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DENSITY-DEPENDENT LIMITS FOR A NONLINEAR REACTION–DIFFUSION MODEL[1]

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A reaction–diffusion model, constructed on a grid by linking nonlinear density-dependent birth and death processes through particle diffusion, is studied by proving two laws of large numbers and a fluctuation theorem under varying assumptions on the density parameter.

1. Introduction. In this paper we study a density-dependent stochastic reaction–diffusion model introduced in Arnold and Theodosopulu (1980). The model is constructed by partitioning $[0,1]^p$ into N^p congruent cells, distributing approximately $N^p l$ particles in the system and allowing particles to diffuse between cells by rate N^2 random walks and to react within cells as density-dependent birth and death processes [Ethier and Kurtz (1986)]. Cell numbers are divided by l, the density parameter, to represent concentrations, and one obtains a step-function-valued Markov process $X_{N,l}(t,r)$, for $t \geq 0$ and $r \in [0,1]^p$, satisfying the stochastic differential equation

$$dX_{N,l}(t,r) = \Big(\Delta_N X_{N,l}(t,r) + R\big(X_{N,l}(t,r)\big)\Big)\,dt + dZ_{N,l}(t,r),$$

where Δ_N is a discrete Laplacian, R is a nonlinear polynomial determined by the birth and death rates and $Z_{N,l}$ is a martingale arising from Dynkin's formula. The equation appears as a spatially discretized and stochastically perturbed version of the reaction–diffusion equation

$$\frac{\partial\psi(t,r)}{\partial t} = \Delta\psi(t,r) + R\big(\psi(t,r)\big).$$

In this paper, we prove two laws of large numbers and a central limit theorem. Letting $\|\cdot\|_{L_2}$ denote the $L_2([0,1]^p)$ norm and setting

$$X_{N,l}(t) = X_{N,l}(t,\cdot), \qquad \psi(t) = \psi(t,\cdot),$$

we show that for each $p \geq 1$ and $T > 0$,

$$\sup_{t\leq T}\|X_{N,l}(t) - \psi(t)\|_{L_2} \to 0$$

in probability if $l \to \infty$ as $N \to \infty$.

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For $p = 1$, this result was proved for the linear model in Blount (1991), which improved Kotelenez (1986a), where it was assumed that $N^2/l \to 0$ as $N \to \infty$. Our result complements Blount (1992), where assuming $\log N/l \to 0$ as $N \to \infty$ allows one to replace the L_2 norm by the supremum norm.

For the case with l constant, $p = 1$, linear birth and quadratic death rates, we prove

$$\sup_{t \le T} \| X_{N,l}(t) - \psi_l(t) \|_{-\alpha} \to 0$$

in probability as $N \to \infty$ for any $\alpha > 0$, where $(H_{-\alpha}, \| \cdot \|_{-\alpha})$ is the Hilbert distribution space defined in Section 5 and $\psi_l(t)$ is the solution of

$$\frac{\partial \psi_l(t,r)}{\partial t} = \Delta \psi_l(t,r) + R\big(\psi_l(t,r)\big) - \frac{d_2}{l} \psi_l(t,r)$$

with d_2 the coefficient of $-x^2$ in $R(x)$. Thus, in the constant density case, convergence to a perturbed form of the previous equation occurs. This result is related to the law of large numbers for a "local" reaction studied in Dittrich (1988a, b) and Kotelenez (1992) for a measure-valued model and in Boldrighini, De Masi and Pellegrinotti (1992) for a grid model similar to the one considered here.

For the central limit theorem, we assume $p = 1$. Assuming $N/l \to 0$ as $N \to \infty$, it was shown in Kotelenez (1988) and Blount (1993) that $(Nl)^{1/2} (X_{N,l} - \psi)$ converges in distribution to the solution of a stochastic partial differential equation

$$dV(t) = \Big(\Delta + R'\big(\psi(t)\big)\Big) V(t)\, dt + dM(t),$$

where $R'(x) = dR(x)/dx$ and M is a Gaussian martingale obtained from $\lim (Nl)^{1/2} Z_{N,l}$.

Here we consider the case $N = l$. Let $\overline{R}(x,y)$ be defined by

$$R(x) - R(y) = R'(y)(x-y) + \overline{R}(x,y)(x-y)^2.$$

Assuming $N = l \to \infty$, we show $N(X_{N,l} - \psi)$ converges in distribution on $D([0,\infty); H_{-\alpha})$, for $\alpha > 1/2$, to the solution of

$$dV(t) = \Big(\Delta + R'\big(\psi(t)\big)\Big) V(t)\, dt + dM(t) + \overline{R}\big(\psi(t), \psi(t)\big) \psi(t)\, dt.$$

Thus, assuming $N = l$ gives a fluctuation limit that is a deterministic perturbation of the previous limit.

If the stochastic model has no reaction but only diffusion of particles, then the model has a stationary distribution where the particle numbers of different cells are independent Poisson random variables. Our results are consistent with the heuristic that, asymptotically, cell numbers should behave like appropriately scaled and independent Poisson random variables due to convergence to local equilibrium distributions. However, we do not attempt to directly use this idea but instead exploit the smoothing effects of the semigroup generated by Δ_N and the martingale structure of the stochastic model.

Section 2 describes the deterministic model; Section 3 describes the stochastic model and develops some basic technical results; Sections 4, 6 and 5 prove (in that order) our previously described results.

2. The deterministic model. For $x \in \mathbf{R}$, let $b(x)$ and $d(x)$ be polynomials with nonnegative coefficients such that $d(0) = 0$, $\deg b(x) < \deg d(x)$, $\deg d(x) \geq 2$, and let

$$R(x) = b(x) - d(x) = \sum_{n=0}^{q} c_n x^n.$$

Let $S = [0,1]^p$ and Δ denote the Laplacian, and for $(t,r) \in (0,\infty) \times S$, let $\psi(t,r)$ be a solution of

$$\begin{aligned} &\frac{\partial}{\partial t}\psi(t,r) = \Delta\psi(t,r) + R\big(\psi(t,r)\big), \\ &\psi(0,r) \geq 0 \quad \text{and } \psi \text{ is periodic in each of the } p \text{ space variables with period 1.} \end{aligned} \tag{2.1}$$

For $x \in \mathbf{R}$, define $\varphi_n(x)$ and $\psi_n(x)$ by

$$\varphi_n(x) = \begin{cases} \sqrt{2}\cos(\pi n x), & \text{for } n > 0 \text{ and even}, \\ 1, & n = 0, \end{cases}$$

$$\psi_n(x) = \sqrt{2}\sin(\pi n x) \qquad \text{for } n > 0 \text{ and even}.$$

For $m = (m_1, \dots, m_p) \in \{0, 2, 4, \dots\}^p$ and $r = (r_1, \dots, r_p) \in S$, let $f_m(r)$ denote functions of the form

$$f_m(r) = \prod_{n=1}^{p} g_{m_n}(r_n),$$

where $g_{m_n} = \varphi_{m_n}$ or ψ_{m_n} · $\{f_m\}$ are eigenfunctions of Δ with eigenvalues

$$-\beta_m = -\pi^2 \sum_{n=1}^{p} m_n^2;$$

they also form an orthonormal basis of $L_2(S)$. Thus, $T(t)$, the semigroup on $L_2(S)$ generated by Δ, is represented by

$$T(t)f = \sum_m \exp(-\beta_m t)\langle f, f_m\rangle f_m,$$

where $f \in L_2(S)$ and $\langle f, g\rangle = \int_S f(r)g(r)\,dr$ is the inner product on $L_2(S)$.

Using variation of constants, any solution of (2.1) can be put in the form

$$\psi(t) = T(t)\psi(0) + \int_0^t T(t-s)R\big(\psi(s)\big)\,ds, \tag{2.2}$$

where $\psi(t) = \psi(t, \cdot)$. A solution of (2.2) is a mild solution of (2.1).

Before defining the stochastic model, we need to suitably discretize Δ, $T(t)$ and $\{f_m\}$. Let N be a positive odd integer and for $k = (k_1, \dots, k_p) \in \{0, 1, \dots, N-1\}^p$, let

$$I_k = \prod_{n=1}^{p} \left[k_n N^{-1}, (k_n + 1)N^{-1}\right) \subset S.$$

Let $H^N \subset L_2(S)$ denote the step functions that are constant on each of the sets I_k and are extended to be periodic in each variable with period 1. Set $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (p coordinates and a 1 in the ith coordinate) and for $f \in H^N$, define

$$\nabla_i^{\pm} f(r) = N^{-1}\Big(f(r \pm N^{-1}e_i) - f(r)\Big),$$
$$\Delta_N f(r) = \sum_{i=1}^{p} -\nabla_i^{+}\nabla_i^{-} f(r) = \sum_{i=1}^{p} N^2\Big(f(r + N^{-1}e_i) - 2f(r) + f(r - N^{-1}e_i)\Big).$$

Let $P_N \colon L_2(S) \to H^N$ be the self-adjoint projection given by

$$P_N f(r) = N^p \int_{I_k} f(r')\,dr' \quad \text{if } r \in I_k.$$

Using P_N first extends Δ_N, $T_N(t)$ and $\nabla_i^{\pm}$ to $L_2(S)$.

If $f_m = \prod_{n=1}^{p} g_{m_n}$ is an eigenfunction of Δ and $m \in \{0, 2, \dots, N-1\}^p$, define

$$f_{m,N}(r) = \prod_{n=1}^{p} g_{m_n}\left(k_n N^{-1}\right) \quad \text{if } r \in I_k.$$

$\{f_{m,N}\}$ form an orthonormal basis of H^N as a subspace of $L_2(S)$ and are eigenfunctions of Δ_N with eigenvalues

$$-\beta_{m,N} = -N^2 \sum_{n=1}^{p} \Big(1 - \cos\left(\pi m_n N^{-1}\right)\Big).$$

Basic calculations show that for $0 < c_1(p) < c_2(p) < \infty$,

$$c_1(p)\beta_m \le \beta_{m,N} \le c_2(p)\beta_m. \tag{2.3}$$

$T_N(t)$, the semigroup generated on $L_2(S)$ by Δ_N, can be represented by

$$T_N(t)f = \sum_m \exp(-\beta_{m,N}t)\langle f, f_{m,N}\rangle f_{m,N}.$$

Let $\|f\| = \langle f, f\rangle^{1/2}$ and $\|f\|_\infty = \sup_{r \in S} |f(r)|$. Then, as is well known, $T(t)$ and $T_N(t)$ are positive contraction maps for both norms.

3. The stochastic model. Let $l > 0$ and b and d be the polynomials appearing in the definition of (2.1). $X(t)$ denotes the H^N-valued Markov process defined as

$$X(t,r) = n_k(t)/l \quad \text{if } r \in I_k,$$

where $\{n_k\}$ are the nonnegative integer-valued components of a jump Markov process with transition rates given by

$$\begin{aligned} n_k &\to n_k + 1 \quad \text{at rate } lb(n_k l^{-1}),\\ n_k &\to n_k - 1 \quad \text{at rate } ld(n_k l^{-1}),\\ (n_k, n_{k\pm e_i}) &\to (n_k - 1, n_{k\pm e_i} + 1) \quad \text{at rate } N^2 n_k,\ 1 \le i \le p. \end{aligned} \tag{3.1}$$

(We assume $n_{k\pm Ne_i} = n_k$, $1 \le i \le p$.)

We take $X(t)$ to be right continuous with left limits and Markov with respect to the filtration $\{F_t^N\}_{t\ge 0}$, where F_t^N is the completion of the σ-field generated by $\{X(s)\}_{s\le t}$. For each N and l we obtain a different process, $X_{N,l}$, but we suppress the subscripts unless necessary to avoid confusion.

