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ASYMPTOTICS OF EXIT TIMES FOR MARKOV JUMP PROCESSES. II: APPLICATIONS TO JACKSON NETWORKS

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We show that a Jackson network relaxes exponentially fast to its steady state by giving a lower bound on the Cheeger constant for the associated Markov process. We also give lower bounds on the mean time until some node of the network overflows.

1. Introduction. This article is the continuation of Iscoe and McDonald (1994) (hereafter cited as Part I), where we studied the distribution of the first hitting time τ of some forbidden set F by a Markov jump process with unique stationary measure π. We now apply these results to a Jackson network having m nodes, where we suppose the buffer at node i is of size $l_i - 1$. We define the forbidden set of this network to be those states where the queue at some node exceeds the buffer size. Applying the results in Part 1, we conclude that the time until overload is approximately exponential with mean $1/\Lambda$ if $l \equiv \min_i l_i$ is large enough [so that $\pi(F)$ is small]. Λ is the Perron–Frobenius eigenvalue associated with the infinitesimal generator $-L$ of the Jackson network killed on the forbidden set. We also obtain an explicit upper bound on the probability of overload during a fixed time period as well as a lower bound on the mean time until overload.

To establish the above results we must show the existence of a gap between the eigenvalue 0 and the rest of the spectrum of the infinitesimal generator of the Jackson network. This is done in Section 3. The existence of this gap is equivalent to the exponentially quick relaxation of the Markov jump process to the steady state. Using the results in Lawler and Sokal (1988) on the Cheeger constant for a jump process, the proof of Theorem 3.1 provides an algorithm for estimating the gap. The essential idea is to compare the Cheeger constant associated with L with the Cheeger constant associated with $\widetilde{L}$, where $-\widetilde{L}$ is the infinitesimal generator of the vector of independent birth and death processes evolving like the marginals at each node of our Jackson network. This is possible because π is the product of the stationary measures of the marginal processes.

The fact that the marginal processes associated with each node of a Jackson network are Markov (birth and death) processes allows one to closely approximate Λ by $\sum_{i=1}^{m} \Lambda_i$, where Λ_i is the Perron–Frobenius eigenvalue associated with the infinitesimal generator of the marginal process at node i killed outside of

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$[0, l_i - 1]$. Λ is asymptotic to $1/E\tau$, where τ is the mean time to exit the complement of F, and Λ_i is asymptotic to $1/E\tau_i$, where τ_i is the mean time for the marginal process at node i to exit $[0, l_i - 1]$. We thus have a partial confirmation (in the reversible case) of the conjecture by Anderson and Frater (1988) that the mean time until the network overflows can be asymptotically bounded below by combining the mean times until the individual nodes overflow like electrical resistances in parallel. Meyn and Frater (1990) have established an exact equality when the expectations are relative to the distribution of the Jackson network immediately after recovery from an overflow. This result does not give the mean length of the idle period; that is, the time until overload starting with empty queues or, what is asymptotically equivalent, from the equilibrium distribution. We have not been able to prove the conjecture in the nonreversible case, but we do give lower bounds based on the general results in Part 1.

2. Definitions, notation and results. We will keep the notation and definitions found in Part 1, with appropriate modifications for the present discrete setting. In addition, we follow the presentation in Brémaud (1981) and consider a *Jackson network* having m stations or nodes. A typical node i receives customers exogenously according to a Poisson process of rate $\overline{\lambda}_i$. When x_i customers are at node i, then the service rate is $\mu_i(x_i)$. When a customer completes his service at node i, he is routed to node j with probability $r_{ij}, j \neq i$ ($r_{ii} = 0$), and leaves the network with probability $r_i := 1 - \sum_{j=1}^{m} r_{ij}$. Such a system is a Markov process $(X_t;\ t \geq 0)$ on $S := \{0, 1, 2, \dots\}^m$, where $\mathbf{x} = (x_1, \dots, x_m) \in S$ denotes the state in which there are x_i customers waiting or being served at node i, $i = 1, \dots, m$. We assume that the network is both *exogenously supplied* and *open*. [The term "*exogenously supplied*" means that each node j is fed (possibly via other nodes) by some node i for which $\overline{\lambda}_i \neq 0$, and the term "open" means that each node i feeds some node j (again possibly via other nodes) for which $r_j \neq 0$.] Denote the possible transitions from one state to another with the following operators:

$$T_{ij}\mathbf{x} := (x_1, \dots, x_i - 1, \dots, x_j + 1, \dots, x_m),$$
$$T_{\cdot j}\mathbf{x} := (x_1, \dots, x_j + 1, \dots, x_m),$$
$$T_{i\cdot}\mathbf{x} := (x_1, \dots, x_i - 1, \dots, x_m).$$

Clearly T_{ij} represents a departure from node i to node j, $T_{\cdot j}$ represents an exogenous arrival at node j and $T_{i\cdot}$ represents a departure from the network from node i.

The process $(X_t;\ t \geq 0)$ is irreducible by our assumption that the network is exogenously supplied and open. We state the main result about the invariant measure of such Jackson networks [see Brémaud (1981), Theorem T7, Chapter V]:

THEOREM 2.1. *For an exogenously supplied and open Jackson network, there exists a unique stationary distribution if and only if the* (*unique*) *solution of the* *traffic equations*,

$$\lambda_i = \overline{\lambda}_i + \sum_{j=1}^{m} \lambda_j r_{ji}, \qquad 1 \le i \le m, \tag{2.1}$$

satisfies the light-traffic condition

$$b_i := \sum_{n=0}^{\infty} \frac{\lambda_i^n}{\prod_{y=1}^{n} \mu_i(y)} < \infty, \qquad 1 \le i \le m.$$

The stationary distribution $\pi(\mathbf{x}), \mathbf{x} = (x_1, \dots, x_m) \in S$, *is then given by the product*

$$\pi(\mathbf{x}) = \prod_{i=1}^{m} \pi_i(x_i),$$

where

$$\pi_i(x_i) := \frac{1}{b_i} \frac{\lambda_i^{x_i}}{\prod_{y=1}^{x_i} \mu_i(y)}.$$

Thus at any instant t, *the queue sizes at different nodes are independent with respect to* π *and the queue at node i has the stationary distribution of a birth and death process having constant birth rate* λ_i *and death rate* $\mu_i(y)$ *at* $y \in \{1, 2, \dots\}$.

We make the following standing assumptions.

H1. We assume that the network is exogenously supplied and open, and that the light-traffic condition holds.

