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LIM INF RESULTS FOR GAUSSIAN SAMPLES AND CHUNG'S FUNCTIONAL LIL[1]

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Lim inf results for Gaussian samples are obtained. In particular, there are applications to rates of convergence for the functional form of Chung's law of the iterated logarithm for Brownian motion.

1. Introduction. Let $\{W(t): t \geq 0\}$ be Brownian motion in R^1 and assume $H_\mu \subseteq C[0,1]$ is the Hilbert space of absolutely continuous functions on $[0,1]$ whose unit ball is the set

$$K = \left\{ f(t) = \int_0^t f'(s)\,ds,\ 0 \leq t \leq 1\colon \int_0^1 |f'(s)|^2\,ds \leq 1 \right\}. \tag{1.1}$$

Here the inner product norm is given by

$$\|f\|_\mu = \left(\int_0^1 |f'(s)|^2\,ds \right)^{1/2}, \qquad f \in H_\mu, \tag{1.2}$$

and $\|f\|_\infty$ denotes the usual sup-norm on $C[0,1]$. If

$$\eta_n(t) = W(nt)/(2nL_2 n)^{1/2}, \qquad 0 \leq t \leq 1,$$

then the functional form of Chung's law of the iterated logarithm, given in [1] and in more refined form in [3], implies for each f in $C[0,1]$ that with probability 1,

$$\liminf_{n\to\infty} L_2 n \|\eta_n - f\|_\infty = \begin{cases} \pi/4 \cdot \left(1 - \|f\|_\mu^2\right)^{-1/2}, & \text{if } \|f\|_\mu < 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.3}$$

Here and throughout $Lx = \max(1, \log_e x)$ and $L_2 x = L(Lx)$.

A natural question that arises is what is the precise behavior when $\|f\|_\mu = 1$?

If f is a piecewise linear or quadratic function, then the results in [1] and [2] yield

$$\liminf_{n\to\infty} (L_2 n)^{2/3} \|\eta_n - f\|_\infty = \gamma(f), \tag{1.4}$$

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where $0 < \gamma(f) < \infty$. Also, Theorem 2 in [4] easily implies that with probability 1,

$$\liminf_{n \to \infty} (L_2 n)^{2/3} \|\eta_n - f\|_\infty < \infty \tag{1.5}$$

for all $f \in K$, and recently more definitive results were obtained by Grill [5]. For example, an interesting consequence of [5] implies that (1.4) holds with $0 < \gamma(f) < \infty$ whenever $\|f\|_\mu = 1$ and f' is of bounded variation on $[0,1]$. Furthermore, if $\|f\|_\mu = 1$, but f' is not of bounded variation on $[0,1]$, then Corollary 2 of [5] states that with probability 1,

$$\liminf_{n \to \infty} (L_2 n)^{2/3} \|\eta_n - f\|_\infty = 0. \tag{1.6}$$

Grill's paper also obtains specific examples of f, with $\|f\|_\mu = 1$, which interpolate between the power $2/3$ in (1.4), and the power 1 in (1.3).

In view of what are now rather well understood rescaling ideas, (1.4) with $0 < \gamma(f) < \infty$ is equivalent to proving that the series

$$\sum_{n \geq 1} P\Big(\big\|W - (2Ln)^{1/2} f\big\|_\infty \leq C(Ln)^{-1/6} \Big) \tag{1.7}$$

converges for some $C > 0$ and diverges for some other constant $C < \infty$. Hence, throughout the remainder of the paper we will concentrate on obtaining information regarding (1.7) for general Gaussian random vectors. This, along with the Borel–Cantelli lemma, allows us to obtain lim inf results for sequences of i.i.d. Gaussian vectors with values in a separable Banach space. When appropriate, we will include remarks regarding the corresponding analogues for scaled Brownian motion.

In Section 2, we recall some basic facts about Gaussian random vectors, and in Section 3 we state the first results of the paper along with some remarks relating to [5] and the remaining sections of the paper. Section 4 contains the proof of Theorem 1, and Section 5 examines the functional

$$I(f, \delta) = \inf_{\|f - g\| \leq \delta} \|g\|_\mu^2.$$

In particular, we examine the asymptotic behavior of $(\|f\|_\mu^2 - I(f,\delta))/\delta$ as $\delta \to 0$ for Gaussian measures. These results, combined with Theorem 1, characterize precisely those points f on the boundary of K, that is, with $\|f\|_\mu = 1$, which are approached slowest by the random sequence $\{X_n/(2Ln)^{1/2}\}$. For example, if μ is Wiener measure on $C[0,1]$, then from (1.1) and (1.2) we have for each f in H_μ that f' is in $L^2[0,1]$, but from (1.6) and the surrounding discussion, f' being of bounded variation is also relevant. However, because f' may fail to exist everywhere on $[0,1]$, to characterize those points on the boundary of K approached most slowly, we will see that one needs to examine if there is a version of f' that has bounded variation on $[0,1]$. By a version of f' we mean a measurable function g defined everywhere on $[0,1]$, and such that $g = f'$

almost everywhere with respect to Lebesgue measure. These subtleties were not discussed in [5], but to be precise (1.6) does require some clarification. That is, if f is piecewise linear, then f' does not exist at those points in $[0, 1]$ where the graph of f has a corner, and hence f' is not of bounded variation. On the other hand, for piecewise linear f with $\|f\|_\mu = 1$ we know (1.4) holds and hence (1.6) fails. Of course, in this example if the graph of f has only finitely many corners, then f' has an obvious version (extension) that is of bounded variation, but what is the general situation? Section 5 clarifies the matter for μ Wiener measure, as well as for other Gaussian measures.

In Section 6 we introduce some comparison principles for two points on the boundary of K to have a common rate of approach by the sequence $\{X_n/(2Ln)^{1/2}\}$. In Section 7 we obtain both the exact constant, as well as the rate of convergence, for some examples in Hilbert space. These results apply to the measure induced on $L^2[0, 1]$ by the usual series representation for Brownian motion, and in this situation the exact constant has a universal form. This is surprising because we doubt that this is the case when μ is Wiener measure on $C[0, 1]$ and distances are computed in the sup-norm.

Throughout we write $a_n \sim b_n$ when $\lim_{n\to\infty} a_n/b_n = 1$, and $a_n \approx b_n$ if there is a constant C, $1 < C < \infty$, such that

$$1/C \le \liminf_{n\to\infty} a_n/b_n \le \limsup_{n\to\infty} a_n/b_n \le C. \tag{1.8}$$

Similar notation is also used for functions.

2. Gaussian random vectors.

Let B denote a real separable Banach space with norm $\|\cdot\|$ and topological dual B^*. If X is a centered Gaussian random vector with values in B and $\mu = \mathcal{L}(X)$, then it is well known that there is a unique Hilbert space $H_\mu \subseteq B$ such that μ is determined by considering the pair (B, H_μ) as an abstract Wiener space (see [6]). For example, if $B = C[0, 1]$ and μ is Wiener measure, then the unique Hilbert space H_μ has unit ball K as in (1.1) with inner product norm given by (1.2).

In general, H_μ can be described as the completion of the range of the mapping $S: B^* \to B$ defined via the Bochner integral

$$Sf = \int_B x f(x)\, d\mu(x), \qquad f \in B^*, \tag{2.1}$$

and the completion is in the inner product norm

$$\langle Sf, Sg\rangle_\mu = \int_B f(x) g(x)\, d\mu(x), \qquad f, g \in B^*. \tag{2.2}$$

Lemma 2.1 in [7] presents the details of this construction along with various properties of the relationship between H_μ and B. In particular, if $\{\alpha_k: k \ge 1\}$ is a sequence in B^* orthonormal in $L^2(\mu)$ such that $\{S\alpha_k: k \ge 1\}$ is a CONS in $H_\mu \subseteq B$, then the operators defined for $d \ge 1$ by

$$\Pi_d(x) = \sum_{k=1}^{d} \alpha_k(x) S\alpha_k \quad \text{and} \quad Q_d(x) = x - \Pi_d(x) \tag{2.3}$$

are continuous mappings from B to B. Furthermore, when restricted to H_μ, Π_d and Q_d are orthogonal projections onto their ranges. It is also known that for μ centered Gaussian, $\lim_{d\to\infty}\|Q_d(x)\| = 0$ with μ-probability 1, and that for $f \in H_\mu$ we can define the stochastic inner product for μ almost all x in B by

$$\langle x,f\rangle^{\sim} = \lim_{d\to\infty}\sum_{k=1}^{d}\alpha_k(x)\langle f,S\alpha_k\rangle_\mu. \tag{2.4}$$

Because $\langle f,S\alpha_k\rangle_\mu = \alpha_k(f)$, it is easy to see that $\langle\cdot,f\rangle^{\sim}$ is $N(0,\sigma^2)$ with $\sigma^2 = \langle f,f\rangle_\mu$. Furthermore, if $f = Sh$ for some $h \in B^*$, then we actually have with μ-probability 1 that

$$\langle x,f\rangle^{\sim} = h(x). \tag{2.5}$$

The Cameron–Martin translation formula for Gaussian measures states that

$$\mu(E+f) = \int_E \exp\{-\tfrac{1}{2}\|f\|_\mu^2 - \langle x,f\rangle^{\sim}\}\,d\mu(x) \tag{2.6}$$

for Borel subsets E of B. This is well known, but a particularly nice proof is contained in Proposition 2.1 of [3]. When the Banach space B is a Hilbert space H, we will frequently identify the dual $B^* = H^*$ with H itself without further comment. Also, it is well known that the support of μ is the closure of H_μ in the B-norm. We denote this by writing $\operatorname{supp}(\mu) = \overline{H}_\mu$.