If $f(t)$ is a right continuous H^N-valued process with left limits, let $\delta f(t) = f(t) - f(t-)$, where $f(0-) = 0$. Note that the first two jump rates in (3.1) determine the rates at which particles react in each cell, and the last rate determines the diffusion of particles by random walks to neighboring cells. Let δX_R and δX_D be the reaction and diffusion jumps of X, respectively, and let

$$\begin{aligned} Z_R(t) &= \sum_{s\le t} \delta X_R(s) - \int_0^t R(X(s))\,ds,\\ Z_D(t) &= \sum_{s\le t} \delta X_D(s) - \int_0^t \Delta_N X(s)\,ds \end{aligned} \tag{3.2}$$

and $|R|(x) = b(x) + d(x)$.

The following result follows from Dynkin's semigroup formula and basic computations. See Kotelenez (1986a) and Blount (1991).

LEMMA 3.1. *If* $\|X(0)\|_\infty \le C(N,l) < \infty$, *then* Z_R *and* Z_D *are* F_t^N *martingales, and, for* $f, g \in H^N$,

$$\sum_{s\le t} \langle \delta Z_R(s), f\rangle\langle \delta Z_R(s), g\rangle - (N^p l)^{-1} \int_0^t \big\langle |R|(X(s)), fg\big\rangle\,ds$$

and

$$\sum_{s\le t} \langle \delta Z_D(s), f\rangle\langle \delta Z_D(s), g\rangle$$
$$- (N^p l)^{-1} \int_0^t \Big\langle X(s), \sum_{i=1}^p \big[(\nabla_i^+ f)(\nabla_i^+ g) + (\nabla_i^- f)(\nabla_i^- g)\big]\Big\rangle\,ds$$

are F_t^N martingales.

Letting $Z(t) = Z_R(t) + Z_D(t)$, from (3.2) we have

$$X(t) = X(0) + \int_0^t \Delta_N X(s)\,ds + \int_0^t R\big(X(s)\big)\,ds + Z(t). \tag{3.3}$$

Using variation of constants, (3.3) can be written as

$$X(t) = T_N(t)X(0) + \int_0^t T_N(t-s)R\big(X(s)\big)\,ds + Y(t), \tag{3.4}$$

where $Y(t) = \int_0^t T_N(t-s)\,dZ(s)$.

Let $Y(t) = Y_R(t) + Y_D(t) = \int_0^t T_N(t-s)\,dZ_R(s) + \int_0^t T_N(t-s)\,dZ_D(s)$. If $J \in \{D, R\}$, then from variation of constants we have

$$Y_J(t) = \int_0^t \Delta_N Y_J(s)\,ds + Z_J(t).$$

If $(f_{m,N}, \beta_{m,N})$ is an eigenpair of Δ_N, let Y_m, Z_m denote $\langle Y_J, f_{m,N}\rangle$, $\langle Z_J, f_{m,N}\rangle$. By the previous equation and Itô's formula, respectively,

$$\begin{aligned} Y_m(t) &= -\int_0^t \beta_{m,N} Y_m(s)\,ds + Z_m(t),\\ Y_m^2(t) &= -2\beta_{m,N}\int_0^t Y_m^2(s)\,ds + 2\int_0^t Y_m(s-)\,dZ_m(s) + \sum_{s\le t}\big(\delta Z_m(s)\big)^2. \end{aligned} \tag{3.5}$$

Applying (3.5) and Lemma 3.1 with $f = g = f_{m,N}$ proves the following lemma.

LEMMA 3.2. *Assume* $\|X(0)\|_\infty \le c(N,l) < \infty$. *Then*:

(a) $E\langle Y_D(t), f_{m,N}\rangle^2 = (N^p l)^{-1} E\int_0^t \langle X(s), \sum_{i=1}^p (\nabla_i^+ f_{m,N})^2 + (\nabla_i^- f_{m,N})^2\rangle \exp(-2\beta_{m,N} \times (t-s))\,ds$.

(b) $E\langle Y_R(t), f_{m,N}\rangle^2 = (N^p l)^{-1} E\int_0^t \langle |R|(X(s)), (f_{m,N})^2\rangle \exp(-2\beta_{m,N}(t-s))\,ds$.

(c) $\langle Y_D(t), f_{m,N}\rangle^2 \le A(f_{m,N})(t)$, *where* $A(f_{m,N})(t)$ *is a submartingale satisfying*

$$EA(f_{m,N})(t) = (N^p l)^{-1} E\int_0^t \left\langle X(s), \sum_{i=1}^p \big(\nabla_i^+ f_{m,N}\big)^2 + \big(\nabla_i^- f_{m,N}\big)^2\right\rangle ds.$$

(d) $\langle Y_R(t), f_{m,N}\rangle^2 \le B(f_{m,N})(t)$, *where* $B(f_{m,N})(t)$ *is a submartingale satisfying*

$$EB(f_{m,N})(t) = (N^p l)^{-1} E\int_0^t \left\langle |R|\big(X(s)\big), f_{m,N}^2\right\rangle ds.$$

To apply Lemma 3.2 we need to have a bound on the moments of X. From

Lemma 3.2 of Kotelenez (1988), with a slight modification of its proof, we have the following lemma.

LEMMA 3.3. *For* $n \geq 1$,

$$\sup_{s \leq t} \|EX^n(s)\|_\infty \leq C\big(t, l, \|EX^n(0)\|_\infty, \rho\big) < \infty,$$

where C *is decreasing in* l *and* ρ *is any number such that* $R(x) < 0$ *for* $x > \rho$.

Let $M = (\log N)^2$ and let n be an integer satisfying $0 \leq n < p^{1/2}N/M$. For a multiindex $m \in \{0, 2, \dots, N-1\}^p$, let $|m| = (\sum_{i=1}^p m_i^2)^{1/2}$ and let $B_n = \{m: nM \leq |m| < (n+1)M\}$. For $n \geq 1$, $\max_{m \in B_n} |m| / \min_{m \in B_n} |m| \leq (n+1)/n \leq 2$. Thus, by (2.3),

$$\max_{m \in B_n} \beta_{m,N} \Big/ \min_{m \in B_n} \beta_{m,N} \leq c(p)$$

for $n \geq 1$. If $|B_n|$ denotes the cardinality of B_n, then $|B_n| \leq L_n$, where $L_n = c(p)M^p(n+1)^{p-1}$. Thus, $L_n/N^p \leq c(p)(\log N)^2/N \to 0$ as $N \to \infty$ and $\sum_n L_n \leq c(p)N^p$. Note that $\{f_{m,N}\} = \bigcup_n \{f_{m,N}: m \in B_n\}$ and $\max_m \beta_{m,N} \leq c(p)N^2$.

LEMMA 3.4.

(a) *Let* τ *be an* $\{F_t^N\}$ *stopping time such that* $\sup_{t \leq T} \|X(t \wedge \tau-)\| \leq b < \infty$. *Then for* $n \geq 1, l \geq l_0, c(p) > 0$ *and* $N \geq N_0(a^2, l_0)$,

$$P\left(\sup_{t \leq T}\left(\sum_{m \in B_n} \langle Y_D(t \wedge \tau), f_{m,N}\rangle^2\right) \geq a^2 L_n/N^p\right) \leq c(p,T)N^2 L_n^{1/2}\big(c(p)a^2 l/b\big)^{-L_n^{1/2}}$$

(b) *If* $\sup_{t \leq T} \| |R|\big(X(t \wedge \tau-)\big)\| \leq b < \infty$, *then the same estimate holds with* Y *in place of* Y_D.

PROOF. In the case $p = 1$, the result is proved as Lemma 3.21(b) in Blount (1991) for the model with linear birth and death polynomials. However, the proof depends only on the covariance structure of Z as determined in Lemma 3.1. Using the facts in the previous paragraph, the proof for $p = 1$ extends to $p > 1$ with only minor notational changes in the proof given in Blount (1991). □

LEMMA 3.5. *Let* τ *be as in Lemma* 3.4. *Then* $\sup_{t \leq T} \|Y_D(t \wedge \tau)\| \to 0$ *in probability if* (N, l) *is any sequence such that* $l \to \infty$ *as* $N \to \infty$.

PROOF. Using the foregoing notation,

$$\|Y_D(t \wedge \tau)\|^2 = \sum_{m \in B_0} \langle Y_D(t \wedge \tau), f_{m,N}\rangle^2 + \sum_{n \geq 1} \sum_{m \in B_n} \langle Y_D(t \wedge \tau), f_{m,N}\rangle^2.$$

By Lemma 3.2(c) and Doob's inequality,

$$P\left(\sup_{t\le T}\sum_{m\in B_0}\langle Y_D(t\wedge\tau),f_{m,N}\rangle^2\ge a^2\right)$$
$$\le a^{-2}(N^pl)^{-1}\sum_{m\in B_0}E\int_0^{T\wedge\tau}\left\langle X(s),\sum_{i=1}^p\left(\nabla_i^+f_{m,N}\right)^2+\left(\nabla_i^-f_{m,N}\right)^2\right\rangle ds$$
$$\le a^{-2}(N^pl)^{-1}c(p)b(\log N)^{2p+4}T\to 0\quad\text{as } N\to\infty$$

(even if l is constant).

Recall $\Sigma_nL_n\le c(p)N^p$. Thus,

$$P\left(\sup_{t\le T}\sum_{n\ge 1}\sum_{m\in B_n}\langle Y_D(t\wedge\tau),f_{m,N}\rangle^2\ge a^2\right)$$
$$\le\sum_{n\ge 1}P\left(\sup_{t\le T}\sum_{m\in B_n}\langle Y_D(t\wedge\tau),f_{m,N}\rangle^2\ge a^2L_n/\big(c(p)N^p\big)\right)$$
$$\le p^{1/2}N(\log N)^{-2}c(p,T)N^2(\log N)^p\left(p^{1/2}N(\log N)^{-2}+1\right)^{(p-1)/2}$$
$$\times\left(c(p)a^2l/b\right)^{-\log N},$$

where we have applied Lemma 3.4 and $c(p)>0$. The last expression converges to 0 if $l\to\infty$ as $N\to\infty$, which completes the proof. □

LEMMA 3.6. *Assume* $\|EX^q(0)\|_\infty\le b<\infty$ [*recall* $q=\deg R(x)$]. *Then* $\sup_{t\le T}\|Y_R(t)\|\to 0$ *in probability if* (N,l) *is any sequence such that* $l\to\infty$ *as* $N\to\infty$.

PROOF. By Lemma 3.2(d), $\|Y_R(t)\|^2\le B(t)$, where $B(t)$ is a submartingale satisfying $EB(t)\le(N^pl)^{-1}E\Sigma_m\int_0^t\langle|R|(X(s)),f_{m,N}^2\rangle\,ds$. Thus, by Lemma 3.3, $EB(T)\le c(T)/l$ and the result follows from Doob's inequality. □

LEMMA 3.7. *Assume* $\|EX^{2q}(0)\|_\infty\le b<\infty$. *Then, indexed by* (N,l) *with* $l\ge l_0>0$, *the distributions of*

$$\left\{\int_0^{\cdot}T_N(\cdot-s)R\big(X(s)\big)\,ds\right\}$$

on $C([0,T];L_2(S))$ *are relatively compact.*

PROOF. Let $g(t)=\int_0^tT_N(t-s)R(X(s))\,ds$. We claim:

(i) $E[\sup_{t\le T}\Sigma_{|m|\ge n}\langle g(t),f_{m,N}\rangle^2]\le c(T)/n^2$,

(ii) $E[\sup_{0\le t\le t+\varepsilon\le T,\,0<\varepsilon\le\varepsilon_0}\langle g(t+\varepsilon)-g(t),f_{m,N}\rangle^2]\le c(T,|m|)(\varepsilon_0^2+\varepsilon_0)$.

The proof will then follow from Ascoli's theorem, Theorem 3.2.2(b) of Ethier and Kurtz (1986) and the fact that $\|f_{m,N}-f_m\|_\infty\to 0$ as $N\to\infty$.