H2. For any node i, the death rates are bounded from above and below away from 0; set: $\mu^i = \sup_y \mu_i(y) < \infty$ and $\mu_i = \inf_{y>0} \mu_i(y) > 0$.

H3. Each of the π_i's has an exponential tail in the sense that

$$\sup_{x \in \mathbb{Z}_+} \frac{\sum_{y > x} \pi_i(y)}{\pi_i(x)} < \infty.$$

In view of the second hypothesis (H2), the third hypothesis (H3) certainly holds if $\lambda_i < \mu_i$ for all i, $1 \le i \le m$.

Thus, in particular, **H2** implies that the transition rate kernel $J(\mathbf{x}, \cdot)$ of the Jackson network is uniformly bounded. Explicitly,

$$M \equiv \sup_{\mathbf{x}} J(\mathbf{x}) \equiv \sup_{\mathbf{x}} J(\mathbf{x}, \{\mathbf{x}\}^c) = \sum_{i=1}^{m} \left[\overline{\lambda}_i + \mu^i\right].$$

Let $-L$ denote the infinitesimal generator of the Jackson network on $L^2(S, \pi)$. [Although S is discrete, we will write $L^2(S, \pi)$ instead of employing the more usual notation $l^2(S,\pi)$ to emphasize the role of π as a measure. However we will abbreviate $\pi(\{\mathbf{x}\})$ to $\pi(\mathbf{x})$, for $\mathbf{x} \in S$.] For any $f \in L^2(S,\pi)$, Lf is given by

$$Lf(\mathbf{x}) = \sum_{j=1}^{m} [f(\mathbf{x}) - f(T_{\cdot j}\mathbf{x})]\overline{\lambda}_j + \sum_{i=1}^{m}\sum_{j=1}^{m} [f(\mathbf{x}) - f(T_{ij}\mathbf{x})]\,\mu_i(x_i)r_{ij}$$

$$+ \sum_{j=1}^{m} [f(\mathbf{x}) - f(T_{j\cdot}\mathbf{x})]\,\mu_j(x_j)r_{j\cdot}. \tag{2.2}$$

The infinitesimal generator $-L_i$ on $L^2(\mathbb{N},\pi_i)$ of the marginal birth and death process associated with the ith node, has constant birth rate λ_i and death rate $\mu_i(y)$ at $y \in \mathbb{N}$ and is determined by

$$L_if(y) := [f(y) - f(y+1)]\lambda_i + [f(y) - f(y-1)]\mu_i(y) \tag{2.3}$$

for $f \in L^2(\mathbb{N},\pi_i)$, $1 \le i \le m$. L_i is self-adjoint on $L^2(\mathbb{N},\pi_i)$; that is, π_i is a reversibility measure for the birth and death process.

Now consider a Jackson network where the maximum queue size permitted at node i is $l_i - 1$. Customers arriving when this maximum has been reached are lost. We can study the onset of overload of this limited model through the one determined by (2.2) as follows. The forbidden states F for the latter are those $(x_1, x_2, \dots, x_m)$ such that $x_i \ge l_i$ for at least one i, and we are interested in the first time τ that we enter these forbidden states from $B := F^c$. The limited model is stochastically equivalent to the one determined by (2.2) up to time τ. Because we are only concerned with the behaviour of the processes on the interval $[0,\tau]$, we shall not make a formal distinction between the two models and will work with the one determined by (2.2). Denote the infinitesimal generator of the killed Jackson network (i.e., killed off B) by $-L^B$. $-L^B$ is a finite matrix because B is a finite set. Hence $-L^B + MI$ is a positive finite matrix. By the Perron–Frobenius theorem [see Seneta (1981)] this matrix has a positive maximum eigenvalue $M + \Lambda(B)$ (of multiplicity 1). Thus L^B has the extreme eigenvalue $\Lambda(B) > 0$ (of multiplicity 1).

Recall the definitions from Part 1 of the killing rate $K^B(\mathbf{x}) := J(\mathbf{x}, B^c)$, $\mathbf{x} \in B$, and the resuscitation rate $R^B(\mathbf{y}) := [\pi(\mathbf{y})]^{-1}\sum_{\mathbf{x}\in B^c}\pi(\mathbf{x})J(\mathbf{x},\mathbf{y})$, $\mathbf{y} \in B$. It will be convenient to renormalize $\pi\,|\,B$, the restriction of π to B, to be a probability [assuming that $\pi(B) > 0$, which we do for the remainder of this article] and to set

$$\widehat{\pi} \equiv \widehat{\pi}^B := [\pi(B)]^{-1}(\pi \mid B), \qquad L^2(\widehat{\pi}) \equiv L^2(B,\widehat{\pi}),$$
$$(\cdot,\cdot)_{\widehat{\pi}} \equiv (\cdot,\cdot)_{L^2(\widehat{\pi})}, \qquad \|\cdot\|_{\widehat{\pi}} \equiv \|\cdot\|_{L^2(\widehat{\pi})}\,.$$

We will employ similar notation for the marginals π_i renormalized on $B_i := [0, l_i - 1]$. We also recall the following three constants:

$$\overline{\kappa} \equiv \overline{\kappa}^B := \sum_{\mathbf{x}\in B} K^B(\mathbf{x})\widehat{\pi}(\mathbf{x}), \quad \kappa_1 := \|K^B - \overline{\kappa}\|_{\widehat{\pi}}, \quad \kappa_2 := \|R^B - \overline{\kappa}\|_{\widehat{\pi}}. \tag{2.4}$$

Inserting the jump rates of the Jackson network we have

$$
\begin{aligned}
\overline{\kappa} &= \sum_{i=1}^{m} \sum_{\mathbf{x}: x_i = l_i - 1} \Big[\overline{\lambda}_i + \sum_{j \neq i} \mu_j(x_j) r_{ji}\Big] \widehat{\pi}(\mathbf{x}) \\
&= \sum_{i=1}^{m} \Big[\overline{\lambda}_i + \sum_{j \neq i} \sum_{x_j \le l_j - 1} \mu_j(x_j) r_{ji} \widehat{\pi}_j(x_j)\Big] \widehat{\pi}_i(l_i - 1) \quad [\text{as } \widehat{\pi}_i(B_i) = 1] \\
&= \sum_{i=1}^{m} \Big[\overline{\lambda}_i + \sum_{j \neq i} \lambda_j r_{ji} \sum_{0 \le x_j \le l_j - 2} \widehat{\pi}_j(x_j)\Big] \widehat{\pi}_i(l_i - 1) \quad (\text{by reversibility}) \\
&= \sum_{i=1}^{m} \lambda_i \widehat{\pi}_i(l_i - 1) - \sum_{i=1}^{m} \sum_{j \neq i} \lambda_j r_{ji} \widehat{\pi}_j(l_j - 1) \widehat{\pi}_i(l_i - 1) \quad [\text{by } (2.1)]
\end{aligned} \tag{2.5}
$$

so that

$$
\overline{\kappa} \le \sum_{i=1}^{m} \lambda_i \widehat{\pi}_i(l_i - 1). \tag{2.6}
$$

