DURRETT, R. (1985). Stochastic growth models: ten problems for the 80's (and 90's). In *Particle Systems, Random Media, and Large Deviations* (R. Durrett, ed.) 87–99. Amer. Math. Soc., Providence, RI.

DURRETT, R. (1988). *Lecture Notes on Particle Systems and Percolation*, Chapter 1. Wadsworth and Brooks/Cole, Pacific Grove, CA.

DURRETT, R. and GRAY, L. (1990). Some peculiar properties of a particle system with sexual reproduction. Preprint. Preliminary version in P. Tautu (1986). Stochastic spatial processes. *Lecture Notes in Math.* **1212**. Springer, New York.

DURRETT, R. and GRIFFEATH, D. (1982). Contact processes in several dimensions. *Z. Wahrsch. Verw. Gebiete* **59** 535–552.

HARRIS, T. (1978). Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378.

HOLLEY, R. (1972). An ergodic theorem for interacting particle systems with attractive interactions. *Z. Wahrsch. Verw. Gebiete* **24** 325–334.

LIGGETT, T. M. (1985). *Interacting Particle Systems.* Springer, New York.

DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE
AND STATISTICS
PURDUE UNIVERSITY, CALUMET
HAMMOND, INDIANA 46323-2094