Consider, for $|m| \neq 0$,

$$\langle g(t), f_{m,N}\rangle^2 = \left(\int_0^t \exp(-\beta_{m,N}(t-s))\left\langle f_{m,N}, R(X(s))\right\rangle ds\right)^2$$
$$\leq \left(\int_0^t \exp(-2\beta_{m,N}(t-s))\, ds\right)\left(\int_0^t \left\langle f_{m,N}, R(X(s))\right\rangle^2 ds\right)$$
$$\leq c|m|^{-2}\int_0^t \left\langle f_{m,N}, R(X(s))\right\rangle^2 ds,$$

where we have applied Cauchy–Schwarz and (2.3). Thus,

$$\sum_{|m|\geq n} \langle g(t), f_{m,N}\rangle^2 \leq cn^{-2}\int_0^t \sum_m \left\langle f_{m,N}, R(X(s))\right\rangle^2 ds$$
$$\leq cn^{-2}\int_0^T \left\langle 1, R^2(X(s))\right\rangle ds$$

and (i) follows by Lemma 3.3.

Consider

$$\left|\langle g(t+\varepsilon) - g(t), f_{m,N}\rangle\right|^2$$
$$= \left|\int_0^{t+\varepsilon} \exp(-\beta_{m,N}(t+\varepsilon-s))\left\langle f_{m,N}, R(X(s))\right\rangle ds\right.$$
$$\left. - \int_0^t \exp(-\beta_{m,N}(t-s))\left\langle f_{m,N}, R(X(s))\right\rangle ds\right|^2$$
$$\leq \frac{1}{2}\left(\int_0^T \left|\left\langle f_{m,N}, R(X(s))\right\rangle\right| ds\right)^2 (\exp(-\varepsilon\beta_{m,N}) - 1)^2$$
$$+ \frac{1}{2}\left(\int_t^{t+\varepsilon} \left|\left\langle f_{m,N}, R(X(s))\right\rangle\right| ds\right)^2$$
$$\leq 2^{p-1}\left(\int_0^T \left\langle 1, R^2(X(s))\right\rangle ds\right)(\beta_{m,N}^2\varepsilon^2 T + \varepsilon)$$

and (ii) follows from (2.3) and Lemma 3.3. □

4. Law of large numbers in L_2. In this section we assume the following:

(A1) $\|EX^{2q}(0)\|_\infty \leq c < \infty$.

(A2) $(N, l(N))$ is any sequence satisfying $l(N) \to \infty$ as $N \to \infty$.

(A3) $\|X(0) - \psi_0\| \to 0$ in probability, where ψ_0 is continuous and satisfies the boundary conditions in (2.1).

After preliminary results and discussion we prove the following theorem.

THEOREM 4.1.

$$\sup_{t \le T} \| X(t) - \psi(t) \| \to 0 \quad \textit{in probability},$$

where ψ *is the unique mild solution of* (2.1) *with* $\psi(0, r) = \psi_0(r)$.

By (A3), $P(\|X(0)\| > e) \to 0$ as $N \to \infty$ for $e > \|\psi_0\|$. Thus, by conditioning on the event $\|X(0)\| \le \|\psi_0\| + 1$, without loss of generality we assume:
(A4) $\|X(0)\| \le \|\psi_0\| + 1$.
Because this implies $X(0) \le c(N, l) < \infty$, the estimates of Section 3 hold.

LEMMA 4.1.

$$\sup_{t \le T} \|Y(t)\| \to 0 \quad \textit{in probability}.$$

PROOF. The result holds for Y_R in place of Y by Lemma 3.6, so we need only prove it for Y_D in place of Y.

Let $\tau = \inf\{t\colon \|Y_D(t)\| \ge a > 0\}$. Then

$$P\Big(\sup_{t \le T} \|Y_D(t)\| > a\Big) \le P\Big(\sup_{t \le T} \|Y_D(t \wedge \tau)\| \ge a\Big),$$

so we may consider $Y_D(t \wedge \tau)$ in place of $Y(t)$.

For any $e > \|\psi_0\| + 1$, let

$$\tau_e = \inf\{t\colon \|X(t)\| \ge e\}.$$

By definition of the possible transitions for X,

$$\|\delta Y_D(t)\| = \|\delta X_D(t)\| \le \|\delta X(t)\| \le \left(2/(N^p l^2)\right)^{1/2},$$

so we may assume $\|Y_D(t \wedge \tau)\| \le a + 1$ and $\|X(t \wedge \tau_e)\| \le e + 1$. Consider

$$\begin{aligned} &P\Big(\sup_{t \le T} \|Y_D(t \wedge \tau)\| \ge a\Big) \\ &\quad \le P\Big(\sup_{t \le T} \|Y_D(t \wedge \tau)\| \ge a, \sup_{t \le T} \|X(t \wedge \tau)\| < e\Big) \\ &\qquad + P\Big(\sup_{t \le T} \|X(t \wedge \tau)\| \ge e\Big). \end{aligned}$$

We have

$$\begin{aligned} &P\Big(\sup_{t \le T} \|Y_D(t \wedge \tau)\| \ge a, \sup_{t \le T} \|X(t \wedge \tau)\| < e\Big) \\ &\quad \le P\Big(\sup_{t \le T} \|Y_D(t \wedge \tau \wedge \tau_e)\| \ge a\Big) \to 0 \end{aligned}$$

by Lemma 3.5. Now choose ρ such that $R(x) < 0$ for $x > \rho$. From (A4), $\|Y_D(t \wedge \tau)\| \le a + 1$, $\int_0^t T_N(t-s)R(X(s))\,ds \le \rho t$, $X(t) \ge 0$ and (3.4), we have, for e large enough (but fixed),

$$P\Bigl(\sup_{t \le T} \|X(t \wedge \tau)\| \ge e\Bigr) \le P\Bigl(\sup_{t \le T} \|Y_R(t)\| \ge e/2\Bigr) \to 0$$

as noted previously. □

LEMMA 4.2. (a) *$\{X_{N,l(N)}\}$ is relatively compact in $D([0,T];L_2(S))$.*
(b) *If $\{X_{N_n}\} \subset \{X_{N,l(N)}\}$ and $X_{N_n} \to f$ in distribution as $N_n \to \infty$, then $P(\sup_{t \le T} \|f(t)\|_\infty \le c(T) < \infty) = 1$ and $f \in C([0,T];L_2(S))$.*

PROOF. (a) follows from Lemma 3.7, Lemma 4.1, (A1)–(A3) and (3.4). Because $\sup_{t \le T} \|T_{N_n}(t)X_{N_n}(0) - T(t)\psi_0\| \to 0$ in probability, this also shows $f \in C([0,T]; L_2(S))$.

Choose ρ such that $R(x) < 0$ for $x > \rho$ and let $g \in L_2(S)$ with $g \ge 0$. By (3.4) and (A4), $0 \le \langle X(t),g\rangle \le (c + \rho T)\langle 1,g\rangle + \langle T_N(t)(X(0) - \psi(0)) + Y(t),g\rangle$. By the continuous mapping theorem, $\langle X_{N_n},g\rangle \to \langle f,g\rangle$ in distribution on $D([0,T];\mathbf{R})$. Thus, by the preceding inequality, Lemma 4.1 and (A3),

$$P\Bigl(0 \le \langle f(t),g\rangle \le (c + \rho T)\langle 1,g\rangle, 0 \le t \le T\Bigr) = 1.$$

If D is a countable dense subset of $L_2(S)$, we have

$$P\Bigl(0 \le \bigl|\langle f(t),g\rangle\bigr| \le (c + \rho T)\bigl\langle 1, |g|\bigr\rangle, 0 \le t \le T, g \in D\Bigr) = 1,$$

and the bound on $\|f(t)\|_\infty$ follows from the identification of the dual of $L_1(S)$ with $L_\infty(S)$. □

PROOF OF THEOREM 4.1. With a continuous initial condition, (2.1) has a unique mild solution by Theorem A.1 of the Appendix of Kotelenez (1986b). Thus, to prove Theorem 4.1 it suffices, by Lemma 4.2(a), to show that $\{X_{N_n}\} \subset \{X_{N,l(N)}\}$ and $X_{N_n} \to f$ in distribution as $N_n \to \infty$ implies f is a mild solution of (2.1).

By applying the Skorohod representation theorem [Theorem 1.8 of Ethier and Kurtz (1986)], we obtain, on some probability space, $\{\widehat{X}_{N_n}\}$ and $\widehat{f}$, where $\widehat{X}_{N_n} = X_{N_n}$ and $\widehat{f} = f$ in distribution and $\widehat{X}_{N_n} \to \widehat{f}$ almost surely.

Let $E_N = (H^N, \|\cdot\|)$ and define

$$F_N: D\bigl([0,T];E_N\bigr) \to D\bigl([0,T];E_N\bigr)$$

by

$$F_N(\gamma)(t) = \gamma(t) - T_N(t)\gamma(0) - \int_0^t T_N(t-s)R\bigl(\gamma(s)\bigr)\,ds.$$

Letting $\widehat{Y}_{N_n} = F_{N_n}(\widehat{X}_{N_n})$, we have

$$\widehat{X}_{N_n}(t) = T_{N_n}(t)\widehat{X}_{N_n}(0) + \int_0^t T_{N_n}(t-s)R\big(\widehat{X}_{N_n}(s)\big)\,ds + \widehat{Y}_{N_n}(t).$$

Because E_N has finite dimension, F_N is continuous and $Y_{N_n} = F_{N_n}(X_{N_n})$ implies that $\widehat{Y}_{N_n} = Y_{N_n}$ in distribution. In particular, $\sup_{t\le T}\|\widehat{Y}_{N_n}(t)\| \to 0$ in probability by Lemma 4.1. Thus, without loss of generality, we delete the $\wedge$ and assume $X_{N_n} \to f$ almost surely. By (3.4), we have with $f(0) = \psi_0$,

$$f(t) = T(t)f(0) + \int_0^t T(t-s)R\big(f(s)\big)\,ds + \varepsilon_{N_n}(t) + \delta_{N_n}(t), \tag{4.1}$$

where

$$\varepsilon_{N_n}(t) = f(t) - X_{N_n}(t) + Y_{N_n}(t) + T_{N_n}(t)X_{N_n}(0) - T(t)f(0)$$

and

$$\delta_{N_n}(t) = \int_0^t T_{N_n}(t-s)R\big(X_{N_n}(s)\big)\,ds - \int_0^t T(t-s)R\big(f(s)\big)\,ds.$$

Consider ε_{N_n}. By our Lemma 4.2(b) and Lemma 3.10.1 of Ethier and Kurtz (1986),

$$\sup_{t\le T}\|f(t) - X_{N_n}(t)\| \to 0 \tag{4.2}$$

almost surely. Thus, by Trotter–Kato and Lemma 4.1, $\sup_{t\le T}\|\varepsilon_{N_n}(t)\| \to 0$ in probability.