Moreover

$$
\kappa_1 \le [M\overline{\kappa}]^{1/2} \quad \text{and} \quad \kappa_2 \le [M\overline{\kappa}]^{1/2}. \tag{2.7}
$$

The inequalities in (2.7) follow from the bound $J(\mathbf{x}) \le M$ and Lemma 2.5 of Part I. It is clear that the three kappas tend to zero as $\min_i(l_i) \to \infty$.

Finally we recall the notion of a *spectral gap*. We remark that in the reversible case, in which L is self-adjoint on $L^2(S, \pi)$, the spectral gap (when positive) is the gap in $\sigma(L) \subset \mathbb{R}_+$ between the simple eigenvalue 0 and the rest of the spectrum.

DEFINITION 2.2. For $-L$ being the infinitesimal generator of the Jackson network on $L^2(\pi)$, and 1 denoting the constant function with value 1,

$$
\operatorname{Gap}(L) := \inf\big\{(f, Lf)_\pi : \|f\|_\pi = 1, (f, 1)_\pi = 0\big\}.
$$

In Section 3, we show that $\operatorname{Gap}(L) > 0$. The following result is then an immediate application of Theorems 2.7 and 2.8 of Part 1.

THEOREM 2.3. *If $\pi(B^c)$ is sufficiently small, then*

$$
\big|P_{\widehat{\pi}}(\tau > t) - \exp\big(-\Lambda(B)t\big)\big| \le \beta(B)\exp\big(-\Lambda(B)t\big), \tag{2.8}
$$

where

$$
|\Lambda(B) - \overline{\kappa}| \le 2\kappa_1\kappa_2/\big[\operatorname{Gap}(L) - \overline{\kappa}\big]
$$

and

$$
E_{\widehat{\pi}}\tau \ge \Big[\overline{\kappa} + 2\kappa_1\kappa_2/\big[\operatorname{Gap}(L) - \overline{\kappa}\big]\Big]^{-1}(1 - \beta(B))
$$

where

$$\beta(B) := \frac{4}{\left(\mathrm{Gap}(L)-\overline{\kappa}\right)^2 - 4\kappa_1\kappa_2}\left[1+\frac{\sqrt{\left(\mathrm{Gap}(L)-\overline{\kappa}\right)^2+4\kappa_2^2}}{\mathrm{Gap}(L)-\overline{\kappa}}\right]\kappa_1\kappa_2. \tag{2.9}$$

Conditions ensuring that $\pi(B^c)$ *is sufficiently small are given in Theorem* 2.8 *in Part* 1. $\overline{\kappa}$, κ_1 *and* κ_2 *are described by* (2.4)–(2.7).

In the reversible case, in which π is, in addition, a reversibility measure for $(X_t;\, t \geq 0)$, Theorem 2.3 can be sharpened.

THEOREM 2.4. *If the Jackson network is reversible, then*

$$E_{\hat{\pi}}\tau \geq \left(\sum_{i=1}^{m}\left[E_{\hat{\pi}_i}\tau_i\right]^{-1}\right)^{-1}\left[1-\min\left(\frac{\overline{\kappa}}{\mathrm{Gap}(L)}, 1\right)\right],$$

where τ_i *is the first time the ith node leaves* B_i. *Also*

$$E_{\hat{\pi}_i}\tau_i = \frac{1}{\left[\Pi_i(l_i-1)\right]^2}\sum_{y=0}^{l_i-1}\frac{\left[\Pi_i(y)\right]^2}{\lambda_i\hat{\pi}_i(y)} \quad \textit{where } \Pi_i(y) := \sum_{s=0}^{y}\pi_i(s).$$

Reversibility is a rather stringent condition for Jackson networks because it means that for all nodes i and j,

$$\overline{\lambda_i} = \lambda_i r_i, \quad \text{and} \quad \lambda_i r_{ij} = \lambda_j r_{ji}.$$

If reversibility is satisfied, L^B is self-adjoint on $L^2(\hat{\pi})$. It is clear that most Jackson networks are not reversible. Thus the results in their full generality will usually be needed.

We illustrate our results by considering a simple case with two nodes 1 and 2 each having single servers with constant service rates μ_1 and μ_2, respectively. Customers finishing service at node 1 jump to the queue at node 2, with probability r_{12}, or leave the system with probability $r_1 = 1 - r_{12}$. Similarly customers finishing service at node 2 jump to the queue at node 1, with probability r_{21}, or leave the system with probability $r_2 = 1 - r_{21}$.

In this example the traffic equations of Theorem 2.1 are

$$\lambda_1 = \overline{\lambda_1} + r_{21}\lambda_2,$$
$$\lambda_2 = \overline{\lambda_2} + r_{12}\lambda_1,$$

so

$$\lambda_1 = \frac{\overline{\lambda_1} + r_{21}\overline{\lambda_2}}{1 - r_{12}r_{21}}, \qquad \lambda_2 = \frac{\overline{\lambda_2} + r_{12}\overline{\lambda_1}}{1 - r_{12}r_{21}},$$

where λ_1 and λ_2 represent the net flow into nodes 1 and 2, respectively. The light-traffic conditions are simply that $\rho_1 := \lambda_1/\mu_1 < 1$ and that $\rho_2 := \lambda_2/\mu_2 < 1$.

By Theorem 2.1 again, this Markov jump process is positive recurrent with respect to the stationary product probability measure $\pi(x_1, x_2) = \pi_1(x_1) \cdot \pi_2(x_2)$, where

$$\pi_1(x_1) = (1 - \rho_1)\rho_1^{x_1}, \qquad \pi_2(x_2) = (1 - \rho_2)\rho_2^{x_2}.$$

An estimate of the gap will be derived at the end of Section 3, for a simple case of this example.

3. Gaps for Jackson networks. Note that although $\pi = \prod_{i=1}^{m} \pi_i$, $L \neq \sum_{i=1}^{m} L_i$. Nevertheless a useful comparison between $\operatorname{Gap}(L)$ and $\operatorname{Gap}(\sum_{i=1}^{m} L_i)$ will be made in the proof of the next theorem.