3. Statement of Theorem 1 and some remarks. The formulation of (1.4), (1.5) and (1.6) in the setting of Gaussian samples is the content of our first theorem. First, however, we recall the I-function defined for $f \in B$ and $\delta \geq 0$ by

$$I(f,\delta) = \inf_{\|f-g\|\leq\delta}\|g\|_\mu^2. \tag{3.1}$$

Here $\|g\|_\mu$ is $+\infty$ for $g \notin H_\mu$ and $\|\cdot\|$ is the norm on B. Of course, $I(f,0) = \|f\|_\mu^2$, which is finite iff $f \in H_\mu$. The analogue of (3.1) when μ is Wiener measure was used extensively in [5] (also see (2.6) in [4]), and we include some further comments on these matters following the statement of Theorem 1. Of course, $I(f,\delta)$ also appears in large deviation literature on Gaussian measures.

THEOREM 1. *Let $X, X_1, X_2, \ldots$ be i.i.d. B-valued centered Gaussian random vectors with $\mu = \mathcal{L}(X)$. Let*

$$\psi(\varepsilon) = \log\mu\big(x\colon \|x\| \leq \varepsilon\big), \qquad \varepsilon > 0, \tag{3.2}$$

and assume $d = d(n)$ is the unique solution to the equation

$$\psi\big(d(2Ln)^{1/2}\big) + \sigma^{-1}\,dL\,n = 0, \tag{3.3}$$

where

$$\sigma^2 = \sup_{\|h\|_{B^*} \le 1} E\big(h^2(X)\big).$$

Then the following hold:

(i) *If* $\|f\|_\mu \le 1$, *then with probability* 1,

$$\liminf_{n\to\infty} \big(d(n)\big)^{-1} \|X_n/(2Ln)^{1/2} - f\| \le 2. \tag{3.4}$$

(ii) *If* $\|f\|_\mu = 1$, $f = Sh$, *where* $h \in B^*$, *and* $\psi(\varepsilon) \le -\varepsilon^{-p}$ *for some* $p > 0$ *and all* $\varepsilon > 0$ *sufficiently small*, *then with probability* 1,

$$\liminf_{n\to\infty} \big(d(n)\big)^{-1} \|X_n/(2Ln)^{1/2} - f\| > 0. \tag{3.5}$$

(iii) *If* $\|f\|_\mu = 1$, $\psi(\varepsilon)$ *is such that for each* $\lambda > 0$, *there is a positive constant* c_λ *satisfying*

$$\psi(\lambda\varepsilon) \ge c_\lambda \psi(\varepsilon) \tag{3.6}$$

for all $\varepsilon > 0$ *sufficiently small*, *and*

$$\lim_{\delta\to 0} \frac{1 - I(f,\delta)}{\delta} = \infty, \tag{3.7}$$

then with probability 1,

$$\liminf_{n\to\infty} \big(d(n)\big)^{-1} \|X_n/(2Ln)^{1/2} - f\| = 0. \tag{3.8}$$

REMARK A. The conclusion in (3.4) is the Gaussian sample analogue of (1.5) because for μ Wiener measure,

$$\psi(\varepsilon) \sim -\pi^2/8\varepsilon^2 \quad \text{as } \varepsilon \to 0, \tag{3.9}$$

and hence

$$d = d(n) \approx (Ln)^{-2/3}. \tag{3.10}$$

Furthermore, when μ is Wiener measure, (3.4), (3.5) and rescaling combine to yield (1.4) with $0 < \gamma(f) < \infty$ whenever f' is of bounded variation on [0, 1]. This follows because it is easy to check that if

$$h(x) = x(1)f'(1) - \int_0^1 x(s)\,df'(s), \tag{3.11}$$

where f' is of bounded variation on [0, 1], then $h \in C[0,1]^*$ and $f = Sh$, where S is determined by Wiener measure on $C[0,1]$.

REMARK B. We formulated Theorem 1 in an attempt to find all the points of K that, modulo constants, are approached slowest by the sequence $\{X_n / (2Ln)^{1/2}\}$, and Theorem 1 shows that

$$E = \left\{ f \in K\colon \|f\|_\mu = 1 \text{ and } f \in SB^* \right\} \tag{3.12}$$

is the correct set no matter which Gaussian measure μ on B is being considered provided $\psi(\varepsilon)$ satisfies the conditions in (ii) and (iii). For example, the conclusions in (i) and (ii) combine to show that E is a subset of the points of K that are approached slowest. Furthermore, for any Gaussian measure μ we show in Section 5 that if $\|f\|_\mu = 1$, then $f \notin SB^*$ is equivalent to the condition (3.7). Hence under the minor assumptions on $\psi(\varepsilon)$, (iii) implies that E is the correct set.

REMARK C. The function $\psi(\varepsilon)$ is hard to compute, but if $\psi(\varepsilon)$ is regularly varying or slowly varying as $\varepsilon \to 0$, then the required assumptions on $\psi(\varepsilon)$ in (ii) and (iii) all hold.

REMARK D. It is easy to check that all of the properties given in (3)–(7) of Lemma 1 in [5] for Wiener measure also hold when μ is an arbitrary centered Gaussian measure on a separable Banach space. For example, the analogue of property (5) in [5] is that if $g \in H_\mu$ and $\|g - f\| \le \delta$, then

$$\langle g, h\rangle_\mu \ge \langle h, h\rangle_\mu, \tag{3.13}$$

where h is the unique vector in H_μ yielding $I(f, \delta)$. Using these properties and the ideas in [5], it is possible to prove the following modification of Theorem 1 in [5].

THEOREM 1*. *Let $X, X_1, X_2, \ldots$ be i.i.d. B-valued centered Gaussian random vectors with values in B and $\mu = \mathcal{L}(X)$. Let $\psi(\varepsilon)$ be as in* (3.2), *and assume that for $\varepsilon > 0$ sufficiently small the function ψ satisfies both of the following*:

$$\varepsilon^{-q} \le -\psi(\varepsilon) \le \varepsilon^{-p} \quad \textit{for some } p \ge q > 0 \tag{3.14}$$

and for each $\alpha \in (0,1)$ there is a $\beta > 0$ such that

$$-\psi(\varepsilon) + \psi(\alpha\varepsilon) \le \psi(\beta\varepsilon). \tag{3.15}$$

If $f \in K$ and $d = d(n)$ is the unique solution of the equation

$$Ln\big(1 - I(f, d)\big) = -\psi\big((2Ln)^{1/2} d\big), \tag{3.16}$$

then with probability 1,

$$1 \le \liminf_{n\to\infty} \big(d(n)\big)^{-1} \big\|X_n/(2Ln)^{1/2} - f\big\| \le 2. \tag{3.17}$$

REMARK E. Because $\psi(\varepsilon)$ is explicitly known in only a few examples and because the conditions on $\psi(\varepsilon)$ in Theorem 1* are more restrictive than those in Theorem 1, we chose to present the proof of Theorem 1. However, when Theorem 1* is applicable, it presents the rate of approach modulo constants for all $f \in K$.

REMARK F. One of the important computations involving $I(f, \delta)$ in [5] is given in Lemma 2 of that article. This is a beautiful result, and in Section 5 we extend it to arbitrary centered Gaussian measures as well as provide some additional facts when μ is Wiener measure. The analogue of Lemma 2 for arbitrary centered Gaussian measures μ is obtained in Proposition 1 of Section 5. It proves the following:

If $\|f\|_\mu = 1$ and $f = Sh$ for $h \in B^*$, then as $\delta \to 0$,

$$I(f, \delta) = 1 - 2\|h\|_{\tilde{B}^*}\delta + o(\delta), \tag{3.18}$$

where $\|h\|_{\tilde{B}^*} = \sup\{|h(x)|: \|x\| \le 1,\ x \in \overline{H}_\mu\}$.

4. Proof of Theorem 1. First assume $0 < \|f\|_\mu \le 1$. Because it is well known that $\psi(\varepsilon)$ is continuous, strictly increasing and $\lim_{\varepsilon \to 0} \psi(\varepsilon) = -\infty$ (when μ has separable support), it is clear that (3.3) has a unique solution and furthermore that $\lim_{n \to \infty} d^2(n)Ln = 0$. Given f, $0 < \|f\|_\mu \le 1$, set $f_n = f(1 - d(n)/\|f\|)$. Then for n large, $\|f - f_n\| = d(n)$, (2.6) and Jensen's inequality imply

$$\begin{aligned} &P\Big(\|X_n/(2Ln)^{1/2} - f\| \le 2d(n)\Big) \\ &\quad \ge P\Big(\|X_n/(2Ln)^{1/2} - f_n\| \le d(n)\Big) \\ &\quad = \exp\{-\|f_n\|_\mu^2 Ln\} \int_{\{x \in B:\, \|x\| \le d(n)(2Ln)^{1/2}\}} \exp\{-\langle x, f_n\rangle^\sim\}\, d\mu(x) \\ &\quad \ge \exp\Big\{-(Ln)\big(1 - d(n)/\|f\|\big)^2\|f\|_\mu^2 + \psi\big(d(n)(2Ln)^{1/2}\big)\Big\}. \end{aligned} \tag{4.1}$$

From (3.3) and $\|f\|_\mu \le 1$ we thus have

$$\begin{aligned} &P\left(\left\|\frac{X_n}{(2Ln)^{1/2}} - f\right\| \le 2d(n)\right) \\ &\quad \ge \exp\left\{-(Ln)\|f\|_\mu^2 + 2Lnd(n)\frac{\|f\|_\mu^2}{\|f\|} + Lnd^2(n)\frac{\|f\|_\mu^2}{\|f\|^2} - \sigma^{-1}d(n)Ln\right\}. \end{aligned} \tag{4.2}$$

Because $d^2(n)Ln \to 0$ and $\|f\| \le \sigma\|f\|_\mu$ with $\|f\|_\mu \le 1$, (4.2) implies

$$P\Big(\|X_n/(2Ln)^{1/2} - f\| \le 2d(n)\Big) \ge 1/n, \tag{4.3}$$

and hence the Borel–Cantelli lemma yields (3.4) if $0 < \|f\|_\mu \le 1$.