Consider

$$\begin{aligned}\delta_{N_n}(t) = \sum_{|m|\le K}\Bigg[&\int_0^t \exp\big(-\beta_{m,N_n}(t-s)\big)\Big\langle f_{m,N_n}, R\big(X_{N_n}(s)\big)\Big\rangle\,ds f_{m,N_n}\\ &- \int_0^t \exp\big(-\beta_m(t-s)\big)\Big\langle f_m, R\big(f(s)\big)\Big\rangle\,ds\, f_m\Bigg]\\ &+ \sum_{|m|>K}\int_0^t \exp\big(-\beta_{m,N_n}(t-s)\big)\Big\langle f_{m,N_n}, R\big(X_{N_n}(s)\big)\Big\rangle\,ds f_{m,N_n}\\ &- \sum_{|m|>K}\int_0^t \exp\big(-\beta_m(t-s)\big)\Big\langle f_m, R\big(f(s)\big)\Big\rangle\,ds f_m.\end{aligned}$$

In the proof of Lemma 3.7, it was shown that

$$E\left[\sup_{t\le T}\left\|\sum_{|m|>K}\int_0^t \exp\big(-\beta_{m,N_n}(t-s)\big)\Big\langle f_{m,N_n}, R\big(X_{N_n}(s)\big)\Big\rangle\,ds\, f_{m,N}\right\|^2\right] \le c(T)/K^2.$$

By the bound on $\|f(t)\|_\infty$ in Lemma 4.2(b), the same argument shows

$$E\left[\sup_{t\le T}\left\|\sum_{|m|>K}\int_0^t \exp(-\beta_m(t-s))\Big\langle f_m, R\big(f(s)\big)\Big\rangle\, ds\, f_m\right\|^2\right] \le c(T)/K^2.$$

For m fixed, $|\beta_{m,N_n}-\beta_m| + \|f_{m,N_n} - f_m\|_\infty \to 0$. Also, $\|f_{m,N}\|_\infty + \|f_m\|_\infty \le c(p)$. By (A1), Lemma 3.3, and Lemma 4.2(b),

$$E\int_0^T \Big\langle 1, \big|R\big(X_N(s)\big)\big| + \big|R\big(f(s)\big)\big|\Big\rangle\, ds \le c(T).$$

Also, by Lemma 4.2(b),

$$\int_0^T \Big\langle 1, \big|R\big(X_N(s)\big)\big| + \big|R\big(f(s)\big)\big|\Big\rangle\, ds$$
$$\le c(T)\left(\int_0^T \big\|X_{N_n}(s) - f(s)\big\|^2\, ds\right)^{1/2}\left(\int_0^T \langle X_{N_n}^{2(q-1)}(s) + 1, 1\rangle\, ds\right)^{1/2}.$$

The right-hand side of the inequality converges to 0 in probability by (4.2), (A1) and Lemma 3.3.

From the discussion in the previous paragraph, it follows that, for m fixed,

$$\sup_{t\le T}\left\|\int_0^t\Big(\exp(-\beta_{m,N_n}(t-s))\Big\langle f_{m,N_n}, R\big(X_{N_n}(s)\big)\Big\rangle f_{m,N_n} - \exp(-\beta_m(t-s))\Big\langle f_m, R\big(f(s)\big)\Big\rangle f_m\Big)\, ds\right\|_\infty$$

converges to 0 in probability. Thus, $\sup_{t\le T}\|\delta_{N_n}(t)\| \to 0$ in probability. By going to a further subsequence if necessary, we obtain

$$\sup_{t\le T}\big\|\delta_{N_n}(t) + \varepsilon_{N_n}(t)\big\| \to 0$$

almost surely and Theorem 4.1 follows from (4.1). □

5. The central limit theorem. In this section we assume $p = 1$ and let $\{H_\alpha\}_{\alpha\in\mathbf{R}}$ denote the decreasing sequence of Hilbert spaces obtained by completion of the trigonometric polynomials in the norm

$$\|f\|_\alpha = \left[\sum_n \big(\langle f,\varphi_n\rangle^2 + \langle f,\psi_n\rangle^2\big)(1+\beta_n)^\alpha\right]^{1/2}.$$

Note $H_0 = L_2(S)$. If $\alpha \ge 0$, $f \in H_{-\alpha}$ and $g \in H_\alpha$, the $L_2(S)$ inner product $\langle\cdot,\cdot\rangle$ extends to

$$\langle f,g\rangle = \sum_n \big(\langle f,\varphi_n\rangle\langle g,\varphi_n\rangle + \langle f,\psi_n\rangle\langle g,\psi_n\rangle\big)$$

and satisfies $|\langle f,g\rangle| \le \|f\|_{-\alpha}\|g\|_{\alpha}$. Also, $\alpha_1 < \alpha_2$ implies $\|f\|_{\alpha_1} \le \|f\|_{\alpha_2}$.

Note $H^N \subset H_0 \subset H_{-\alpha}$ for $\alpha \ge 0$. For $\alpha \in \mathbf{R}$ and $f \in H^N$, let

$$\|f\|_{\alpha,N} = \left[\sum_n \left(\langle f,\varphi_{n,N}\rangle^2 + \langle f,\psi_{n,N}\rangle^2\right)(1+\beta_{n,N})^{\alpha}\right]^{1/2},$$

where $\varphi_{n,N}$ and $\psi_{n,N}$ are the discretized versions of φ_m and ψ_m from Section 2. If $\alpha \ge 0$ and $f,g \in H^N$, then $|\langle f,g\rangle| \le \|f\|_{-\alpha,N}\|g\|_{\alpha,N}$.

Note $\|f\|_{0,N} = \|f\|_0$ and, more generally, basic calculations [Blount (1987)] show that for $f \in H^N$ and $\alpha \ge 0$ there exist positive constants $c_1(\alpha)$ and $c_2(\alpha)$ with

$$c_1(\alpha)\|f\|_{-\alpha,N} \le \|f\|_{-\alpha} \le c_2(\alpha)\|f\|_{-\alpha,N}. \tag{5.1}$$

The following basic facts will be needed for computations:
Assume $m,n \ne 0$.

(a) $$\varphi_{m,N}\varphi_{n,N} = 2^{-1/2}[\varphi_{m+n,N} + \varphi_{m-n,N}],$$

(b) $$\psi_{m,N}\psi_{n,N} = 2^{-1/2}[\varphi_{m-n,N} - \varphi_{m+n,N}],$$

(c) $$\varphi_{m,N}\psi_{m,N} = 2^{-1/2}[\psi_{m+n,N} - \psi_{m-n,N}]$$

Let $a_{m,N} = N[\cos(\pi m N^{-1}) - 1]$ and $b_{m,N} = N\sin(\pi m N^{-1})$.

(d) $$\nabla^{\pm}\varphi_{m,N} = a_{m,N}\varphi_{m,N} \pm b_{m,N}\psi_{m,N},$$

(e) $$\nabla^{\pm}\psi_{m,N} = \pm b_{m,N}\varphi_{m,N} + a_{m,N}\psi_{m,N},$$

(f) $$a_{m,N}^2 + b_{m,N}^2 = \beta_{m,N}$$

(g) $$\varphi_{m,N} = \varphi_{2N-m,N}, \qquad \psi_{m,N} = -\psi_{2N-m,N},$$
$$a_{m,N} = a_{2N-m,N}, \qquad b_{m,N} = -b_{2N-m,N}.$$

Note that multiplying $\varphi_{m,N}\varphi_{n,N}$, and so forth gives rise to subscripts outside our original domain of definition, but (g) shows this is not a problem.

Fix $\alpha > 1/2$ and, for the remainder of this section, assume:

(A1) $N = l$.

(A2) $\|X(0) - \psi(0)\|_{\infty} \to 0$ in probability.

(A3) $P((Nl)^{1/4}\|X(0) - \psi(0)\|_0 \le c) \to 1$ for some $c < \infty$.

(A4) $(Nl)^{1/2}(X(0) - \psi(0)) \to V_0$ in distribution on $H_{-\alpha}$.

(A5) $d^k/dr^k\psi(0,r)$ is continuous for some $k \ge \max(4,\alpha)$.

Recall ψ is a solution of (2.1) and $R(x)$ the reaction polynomial. For $\gamma \in [-k,k]$ it is shown in the Appendix of Kotelenez (1986b) that $U(t,s)$, the evolution system [see Pazy (1983)] generated on H_γ by $\Delta + R'(\psi(t))$, satisfies both

$$\begin{aligned} &\|U(t,s)f\|_{\gamma} \le \|f\|_{\gamma}\exp\bigl(c(T)(t-s)\bigr) \quad \text{for } s \le t \le T,\\ &\|\psi(t)f\|_{\gamma} \le c(T)\|f\|_{\gamma}. \end{aligned} \tag{5.2}$$

For $\beta > 3/2$, let M denote the $C([0,\infty); H_{-\beta})$-valued Gaussian martingale with characteristic functional

$$E\Big[\exp\big(i\langle M(t),\varphi\rangle\big)\Big] = \exp\left(-\int_0^t \left(\langle \psi(s),(\dot{\varphi})^2\rangle - \tfrac{1}{2}\Big\langle |R|\big(\psi(s)\big),\varphi^2\Big\rangle\right) ds\right)$$

for $\varphi \in H_\beta$. [See Kotelenez (1988) for further discussion of M]. We assume V_0 independent of M. Let

$$\overline{R}(x,y) = \sum_{i=2}^{m} c_i \left(\sum_{j=1}^{i-1} j x^{i-1-j} y^{j-1}\right),$$

where $\{c_i\}_2^m$ are coefficients of $R(x)$, and consider the stochastic partial differential equation

$$dV(t) = \Big(\Delta + R'\big(\psi(t)\big)\Big)V(t)\,dt + \overline{R}\big(\psi(t),\psi(t)\big)\psi(t)\,dt + dM(t),\ V(0) = V_0. \tag{5.3}$$

It follows from Kotelenez (1988) that (5.3) has a unique mild solution $V \in C([0,\infty); H_{-\alpha})$ a.s. and given by

$$V(t) = U(t,0)V(0) + \int_0^t U(t,s)\,dM(s) + \int_0^t U(t,s)\overline{R}\big(\psi(s),\psi(s)\big)\psi(s)\,ds. \tag{5.4}$$

In this section we prove the following result:

THEOREM 5.1. *$(Nl)^{1/2}(X-\psi) \to V$ in distribution on $D([0,\infty); H_{-\alpha})$.*

The proof will be given after preliminary results and discussion.

Let $h = h_N$ denote the H^N-valued solution of the ordinary differential equation

$$\frac{\partial h(t,r)}{\partial t} = \Delta_N h(t,r) + R\big(h(t,r)\big), \qquad h(0,r) = P_N\psi(0,r). \tag{5.5}$$

h is a spatially discretized version of (2.1).

Fix ρ such that $R(x) < 0$ for $x > \rho$. We may assume $\psi(0) < \rho$, and from the Appendix of Kotelenez (1986b), we have

$$0 \le \psi(t,r), h(t,r) < \rho \tag{5.6}$$

and

$$\sup_{t\le T} \|\psi(t) - h(t)\|_\infty \le c(T)N^{-1}. \tag{5.7}$$

It is shown in the proof of Lemma 6.4 of Blount (1993) that

$$\sup_{t\le T} N\|h(t) - \psi(t)\|_{-\alpha} \to 0. \tag{5.8}$$

Let $U_N(t,s)$ denote the evolution system generated on H^N by $\Delta_N + R'(h(t))$. From Lemma 4.1 of Blount (1993) we have, for $\gamma \in [0,k]$ and $f \in H^N$,

$$\begin{aligned} \|U_N(t,s)f\|_{-\gamma} &\le \|f\|_{-\gamma}\exp\bigl(c(T)(t-s)\bigr) \quad \text{for } s \le t \le T, \\ \|h(t)f\|_{-\gamma} &\le c(T)\|f\|_{-\gamma}. \end{aligned} \tag{5.9}$$

By (A1), (A2), (5.7) and Theorem 4.1 of Blount (1992),

$$\sup_{t \le T}\bigl(\|h(t) - X(t)\|_\infty + \|\psi(t) - X(t)\|_\infty\bigr) \to 0 \tag{5.10}$$

in probability. Using (3.3) and (5.5), we can write

$$\begin{aligned} X(t) - h(t) = X(0) - h(0) &+ \int_0^t \Bigl(\Delta_N + R'\bigl(h(s)\bigr)\Bigr)\bigl(X(s) - h(s)\bigr)\,ds \\ &+ \int_0^t \overline{R}\bigl(X(s), h(s)\bigr)\bigl(X(s) - h(s)\bigr)^2\,ds + Z(t). \end{aligned} \tag{5.11}$$

By variation of constants, we can write

$$\begin{aligned} N\bigl(X(t) - h(t)\bigr) &= U_N(t,0)N\bigl(X(0) - h(0)\bigr) + \int_0^t U_N(t,s)N\,dZ(s) \\ &\quad + \int_0^t U_N(t,s)\overline{R}\bigl(X(s), h(s)\bigr)N\bigl(X(s) - h(s)\bigr)^2\,ds. \end{aligned} \tag{5.12}$$

LEMMA 5.1. *If* $\log N/l \to 0$ *as* $N \to \infty$, *then* (A2), (A4) *and* (A5) *imply that*

$$(Nl)^{1/2}\Bigl[U_N(0,\cdot)\bigl(X(0) - h(0)\bigr) + \int_0^{\cdot} U_N(\cdot,s)\,dZ(s)\Bigr]$$

converges in distribution on $D([0,\infty); H_{-\alpha})$ *to* $U(\cdot,0)V_0 + \int_0^{\cdot} U(\cdot,s)\,dM(s)$.