THEOREM 3.1. *Under the standing assumptions*, $\operatorname{Gap}(L) > 0$, *where* $-L$ *is the infinitesimal generator of the Jackson network.*

PROOF. By Theorem 2.3 in Lawler and Sokal (1988) we have, for any stationary Markov jump process, on a measurable state space $(S, \mathcal{S})$ with bounded infinitesimal generator $-L$ and stationary measure π,

$$\begin{aligned} \operatorname{Gap}(L) &\equiv \inf\{(f, Lf)_\pi : \|f\|_\pi = 1 \text{ and } (f, \mathbf{1})_\pi = 0\} \\ &\geq \mathbf{k}^2/8M, \end{aligned}$$

where $M = \pi\text{-ess sup } J(x, \{x\}^c) < \infty$ and $\mathbf{k}$ is Cheeger's isoperimetric constant defined by

$$\mathbf{k} := \inf_{\substack{A \in \mathcal{S} \\ \pi(A) > 0}} \mathbf{k}(A)$$

with

$$\mathbf{k}(A) := \frac{\int_{\mathbf{x} \in A} \pi(dx) J(\mathbf{x}, A^c)}{\pi(A)\pi(A^c)}.$$

Recall that for the Jackson network, $M = \sum_{i=1}^{m} [\lambda_i + \mu^i]$.

Define $\tilde{J}$ to be the transition rate kernel associated with the vector $(\tilde{X}_t;\ t \geq 0)$ of m independent, stationary birth and death processes, having infinitesimal generator $-\tilde{L} := \sum_{i=1}^{m} -L_i$, with L_i given by (2.3). J will continue to denote the transition rate kernel associated with the Jackson network, whose infinitesimal generator $-L$ is described at (2.1). In order to establish the proposition, it suffices to find a constant $\nu > 0$ such that for all $A \subset S$,

$$\sum_{\mathbf{x} \in A} \pi(\mathbf{x}) J(\mathbf{x}, A^c) \geq \nu \sum_{\mathbf{x} \in A} \pi(\mathbf{x}) \tilde{J}(\mathbf{x}, A^c). \tag{3.1}$$

Indeed, because π is also a stationary distribution for $(\tilde{X}_t;\ t \geq 0)$, then $\mathbf{k} \geq \nu \tilde{\mathbf{k}}$ where $\tilde{\mathbf{k}}$ is the Cheeger constant for $(\tilde{X}_t;\ t \geq 0)$. Now by Theorem 2.6 in Liggett (1989), for such a process,

$$\operatorname{Gap}(\tilde{L}) = \inf_i \operatorname{Gap}(L_i).$$

Using hypothesis **H3**, that each of the π_i's has an exponential tail, we may apply Corollary 3.8 in Liggett (1989) to conclude that $\mathrm{Gap}(L_i) > 0$ and hence that $\mathrm{Gap}(\widetilde{L}) > 0$. Next, Theorem 2.1 in Lawler and Sokal (1988), applied to $\widetilde{L}$, yields $\widetilde{\mathbf{k}} \geq \mathrm{Gap}(\widetilde{L}) > 0$. Thus $\mathbf{k} > 0$ and hence $\mathrm{Gap}(L) > 0$.

If we assume that $\lambda := \min(\overline{\lambda}_i/\lambda_i;\ 1 \leq i \leq m) > 0$ and $\mu := \min(r_i; 1 \leq i \leq m) > 0$, then we can simply take $\nu = \min(\lambda, \mu)$, for then $\overline{\lambda}_i \geq \nu\lambda_i$ and $\mu_i(x_i)r_i \geq \nu\mu_i(x_i)$ for all i's and x_i's. Clearly then $J(\mathbf{x}, A^c) \geq \nu\widetilde{J}(\mathbf{x}, A^c)$ for all $A \in \mathcal{S}$ and all $\mathbf{x} \in A$ [we have simply dropped the terms $\mu_i(x_i)r_{ij}$ in the comparison]. Thus (3.1) is easily satisfied in this case. Unfortunately, the network may be designed so that customers all arrive at one node and all leave from another. In this case some, but not all, of the $\overline{\lambda}_i$ are zero and some, but not all, of the r_i are zero. There is a gap nevertheless. We shall again be able to verify that (3.1) holds for some positive constant ν, but it is convenient to describe the comparison in terms of an intermediate vector-valued birth and death process whose jump kernel will be denoted by $Q \equiv Q^+ + Q^-$. We first construct the birth part Q^+.

Fix some $\mathbf{x} \in S$. We shall define $Q^+(\mathbf{x}, T_{\cdot i}\mathbf{x})$ in such a way as to be actually independent of $\mathbf{x}$. Consider the polytope $\mathcal{P}_+(\mathbf{x}) = \{\mathbf{x}\} \cup \{T_{\cdot k}\mathbf{x} : 1 \leq k \leq m\}$. Define $\widetilde{C}_{\cdot i}(\mathbf{x})$ to be the set of all nonself-intersecting *probable* paths on $\mathcal{P}_+(\mathbf{x})$ that start at $\mathbf{x}$ and end at $T_{\cdot i}\mathbf{x}$. (By "probable," we simply mean that the transition from any point on the path to the next point has positive probability of occurring.) In terms of the nodes in the network: A typical path $\mathbf{t} \in \widetilde{C}_{\cdot i}(\mathbf{x})$ consists of $\mathbf{x}$ followed by an exogenous arrival at some node, say a, followed by the departure from a into some node b, et cetera, and finally a transition from node d, say, to node i. Because the Jackson network is exogenously supplied, there must exist such probable paths; of course $\mathbf{t} = (\mathbf{x}, T_{\cdot i}\mathbf{x})$ is admissible if $\overline{\lambda}_i > 0$. Note that the transitions corresponding to the generic path described above are

$$T_{\cdot a}, T_{ab}, T_{bc}, \dots, T_{di}.$$

For such a path $\mathbf{t}$, define

$$\lambda(\mathbf{t}) := \min(\overline{\lambda}_a, \lambda_a r_{ab}, \lambda_b r_{bc}, \dots, \lambda_d r_{di}).$$