If $\|f\|_\mu = 0$, then (3.3) immediately implies

$$\begin{aligned} P\Big(\|X_n/(2Ln)^{1/2} - f\| \le d(n)\Big) &= \exp\Big\{\psi\big(d(n)(2Ln)^{1/2}\big)\Big\} \\ &= \exp\{-\sigma^{-1}d(n)Ln\} \\ &\ge 1/n \end{aligned}$$

as $d(n) \to 0$. Again the Borel–Cantelli lemma yields (3.4), so Theorem 1(i) is proved.

To prove Theorem 1(ii), assume $\|f\|_\mu = 1$ and $f = Sh$ for $h \in B^*$. Then

$$\begin{aligned} &P\Big(\big\|X_n/(2Ln)^{1/2} - f\big\| \le \alpha d(n)\Big) \\ &\quad = \exp\{-Ln\} \int_{\{\|x\| \le \alpha d(n)(2Ln)^{1/2}\}} \exp\Big\{-\big\langle x, (2Ln)^{1/2} f\big\rangle^{\sim}\Big\}\, d\mu(x) \\ &\quad \le \exp\Big\{-Ln + 2\alpha d(n) Ln \|h\|_{B^*} + \psi\big(\alpha d(n)(2Ln)^{1/2}\big)\Big\} \end{aligned} \tag{4.4}$$

because $\langle x, f\rangle^{\sim} = h(x)$ with probability 1 when $f = Sh$, $h \in B^*$. Now $0 < \alpha < 1$ implies

$$\psi\big(\alpha d(n)(2Ln)^{1/2}\big) \le \psi\big(d(n)(2Ln)^{1/2}\big),$$

and hence (3.3) and (4.4) combine to yield

$$P\Big(\big\|X_n/(2Ln)^{1/2} - f\big\| \le \alpha d(n)\Big) \le \exp\Big\{-Ln + d(n)Ln\big(2\alpha\|h\|_{B^*} - \sigma^{-1}\big)\Big\}. \tag{4.5}$$

Because $\|f\|_\mu = 1$ and $\langle x, f\rangle_\mu = h(x)$ we have $\|h\|_{B^*} \ge \sigma^{-1}$, so taking $\alpha = \sigma^{-1}/(4\|h\|_{B^*})$ we have $0 < \alpha < 1$, and (4.5) implies

$$P\Big(\big\|X_n/(2Ln)^{1/2} - f\big\| \le \alpha d(n)\Big) \le \exp\{-Ln - 2^{-1}\sigma^{-1} d(n) Ln\}. \tag{4.6}$$

This sequence of probabilities converges if $d(n)Ln \ge (Ln)^q$ for some $q > 0$, and hence the Borel–Cantelli lemma will yield (3.5) with limiting constant at least $\sigma^{-1}/(4\|h\|_{B^*})$. Thus it remains to show $d(n)Ln \ge (Ln)^q$ for some $q > 0$.

Because we are assuming $\psi(\varepsilon) \le -\varepsilon^{-p}$ for some $p > 0$, we thus have from (3.3) that

$$d(n) \ge C(Ln)^{-(p+2)/(2(p+1))},$$

where C is some positive constant. Hence $d(n)Ln \ge C(Ln)^{p/(2(p+1))}$ and (3.5) holds.

Now assume (3.6) and (3.7) hold. Arguing as in [5], Lemma 1, there is a unique element g_n in H_μ such that for $\lambda > 0$, $\|g_n - f\| = \lambda d(n)$ and

$$I(f, \lambda d(n)) = \|g_n\|_\mu^2. \tag{4.7}$$

Hence

$$\begin{aligned} &P\Big(\big\|X_n/(2Ln)^{1/2} - f\big\| \le 2\lambda d(n)\Big) \\ &\quad \ge P\Big(\big\|X_n/(2Ln)^{1/2} - g_n\big\| \le \lambda d(n)\Big) \\ &\quad \ge \exp\Big\{-\|g_n\|_\mu^2 Ln + \psi\big(\lambda d(n)(2Ln)^{1/2}\big)\Big\} \\ &\quad \ge \exp\Big\{-\|g_n\|_\mu^2 Ln + c_\lambda \psi\big(d(n)(2Ln)^{1/2}\big)\Big\} \\ &\quad = \exp\Big\{-I\big(f, \lambda d(n)\big)Ln - c_\lambda \sigma^{-1} d(n) Ln\Big\} \\ &\quad \ge \exp\Big\{-\big(1 - M\lambda d(n)\big)Ln - c_\lambda \sigma^{-1} d(n) Ln\Big\}, \end{aligned} \tag{4.8}$$

where the second inequality is by the Cameron–Martin formula and Jensen's inequality, the third inequality follows from (3.6) because $d(n)(2Ln)^{1/2} \to 0$, the equality is by (4.7) and (3.3), and the last inequality is by (3.7) with M arbitrarily large as $d(n) \to 0$. Thus as $n \to \infty$ and $M\lambda > c_\lambda \sigma^{-1}$, we have

$$P\Big(\|X_n/(2Ln)^{1/2} - f\| \le 2\lambda d(n)\Big) \ge 1/n.$$

Applying the Borel–Cantelli lemma again we obtain with probability 1 that

$$\liminf_{n\to\infty} \big(d(n)\big)^{-1} \|X_n/(2Ln)^{1/2} - f\| \le 2\lambda.$$

Because $\lambda > 0$ is arbitrary, this completes the proof of (3.8). Hence Theorem 1 is proved. □

5. Some remarks on $I(f,\delta)$. Here we characterize SB^* in terms of the asymptotic behavior of $I(f,\delta)$ as $\delta \to 0$ for arbitrary Gaussian measures μ on a separable Banach space. The basic results are contained in Propositions 1 and 2, and show that for $\|f\|_\mu = 1$ we have $f \in SB^*$ iff

$$\limsup_{\delta\to 0} \frac{1 - I(f,\delta)}{\delta} < \infty.$$

In particular, this characterization is valid for Wiener measure on $C[0,1]$, and combined with the discussion below it adds some precision to the results in Lemma 2 and Corollary 2 of [5]. Now we turn to a lemma of use in our understanding of the Brownian motion case.

LEMMA 1. *If μ is Wiener measure on $B = C[0,1]$ and $f \in H_\mu$, then*

(5.1) *$f \in SB^*$ iff f' has a version λ such that $V(\lambda) < \infty$.*

PROOF. If f' has a version λ such that $V(\lambda) < \infty$, define $h \in B^*$ by

$$h(x) = x(1)\lambda(1) - \int_0^1 x(s)\,d\lambda(s).$$

Then $Sh(t) = \int_0^t \lambda(s)\,ds$ and because λ is a version of f' we have $\lambda = f'$ a.e. Hence $Sh(t) = \int_0^t f'(s)\,ds$, but $f \in H_\mu$ implies f is absolutely continuous with $f(0) = 0$, so $\int_0^t f'(s)\,ds = f(t)$. Hence $Sh = f$ as required.

If $f \in SB^*$, then $f(t) = Sh(t)$. Now $h \in B^*$ implies there is a function of bounded variation λ_h on $[0,1]$ such that

$$h(x) = \int_0^1 x(s)\,d\lambda_h(s). \tag{5.2}$$

In computing Sh we are free to assume λ_h is right continuous at zero because $x(0) = 0$ with μ-measure 1. Similarly, because $x(s)$ is continuous with

μ-measure 1, we are free to assume λ_h is left continuous on $(0, 1)$. Hence we do this and define

$$\lambda(1) = \lambda_h(1) - \lim_{t \to 1^-} \lambda_h(t),$$
$$\lambda(s) = -\lambda_h(s) + \lim_{t \to 1^-} \lambda_h(t) + \lambda(1), \qquad 0 < s < 1, \tag{5.3}$$
$$\lambda(0) = \lim_{s \to 0^+} \lambda(s).$$

Then λ is of bounded variation on $[0, 1]$, right continuous at zero, left continuous on $(0, 1]$ and

$$h(x) = x(1)\lambda(1) - \int_0^1 x(s)\,d\lambda(s). \tag{5.4}$$

Using (5.4) for $h(x)$ we get $f(t) = Sh(t) = \int_0^t \lambda(s)\,ds$ as above, and hence $f'(t) = \lambda(t)$ a.e. Because λ is of bounded variation, (5.1) is proved. □

The difference between the assumption that f' has a version that is of bounded variation and in saying f' is of bounded variation is revealed in the next lemma. For this it is useful to define $\mathcal{G} = \{f \in H_\mu\colon V(f') < \infty\}$, and we emphasize that for $f \in \mathcal{G}$ we are assuming f' is defined everywhere on $[0, 1]$ and of bounded variation as well. We also recall from Lemma 1 and its proof that $f = Sh$ for $h \in B^*$ implies f' has a version $\lambda \in \mathcal{F}$, where $\mathcal{F}$ denotes the functions of bounded variation on $[0, 1]$ that are normalized to be right continuous at zero and left continuous on $[0, 1]$. Furthermore, by (5.4), h can be written as

$$h(x) = x(1)\lambda(1) - \int_0^1 x(s)\,d\lambda(s), \tag{5.5}$$

and $f(t) = Sh(t) = \int_0^t \lambda(s)\,ds$, $0 \le t \le 1$. Hence

$$SB^* = \left\{f(t) = \int_0^t \lambda(s)\,ds,\ 0 \le t \le 1\colon \lambda \in \mathcal{F}\right\},$$

and the linear functional h defined in (5.5) is such that $\|h\|_{B^*} = |\lambda(1)| + V(\lambda)$.

Now $\mathcal{G} \subseteq SB^*$ because $f \in \mathcal{G}$ implies $f = Sh$, where $h(x) = x(1)f'(1) - \int_0^1 x(s)\,df'(s)$, and the lemma below indicates f' is smooth. Hence (3.18) implies Lemma 2 of [5] with $\|h\|_{B^*} = |f'(1)| + V(f')$. However, $\mathcal{G}$ is a proper subset of SB^* as can be seen from the following lemma.

LEMMA 2. *If $f \in \mathcal{G}$, then f' is continuous on* $[0, 1]$.