PROOF. This was shown in the proof of Lemma 6.3 of Blount (1993) assuming $N/l \to 0$ as $N \to \infty$. However, the stronger assumption on l was only used to eliminate the last term in (5.12) [with $(Nl)^{1/2}$ in place of N], and the proof of Lemma (5.1) only requires (A4) and (5.10), which holds by Theorem 4.1 of Blount (1992). □

By Lemma 5.1, (5.12) and (5.8), Theorem 5.1 will follow if we can show that the last term in (5.12) converges in probability to the corresponding term in (5.4).

By (A2), (A3) and (5.6), we may, by conditioning on $X(0)$, assume that

$$0 \le X(0) < \rho \tag{5.13}$$

and

$$N\|X(0) - h(0)\|_0^2 \le c < \infty. \tag{5.14}$$

Let $\tau_1 = \inf\{t\colon \|X(t)\|_\infty \geq \rho\}$. Because $\|\delta X(t)\|_\infty \leq l^{-1}$, we may assume

$$\|X(t \wedge \tau_1)\|_\infty < \rho + 1. \tag{5.15}$$

LEMMA 5.2. *There exists $a = a(\rho, T) < \infty$ such that*

$$P\Big(\sup_{t \leq T} N\|Y(t \wedge \tau_1)\|_0^2 \geq a^2\Big) \to 0.$$

PROOF. We use the notation developed for the proof of Lemma 3.5. For any $a > 0$, it follows from Lemma 3.2(c) and (d) that

$$P\bigg(\sup_{t \leq T} N \sum_{m \in B_0} \big\langle Y(t \wedge \tau_1), f_{m,N}\big\rangle^2 \geq a^2\bigg)$$
$$\leq c(T, \rho) a^{-2} l^{-1} (\log N)^6 \to 0 \quad \text{as } N = l \to \infty.$$

By Lemma 3.4(b), exactly as in the proof of Lemma 3.5,

$$P\bigg(\sup_{t \leq T} N \sum_{m \notin B_0} \big\langle Y(t \wedge \tau_1), f_{m,N}\big\rangle^2 \geq a^2\bigg)$$
$$\leq c(T) N^3 (\log N)^{-1} \big(ca^2 l/(N\rho)\big)^{-\log N}$$
$$= c(T) N^3 (\log N)^{-1} \big(ca^2/\rho\big)^{-\log N} \quad \text{because } N = l.$$

For $a = a(\rho, T)$ large enough, the last expression converges to 0 as $N \to \infty$. □

Choose an $a = a(\rho, T)$ as given by Lemma 5.2 and let $\tau_2 = \inf\{t\colon N\|Y(t \wedge \tau_1)\|_0^2 \geq a\}$. By Lemma 5.2 and (5.10), $\tau = \tau_1 \wedge \tau_2$ satisfies

$$P(\tau \leq T) \to 0 \quad \text{as } N \to \infty. \tag{5.16}$$

Let

$$\overline{X}(t) = \begin{cases} X(t), & t < \tau, \\ X(t \wedge \tau) + \int_{t \wedge \tau}^t \Delta_N \overline{X}(s)\, ds + \displaystyle\int_{t \wedge \tau}^t R\big(\overline{X}(s)\big)\, ds, & \tau \leq t < \infty. \end{cases}$$

$\overline{X}$ evolves as X until time τ and afterward, if $\tau < \infty$, according to the solution of (5.5) beginning at $X(\tau)$. By (5.16),

$$P\big(X(t) \neq \overline{X}(t), 0 \leq t \leq T\big) \to 0 \quad \text{as } N \to \infty, \tag{5.17}$$

so we may restrict our attention to $\overline{X}$.

Letting $\overline{Z}(t) = Z(t \wedge \tau)$, we have

$$\overline{X}(t) = X(0) + \int_0^t \Delta_N \overline{X}(s)\, ds + \int_0^t R\big(\overline{X}(s)\big)\, ds + \overline{Z}(t).$$

By variation of constants,

$$\overline{X}(t) = T_N(t)X(0) + \int_0^t T_N(t-s)R\big(\overline{X}(s)\big)\,ds + \overline{Y}(t), \tag{5.18}$$

where

$$\overline{Y}(t) = \int_0^t T_N(t-s)\,d\overline{Z}(s). \tag{5.19}$$

If $\tau < \infty$ and $t \geq \tau$, then

$$\overline{X}(t) = T_N(t-\tau)X(\tau) + \int_\tau^t T_N(t-s)R\big(\overline{X}(s)\big)\,ds.$$

By (5.15), $0 \leq X(\tau) < \rho + 1$ and it follows from the properties of the deterministic system [Kotelenez (1986b)] that

$$0 \leq \overline{X}(t) < \rho + 1 \quad \text{for } t \geq 0. \tag{5.20}$$

(5.18), (5.20), (5.6) and variation of constants applied to (5.5) imply that

$$\begin{aligned}\sup_{t\leq T} \|\overline{X}(t) - h(t)\|_0 &\leq \exp\big(c(\rho)T\big)\Big(\|X(0)-h(0)\|_0 + \sup_{t\leq T}\|\overline{Y}(t)\|_0\Big)\\ &\leq \exp\big(c(\rho)T\big)\Big(\|X(0)-h(0)\|_0 + \sup_{t\leq T}\|Y(t\wedge\tau_2)\|_0\Big),\end{aligned} \tag{5.21}$$

because $\overline{Y}(t) = T_N(t - t\wedge\tau)Y(t\wedge\tau)$ and $\tau \leq \tau_2$. Note $N\|\delta Y(t)\|_0^2 = N\|\delta X(t)\|_0^2 \leq N(2/(Nl^2)) = 2l^{-2}$. Thus we may assume $\sup_{t\leq T} N\|Y(t\wedge\tau_2)\|_0^2 \leq a(\rho, T)$, and by (5.21) and (5.14), we have

$$\sup_{t\leq T} N\|\overline{X}(t) - h(t)\|_0^2 \leq a(\rho, T) < \infty. \tag{5.22}$$

REMARK 5.1. By (5.17) and our previous discussion, it suffices to show that

$$N\int_0^\cdot U_N(\cdot, t)\overline{R}\big(\overline{X}(t), h(t)\big)\big(\overline{X}(t) - h(t)\big)^2\,dt$$

converges in probability on $D([0,\infty); H_{-\alpha})$ to

$$\int_0^\cdot U(\cdot, t)\overline{R}\big(\psi(t), \psi(t)\big)\psi(t)\,dt.$$

Consider

$$\int_0^{\hat{t}} U_N(\hat{t}, t)\overline{R}(X(t), h(t))N\big(\overline{X}(t) - h(t)\big)^2\,dt$$

for $\hat{t} \le T$. Note

$$\|f\|_{-\alpha} \le \langle 1, |f|\rangle c\left(\sum_{1}^{\infty} m^{-2\alpha}\right)^{1/2} = c(\alpha)\langle 1, |f|\rangle$$

because $\alpha > \frac{1}{2}$. Thus, by (5.9), (5.10), (5.17) and (5.22),

$$\sup_{t \le \hat{t} \le T} \left\| U_N(\hat{t}, t)\Big(\overline{R}\big(\overline{X}(t), h(t)\big) - \overline{R}\big(h(t), h(t)\big)\Big) N\big(\overline{X}(t) - h(t)\big)^2 \right\|_{-\alpha} \le c(T, \alpha) \sup_{t \le T} \|\overline{X}(t) - h(t)\|_\infty \to 0$$

in probability, and we may replace $\overline{R}(\overline{X}, h)$ by $\overline{R}(h, h)$.

Let

$$P_{N,n} f = \sum_{m \le n} \langle f, \varphi_{m,N}\rangle \varphi_{m,N} + \langle f, \psi_{m,N}\rangle \psi_{m,N}$$

and

$$P_n f = \sum_{m \le n} \langle f, \varphi_m\rangle \varphi_m + \langle f, \psi_m\rangle \psi_m.$$

By (5.1), $f \in H^N$ implies

$$\|(I - P_{N,n}) f\|_{-\alpha} \le c\langle 1, |f|\rangle \left(\sum_{m > n} m^{-2\alpha}\right)^{1/2}.$$

Thus, (5.9) and (5.22) imply

$$\lim_{n \to \infty} \sup_{t \le \hat{t} \le T} \left\| U_N(\hat{t}, t)\overline{R}\big(h(t), h(t)\big)(I - P_{N,n}) N\big(\overline{X}(t) - h(t)\big)^2 \right\|_{-\alpha} = 0$$

and, similarly,

$$\lim_{n \to \infty} \sup_{t \le \hat{t} \le T} \left\| U(\hat{t}, t)\overline{R}\big(\psi(t), \psi(t)\big)(I - P_n)\psi(t) \right\|_{-\alpha} = 0.$$

This implies we need only consider

$$\langle N(\overline{X} - h)^2, e_{m,N}\rangle e_{m,N}$$

for $e_{m,N} = \varphi_{m,N}$ or $\psi_{m,n}$ and m fixed in place of $N(\overline{X} - h)^2$.

If $e_m = \varphi_m$ or ψ_m, the subsequent Lemma 5.3 shows that $\langle N(\overline{X}(t) - h(t))^2, e_{m,N}\rangle \to \langle \psi(t), e_m\rangle$ in probability for each fixed $t > 0$. Assume for now this is true. By (5.22), (5.6) and the bounded convergence theorem (applied twice), this implies

$$\int_0^T E\left(\left| \left\langle N\big(\overline{X}(t) - h(t)\big)^2, e_{m,N}\right\rangle - \langle \psi(t), e_m\rangle \right|\right) dt \to 0. \tag{5.23}$$

By a standard Trotter–Kato type argument,

$$\sup_{t\le\hat{t}\le T}\ \sup_{f\in A}\left\|\left(U_N(\hat{t},t)-U(\hat{t},t)\right)f\right\|_0\to 0$$

for compact sets $A\subset H_0$. Because $\psi\in C([0,\infty);C([0,1]))$, this implies

$$\sup_{t\le\hat{t}\le T}\left\|\left(U_N(\hat{t},t)-U(\hat{t},t)\right)\overline{R}\left(\psi(t),\psi(t)\right)e_m\right\|_0\to 0.$$

Also, $\|e_m-e_{m,N}\|_\infty+\sup_{t\le T}\|\psi(t)-h(t)\|_\infty\to 0$ as $N\to\infty$. Remark 5.1, the discussion following Remark 5.1 and simple calculations complete the proof of Theorem 5.1, assuming Lemma 5.3, which we now state and prove.