Clearly $\lambda(\mathbf{t})$ does not depend on $\mathbf{x}$. We reduce $\widetilde{C}_{\cdot i}(\mathbf{x})$ to a possibly smaller collection, $C_{\cdot i}(\mathbf{x})$, of paths such that no two paths in $C_{\cdot i}(\mathbf{x})$ have any transitions in common, as follows. Fix a path $\mathbf{t}_1 \in \widetilde{C}_{\cdot i}(\mathbf{x})$ with $\lambda(\mathbf{t}_1)$ maximal and consider the set of paths

$$C^{(1)}_{\cdot i}(\mathbf{x}) := \{\mathbf{t} \in \widetilde{C}_{\cdot i}(\mathbf{x}){:}\ \mathbf{t} \text{ has some transition in common with } \mathbf{t}_1\}.$$

Next choose any $\mathbf{t}_2 \in \widetilde{C}_{\cdot i}(\mathbf{x}) \backslash C^{(1)}_{\cdot i}(\mathbf{x})$ with $\lambda(\mathbf{t}_2)$ maximal (in the latter set). Set

$$C^{(2)}_{\cdot i}(\mathbf{x}) := \{\mathbf{t} \in \widetilde{C}_{\cdot i}(\mathbf{x}) \backslash C^{(1)}_{\cdot i}(\mathbf{x}){:}\ \mathbf{t} \text{ has some transition in common with } \mathbf{t}_2\}.$$

We then choose any $\mathbf{t}_3 \in \widetilde{C}_{\cdot i}(\mathbf{x}) \backslash [C^{(1)}_{\cdot i}(\mathbf{x}) \cup C^{(2)}_{\cdot i}(\mathbf{x})]$ with $\lambda(\mathbf{t}_3)$ maximal (in the latter set), and so on until we arrive at $\widetilde{C}_{\bullet i}(\mathbf{x})\setminus\bigcup_{j=1}^{n} C_{\bullet i}^{(j)}(\mathbf{x}) = \emptyset$ for some n. Then $C_{\bullet i}(\mathbf{x}) := \{\mathbf{t}_j\}_{j=1}^{n}$. Define the transition rate

$$Q^{+}(i) \equiv Q^{+}(\mathbf{x}, T_{\bullet i}\mathbf{x}) := \frac{1}{2m} \sum_{\mathbf{t}\in C_{\bullet i}(\mathbf{x})} \lambda(\mathbf{t}),$$

which is then also independent of $\mathbf{x}$; $Q^{+}(\mathbf{x},\mathbf{y}) := 0$ unless $\mathbf{y} = T_{\bullet i}\mathbf{x}$, for some $i, 1 \le i \le m$.

Now for $A \subset S$ we claim that

$$\tfrac{1}{2}\sum_{\mathbf{x}\in A} \pi(\mathbf{x})J(\mathbf{x}, A^c) \ge \sum_{\mathbf{x}\in A} \pi(\mathbf{x})Q^{+}(\mathbf{x}, A^c). \tag{3.2}$$

(The reason for including the factors of 1/2 above will become apparent at the end of the proof.) Indeed, if $\mathbf{x} \in A$ such that $Q^{+}(\mathbf{x},A^c) > 0$, then for each i, $1 \le i \le m$, such that $T_{\bullet i}\mathbf{x} \in A^c$, $Q^{+}(\mathbf{x}, T_{\bullet i}\mathbf{x}) > 0$. For each $\mathbf{t} \in C_{\bullet i}(\mathbf{x})$, $\mathbf{x} \in A$ and $T_{\bullet i}\mathbf{x} \in A^c$. It follows that at least one of the transitions determining $\mathbf{t}$, say T_{jk}, crosses from A to A^c (i.e., $T_{\bullet j}\mathbf{x} \in A$, $T_{\bullet k}\mathbf{x} \in A^c$) and consequently contributes the term $\pi(\mathbf{y})\mu_j(y_j)r_{jk}$, with $\mathbf{y} = T_{\bullet j}\mathbf{x}$, to the sum $\sum_{\mathbf{y}\in A}\pi(\mathbf{y})J(\mathbf{y},A^c)$. By reversibility $[\mu_j(x_j+1)\pi_j(x_j+1) = \lambda_j\pi(x_j)]$,

$$\pi(\mathbf{y})\mu_j(y_j)r_{jk} = \pi(T_{\bullet j}\mathbf{x})\mu_j(x_j+1)r_{jk} = \pi(\mathbf{x})\lambda_j r_{jk} \ge \pi(\mathbf{x})\lambda(\mathbf{t}).$$

Note that because no two paths in $C_{\bullet i}(\mathbf{x})$ have any transitions in common, $\mathbf{y} = T_{\bullet j}\mathbf{x}$ is only being used at most once in this way for each i; hence at most m times in all. This is also true in the case that the initial transition $\mathbf{x} \to T_{\bullet a}\mathbf{x}$ is being considered, for which the contribution to the previous sum is

$$\pi(\mathbf{x})\overline{\lambda}_a \ge \pi(\mathbf{x})\lambda(\mathbf{t}).$$

Thus

$$\begin{aligned}\sum_{\mathbf{x}\in A} \pi(\mathbf{x})Q^{+}(\mathbf{x},A^c) &= \left(\frac{1}{2m}\right)\sum_{\mathbf{x}\in A}\ \sum_{i:\,T_{\bullet i}\mathbf{x}\in A^c}\ \sum_{\mathbf{t}\in C_{\bullet i}(\mathbf{x})} \lambda(\mathbf{t})\pi(\mathbf{x})\\ &\le \left(\frac{1}{2m}\right)\sum_{\mathbf{y}\in A} m\pi(\mathbf{y})J(\mathbf{y},A^c)\end{aligned}$$

and the claim is established.