PROOF. If f' is of bounded variation on $[0, 1]$, then $f'(t)$ exists at every point of $[0, 1]$. Of course, $f'(0)$ is a right-hand derivative and $f'(1)$ is a left-hand derivative. Furthermore, f' has right- and left-hand limits at all interior points of $[0, 1]$, a right-hand limit at 0 and a left-hand limit at 1. Because $f \in \mathcal{G}$ implies $f(0) = 0$, the mean value theorem applies and $f'(0) = \lim_{s \to 0} f'(\tau(s))$, where $0 \le \tau(s) \le s$.

Hence $f'(0) = \lim_{s\to 0} f'(s)$ and f' is right continuous at 0. Similarly, f' is left continuous at 1 and both left and right continuous at each interior point, so f' is continuous on $[0,1]$. □

Now we turn to the proof of (3.18) for arbitrary centered Gaussian measures. Our first task is a necessary lemma.

LEMMA 3. *Let* $J(f,\delta) = \inf\{\|x\|_\mu: \|f-x\| \le \delta\}$. *Then, for* $f \in \overline{H}_\mu$,

$$J(f,\delta) = \sup\{g(f) - \delta\|g\|_{\tilde{B}^*}: g \in B^*,\ \|g\|_\mu \le 1\}, \tag{5.6}$$

where $\|g\|_\mu^2 = \int_B g^2(x)\,d\mu(x)$ *and* $\|g\|_{\tilde{B}^*} = \sup\{|g(x)|: \|x\| \le 1,\ x \in \overline{H}_\mu\}$.

PROOF. First observe that if $g \in B^*$, $\|g\|_\mu \le 1$, then $\|f-x\| \le \delta$ and $f, x \in \overline{H}_\mu$ imply $|g(f) - g(x)| \le \delta\|g\|_{\tilde{B}^*}$. Hence for $f \in \overline{H}_\mu$, $x \in \overline{H}_\mu$,

$$\|x\|_\mu \ge g(x) \ge g(f) - \delta\|g\|_{\tilde{B}^*}.$$

Because $\|x\|_\mu = \infty$ for $x \notin H_\mu$, taking the sup over g then gives

$$J(f,\delta) \ge \sup\{g(f) - \delta\|g\|_{\tilde{B}^*}: g \in B^*,\ \|g\|_\mu \le 1\}.$$

If $J(f,\delta) = 0$ the result now holds by taking $g = 0$, so assume $J(f,\delta) > 0$ and take $0 < a < J(f,\delta)$. Then $aK \cap \{x: \|f-x\| \le \delta\} = \emptyset$. Because K is compact, the Hahn–Banach theorem implies there is a $g \in B^*$ such that $|g(x)| \le a$ whenever $x \in aK$ and $g(x) > a$ whenever $\|f-x\| \le \delta$ and $x \in \overline{H}_\mu$. Furthermore, because $f \in \overline{H}_\mu$, the Hahn–Banach theorem allows us to have $\|g\|_{B^*} = \|g\|_{\tilde{B}^*}$. Thus $\|g\|_\mu \le 1$ and $g(f+u) > a$ whenever $\|u\| \le \delta$. Hence $g(f) - \delta\|g\|_{\tilde{B}^*} = g(f) - \delta\|g\|_{B^*} > a$. As $a < J(f,\delta)$ is arbitrary, Lemma 3 now holds. □

The proof of (3.18) is the following proposition.

PROPOSITION 1. *Let* $J(f,\delta) = \inf\{\|x\|_\mu: \|f-x\| \le \delta\}$. *Then, when* $\|f\|_\mu = 1$ *and* $f = Sh$ *for* $h \in B^*$, *it follows that*

$$J(f,\delta) = 1 - \delta\|h\|_{\tilde{B}^*} + o(\delta), \tag{5.7}$$

where $\|h\|_{\tilde{B}^*} = \sup\{|h(x)|: \|x\| \le 1,\ x \in \overline{H}_\mu\}$. *In particular, for* $\|f\|_\mu = 1$ *and* $f = Sh$,

$$\lim_{\delta\to 0} \frac{1 - I(f,\delta)}{\delta} = 2\|h\|_{\tilde{B}^*}. \tag{5.8}$$

PROOF. First we deduce (5.7) from Lemma 3. Because $f = Sh \in H_\mu$ with $\|f\|_\mu = 1$, $h \in B^*$, the special choice $g = h$ shows that

$$J(f,\delta) \ge h(f) - \delta\|h\|_{\tilde{B}^*} = 1 - \delta\|h\|_{\tilde{B}^*}. \tag{5.9}$$

Next, consider $g \in B^*$, $\|g\|_\mu \le 1$ and

$$g(f) - \delta\|g\|_{\tilde{B}^*} \ge 1 - \delta\|h\|_{\tilde{B}^*}. \tag{5.10}$$

Then $g(f) \le 1$ and we see that $\|g\|_{\tilde{B}^*} \le \|h\|_{\tilde{B}^*}$. Also $g(f) \ge 1-\delta\|h\|_{\tilde{B}^*}$, so, because $\|g\|_\mu \le 1$, $h(f) = 1$, we have

$$\begin{aligned} \|f-h\|_\mu^2 &= \int_B (g-f)^2\,d\mu \\ &= \int_B g^2\,d\mu + \int_B h^2\,d\mu - 2\int_B gh\,d\mu \\ &\le 2\Big(1 - \int_B gh\,d\mu\Big) \\ &= 2\big(1-g(f)\big) \\ &\le 2\delta\|h\|_{\tilde{B}^*}. \end{aligned} \tag{5.11}$$

Now if $g_k \in B^*$, $\|g_k\|_{\tilde{B}^*} \le 2\|h\|_{\tilde{B}^*}$ and $\|g_k - h\|_\mu^2 \to 0$, then

$$\lim_{k\to\infty} g_k(x) = h(x) \tag{5.12}$$

for all $x \in \overline{H}_\mu$ because the support of μ is $\overline{H}_\mu$.

Now take $\delta_i \to 0$ and choose a sequence $g_i \in B^*$, $\|g_i\|_\mu \le 1$, such that

$$g_i(f) - \delta_i\|g_i\|_{\tilde{B}^*} > J(f,\delta_i) - \delta_i\|h\|_{\tilde{B}^*} \ge 1 - 2\delta_i\|h\|_{\tilde{B}^*}. \tag{5.13}$$

By Lemma 3 such a sequence $\{g_i\}$ exists, and we have $\|g_i\|_{\tilde{B}^*} \le 2\|h\|_{\tilde{B}^*}$. Then by (5.11) and (5.12), we have $\lim_{i\to\infty} g_i(x) = h(x)$ for all $x \in \overline{H}_\mu$. Thus given $x \in \overline{H}_\mu$ and any $\varepsilon > 0$ there is an i_0 sufficiently large so that $g_i(x) \ge h(x) - \varepsilon$ for all $i \ge i_0$. Taking $\|x\| \le 1$, $x \in \overline{H}_\mu$, $h(x) \ge \|h\|_{\tilde{B}^*} - \varepsilon$, we thus have for $i \ge i_0$ that $g_i(x) \ge \|h\|_{\tilde{B}^*} - 2\varepsilon$. Thus for $i \ge i_0$,

$$g_i(f) - \delta_i\|g_i\|_{\tilde{B}^*} \le 1 - \delta_i\|h\|_{\tilde{B}^*} + 2\delta_i\varepsilon.$$

Thus for $f = Sh$, $\|f\|_\mu = 1$, we have

$$J(f,\delta_i) \le 1 - \delta_i\|h\|_{\tilde{B}^*} + 2\delta_i\varepsilon. \tag{5.14}$$

Because $\varepsilon > 0$ can be made arbitrarily small and $\delta_i \to 0$ is an arbitrary sequence, (5.14) and (5.9) combine to give (5.7).

To verify (5.8) is easy because $1 - I(f,\delta) = (1-J(f,\delta))(1+J(f,\delta))$ and $\lim_{\delta\to 0} J(f,\delta) = \|f\|_\mu = 1$. The continuity of $J(f,\delta)$ in δ is well known, and can be proved as in Lemma 1 of [4], where $I(f,\delta)$ is proved to be continuous for $\delta \in [0,\infty)$. □

For a converse to Proposition 1, we have the following proposition.

PROPOSITION 2. *If* $\|f\|_\mu = 1$ *and there exists* $M < \infty$ *such that* $J(f,\delta) \geq 1 - M\delta$ *for all* $\delta > 0$ *sufficiently small, then* $f \in SB^*$. *In particular, if* $\|f\|_\mu = 1$, $f \notin SB^*$, *then*

$$\lim_{\delta \to 0} \frac{1 - I(f,\delta)}{\delta} = \lim_{\delta \to 0} \frac{1 - J(f,\delta)}{\delta} = +\infty. \tag{5.15}$$

PROOF. By Lemma 3 there exists $g \in B^*$, $\|g\|_\mu \leq 1$, such that

$$1 - \delta\|g\|_{\tilde{B}^*} \geq g(f) - \delta\|g\|_{\tilde{B}^*} \geq J(f,\delta) - M\delta \geq 1 - 2M\delta.$$

Thus $\|g\|_{\tilde{B}^*} \leq 2M$. As $\delta \to 0$, $g(f) \to 1$, so a weak-star cluster point h of g as $\delta \to 0$ (which therefore converges uniformly on compact subsets of B) satisfies $\|h\|_\mu \leq 1$, $h(f) = 1$. Thus $f = Sh$ and the proposition is proved. □

REMARK G. If μ is Wiener measure, we see that if $\|f\|_\mu = 1$, then $f \in SB^*$ iff

$$\lim_{\delta \to 0} \frac{1 - I(f,\delta)}{\delta} < \infty.$$

Hence from Theorem 1, the set of points approached slowest by $\{X_n/(2Ln)^{1/2}: n \geq 1\}$ is precisely the set E given in (3.12). Furthermore, by Lemma 1 above,

$$E = \{f \in K: \|f\|_\mu = 1 \text{ and } f' \text{ has a version of bounded variation on } [0,1]\}.$$

Of course, by standard rescaling arguments, E is also the set approached slowest by the scaled Brownian motion in Chung's functional LIL.