LEMMA 5.3. *$\langle N(\overline{X}(t)-h(t))^2,e_{m,N}\rangle\to\langle\psi(t),e_m\rangle$ in probability for each fixed $t>0$.*

PROOF. By variation of constants,

$$\begin{aligned}\overline{X}(t)-h(t)&=T_N(t)\left(X(0)-h(0)\right)+\int_0^t T_N(t-s)R'\left(h(s)\right)\left(\overline{X}(s)-h(s)\right)ds\\&\quad+\int_0^t T_N(t-s)\overline{R}\left(\overline{X}(s),h(s)\right)\left(\overline{X}(s)-h(s)\right)^2ds\\&\quad+\int_0^t T_N(t-s)\,d\overline{Z}(s).\end{aligned}$$

For $0\le\hat{t}\le t$ with $t>0$ fixed, let

$$\begin{aligned}g(\hat{t})&=T_N(t)\left(X(0)-h(0)\right)+\int_0^{\hat{t}} T_N(t-s)R'\left(h(s)\right)\left(\overline{X}(s)-h(s)\right)ds\\&\quad+\int_0^{\hat{t}} T_N(t-s)\overline{R}\left(\overline{X}(s),h(s)\right)\left(\overline{X}(s)-h(s)\right)^2ds\\&\quad+\int_0^{\hat{t}} T_N(t-s)\,d\overline{Z}(s).\end{aligned}$$

Applying Itô's formula to $g^2(\hat{t})$ and noting that $g(t)=\overline{X}(t)-h(t)$, we obtain

$$\begin{aligned}\left(\overline{X}(t)-h(t)\right)^2&=\left(T_N(t)\left(X(0)-h(0)\right)\right)^2\\&\quad+2\int_0^t\left(T_N(t-s)\left(\overline{X}(s)-h(s)\right)\right)\\&\qquad\times\left(T_N(t-s)R'\left(h(s)\right)\left(\overline{X}(s)-h(s)\right)\right)ds\\&\quad+2\int_0^t\left(T_N(t-s)\left(\overline{X}(s)-h(s)\right)\right)\\&\qquad\times\left(T_N(t-s)\overline{R}\left(\overline{X}(s),h(s)\right)\left(\overline{X}(s)-h(s)\right)^2\right)ds\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^t \Big(T_N(t-s)\big(\overline{X}(s-)-h(s)\big)\Big)\big(T_N(t-s)\,d\overline{Z}(s)\big)\\
&+\sum_{s\le t}\big(T_N(t-s)\,\delta\overline{Z}(s)\big)^2\\
&=g_1(t)+g_2(t)+g_3(t)+g_4(t)+g_5(t),
\end{aligned}
$$

respectively.

Consider

$$
\begin{aligned}
&N\Big|\big\langle\varphi_{m,N},g_1(t)\big\rangle\Big|\\
&\quad\le\sqrt{2}N\big\langle 1,g_1(t)\big\rangle\\
&\quad=\sqrt{2}N\sum_n\exp\big(-\beta_{n,N}t\big)\Big(\big\langle X(0)-h(0),\varphi_{n,N}\big\rangle^2+\big\langle X(0)-h(0),\psi_{n,N}\big\rangle^2\Big).\\
&\quad\le\sqrt{2}N\sum_{n\le K}\Big(\big\langle X(0)-h(0),\varphi_{n,N}\big\rangle^2+\big\langle X(0)-h(0),\psi_{n,N}\big\rangle^2\Big)\\
&\qquad+\sqrt{2}N\|X(0)-h(0)\|_0^2\sum_{n\ge K}\exp(-\beta_{n,N}t).
\end{aligned}
\tag{5.24}
$$

By (A1), (A4), and (5.8), $N\|X(0)-h(0)\|_{-\alpha}$ is bounded in probability. Thus, by (5.1), $N^2(\langle X(0)-h(0),\varphi_{n,N}\rangle^2+\langle X(0)-h(0),\psi_{n,N}\rangle^2)$ is bounded in probability for each n, and the first sum converges to 0 in probability for any fixed K. By (5.14) and (2.3) the second sum goes to 0 as $K\to\infty$.

Consider

$$
\begin{aligned}
N\big\langle 1,g_2(t)\big\rangle=2N\int_0^t\sum_n&\exp\big(-2\beta_{n,N}(t-s)\big)\big\langle\overline{X}(s)-h(s),e_{n,N}\big\rangle\\
&\times\Big\langle R'\big(h(s)\big)\big(\overline{X}(s)-h(s)\big),e_{n,N}\Big\rangle\,ds,
\end{aligned}
$$

where we use $e_{n,N}$ to denote $\varphi_{n,N}$ or $\psi_{n,N}$. By (5.12), Lemma 5.1, (5.22), (5.1), (5.6), (5.15) and (5.17), $\sup_{t\le N}N\|\overline{X}(t)-h(t)\|_{-\alpha}$ is bounded in probability. Because $\int_0^t\exp(-\beta_{n,N}(t-s))\,ds\le c(1+n^2)^{-1}$, the results and facts given in the previous sentence show $N\langle 1,g_2(t)\rangle\to 0$ in probability. Applying some of the basic facts listed after (5.1) shows $N\langle e_{m,N},g_2(t)\rangle\to 0$ in probability with essentially the same proof.

Consider

$$
N\Big|\big\langle e_{m,N},g_3(t)\big\rangle\Big|\le c\sup_{t\le T}\|\overline{X}(t)-h(t)\|_\infty\cdot\sup_{t\le T}N\|\overline{X}(t)-h(t)\|_0^2\to 0
$$

in probability by (5.10), (5.17) and (5.22).

Consider $N\langle\varphi_{m,N},g_4(t)\rangle$ and let

$$
f_m(t,s)=T_N(t-s)\Big[\varphi_{m,N}\Big(T_N(t-s)\big(\overline{X}(s)-h(s)\big)\Big)\Big].
$$

Because $(\delta Z_D)(\delta Z_R) \equiv 0$, Lemma 3.1 with $N = l$ and Itô's formula imply that

$$\begin{aligned}
&E\Big[N^2\langle\varphi_{m,N}, g_4(t)\rangle^2\Big] \\
&\quad = N^2 E\left[\sum_{s\le t\wedge\tau}\Big\langle\varphi_{m,N}, \Big(T_N(t-s)\big(\overline{X}(s-)-h(s)\big)\Big)\big(T_N(t-s)\delta Z(s)\big)\Big\rangle^2\right] \\
&\quad = E\left[\int_0^{t\wedge\tau}\left[\Big\langle X(s), \big(\nabla^+ f_m(t,s)\big)^2 + \big(\nabla^- f_m(t,s)\big)^2\Big\rangle + \big\langle |R|\big(X(s)\big), f_m^2(t,s)\big\rangle\right] ds\right] \\
&\quad \le c\int_0^t E\big(\|\nabla^+ f_m(t,s)\|_0^2 + \|f_m(t,s)\|_0^2\big) ds \\
&\quad \le c\int_0^t \sum_n \exp\big(-cn^2(t-s)\big) E\Big(\langle\overline{X}(s)-h(s), \varphi_{n,N}\rangle^2 + \langle\overline{X}(s)-h(s), \psi_{n,N}\rangle^2\Big) \\
&\qquad\qquad \times (n+m+1)^2 ds,
\end{aligned}$$

where we have used the basic facts listed after (5.1) to obtain the last inequality. Equation (5.18) and variation of constants applied to (5.5) show

$$\begin{aligned}
\langle\overline{X}(t)-h(t), e_{n,N}\rangle &= \exp(-\beta_{n,N}t)\langle\overline{X}(0)-h(0), e_{n,N}\rangle \\
&\quad + \int_0^t \exp\big(-\beta_{n,N}(t-s)\big)\Big\langle R\big(\overline{X}(s)\big) - R\big(h(s)\big), e_{n,N}\Big\rangle ds \\
&\quad + \int_0^t \exp\big(-\beta_{n,N}(t-s)\big)\, d\langle\overline{Z}(s), e_{n,N}\rangle.
\end{aligned}$$

Applying Lemma 3.1 to the last term we obtain

$$\begin{aligned}
E\langle\overline{X}(t)-h(t), e_{n,N}\rangle^2 &\le 3^{-1}\exp\big(-cn^2 t\big) E\Big(\langle X(0)-h(0), e_{n,N}\rangle^2\Big) \\
&\quad + c(1+n^2)^{-2} E\Big(\sup_{s\le T}\|\overline{X}(s)-h(s)\|_\infty^2\Big) + c(T)(Nl)^{-1}.
\end{aligned}$$

From this and our previous inequality, we have

$$\begin{aligned}
&E\Big[N^2\langle\varphi_{m,N}, g_4(t)\rangle^2\Big] \\
&\quad \le c(m,T)\left[E\|X(0)-h(0)\|_0^2 + E\Big(\sup_{s\le T}\|\overline{X}(s)-h(s)\|_\infty^2\Big) + l^{-1}\right] \to 0
\end{aligned}$$

by previously noted results, and the same holds with $\psi_{m,N}$ in place of $\varphi_{m,N}$.

Consider $Ng_5(t) = N\sum_{s\leq t}(T_N(t-s)\delta\overline{Z}_D(s))^2 + N\sum_{s\leq t}(T_N(t-s)\delta\overline{Z}_R(s))^2$:

$$
\begin{aligned}
E&\left|N\sum_{s\leq t}\left\langle e_{m,N}, \left(T_N(t-s)\delta\overline{Z}_R(s)\right)^2\right\rangle\right| \\
&\leq \sqrt{2}NE\sum_{s\leq t\wedge\tau}\left\langle 1, \left(T_N(t-s)\delta Z_R(s)\right)^2\right\rangle \\
&= \sqrt{2}NE\sum_n (Nl)^{-1}\int_0^{t\wedge\tau}\exp\left(-\beta_{n,N}(t-s)\right)\left\langle |R|\left(X(s)\right), \varphi_{n,N}^2 + \psi_{n,N}^2\right\rangle ds \\
&\leq cl^{-1}\sum_n\left(1+n^2\right)^{-1} \to 0,
\end{aligned}
$$

where the last equality followed from Lemma 3.1. So we need only consider

$$N\sum_{s\leq t}\left(T_N(t-s)\delta\overline{Z}_D(s)\right)^2.$$

For $0 \leq \mu \leq t$ with $t > 0$ fixed and $f \in D([0,\infty); H_0)$, let

$$
\begin{aligned}
F(f)(t,\mu) = (Nl)^{-1}&\sum_{m,n}\int_0^\mu \exp\left[(-\beta_{m,N} - \beta_{n,N})(t-s)\right] \\
&\times\Big[\left\langle f(s), (\nabla^+\varphi_{m,N})(\nabla^+\varphi_{n,N}) + (\nabla^-\varphi_{m,N})(\nabla^-\varphi_{n,N})\right\rangle\varphi_{m,N}\varphi_{n,N} \\
&+ \left\langle f(s), (\nabla^+\psi_{m,N})(\nabla^+\psi_{n,N}) + (\nabla^-\psi_{m,N})(\nabla^-\psi_{n,N})\right\rangle\psi_{m,N}\psi_{n,N} \\
&+ 2\left\langle f(s), (\nabla^+\varphi_{m,N})(\nabla^+\psi_{m,N}) + (\nabla^-\varphi_{m,N})(\nabla^-\psi_{n,N})\right\rangle\varphi_{m,N}\psi_{n,N}\Big] ds.
\end{aligned}
\tag{5.25}
$$

For k fixed and $0 \leq \mu \leq t$, Lemma 3.1 implies

$$m(\mu) = N\sum_{s\leq\mu\wedge\tau}\left\langle e_{k,N}, \left(T_N(t-s)\delta Z_D(s)\right)^2\right\rangle - N\left\langle e_{k,N}, F(X)(t,\mu\wedge\tau)\right\rangle$$

is a mean 0 martingale. Note $\langle 1, (T_N(t-s)\delta Z_D(s))^2\rangle \leq \langle 1, (\delta Z_D(s))^2\rangle \leq 2(Nl^2)^{-1}$. Thus,

$$
\begin{aligned}
Em^2(\mu) &= EN^2\sum_{s\leq\mu\wedge\tau}\left\langle e_{k,N}, \left(T_N(t-s)\delta Z_D(s)\right)^2\right\rangle^2 \\
&\leq 4N^2\left(Nl^2\right)^{-1}E\sum_{s\leq\mu\wedge\tau}\left\langle 1, \left(T_N(t-s)\delta Z_D(s)\right)^2\right\rangle \\
&\leq 4N^2\left(Nl^2\right)^{-1}cl^{-1} = cN^{-2}
\end{aligned}
$$

because $N = l$ and where the last inequality follows from Lemma 3.1. Thus, $Em^2(t) \to 0$ as $N \to \infty$ and the proof will be complete after showing $N\langle e_{k,N}, F(X)(t, t\wedge\tau)\rangle \to \langle e_k, \psi(t)\rangle$ in probability.