Similarly we construct the death part Q^{-}. For $\mathbf{0} \ne \mathbf{x} \in S$ and $1 \le i \le m$ such that $T_{i\bullet}\mathbf{x} \in S$, set

$$\mathcal{P}_{-}^{(i)}(\mathbf{x}) := \mathcal{P}_{+}(T_{i\bullet}\mathbf{x}) = \{\mathbf{x}\} \cup \{T_{i\bullet}\mathbf{x}\} \cup \{T_{ij}\mathbf{x}\colon j \ne i\}.$$

Define $\widetilde{C}_{i\bullet}(\mathbf{x})$ to be the set of all probable non-self-intersecting paths on $\mathcal{P}_{-}^{(i)}(\mathbf{x})$, having common initial point $\mathbf{x}$ and terminal point $T_{i\bullet}\mathbf{x}$. The network is open. There must exist such paths where each transition has a positive jump rate. Of course the path $(\mathbf{x}, T_{i\bullet}\mathbf{x}) \in \widetilde{C}_{i\bullet}(\mathbf{x})$ if $r_i \ne 0$. The transitions

$$T_{ia}, T_{ab}, T_{bc}, \dots, T_{d\bullet}$$

yield a typical $\mathbf{t} \in \widetilde{C}_{i\cdot}(\mathbf{x})$ starting with a transition from node i to node a, then one to node b, et cetera, and finally ending with a departure from the network through node d. For such a path, define

$$\mu(\mathbf{t}) := \min\left(\mu_i r_{ia}, \frac{\mu_i}{\lambda_i}\lambda_a r_{ab}, \dots, \frac{\mu_i}{\lambda_i}\lambda_d r_d\right).$$

If the transition $T_{i\cdot}$ is possible, that is, $r_i \neq 0$, then $\mu(\mathbf{t}) := \mu_i r_i$. Clearly $\mu(\mathbf{t})$ does not depend on $\mathbf{x}$. Reduce $\widetilde{C}_{i\cdot}(\mathbf{x})$ to a possibly smaller collection of paths $C_{i\cdot}(\mathbf{x})$, in which no two paths have any transitions in common, as we did with $\widetilde{C}_{\cdot i}(\mathbf{x})$ and $C_{\cdot i}(\mathbf{x})$, by maximizing $\mu(\mathbf{t})$ at each stage instead of $\lambda(\mathbf{t})$. Define the transition rate

$$Q^-(i) \equiv Q^-(\mathbf{x}, T_{i\cdot}\mathbf{x}) := \left(\frac{1}{2m}\right) \sum_{\mathbf{t} \in C_{i\cdot}(\mathbf{x})} \mu(\mathbf{t}),$$

which is then also independent of $\mathbf{x}$; $Q^-(\mathbf{x}, \mathbf{y}) := 0$ unless $\mathbf{y} = T_{i\cdot}\mathbf{x}$ for some i, $1 \leq i \leq m$.

Now for $A \subset S$ we claim that

$$\tfrac{1}{2} \sum_{\mathbf{x} \in A} \pi(\mathbf{x}) J(\mathbf{x}, A^c) \geq \sum_{\mathbf{x} \in A} \pi(\mathbf{x}) Q^-(\mathbf{x}, A^c). \tag{3.3}$$

Indeed, if $\mathbf{x} \in A$ such that $Q^+(\mathbf{x}, A^c) > 0$, then for each i, $1 \leq i \leq m$, such that $T_{i\cdot}\mathbf{x} \in A^c$, $Q^-(\mathbf{x}, T_{i\cdot}\mathbf{x}) > 0$. For each $\mathbf{t} \in C_{i\cdot}(\mathbf{x})$, $\mathbf{x} \in A$ and $T_{i\cdot}\mathbf{x} \in A^c$. It follows that at least one of the transitions determining $\mathbf{t}$, say T_{jk}, crosses from A to A^c (i.e., $T_{ij}\mathbf{x} \in A$ and $T_{ik}\mathbf{x} \in A^c$) and consequently contributes the term $\pi(\mathbf{y})\mu_j(y_j)r_{jk}$, with $\mathbf{y} = T_{ij}\mathbf{x}$, to the sum $\sum_{\mathbf{y} \in A} \pi(\mathbf{y}) J(\mathbf{y}, A^c)$. By reversibility,

$$\pi(\mathbf{y})\mu_j(y_j)r_{jk} = \pi(T_{ij}\mathbf{x})\mu_j(x_j + 1)r_{jk} = \pi(\mathbf{x})\big(\mu_i(x_i)/\lambda_i\big)\lambda_j r_{jk} \geq \pi(\mathbf{x})\mu(\mathbf{t}).$$

If the transition $T_{d\cdot}$ is considered, then it contributes the term $\pi(\mathbf{y})\mu_d(y_d)r_d$, where $\mathbf{y} = T_{id}\mathbf{x}$, to the sum $\sum_{\mathbf{y} \in A} \pi(\mathbf{y}) J(\mathbf{y}, A^c)$, in which case

$$\pi(\mathbf{y})\mu_d(y_d)r_d = \pi(T_{id}\mathbf{x})\mu_d(x_d + 1)r_d = \pi(\mathbf{x})\big(\mu_i(x_i)/\lambda_i\big)\lambda_d r_d \geq \pi(\mathbf{x})\mu(\mathbf{t}).$$

If $r_i \neq 0$, so that $T_{i\cdot}$ is a possible transition, then in the sum $\sum_{\mathbf{y} \in A} \pi(\mathbf{y}) J(\mathbf{y}, A^c)$ there is a term $\pi(\mathbf{x})\mu_i(x_i)r_i \geq \pi(\mathbf{x})\mu(\mathbf{t})$, as well. Because no two paths in $C_{i\cdot}(\mathbf{x})$ have any transitions in common, $\mathbf{y}$ (in either case above) is only being used at most once in this way for each i; hence, at most m times in all. Thus

$$\begin{aligned}\sum_{\mathbf{x} \in A} \pi(\mathbf{x}) Q^-(\mathbf{x}, A^c) &= \left(\frac{1}{2m}\right) \sum_{\mathbf{x} \in A} \sum_{i: T_{i\cdot}\mathbf{x} \in A^c} \sum_{\mathbf{t} \in C_{i\cdot}(\mathbf{x})} \mu(\mathbf{t})\pi(\mathbf{x}) \\ &\leq \left(\frac{1}{2m}\right) \sum_{\mathbf{y} \in A} m\pi(\mathbf{y}) J(\mathbf{y}, A^c)\end{aligned}$$

and the claim is established.