6. Some comparison results. Theorem 1* implies that if $\|f\|_\mu = 1$, then the normalizing sequence $d(n)$ defined by (3.6) gives the correct rate of convergence in (3.17). The examples in [5], and Theorem 1(iii) show that if $\|f_1\|_\mu = 1$ and $\|f_2\|_\mu = 1$, then the sequences $d_1(n)$ for f_1 and $d_2(n)$ for f_2 may be considerably different. Our next result gives conditions when the sequences $\{d_1(n)\}$ and $\{d_2(n)\}$ are comparable.

THEOREM 2. *Let* μ *be a centered Gaussian measure on B with* ψ *as in* (3.2). *Assume* (3.14) *and* (3.15) *hold and for* $x_n > 0, y_n > 0$ *such that* $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} x_n/y_n = 0$ *we have*

$$\lim_{n \to \infty} \psi(x_n)/\psi(y_n) = \infty. \tag{6.1}$$

If $f_1, f_2 \in H_\mu$ *with* $\|f_1\|_\mu = \|f_2\|_\mu = 1$ *and* $f_1 - f_2 \in SB^*$ *or* $f_1 + f_2 \in SB^*$, *then*

$$d_1 \approx d_2. \tag{6.2}$$

REMARK H. If $\psi(\varepsilon) \approx -\varepsilon^{-p}$ for $0 < p < \infty$, then (3.14), (3.15) and (6.1) all hold. In particular, Theorem 2 applies to Brownian motion and the Brownian bridge.

To prove this result it is useful to have the following lemmas. The first is a simple generalization of part of Lemma 1 in [5], so we omit the proof.

LEMMA 4. *If* $f \in H_\mu$ *and* $\delta \le \sigma \|f\|_\mu$ *where* $\sigma^2 = \sup_{\|h\|_{B^*} \le 1} \int_B h^2(x)\, d\mu(x)$, *then*

$$\|f\|_\mu^2 - I(f,\delta) \ge \delta \|f\|_\mu / \sigma. \tag{6.3}$$

LEMMA 5. *If* $f_1, f_2 \in H_\mu$, $\|f_1\|_\mu = \|f_2\|_\mu = 1$ *and* $f_1 - f_2 \in SB^*$ *or* $f_1 + f_2 \in SB^*$, *then as* $\delta \to 0$,

$$1 - I(f_1,\delta) \approx 1 - I(f_2,\delta). \tag{6.4}$$

PROOF. Because $I(f,\delta) = I(-f,\delta)$ we need only show the result for $f_1 - f_2 \in SB^*$. Hence let $f_1 - f_2 = Sh$ for $h \in B^*$. Then

$$\begin{aligned} I(f_2,\delta) &= \inf_{\substack{g \in H_\mu \\ \|g\| \le \delta}} \left(1 - 2\langle f_2, g\rangle_\mu + \|g\|_\mu^2\right) \\ &= \inf_{\substack{g \in H_\mu \\ \|g\| \le \delta}} \left(1 - 2\langle f_1 - Sh, g\rangle_\mu + \|g\|_\mu^2\right) \\ &= \inf_{\substack{g \in H_\mu \\ \|g\| \le \delta}} \left(1 - 2\langle f_1, g\rangle_\mu + \|g\|_\mu^2 + 2\langle Sh, g\rangle_\mu\right) \end{aligned} \tag{6.5}$$

and

$$I(f_1,\delta) = \inf_{\substack{g \in H_\mu \\ \|g\| \le \delta}} \left(1 - 2\langle f_1, g\rangle_\mu + \|g\|_\mu^2\right). \tag{6.6}$$

Combining (6.5) and (6.6) and using the fact that

$$|\inf A - \inf B| \le \sup |A - B|,$$

we have

$$|I(f_1,\delta) - I(f_2,\delta)| \le 2\|h\|_{B^*}\delta.$$

Hence for $\delta < \sigma$,

$$\begin{aligned} \left|\frac{1 - I(f_2,\delta)}{1 - I(f_1,\delta)}\right| &\le 1 + \frac{|I(f_1,\delta) - I(f_2,\delta)|}{1 - I(f_1,\delta)} \\ &\le 1 + \frac{2\|h\|_{B^*}\delta}{\delta/\sigma} \\ &= 1 + 2\sigma\|h\|_{B^*} \end{aligned} \tag{6.7}$$

by Lemma 4 because $\|f_1\|_\mu = \|f_2\|_\mu = 1$. Similarly, we have for $\delta < \sigma$,

$$\left|\frac{1 - I(f_1,\delta)}{1 - I(f_2,\delta)}\right| \le 1 + 2\sigma\|h\|_{B^*}. \tag{6.8}$$

Hence (6.4) follows from (6.7) and (6.8). □

PROOF OF THEOREM 2. By (3.16) of Theorem 1* we have $d_1(n)$ and $d_2(n)$ such that

$$\frac{1-I(f_1,d_1(n))}{1-I(f_2,d_2(n))}=\frac{\psi((2Ln)^{1/2}d_1(n))}{\psi((2Ln)^{1/2}d_2(n))}. \tag{6.9}$$

If $\liminf_{n\to\infty} d_1(n)/d_2(n) = 0$, then arguing along a subsequence where the lim inf is 0, the right-hand term of (6.9) goes to infinity by (6.1) because $\lim_{n\to\infty}(2Ln)^{1/2}d_i(n)=0$ for $i = 1, 2$. However, along this subsequence the left-hand term of (6.9) is bounded because

$$\begin{aligned}\left|\frac{1-I(f_1,d_1)}{1-I(f_2,d_2)}\right| &= \left|\frac{1-I(f_1,d_1)}{1-I(f_1,d_2)}\right|\left|\frac{1-I(f_1,d_2)}{1-I(f_2,d_2)}\right|\\ &\le C\left|\frac{1-I(f_1,d_1)}{1-I(f_1,d_2)}\right|\\ &\le C,\end{aligned}$$

where the first inequality is by Lemma 5 and the second inequality follows from $d_1 \le d_2$ along the subsequence. Hence $\liminf_{n\to\infty} d_1(n)/d_2(n) > 0$. Similarly, we also have $\limsup_{n\to\infty} d_1(n)/d_2(n) < \infty$. Therefore, $d_1 \approx d_2$ and Theorem 2 is proved. □

Next we turn to the Hilbert space case with $\mu = \mathcal{L}(X)$, where

$$X=\sum_{n\ge 1}\lambda_n\xi_n\phi(n). \tag{6.10}$$

In (6.10), $\lambda_n > 0$, $\{\xi_n: n \ge 1\}$ are independent $N(0,1)$, $\{\phi(n): n \ge 1\}$ is an orthonormal sequence in H and $\sum_{n\ge 1}\lambda_n^2 < \infty$. In this setting,

$$H_\mu=\left\{a=\sum_{n\ge 1}a_n\phi(n): \sum_{n\ge 1}a_n^2/\lambda_n^2<\infty\right\},$$

with $\langle a,h\rangle_\mu = \sum_{n\ge 1}a_nh_n/\lambda_n^2$ when $a = \sum_{n\ge 1}a_n\phi(n)$ and $h = \sum_{n\ge 1}h_n\phi(n)$. Hence the sequence $\{\lambda_n\phi(n): n \ge 1\}$ is a complete orthonormal sequence in H_μ and $a = Sh = E(Xh(X))$ implies $a_n = h_n\lambda_n^2$, $n \ge 1$, where $a = \sum_{n\ge 1}a_n\phi(n)$ and $h = \sum_{n\ge 1}h_n\phi(n)$ by identifying H and its dual space H^*. Thus $a = Sh$, $h \in H^*$, implies $\sum_{n\ge 1}|a_n/\lambda_n^2|^2 < \infty$.

Let $a = \sum_{n\ge 1}a_n\phi(n)$ and $b = \sum_{n\ge 1}b_n\phi(n)$ with $\|a\|_\mu = \|b\|_\mu = 1$. Now $\|a\|_\mu^2 = \sum_{n\ge 1}a_n^2/\lambda_n^2$, $\|b\|_\mu^2 = \sum_{n\ge 1}b_n^2/\lambda_n^2$ and by direct computation it is easy to check by identifying H and H^* that

$$a-b\in SB^* \quad\text{iff}\quad \sum_{n\ge 1}(a_n-b_n)^2/\lambda_n^4<\infty \tag{6.11}$$

and

(6.12) $$a+b \in SB^* \quad \text{iff} \sum_{n\geq 1}(a_n+b_n)^2/\lambda_n^4 < \infty.$$

Hence, under the conditions of Theorem 2, we have that if $\Sigma_{n\geq 1}(a_n - b_n)^2/\lambda_n^4 < \infty$ or $\Sigma_{n\geq 1}(a_n+b_n)^2/\lambda_n^4 < \infty$, then $d_a \approx d_b$. It is also possible to show under these conditions that if $\{a_n\}$ and $\{b_n\}$ satisfy $\|a\|_\mu = \|b\|_\mu = 1$ and

(6.13) $$\sum_{n\geq 1} \frac{|a_n^2 - b_n^2|}{\lambda_n^4} < \infty,$$

then $d_a \approx d_b$. The proof of this is computational, so we do not include the details. Finally, to finish this section we include a remark on the relationship of (6.11), (6.12) and (6.13).