We consider $\langle \varphi_{k,N}, F(f)(t,\mu)\rangle$ using the basic facts listed after (5.1). Using facts (c) and (g), it follows that $\langle \varphi_{k,N}, \varphi_{m,N}\psi_{n,N}\rangle \equiv 0$, so we need only consider the first two inner products on the right-hand side of (5.25). Note that only terms with $m, n \neq 0$ contribute to the sum. The remaining two inner products sum to

$$\langle f(s), \varphi_{m-n,N}\rangle 2(a_{m,N}a_{n,N} + b_{m,N}b_{n,N})\varphi_{m-n,N} + \langle f(s), \varphi_{m+n,N}\rangle 2(a_{m,N}a_{n,N} - b_{m,N}b_{n,N})\varphi_{m+n,N},$$

after applying (a), (b), (d) and (e).
Fact (g) shows that

$$\langle \varphi_{k,N}, \varphi_{m-n,N}\rangle = \begin{cases} 1, & \text{if } |m-n| = k, \\ 0, & \text{else,} \end{cases}$$

$$\langle \varphi_{k,N}, \varphi_{m+n,N}\rangle = \begin{cases} 1, & \text{if } m+n = k \text{ or } m+n = 2N-k, \\ 0, & \text{else.} \end{cases}$$

From this we obtain

$$\begin{aligned}\langle \varphi_{k,N}, F(f)(t,u)\rangle &= 2(Nl)^{-1}\int_0^u \sum_{m \le N} \exp\bigl(-(\beta_{m,N} + \beta_{m+k,N})(t-s)\bigr) \\ &\quad \times 2(a_{m,N}a_{m+k,N} + b_{m,N}b_{m+k,N})\langle f(s), \varphi_{k,N}\rangle\, ds \\ &\quad + (Nl)^{-1}\int_0^u \sum_{\substack{m+n=k \\ m+n=2N-k \\ m,n\le N}} \exp\bigl(-(\beta_{m,N} + \beta_{n,N})(t-s)\bigr) \\ &\quad \times 2(a_{m,N}a_{n,N} - b_{m,N}b_{n,N})\langle f(s), \varphi_{k,N}\rangle\, ds.\end{aligned}$$

Basic calculations show

$$\frac{2(a_{m,N}a_{m+k,N} + b_{m,N}b_{m+k,N})}{\beta_{m,N} + \beta_{m+k,N}} = 1 + c(N,m,k)$$

$$\text{where } |c(N,m,k)| \le ck(m+k)^{-1}.$$

From (2.3), basic facts (f) and (g) and the previous two equations, it follows that

$$\langle \varphi_{k,N}, F(f)(t,\mu)\rangle = 2(Nl)^{-1}\left[\sum_{m+k\le N}\int_0^\mu \exp\bigl(-(\beta_{m,N} + \beta_{m+k,N})(t-s)\bigr) \times (\beta_{m,N} + \beta_{m+k,N})\langle f(s), \varphi_{k,N}\rangle\, ds\right] + \varepsilon_{N,k}(f,\mu), \tag{5.26}$$

where $|\varepsilon_{N,k}(f,\mu)| \le c(k)(\log N)(Nl)^{-1}\sup_{s\le\mu}|\langle f(s), \varphi_{k,N}\rangle|$.

Analogous calculations show (5.26) holds with $\psi_{k,N}$ in place of $\varphi_{k,N}$. Using (5.6), (5.10), (5.16), $N = l$, (5.26) and $\|\varphi_{k,N} - \varphi_k\|_\infty \to 0$ shows the limit of $N\langle \varphi_{k,N}, F(X)(t, t\wedge\tau)\rangle$ is the same as

$$\lim_{N\to\infty} 2N^{-1}\sum_{m+k\le N}\int_0^t \exp\bigl(-(\beta_{m,N} + \beta_{m+k,N})(t-s)\bigr)(\beta_{m,N} + \beta_{m+k,N})\langle \psi(s), \varphi_k\rangle\, ds.$$

However, this limit is $\langle\psi(t),\varphi_k\rangle$ by (2.3) and the continuity of $\langle\psi(\cdot),\varphi_k\rangle$. The same holds for $\psi_{k,N},\psi_k$. □

6. Law of large numbers with a constant density parameter. Let $R(x) = b(x) - d(x)$, where $b(x) = b_1x + b_0$, $d(x) = d_2x^2 + d_1x$, $d_2 > 0$ and b_1, b_0, $d_1 \geq 0$. For any value of l, the density parameter, let

$$R_l(x) = R(x) - l^{-1}d_2x. \tag{6.1}$$

Let ψ_l denote the solution of (2.1) with $S = [0,1]$, $\psi_l(0) = \psi(0)$ continuous and R_l in place of R. X is the stochastic model with reaction polynomial $R(x)$. In this section, we assume:

(A1) $P(\|X(0)\|_0 \leq c) \to 1$ for some $c < \infty$.

(A2) $|\langle X(0)-\psi(0),\varphi_m\rangle| + |\langle X(0)-\psi(0),\psi_m\rangle| \to 0$ in probability for each fixed m as $N\to\infty$.

(A3) l is constant.

THEOREM 6.1. (A1)–(A3) *imply that*

$$\sup_{t\leq T}\|X(t)-\psi_l(t)\|_{-\alpha}\to 0$$

in probability as $N\to\infty$ *for any* $\alpha > 0$.

The proof of Theorem 6.1 will follow from preliminary results and discussion. (A1) and (A2) imply

$$\|X(0)-\psi(0)\|_{-\alpha}\to 0 \quad \text{in probability } \forall\,\alpha > 0 \tag{6.2}$$

and by conditioning on $X(0)$, we may assume

$$P\big(\|X(0)\|_0 \leq c\big) = 1 \quad \text{for } c < \infty. \tag{6.3}$$

LEMMA 6.1. *There exists* $a = a(l,T) < \infty$ *such that* $P(\sup_{t\leq T}\|X(t)\|_0 < a)\to 1$ *as* $N\to\infty$.

PROOF. Let X^b denote the stochastic model where there is no "death" reaction ($d\equiv 0$), but only diffusion, immigration and "births" according to $b(x) = b_1x + b_0$. If $X^b(0) = X(0)$, we may couple X and X^b so that $P(X(t)\leq X^b(t), t\geq 0) = 1$. This may be done as follows. Run X and consider the initial particles as painted white. Between deaths, particles independently do random walks and give birth, with the offspring placed in the same cell as the parent. Particles also are independently added to the system as determined by b_0. Each time a white particle dies, paint the particle black, but let it remain in the system where it may move and give birth independently of other particles according to the same rules. However, its offspring are also painted black and black particles do not interact with white ones. X is determined by the number of white particles in each cell and X^b is determined by the total number of particles in each cell, black and white. By construction, $X \leq X^b$. Because this implies $\|X(t)\|_0 \leq \|X^b(t)\|_0$, it suffices to prove the statement of the lemma for X^b. We have

$$X^b(t) = T_N(t)X^b(0) + b_1 \int_0^t T_N(t-s)X^b(s)\,ds + b_0 t + Y^b(t),$$

which implies

$$\|X^b(t)\|_0 \leq \|X^b(0)\|_0 + b_1 \int_0^t \|X^b(s)\|_0\,ds + b_0 t + \|Y^b(t)\|_0. \tag{6.4}$$

By (6.3), (6.4) and Gronwall's inequality,

$$\sup_{t\leq T} \|X^b(t)\|_0 \leq \exp(b_1 T)\left(c + b_0 T + \sup_{t \leq T} \|Y^b(t)\|_0\right),$$

so it suffices to prove the result for $\sup_{t\leq T} \|Y^b(t)\|_0$. Let $\tilde{\tau} = \inf\{t: \|Y^b(t)\|_0 \geq a\}$. Using Gronwall's inequality again,

$$\sup_{t\leq T} \|X^b(t\wedge\tilde{\tau}-)\| \leq \exp(b_1 T)(c + b_0 T + a). \tag{6.5}$$

We use the notation developed for the proof of Lemma 3.5. Consider

$$\begin{aligned} P\left(\sup_{t\leq T}\|Y^b(t)\|_0 > a\right) &\leq P\left(\sup_{t\leq T}\sum_{m\in B_0}\langle Y^b(t\wedge\tilde{\tau}), f_{m,N}\rangle^2 \geq a^2/2\right) \\ &\quad + P\left(\sup_{t\leq T}\sum_{m\notin B_0}\langle Y^b(t\wedge\tilde{\tau}), f_{m,N}\rangle^2 \geq a^2/2\right) \\ &\leq (Nl)^{-1}\left[C(T)(1+a)(\log N)^6\right]a^{-2} \\ &\quad + C(T)N^3(\log N)^{-1}\left(C_1(T)a^2 l/(1+a)\right)^{-\log N}, \end{aligned}$$

where $C_1(T) > 0$ and we have applied Lemma 3.2 (c) and (d) to the sum over B_0 and Lemma 3.4(b) to the sum over $m \notin B_0$. The first term on the right converges to 0 for any a, and the second term converges to 0 for $a = a(l, T)$ chosen large enough. $\square$

Fix $T > 0$ and $a = a(l,T) < \infty$ satisfying Lemma 6.1 and let $\tau = \inf\{t: \|X(t)\|_0 \geq a\}$. As in Section 5, let $\overline{X}$ denote the solution of

$$\overline{X}(t) = X(0) + \int_0^t \Delta_N \overline{X}(s)\,ds + \int_0^t R(\overline{X}(s))\,ds + \overline{Z}(t), \tag{6.6}$$

where $\overline{Z}(t) = Z(t\wedge\tau)$. We have

$$P(\overline{X}(t) \neq X(t), 0 \leq t \leq T) \leq P(\tau \leq T) \to 0, \tag{6.7}$$

so we consider $\overline{X}$ in place of X.

Note $\|\delta X(t)\|_0 \le (2/(Nl^2))^{1/2} \to 0$ as $N \to \infty$. Thus, we may assume $\|X(t\wedge\tau)\|_0 \le a+1$ and $\|\overline{X}(t)\|_0 = \|X(t)\|_0 \le a+1$ for $t \le \tau$. If $t < \infty$ and $t \ge \tau$, then

$$0 \le \overline{X}(t) = T_N(t-\tau)X(\tau) + \int_\tau^t T_N(t-s)R\big(\overline{X}(s)\big)ds$$
$$\le T_N(t-\tau)X(\tau) + \int_\tau^t T_N(t-s)b\big(\overline{X}(s)\big)ds.$$

This implies

$$\|\overline{X}(t)\|_0 \le (a+1) + b_1\int_\tau^t \|\overline{X}(s)\|_0\,ds + b_0(t-\tau).$$

By Gronwall's inequality and our previous discussion, this shows

$$\sup_{t\le T}\|\overline{X}(t)\|_0 \le \overline{a}(l,T) < \infty. \tag{6.8}$$

Applying variation of constants, we have

$$\overline{X}(t) = T_N(t)X(0) + \int_0^t T_N(t-s)R\big(\overline{X}(s)\big)ds + \overline{Y}(t), \tag{6.9}$$

where

$$\overline{Y}(t) = \overline{Y}_R(t) + \overline{Y}_D(t) = \int_0^t T_N(t-s)\,d\overline{Z}_R(s) + \int_0^t T_N(t-s)\,d\overline{Z}_D(s).$$

Let $V(t) = T_N(t)\psi(0) + \int_0^t T_N(t-s)R(\overline{X}(s))\,ds$. Then

$$\overline{X}(t) = V(t) + T_N(t)\big(X(0)-\psi(0)\big) + \overline{Y}(t) \tag{6.10}$$

and

$$\begin{aligned} V(t) = {} & T_N(t)\psi(0) + \int_0^t T_N(t-s)R\big(V(s)\big)ds - d_2\int_0^t T_N(t-s)\overline{Y}_D^2(s)\,ds \\ & + \int_0^t T_N(t-s)\varepsilon(s)\,ds, \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} \varepsilon(s) = {} & -d_2\Big[2\overline{Y}_D(s)\overline{Y}_R(s) + \overline{Y}_R^2(s) + \Big(T_N(s)\big(X(0)-\psi(0)\big)\Big)^2\Big] \\ & + (b_1-d_1)\Big[T_N(s)\big(X(0)-\psi(0)\big) + \overline{Y}(s)\Big] \\ & - 2d_2\Big[T_N(s)\big(X(0)-\psi(0)\big)\big(V(s)+\overline{Y}(s)\big) + V(s)\overline{Y}(s)\Big]. \end{aligned} \tag{6.12}$$

LEMMA 6.2. (a) $\sup_{t\le T}\|\overline{Y}(t)\|_0 \le a(l,T) < \infty$.