Adding (3.2) and (3.3) yields

$$\sum_{\mathbf{x}\in A}\pi(\mathbf{x})J(\mathbf{x},A^c)\geq\sum_{\mathbf{x}\in A}\pi(\mathbf{x})Q(\mathbf{x},A^c) \tag{3.4}$$

with $Q := Q^+ + Q^-$. Now Q is the transition rate kernel of some multidimensional birth and death process, but π is not necessarily a stationary distribution for it. As such we make one final minorization, which will yield the desired comparison of J with $\widetilde{J}$. Redefine $\lambda := \min(Q^+(i)/\lambda_i; 1 \leq i \leq m)$, $\mu := \min(Q^-(i)/\mu^i; 1 \leq i \leq m)$ and $\nu := \min(\lambda,\mu) > 0$. Then for all $A \subset S$ and $\mathbf{x} \in A$, we have $Q(\mathbf{x}, A^c) \geq \nu\widetilde{J}(\mathbf{x}, A^c)$ so that (3.1) is then satisfied, by (3.4). □

Let us return to our simple example of a Jackson network with two nodes, which was described at the end of Section 2, and assume that $\overline{\lambda}_2 = 0$ and $r_{21} = 0$. Therefore, $\lambda_1 = \overline{\lambda}_1$, $\lambda_2 = \overline{\lambda}_1 r_{12}$, $r_2 = 1$ and $M = \overline{\lambda}_1 + \mu_1 + \mu_2$. Note that this process is not reversible. Let us estimate Gap(L) by using the algorithm described in the proof of Theorem 3.1. We begin by calculating the birth and death rates $Q^+(i), Q^-(i), i = 1,2$, for the intermediate process. First, the transition [by $(X_t;\ t \geq 0)$] from $\mathbf{x} \equiv (x_1,x_2)$ to (x_1+1,x_2) can only occur by an arrival at node 1 at the rate $\overline{\lambda}_1$, so $Q^+(1) = \overline{\lambda}_1/4$. Next, the transition from (x_1,x_2) to (x_1,x_2+1) can only occur on $\mathcal{P}_+(\mathbf{x})$ by an arrival at node 1 followed by a departure from node 1 to node 2; that is, we have only one path $\mathbf{t} \in C_{\cdot 2}(\mathbf{x})$, namely,

$$(x_1,x_2)\to(x_1+1,x_2)\to(x_1,x_2+1).$$

The rates of these two transitions are $\overline{\lambda}_1$ and $\mu_1 r_{12}$, so

$$Q^+(2)=\tfrac{1}{4}\min(\overline{\lambda}_1,\lambda_1 r_{12})=\overline{\lambda}_1 r_{12}/4.$$

The transition from $\mathbf{x} \equiv (x_1,x_2)$ to (x_1-1,x_2) can only occur on $\mathcal{P}_-^{(1)}(\mathbf{x})$ in two ways, which happen to have no transitions in common; namely, a departure from node 1 out of the network, at the rate $\mu_1 r_1$, or a departure from node 1 to node 2, followed by a departure from node 2 at the rates $\mu_1 r_{12}$ and μ_2, respectively. Hence, denoting the corresponding paths that comprise $C_{1\cdot}(\mathbf{x})$ by $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively,

$$\begin{aligned}Q^-(1)&=\frac{1}{4}[\mu(\mathbf{t}_1)+\mu(\mathbf{t}_2)]\\&=\frac{1}{4}\left[\mu_1 r_1+\min\left(\mu_1 r_{12},\frac{\mu_1}{\overline{\lambda}_1}(\overline{\lambda}_1 r_{12})\right)\right]\quad\text{(because } \lambda_1=\overline{\lambda}_1 \text{ and } r_2=1)\\&=\frac{1}{4}\mu_1[r_1+r_{12}]=\frac{\mu_1}{4}.\end{aligned}$$

Finally the transition from (x_1,x_2) to (x_1,x_2-1) can only occur on $\mathcal{P}_-^{(2)}(\mathbf{x})$ by a departure from node 2, and this occurs at the rate μ_2. Hence $Q^-(2) = \mu_2/4$.

With $\nu := \min(Q^+(1)/\lambda_1, Q^+(2)/\lambda_2, Q^-(1)/\mu_1, Q^-(2)/\mu_2) = 1/4$, we have from the proof of Theorem 3.1 that

$$\begin{aligned} \text{Gap}(L) &\geq [\nu^2/8M]\,[\text{Gap}(\widetilde{L})]^2 \\ &= \Big[\min\big(\text{Gap}(L_1), \text{Gap}(L_2)\big)\Big]^2 \Big/ \big[128(\overline{\lambda}_1 + \mu_1 + \mu_2)\big], \end{aligned} \tag{3.5}$$

where L_1 and L_2 are given at (2.3) and correspond to $M/M/1$ queues with arrival rates $\overline{\lambda}_1$ and $\overline{\lambda}_1 r_{12}$, respectively, and loads $\rho_1 = \overline{\lambda}_1/\mu_1$ and $\rho_2 = \overline{\lambda}_1 r_{12}/\mu_2$, respectively. Now by Theorem 3.7 in Liggett (1988), an $M/M/1$ queue with arrival rate λ and load ρ has a positive Gap bounded below by $1/[2c(1+2b)]$, where $c = \rho/[(1-\rho)\lambda]$ and $b = \rho/(1-\rho)$. Substitution of these underestimates of $\text{Gap}(L_1)$ and $\text{Gap}(L_2)$ into (3.5) provides an explicit positive lower bound for $\text{Gap}(L)$. □

4. On the mean idle period. Let $-L_i^{B_i}$ denote the infinitesimal generator of the birth and death processes associated with the ith node of the Jackson network killed outside of $B_i = \{0, 1, \dots, l_i - 1\}$. Recall that $\widehat{\pi}_i$ denotes the restriction of π_i to B_i, renormalized to be a probability. Let $\Lambda(B_i)$ be the Perron–Frobenius eigenvalue of $L_i^{B_i}$ and let f_i be an associated right eigenfunction. Suppose f_i is normalized so that $\sum_{x_i} f_i(x_i)^2 \widehat{\pi}_i(x_i) = 1$ and extended as zero off B_i. Let

$$f(\mathbf{x}) = f_1(x_1) \cdot f_2(x_2) \cdots f_m(x_m) \tag{4.1}$$

and

$$b = (f, L^B f)_{\widehat{\pi}}. \tag{4.2}$$

Define,

$$e_i := \sum_{x_i} f_i(x_i)\big[f_i(x_i) - f_i(x_i+1)\big]\lambda_i \widehat{\pi}_i(x_i). \tag{4.3}$$

Note that $e_i \geq 0$ since f_i is decreasing. (Apply the minimum principle to the eigenfunction equation on every $[0, k]$, $k \leq l$.)