REMARK I. The series conditions in (6.11), (6.12) and (6.13) are all distinct. That is, assume $\lambda_{2n} = \lambda_{2n-1}$ for $n \geq 1$, $\Sigma_{n\geq 1}\lambda_n^2 < \infty$, let $a_{2n} = C(\lambda_{2n}^2 + \lambda_{2n}^4)$, $a_{2n-1} = C(\lambda_{2n}^2 - \lambda_{2n}^4)$, $b_{2n} = C(\lambda_{2n}^4 - \lambda_{2n}^2)$ and $b_{2n-1} = C(\lambda_{2n}^2 + \lambda_{2n}^4)$, where C is a proper constant such that $\|a\|_\mu = \|b\|_\mu = 1$. Then (6.13) holds, but (6.11) and (6.12) fail. Similarly, if $\lambda_{2n} = \lambda_{2n-1}$ with $a_{2n} = C(\lambda_{2n} + \lambda_{2n}^3)$, $a_{2n-1} = C(\lambda_{2n} - \lambda_{2n}^3)$, $b_{2n} = C(\lambda_{2n} - \lambda_{2n}^3)$ and $b_{2n-1} = C(\lambda_{2n} + \lambda_{2n}^3)$, then (6.13) fails, but (6.11) holds.

7. Exact constants. Up to this point, we have mainly been interested in finding the points of K that, modulo a constant, are approached slowest by the sequence $\{X_n/(2Ln)^{1/2}\}$. As noted in Remark B in Section 3, the set $\{f\colon \|f\|_\mu = 1 \text{ and } f \in SB^*\}$ is the correct set for many Gaussian measures and $d(n)$ as defined by (3.3) is the right convergence rate. To go beyond the convergence rate and to determine the exact constant is much harder and very little is known. The only known results are for piecewise linear functions [1] and quadratic functions [2] when μ is Wiener measure on $B = C[0,1]$. Note that these two special classes of functions are both in SB^*.

In this section we give the exact constant for all the points $a \in SH^*$, $\|a\|_\mu = 1$, for a class of Gaussian measures (including the Wiener measure) on a Hilbert space H. First we give the following key estimates, which are of independent interest. The method of proof is similar to the one used in [9]. In contrast to the previous sections, we replace λ_n^2 by λ_n for notational simplicity. To aid in this distinction we index all sums by k rather than n.

THEOREM 3. *Let $\mu = \mathcal{L}(X)$, $X = \Sigma_{k\geq 1}\lambda_k^{1/2}\xi_k e_k$ be a centered Gaussian measure on a separable Hilbert space H, where*

(7.1) $$\lambda_k = \Lambda(k+\lambda)^{-\alpha}, \qquad \alpha > 1,\ \lambda > -1,\ \Lambda > 0,$$

$\{\xi_k\colon n \geq 1\}$ are independent $N(0,1)$, and $\{e_k\colon k \geq 1\}$ is a orthonormal sequence in H. Let

(7.2) $$K_1 = \Lambda^{1/\alpha}2^{-(\alpha-1)/\alpha}(\pi/\alpha)\left(\sin \pi/\alpha\right)^{-1}.$$

Then for any $a = \Sigma_{k\geq 1} a_k e_k \in SH^*$ (*i.e.*, $\|S^{-1}a\|^2_{H^*} = \Sigma_{k\geq 1} a_k^2/\lambda_k^2 < \infty$),

$$P\big(\|X - ta\|^2 \leq \theta t^{-2(\alpha-1)/(\alpha+1)}\big)$$

$$\sim K_2 t^{-1/(\alpha+1)} \exp\left\{\theta t^{-2(\alpha-1)/(\alpha+1)}\gamma - \frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\gamma) - \sum_{k\geq 1}\frac{t^2 a_k^2\gamma}{1+2\lambda_k\gamma}\right\}$$

as $t \to \infty$, *where*

$$K_2 = (2\pi)^{-1/2}\left(\frac{1}{2x_0}\sum_{k\geq 1}\frac{a_k^2}{\lambda_k^2} + \frac{\alpha-1}{\alpha}K_1 x_0^{1/\alpha}\right)^{-1/2}, \tag{7.3}$$

x_0 *is the unique positive solution of the equation*

$$4\theta x^2 - 4K_1 x^{(\alpha+1)/\alpha} - \sum_{k\geq 1}\frac{a_k^2}{\lambda_k^2} = 0 \tag{7.4}$$

and $\gamma > 0$ *is the unique solution of the following equation for t large*:

$$\theta t^{-2(\alpha-1)/(\alpha+1)} = \frac{1}{\gamma} + t^2\sum_{k\geq 1}\frac{a_k^2}{(1+2\lambda_k\gamma)^2} + \sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma}. \tag{7.5}$$

Before we give the proof of Theorem 3, we need some lemmas.

LEMMA 6. *Let* λ_k *be defined as in* (7.1). *Then as* $\gamma \to \infty$,

$$\sum_{k\geq 1}\frac{\lambda_k}{1+2\lambda_k\gamma} \sim K_1\gamma^{-(\alpha-1)/\alpha}, \tag{7.6}$$

$$\frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\gamma) \sim \alpha K_1\gamma^{1/\alpha}, \tag{7.7}$$

$$T = \sum_{k\geq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2 \sim \frac{\alpha-1}{2\alpha}K_1\gamma^{1/\alpha}, \tag{7.8}$$

where K_1 *is as in* (7.2).

PROOF. All of these are easy to check. Related details can be found in ([8], Lemma 7). □

LEMMA 7. *Given* $\theta > 0$, *let* $\gamma > 0$ *be the unique solution of* (7.5) *for t large. Then as* $t \to \infty$,

$$\gamma \sim x_0 t^{2\alpha/(\alpha+1)}, \tag{7.9}$$

where x_0 *is the unique positive solution of* (7.4).

PROOF. If $\gamma = \gamma(t) \to \infty$ as $t \to \infty$, then by using D.C.T., as $t \to \infty$,

$$t^2 \sum_{k \geq 1} \frac{a_k^2}{(1 + 2\lambda_k \gamma)^2} \sim \frac{t^2}{4\gamma^2} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2}. \tag{7.10}$$

Hence the statement of the lemma follows from (7.6) and (7.10). □

PROOF OF THEOREM 3. The Laplace transform of the random variable $\|X - ta\|^2$ is given by

$$\int_H \exp\big(-s\|x - ta\|^2\big)\mu(dx) = \exp\left\{-s \sum_{k \geq 1} \frac{t^2 a_k^2}{1 + 2\lambda_k s} - \frac{1}{2} \sum_{k \geq 1} \log(1 + 2\lambda_k s)\right\}.$$

Furthermore, if $f(s,t) = P(\|X - sa\| \leq t)$ for $a \in H_\mu$ and $\mu = \mathcal{L}(X)$, then the Cameron–Martin formula in (2.6) gives

$$f(s,t) = \int_H I\big(x\colon \|x\| \leq t\big) \exp\left\{-\frac{s^2}{2}\|a\|_\mu^2 - s\langle x, a\rangle^\sim\right\} d\mu(x)$$

for $-\infty < s < \infty$ and $t \geq 0$. Because it is known that $\mu(x\colon \|x\| = t) = 0$ for all $t \geq 0$ and $\langle x, a\rangle^\sim$ is $N(0, \|a\|_\mu^2)$, the D.C.T. easily implies $f(s,t)$ is jointly continuous in s and t. Hence using the inversion formula (see, e.g., [10], page 107) and the calculations below, we have for every $\gamma > 0$,

$$P\big(\|X - ta\|^2 \leq \theta t^{-2(\alpha - 1)/(\alpha + 1)}\big) = \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp\big(\Phi(s)\big)\, d\sigma, \tag{7.11}$$

where $s = \gamma + i\sigma$ and

$$\Phi(s) = -\log s + \theta t^{-2(\alpha - 1)/(\alpha + 1)} s - s \sum_{k \geq 1} \frac{t^2 a_k^2}{1 + 2\lambda_k s} - \frac{1}{2} \sum_{k \geq 1} \log(1 + 2\lambda_k s).$$

Note that the function $\Phi'(s)$ has a zero at $(\gamma, 0)$ where $\gamma = \gamma(t)$ is given by

$$\theta t^{-2(\alpha - 1)/(\alpha + 1)} = \frac{1}{\gamma} + t^2 \sum_{k \geq 1} \frac{a_k^2}{(1 + 2\lambda_k \gamma)^2} + \sum_{k \geq 1} \frac{\lambda_k}{1 + 2\lambda_k \gamma}. \tag{7.12}$$

Hence we take $\gamma > 0$ in (7.11) to be the unique solution of (7.12) for t large.

Now we rewrite (7.11) as a sum

$$P\big(\|X - ta\|^2 \leq \theta t^{-2(\alpha - 1)/(\alpha + 1)}\big) = I_1 + I_2 + I_3, \tag{7.13}$$

where

$$I_1 = \frac{1}{2\pi} \int_{|\sigma| > \gamma T^{-2/5}} \exp\big(\Phi(s)\big)\, d\sigma,$$

$$I_2 = \frac{1}{2\pi}\int_{|\sigma|<\gamma T^{-2/5}} \exp\bigl(\operatorname{Re}\Phi(s)\bigr)\Bigl(\exp\bigl(i\operatorname{Im}\Phi(s)\bigr)-1\Bigr)\,d\sigma,$$

$$I_3 = \frac{1}{2\pi}\int_{|\sigma|<\gamma T^{-2/5}} \exp\bigl(\operatorname{Re}\Phi(s)\bigr)\,d\sigma,$$

$$T = \sum_{k\geq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2.$$

We will show that I_3 is the dominating term.