(b) $\sup_{t\le T}\|\overline{Y}(t)\|_{-\alpha} \to 0$ *in probability for* $\alpha > 0$.

(c) $\{\overline{X}_N\}$ *is relatively compact in* $D([0,T]; H_{-\alpha})$ *for any* $\alpha > 0$ *and any distributional limit* g *of* $\{\overline{X}_N\}$ *satisfies* $g \in C([0,T]; C([0,1]))$ *almost surely.*

(d) *For* $e_{m,N} = \varphi_{m,N}$ *or* $\psi_{m,N}$,

$$\sup_{t\le T}\left|\left\langle \int_0^t T_N(t-s)\varepsilon(s)\,ds, e_{m,N}\right\rangle\right| \to 0 \quad \textit{in probability}.$$

PROOF. Let

$$\begin{aligned}\widetilde{V}(t) &= T(t)\psi(0) + \int_0^t T(t-s)R\big(\overline{X}(s)\big)\,ds\\ &= T(t)\psi(0) + \sum_m \int_0^t \exp\big(-\beta_m(t-s)\big)\\ &\qquad\times\Big(\Big\langle\varphi_m, R\big(\overline{X}(s)\big)\Big\rangle\varphi_m + \Big\langle\psi_m, R\big(\overline{X}(s)\big)\Big\rangle\psi_m\Big)ds.\end{aligned}$$

By (6.8),

$$\begin{aligned}&\sup_{t\le T}\left\|\int_0^t \exp\big(-\beta_m(t-s)\big)\Big(\Big\langle\varphi_m, R\big(\overline{X}(s)\big)\Big\rangle\varphi_m + \Big\langle\psi_m, R\big(\overline{X}(s)\big)\Big\rangle\psi_m\Big)\right\|_\infty\\ &\quad\le a(l,T)(1+m^2)^{-1} \text{ and for } 0\le t_1\le t_2\le T,\\ &\quad\int_{t_1}^{t_2}\Big(\Big|\Big\langle\varphi_m, R\big(\overline{X}(s)\big)\Big\rangle\Big| + \Big|\Big\langle\psi_m, R\big(\overline{X}(s)\big)\Big\rangle\Big|\Big)\,ds \le a(l,T)(t_2-t_1).\end{aligned}$$

Because $\sum_m(1+m^2)^{-1} < \infty$, $\{\widetilde{V}\}$ is relatively compact in $C([0,T]; C([0,1]))$ by Ascoli's theorem. Also, (6.8) and $\|\varphi_{m,N}-\varphi_m\|_\infty + \|\psi_m - \psi_{m,N}\|_\infty + |\beta_{m,N}-\beta_m| \to 0$ imply $\sup_{t\le T}\|V(t)-\widetilde{V}(t)\|_\infty \to 0$ in probability using calculations similar to the previous calculations for V. This proves (c) with V in place of $\overline{X}$, and (c) will follow from (b) and (6.10).

Estimates similar to those in the preceding text with V in place of $\widetilde{V}$ show $\sup_{t\le T}\|V(t)\|_\infty \le a(l,T)$. By (6.8), (6.10) and our assumptions, (a) holds. (b) follows from (a), (6.8), Lemma 3.2(c) and (d) (with $\overline{Z}$ in place of Z) and (5.1).

Now we prove (d). Recall the norm $\|\cdot\|_{\alpha,N}$ defined in Section 5. The foregoing estimates using V in place of $\widetilde{V}$ show, for $t > 0$, $\|V(t)\|_{\alpha,N} \le c(t,l,\alpha) < \infty$ for $0 < \alpha < \frac{1}{2}$. Fix $\alpha \in (0,\frac{1}{2})$. Then $|\langle e_{m,N}V(t), \overline{Y}(t)\rangle| \le c(m,t,l)\|\overline{Y}(t)\|_{-\alpha} \to 0$ in probability by (b). Also, $|\langle e_{m,N}V(t), \overline{Y}(t)\rangle| \le \sqrt{2}\|V(t)\|_0\|\overline{Y}(t)\|_0 \le a(l,T)$. Thus,

$$\begin{aligned}&E\left(\sup_{t\le T}\left|\left\langle e_{m,N}, \int_0^t T_N(t-s)\big(V(s)\overline{Y}(s)\big)ds\right\rangle\right|\right)\\ &\quad\le \int_0^T E\big|\langle e_{m,N}, V(s)\overline{Y}(s)\rangle\big|\,ds \to 0.\end{aligned}$$

In a similar way, the remaining terms in (6.12) can be dealt with using L_2 estimates, (a) and the following facts:

(i) $\|T_N(s)(X(0)-\psi(0))\|_0 \le \|X(0)-\psi(0)\|_0 \le c$,

(ii) $\|T_N(s)(X(0)-\psi(0))\|_0 \to 0$ in probability for $s>0$ by our assumptions,

(iii) $E\|\overline{Y}_R(t)\|_0^2 \le c(t,l)N^{-1}$ by Lemma 3.2(b) (with $\overline{Z}$ in place of Z),

(iv) $E\|\overline{Y}_D(t)\|_0^2 \le c(t,l)$ by Lemma 3.2(b). □

Let g be a distributional limit of $\overline{X}$. As in the proof of Theorem 4.1, we may apply the Skorohod representation theorem to assume without loss of generality that $\overline{X} \to g$ a.s. in $D([0,T]; H_{-\alpha})$ for any $\alpha>0$. Also, by (6.10) and the proof of Lemma 6.2(c), we may assume

$$\sup_{t\le T}\|V(t)-g(t)\|_\infty \to 0 \quad \text{a.s.} \tag{6.13}$$

We need a final lemma before completing the proof of Theorem 6.1.

LEMMA 6.3. *For $e_{m,N}=\varphi_{m,N}$ or $\psi_{m,N}$,*

$$\sup_{t\le T}\left|\left\langle e_{m,N}, \int_0^t T_N(t-s)\Big(\overline{Y}_D^2(s)-l^{-1}g(s)\Big)\,ds\right\rangle\right| \to 0$$

in probability.

PROOF. Recall $\overline{Y}_D(t)=\int_0^t T_N(t-s)\,d\overline{Z}_D(s)$. By Itô's formula,

$$\begin{aligned}\overline{Y}_D^2(t) &= 2\int_0^t \big(T_N(t-s)\overline{Y}_D(s-)\big)\big(T_N(t-s)\,d\overline{Z}_D(s)\big)\\ &\quad+\sum_{s\le t}\big(T_N(t-s)\,\delta\overline{Z}_D(s)\big)^2.\end{aligned} \tag{6.14}$$

It follows from Lemma 3.1 that

$$\begin{aligned}&E\left[\left\langle e_{m,N}, \int_0^t \big(T_N(t-s)\overline{Y}_D(s-)\big)\big(T_N(t-s)\,d\overline{Z}_D(s)\big)\right\rangle^2\right]\\ &\quad=(Nl)^{-1}E\int_0^{t\wedge\tau}\Big\langle X(s),\\ &\qquad\qquad\sum_{i=+,-}\Big[\nabla^i T_N(t-s)\big(e_{m,N}T_N(t-s)\overline{Y}_D(s)\big)\Big]^2\Big\rangle\,ds.\end{aligned} \tag{6.15}$$

For $s \le t \wedge \tau$, consider (note $\|X(s)\|_\infty \le N^{1/2}\|X(s)\|_0$)

$$\begin{aligned}
&\left\langle X(s), \left(\nabla^+ T_N(t-s)\left(e_{m,N} T_N(t-s)\overline{Y}_D(s)\right)\right)^2\right\rangle \\
&\quad \le N^{1/2}\|X(s)\|_0 \left\|\nabla^+ T_N(t-s)\left(e_{m,N} T_N(t-s)\overline{Y}_D(s)\right)\right\|_0^2 \\
&\quad \le N^{1/2}\|X(s)\|_0 \left\|\nabla^+ \left(e_{m,N} T_N(t-s)\overline{Y}_D(s)\right)\right\|_0^2 \\
&\quad \le a(l,T) N^{1/2} \sum_n (n+m+1)^2 \exp\left(-cn^2(t-s)\right) \\
&\qquad \times \left(\left\langle \overline{Y}_D(s), \varphi_{n,N}\right\rangle^2 + \left\langle \overline{Y}_D(s), \psi_{n,N}\right\rangle^2\right),
\end{aligned}$$

where the last inequality follows after applying the basic facts listed after (5.1). By the proof of Lemma 3.2(a), $E(\langle \overline{Y}_D(s), \varphi_{n,N}\rangle^2 + \langle \overline{Y}_D(s), \psi_{n,N}\rangle^2) \le c(Nl)^{-1}$ and the expectation on the left side of (6.15) is dominated by $a(l,T,m)N^{-1/2}$. This shows we need only consider the last term on the right-hand side of (6.14) to prove the lemma.

Recall $F(f)(t,u)$ defined in (5.25). It was shown in the proof of Lemma 5.3 that

$$\begin{aligned}
&E\left[\left\langle e_{m,N}, F(X)(t, t\wedge\tau) - \sum_{s\le t} \left(T_N(t-s)\,\delta\overline{Z}_D(s)\right)^2\right\rangle^2\right] \\
&\quad = E\left[\sum_{s\le t} \left\langle e_{m,N}, \left(T_N(t-s)\delta\overline{Z}_D(s)\right)^2\right\rangle^2\right] \\
&\quad \le 4\left(Nl^2\right)^{-1} E\left[\sum_{s\le t} \left\langle 1, \left(T_N(t-s)\delta\overline{Z}_D(s)\right)^2\right\rangle\right] \\
&\quad = 4\left(Nl^2\right)^{-1} E\left[\|\overline{Y}_D(t)\|_0^2\right] \le a(l,T)\left(Nl^3\right)^{-1} \to 0,
\end{aligned}$$

where the last line follows from (6.14) and Lemma 3.2(a), respectively. The proof is completed by basic calculations using (5.26), (6.10), (6.13), Lemma 3.2(c), (6.7) and the continuity of g. □

PROOF OF THEOREM 6.1. By (6.11), (6.13), Lemma 6.2(c) and (d) and Lemma 6.3, we have $P(\langle h(t), f\rangle = 0 \text{ for all } t \ge 0, f \in \{\varphi_m, \psi_m\}_{m=0}^\infty) = 1$, where $h(t) = g(t) - T(t)\psi(0) - \int_0^t T(t-s)R_l(g(s))\,ds$. Because $h \in C([0,T]; C([0,1]))$ a.s., it follows that $h \equiv 0$ a.s. and the proof is complete by previously noted uniqueness results. □

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