PROPOSITION 4.1. *For f as in* (4.1),

$$L^B f = \left(\sum_{i=1}^{m} \Lambda(B_i)\right) f + E,$$

where

$$E(\mathbf{x}) := \sum_{j=1}^{m} \big[f_j(x_j) - f_j(x_j+1)\big] \sum_{i=1}^{m} \big[f_i(x_i - 1)\mu_i(x_i) - \lambda_i f_i(x_i)\big] r_{ij} \prod_{k \neq i,j} f_k(x_k).$$

PROOF. For $\mathbf{x} \in B$,

$$L^B f(\mathbf{x}) = \sum_{i=1}^{m} [f(\mathbf{x}) - f(T_{\cdot i}\mathbf{x})]\overline{\lambda}_i + \sum_{i=1}^{m}\sum_{j=1}^{m} [f(\mathbf{x}) - f(T_{ij}\mathbf{x})]\mu_i(x_i)r_{ij}$$

$$+ \sum_{i=1}^{m} [f(\mathbf{x}) - f(T_{i\cdot}\mathbf{x})]\mu_i(x_i)r_i$$

$$= \sum_{i=1}^{m} [f_i(x_i) - f_i(x_i+1)]\overline{\lambda}_i \prod_{k \neq i} f_k(x_k) \tag{4.4}$$

$$+ \sum_{i,j=1}^{m} [f_i(x_i)f_j(x_j) - f_i(x_i-1)f_j(x_j+1)]\mu_i(x_i)r_{ij} \prod_{k \neq i,j} f_k(x_k) \tag{4.5}$$

$$+ \sum_{i=1}^{m} [f_i(x_i) - f_i(x_i-1)]\mu_i(x_i)r_i \prod_{k \neq i} f_k(x_k). \tag{4.6}$$

By factoring,

$$\sum_{i,j=1}^{m} [f_i(x_i)f_j(x_j) - f_i(x_i-1)f_j(x_j+1)]\mu_i(x_i)r_{ij} \prod_{k \neq i,j} f_k(x_k)$$

$$= \sum_{i=1}^{m} [f_i(x_i) - f_i(x_i-1)]\mu_i(x_i)\sum_{j=1}^{m} r_{ij} \prod_{k \neq i} f_k(x_k) \tag{4.7}$$

$$+ \sum_{i=1}^{m} f_i(x_i-1)\mu_i(x_i)\sum_{j=1}^{m} r_{ij}[f_j(x_j) - f_j(x_j+1)] \prod_{k \neq i,j} f_k(x_k). \tag{4.8}$$

The sum of expressions (4.6) and (4.7) is

$$\sum_{i=1}^{m} [f_i(x_i) - f_i(x_i-1)]\mu_i(x_i) \prod_{k \neq i} f_k(x_k). \tag{4.9}$$

Using the fact that $\lambda_j = \Sigma_{i=1}^{m}\lambda_i r_{ij} + \overline{\lambda_j}$, we see the sum of expressions (4.8) and (4.4) is

$$\sum_{i=1}^{m} [f_i(x_i) - f_i(x_i+1)]\lambda_i \prod_{k \neq i} f_k(x_k) \tag{4.10}$$

$$+ \sum_{j=1}^{m} [f_j(x_j) - f_j(x_j+1)] \sum_{i=1}^{m} [f_i(x_i-1)\mu_i(x_i) - \lambda_i f_i(x_i)]r_{ij} \prod_{k \neq i,j} f_k(x_k). \tag{4.11}$$

Finally, adding expressions (4.9) and (4.10) we conclude that

$$L^B f = \sum_{i=1}^{m} (L_i^{B_i} f_i) \prod_{k \neq i} f_k + E \tag{4.12}$$

$$= \left(\sum_{i=1}^{m} \Lambda(B_i) \right) f + E. \tag{4.13}$$

□

PROPOSITION 4.2. *With b as in* (4.2),

$$b = \sum_{i=1}^{m} \Lambda(B_i) - e,$$

where

$$e := \sum_{i,j=1}^{m} \frac{r_{ij}}{\lambda_j} e_i e_j \geq 0.$$

PROOF. By Proposition 4.1 it suffices to calculate

$$\begin{aligned}
(f,E)_{\widehat{\pi}} &= \sum_{i,j=1}^{m} r_{ij} \sum_{x_j} \big[f_j(x_j) - f_j(x_j+1)\big] f_j(x_j) \widehat{\pi}_j(x_j) \\
&\quad \times \sum_{x_i} \big[f_i(x_i - 1)\mu_i(x_i) - \lambda_i f_i(x_i)\big] f_i(x_i) \widehat{\pi}_i(x_i) \\
&= \sum_{i,j=1}^{m} r_{ij} \sum_{x_j} \big[f_j(x_j) - f_j(x_j+1)\big] f_j(x_j) \widehat{\pi}_j(x_j) \\
&\quad \times \sum_{x_i} \big[f_i(x_i - 1) f_i(x_i) \mu_i(x_i) \widehat{\pi}_i(x_i) - \lambda_i\big] \\
&= \sum_{i,j=1}^{m} r_{ij} \sum_{x_j} \big[f_j(x_j) - f_j(x_j+1)\big] f_j(x_j) \widehat{\pi}_j(x_j) \\
&\quad \times \sum_{x_i} f_i(x_i) \big[f_i(x_i+1) - f_i(x_i)\big] \lambda_i \widehat{\pi}_i(x_i) \\
&= - \sum_{i,j=1}^{m} \frac{r_{ij}}{\lambda_j} e_i e_j,
\end{aligned}$$

where the last equality follows from (4.3). □

PROOF OF THEOREM 2.4. By the Rayleigh–Ritz principle $\Lambda(B) \leq b$, where b is given in (4.2). Moreover, by Proposition 4.2, $b \leq \sum_{i=1}^{m} \Lambda(B_i)$ because e is positive. $L_i^{B_i}$ is a self-adjoint operator on $L^2(\widehat{\pi})$, so applying Corollary 2.14 of Part 1 gives $\Lambda(B_i) \leq [E_{\widehat{\pi}_i} \tau_i]^{-1}$. Finally L is self-adjoint by hypothesis, so applying the corollary again we have

$$[E_{\widehat{\pi}} \tau]^{-1} \leq \Lambda(B) \left[1 - \min\left(\frac{\overline{\kappa}}{\mathrm{Gap}(L)}, 1 \right) \right]^{-1}.$$

The expression for $E_{\hat{\pi}_i}\tau_i$ can be found at (3.12) in Iscoe, McDonald and Qian (1993) and this gives the result. □

Numerical evidence suggests that $(\sum_{i=1}^{m} 1/E_{\hat{\pi}_i}\tau_i)^{-1}$ does provide a lower bound for the exit time for a Jackson network, at least asymptotically, even in the non-self-adjoint case. It is naive, however, to expect that the Perron–Frobenius eigenfunctions can be well-approximated by a product asymptotically. Finding an approximate Perron–Frobenius eigenfunction that is asymptotically exact near the boundary of the forbidden set is the subject of a future paper.

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