Let us rewrite $\operatorname{Re}\Phi(s) = A(\gamma) + B(\gamma,\sigma)$ by using $\log(a+ib) = \log\sqrt{a^2+b^2} + i\arctan(b/a)$. Here

$$A(\gamma) = \theta t^{-2(\alpha-1)/(\alpha+1)}\gamma - \log\gamma - \frac{1}{2}\sum_{k\geq 1}\log(1+2\lambda_k\gamma) - \sum_{k\geq 1}\frac{t^2a_k^2\gamma}{1+2\lambda_k\gamma} \tag{7.14}$$

and

$$\begin{aligned} B(\gamma,\sigma) = {} & -\frac{1}{2}\log\left(1+\left(\frac{\sigma}{\gamma}\right)^2\right) - \frac{1}{4}\sum_{k\geq 1}\log\left(1+\left(\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\right)^2\right) \\ & - \sigma^2\sum_{k\geq 1}\frac{2t^2a_k^2\lambda_k}{(1+2\lambda_k\gamma)^3+(2\lambda_k\sigma)^2(1+2\lambda_k\gamma)}. \end{aligned} \tag{7.15}$$

Then

$$|I_1| \leq \frac{1}{2\pi}\exp\bigl(A(\gamma)\bigr)\int_{|\sigma|>\gamma T^{-2/5}}\exp\bigl(B(\gamma,\sigma)\bigr)\,d\sigma.$$

Because λ_k is decreasing, we have by omitting the first and the third terms in (7.15),

$$\begin{aligned} & \int_{|\sigma|>\gamma T^{-2/5}}\exp\bigl(B(\gamma,\sigma)\bigr)\,d\sigma \\ & \quad \leq \int_{|\sigma|>\gamma T^{-2/5}}\prod_{k\geq 1}\left(1+\left(\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\right)^2\right)^{-1/4}d\sigma \\ & \quad \leq \prod_{k\geq 5}\left(1+T^{-4/5}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right)^{-1/4}\int_{-\infty}^{+\infty}\left(1+\left(\frac{2\lambda_4\sigma}{1+2\lambda_4\gamma}\right)^2\right)^{-1}d\sigma \\ & \quad \leq C\gamma\exp\left\{-\frac{1}{4}\sum_{k\geq 5}\log\left(1+T^{-4/5}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right)\right\} \\ & \quad \leq C\gamma\exp\left\{-\frac{1}{4}\sum_{k\geq 5}\left(T^{-4/5}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2-\frac{1}{2}T^{-8/5}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^4\right)\right\} \\ & \quad \leq C\gamma\exp\left\{-T^{-4/5}\sum_{k\geq 5}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2+\frac{1}{2}T^{-8/5}\sum_{k\geq 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2\right\} \\ & \quad \leq C\gamma\exp\bigl\{-T^{1/5}\bigr\}. \end{aligned}$$

Therefore,

$$|I_1| \le C\gamma e^{A(\gamma)} \exp\{-T^{1/5}\}. \tag{7.16}$$

Now turning to I_2, we have $|\exp(i\,\mathrm{Im}\,\Phi(s)) - 1| \le |\mathrm{Im}\,\Phi(s)|$ and

$$\begin{aligned}
\mathrm{Im}\,\Phi(s) &= \theta t^{-2(\alpha-1)/(\alpha+1)}\sigma - \arctan\frac{\sigma}{\gamma} - \frac{1}{2}\sum_{k\ge 1}\arctan\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\\
&\quad - \sigma\sum_{k\ge 1}\frac{t^2a_k^2}{(1+2\lambda_k\gamma)^2+(2\lambda_k\sigma)^2}\\
&= \left(\frac{\sigma}{\gamma} - \arctan\frac{\sigma}{\gamma}\right) + \frac{1}{2}\sum_{k\ge 1}\left(\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma} - \arctan\frac{2\lambda_k\sigma}{1+2\lambda_k\gamma}\right)\\
&\quad + \sigma\sum_{k\ge 1}\left(\frac{t^2a_k^2}{(1+2\lambda_k\gamma)^2} - \frac{t^2a_k^2}{(1+2\lambda_k\gamma)^2+(2\lambda_k\sigma)^2}\right)
\end{aligned}$$

by plugging in (7.12). Thus we have by using the inequality $x - \arctan x \le x^3/3$, $x > 0$ and $1/x^2 - 1/(x^2+y^2) \le y^2/x^4$,

$$\begin{aligned}
|\mathrm{Im}\,\Phi(s)| &\le \frac{1}{3}\left|\frac{\sigma}{\gamma}\right|^3 + \frac{1}{6}\left|\frac{\sigma}{\gamma}\right|^3\sum_{k\ge 1}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^3 + \frac{1}{4}\left|\frac{\sigma}{\gamma}\right|^3\frac{t^2}{\gamma}\sum_{k\ge 1}\frac{a_k^2}{\lambda_k^2}\left(\frac{2\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^4\\
&\le \frac{1}{3}T^{-6/5} + \frac{2}{3}T^{-6/5}\sum_{k\ge 1}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2 + \frac{1}{4}T^{-6/5}\frac{t^2}{\gamma}\sum_{k\ge 1}\frac{a_k^2}{\lambda_k^2}\\
&\le CT^{-1/5}
\end{aligned}$$

for $|\sigma| \le \gamma T^{-2/5}$. Hence

$$|I_2| \le CT^{-1/5}\int_{|\sigma|<\gamma T^{-2/5}} \exp\big(\mathrm{Re}\,\Phi(s)\big)\,d\sigma. \tag{7.17}$$

Next we turn to the dominating term I_3. Let

$$\beta = \beta(\gamma) = 1 + t^2\sum_{k\ge 1}\frac{a_k^2(2\lambda_k\gamma)^2}{\lambda_k(1+2\lambda_k\gamma)^3} + 2T.$$

Then by using (7.8) and (7.9),

$$\begin{aligned}
\beta &\sim \frac{t^2}{2\gamma}\sum_{k\ge 1}\frac{a_k^2}{\lambda_k^2} + 2T\\
&\sim \left(\frac{1}{2x_0}\sum_{k\ge 1}\frac{a_k^2}{\lambda_k^2} + \frac{\alpha-1}{\alpha}K_1x_0^{1/\alpha}\right)t^{2/(\alpha+1)} \qquad (7.18)\\
&\sim CT.
\end{aligned}$$

Using the inequality $x - x^2/2 \le \log(1+x) \le x$, we have from (7.15) that

$$-\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2 \beta(\gamma) \le B(\gamma,\sigma) \le -\frac{1}{2}\left(\frac{\sigma}{\gamma}\right)^2 \left(1 - \left(\frac{\sigma}{\gamma}\right)^2\right)\beta(\gamma).$$

Hence by the change of variables,

$$\begin{aligned}
&\frac{\gamma}{\sqrt{\beta}} \int_{|u| < T^{-2/5}\sqrt{\beta}} \exp(-u^2/2)\, du \\
&\quad \le \int_{|\sigma| < \gamma T^{-2/5}} \exp(B(\gamma,\sigma))\, d\sigma \\
&\quad \le \frac{\gamma}{\sqrt{\beta}} \int_{|u| < T^{-2/5}\sqrt{\beta}} \exp\left(\frac{-u^2}{2}\right) du \times \exp\left\{\frac{1}{2} T^{-8/5}\beta\right\}.
\end{aligned}$$

Note that from (7.18), $T^{-8/5}\beta \approx T^{-3/5}$ and $T^{-2/5}\sqrt{\beta} \approx T^{1/5}$. Hence,

$$\int_{|\sigma| < \gamma T^{-2/5}} \exp(B(\gamma,\sigma))\, d\sigma \sim \frac{\gamma}{\sqrt{\beta}} \int_{|u| < T^{-2/5}\sqrt{\beta}} \exp\left(\frac{-u^2}{2}\right) du \sim \frac{\sqrt{2\pi}\gamma}{\sqrt{\beta}}.$$

Thus,

$$(7.19)\quad I_3 = \frac{1}{2\pi}\exp(A(\gamma)) \int_{|\sigma| < \gamma T^{-2/5}} \exp(B(\gamma,\sigma))\, d\sigma \sim \frac{\gamma}{\sqrt{2\pi\beta}}\exp(A(\gamma)).$$

Combining (7.13), (7.16), (7.17), (7.18) and (7.19), we have as $t \to \infty$,

$$\begin{aligned}
&P(\|X - ta\|^2 \le \theta \cdot t^{-2(\alpha-1)/(\alpha+1)}) \\
&\quad \sim \frac{\gamma}{\sqrt{2\pi\beta}}\exp(A(\gamma)) \\
&\quad = \frac{1}{\sqrt{2\pi\beta}}\exp\left\{\theta t^{-2(\alpha-1)/(\alpha+1)}\gamma - \frac{1}{2}\sum_{k\ge1}\log(1+2\lambda_k\gamma) - \sum_{k\ge1}\frac{t^2 a_k^2 \gamma}{1+2\lambda_k\gamma}\right\} \\
&\quad \sim K_2 t^{-1/(\alpha+1)}\exp\left\{\theta t^{-2(\alpha-1)/(\alpha+1)}\gamma - \frac{1}{2}\sum_{k\ge1}\log(1+2\lambda_k\gamma) - \sum_{k\ge1}\frac{t^2 a_k^2 \gamma}{1+2\lambda_k\gamma}\right\},
\end{aligned}$$

where K_2 is as in (7.3). This completes the proof of Theorem 3. □

The asymptotic expression given in our theorem is still implicit in terms of t. With a little bit of extra work, we can obtain the following easy to use, but weaker asymptotic expression.

COROLLARY 1. *Under the assumptions in Theorem* 3, *if* $\|a\|_\mu = \sum_{k\ge1} a_k^2/\lambda_k = 1$ *and* x_0 *is defined as in* (7.4), *then as* $t \to \infty$,

$$\log\left(\exp(t^2/2)P(\|X - ta\|^2 \le \theta t^{-2(\alpha-1)/(\alpha+1)})\right) \sim G(\theta) t^{2/(\alpha+1)},$$

provided $G(\theta) \neq 0$, *where*

$$G(\theta) = \frac{1}{2x_0} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} - (\alpha - 1) K_1 x_0^{1/\alpha}. \tag{7.20}$$

PROOF. We need to give more detailed estimates for the exponents in Theorem 3. Note that $\gamma \sim x_0 t^{2\alpha/(\alpha+1)}$ from Lemma 7. Hence as $t \to \infty$,

$$\theta t^{-2(\alpha-1)/(\alpha+1)} \gamma \sim \theta x_0 t^{2/(\alpha+1)}, \tag{7.21}$$

and by (7.7),

$$\frac{1}{2} \sum_{k \geq 1} \log(1 + 2\lambda_k \gamma) \sim \alpha K_1 \gamma^{1/\alpha} \sim \alpha K_1 x_0^{1/\alpha} t^{2/(\alpha+1)}. \tag{7.22}$$

By using $\sum_{k \geq 1} a_k^2 / \lambda_k = 1$, we have

$$\sum_{k \geq 1} \frac{t^2 a_k^2 \gamma}{1 + 2\lambda_k \gamma} = \frac{t^2}{2} - \frac{t^2}{4\gamma} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} \frac{2\lambda_k \gamma}{1 + 2\lambda_k \gamma} \tag{7.23}$$

and as $t \to \infty$,

$$\frac{t^2}{4\gamma} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} \frac{2\lambda_k \gamma}{1 + 2\lambda_k \gamma} \sim \frac{t^2}{4\gamma} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} \sim \frac{1}{4x_0} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} t^{2/(\alpha+1)}. \tag{7.24}$$

Now by Theorem 3 and (7.21), (7.22), (7.23) and (7.24), we obtain

$$\begin{aligned}
&\log\left(\exp\left(\frac{t^2}{2}\right) P\left(\|X - ta\|^2 \leq \theta t^{-2(\alpha-1)/(\alpha+1)}\right)\right) \\
&\quad \sim \theta t^{-2(\alpha-1)/(\alpha+1)} \gamma - \frac{1}{2} \sum_{k \geq 1} \log(1 + 2\lambda_k \gamma) + \frac{t^2}{4\gamma} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} \frac{2\lambda_k \gamma}{1 + 2\lambda_k \gamma} \\
&\quad \sim \left(\theta x_0 - \alpha K_1 x_0^{1/\alpha} + \frac{1}{4x_0} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} \right) t^{2/(\alpha+1)} \\
&\quad = \left(\frac{1}{2x_0} \sum_{k \geq 1} \frac{a_k^2}{\lambda_k^2} - (\alpha - 1) K_1 x_0^{1/\alpha} \right) t^{2/(\alpha+1)} \\
&\quad = G(\theta) t^{2/(\alpha+1)},
\end{aligned}$$

provided $G(\theta) \neq 0$, where the first equality above follows from (7.4). This finishes the proof. □

Our next lemma gives more information about the function $G(\theta)$. This is critical when we are using Corollary 1 to prove limit theorems.

LEMMA 8. *The function* $G(\theta)$, $\theta > 0$, *is strictly increasing and its unique zero is*

$$\theta_0 = \frac{\alpha+1}{4(\alpha-1)}\big(2K_1(\alpha-1)\big)^{2\alpha/(\alpha+1)}\left(\sum_{k\geq 1} a_k^2/\lambda_k^2\right)^{-(\alpha-1)/(\alpha+1)}. \tag{7.25}$$

PROOF. We only need to show x_0 is a strictly decreasing function of θ, where x_0 is the unique positive solution of (7.4). The constant θ_0 given in (7.25) can be solved from the equation $G(\theta) = 0$ and (7.4).

Let $f(t)$, $t > 0$, be the unique positive solution of the equation $tf^2 - af^{(\alpha+1)/\alpha} - b = 0$, where $a > 0$, $b > 0$. Implicit differentiation gives us

$$f^2 + \frac{\alpha+1}{\alpha}f^{-1}\left(\frac{2\alpha}{\alpha+1}tf^2 - af^{(\alpha+1)/\alpha}\right)f' = 0.$$

Note that

$$\frac{2\alpha}{\alpha+1}tf^2 - af^{(\alpha+1)/\alpha} > tf^2 - af^{(\alpha+1)/\alpha} = b > 0.$$

Thus we have $f' < 0$ because $f > 0$. This finishes the proof when we take $a = K_1$ and $b = 4^{-1}\sum_{k\geq 1} a_k^2/\lambda_k^2$. $\square$

Now we provide the convergence rate and exact constant in the setting of this section. Because $\psi(\varepsilon)$ of Theorem 1 is known for these examples to satisfy

$$\psi(\varepsilon) \sim -2^{-1}\Lambda^{1/(\alpha-1)}(\alpha-1)\left(\frac{\pi}{\alpha}\Big/\sin\frac{\pi}{\alpha}\right)^{\alpha/(\alpha-1)}\varepsilon^{-2/(\alpha-1)}$$

as $\varepsilon \to 0$ (see [8]), the convergence rate can easily be computed by Theorem 1 to be

$$d(n) \approx (Ln)^{-\alpha/(\alpha+1)}. \tag{7.26}$$

Hence all that remains is the exact constant, and this is the next result.

THEOREM 4. *Let* $X, X_1, X_2, \ldots$ *be i.i.d. H-valued centered Gaussian random vectors with* $\mu = \mathcal{L}(X)$, *where X is defined as in Theorem* 3. *If* $a \in SH^*$ *and* $\|a\|_\mu = 1$, *then*

$$\liminf_{n\to\infty}(Ln)^{\alpha/(\alpha+1)}\left\|\frac{X_n}{\sqrt{2Ln}} - a\right\| = K_{a,\alpha} \quad a.s.,$$

where

$$K_{a,\alpha} = (\alpha+1)^{1/2}(\alpha-1)^{(\alpha-1)/2(\alpha+1)}\Lambda^{1/(\alpha+1)} \times\left(\frac{4\alpha}{\pi}\sin\frac{\pi}{\alpha}\right)^{-\alpha/(\alpha+1)}\cdot\|S^{-1}a\|^{-(\alpha-1)/(\alpha+1)}$$

and $\|S^{-1}a\|^2 = \sum_{k\geq 1} a_k^2/\lambda_k^2$.

PROOF. From Corollary 1, if $a \in SH^*$ and $\|a\|_\mu = 1$, then for every $\varepsilon > 0$, $G(\theta) > 0$ and t sufficiently large,

$$\begin{aligned} P(\|X - ta\|^2 &\leq \theta t^{-2(\alpha-1)/(\alpha+1)}) \\ &\geq \exp\{-t^2/2 + G(\theta)t^{2/(\alpha+1)}(1-\varepsilon)\}. \end{aligned} \tag{7.27}$$

If $\theta = (1+\varepsilon)\theta_0$, then $G((1+\varepsilon)\theta_0) > 0$ by Lemma 8. Hence (7.27) will give with $t = (2Ln)^{1/2}$ and n large that

$$\begin{aligned} P\left(\left\|\frac{X_n}{\sqrt{2Ln}} - a\right\|^2 \leq (1+\varepsilon)\theta_0(\sqrt{2Ln})^{-2(\alpha-1)/(\alpha+1)-2}\right) \\ \geq \exp\left\{-Ln + G\big((1+\varepsilon)\theta_0\big)(2Ln)^{1/(\alpha+1)}(1-\varepsilon)\right\}. \end{aligned} \tag{7.28}$$

Hence if $0 < \varepsilon < 1$, (7.28) and the Borel–Cantelli lemma imply

$$\liminf_{n\to\infty}(Ln)^{\alpha/(\alpha+1)}\left\|\frac{X_n}{\sqrt{2Ln}} - a\right\| \leq (1+\varepsilon)^{1/2}\theta_0^{1/2}2^{-\alpha/(\alpha+1)} \quad \text{a.s.} \tag{7.29}$$

Similarly, Corollary 1 and $G((1-\varepsilon)\theta_0) < 0$ imply for all $0 < \varepsilon < 1$ that

$$\begin{aligned} P\left(\left\|\frac{X_n}{\sqrt{2Ln}} - a\right\|^2 \leq (1-\varepsilon)\theta_0(\sqrt{2Ln})^{-2(\alpha-1)/(\alpha+1)-2}\right) \\ \leq \exp\left\{-Ln + G\big((1-\varepsilon)\theta_0\big)(2Ln)^{1/(\alpha+1)}(1+\varepsilon)\right\}. \end{aligned} \tag{7.30}$$

Hence the Borel–Cantelli lemma implies

$$\liminf_{n\to\infty}(Ln)^{\alpha/(\alpha+1)}\left\|\frac{X_n}{\sqrt{2Ln}} - a\right\| \geq (1-\varepsilon)^{1/2}\theta_0^{1/2}2^{-\alpha/(\alpha+1)} \quad \text{a.s.} \tag{7.31}$$

Because $\varepsilon > 0$ was arbitrarily small, this gives the theorem because $K_{a,\alpha} = \theta_0^{1/2}2^{-\alpha/(\alpha+1)}$. □

Let $\{\xi_k: k \geq 1\}$ be i.i.d. $N(0,1)$ random variables and consider the orthonormal functions $\{\phi_k: k \geq 1\}$ in $H = L^2[0,1]$ given by $\phi_k(t) = \sqrt{2}\sin((2k-1)\pi t/2)$ for $k \geq 1$. Then it is well known that the series

$$X = W(t) = \sum_{k\geq 1}\frac{1}{\pi(k-1/2)}\xi_k\phi_k(t) \tag{7.32}$$

represents Brownian motion for $0 \leq t \leq 1$, and we define the probability measure $\mu = L(X)$ induced by X on $H = L^2[0,1]$ to be Wiener measure on $L^2[0,1]$. Applying the above results to X, we have the following corollary.

COROLLARY 2. *Let $H = L^2[0,1]$ and assume $X, X_1, X_2, \ldots$ are i.i.d. H-valued centered Gaussian random vectors with $\mu = \mathcal{L}(X)$ being Wiener measure on H. If $a \in SH^*$ and $\|a\|_\mu = 1$, then*

$$\liminf_{n\to\infty} (Ln)^{2/3}\left\|\frac{X_n}{\sqrt{2Ln}} - a\right\| = \frac{\sqrt{3}}{4}\left\|S^{-1}a\right\|^{-1/3} \quad a.s.$$

PROOF. Because $\mu = L(X)$ is Wiener measure in $H = L^2[0,1]$, we have $X = W(t)$ as in (7.32). Thus by taking $\Lambda = \pi^{-2}$, $\lambda = -1/2$ and $\alpha = 2$ in Theorem 4, we obtain the claim of Corollary 2. □

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