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ON CHUNG'S LAW OF THE ITERATED LOGARITHM FOR SOME STOCHASTIC INTEGRALS[1]

BY BRUNO RÉMILLARD

Université du Québec à Trois-Rivières

We prove that there exists a constant $a(A) \in (0, \infty)$ such that $\liminf_{t \to \infty} (\log\log t/t)\sup_{0 \le s \le t} |\int_0^s \langle AW_u, dW_u\rangle| = a(A)$ with probability 1, where A is a skew-symmetric $d \times d$ matrix, $A \neq 0$, and $\{W_t\}_{t \ge 0}$ is a d-dimensional Wiener process.

1. Introduction. Let $\{B_t\}_{t \ge 0}$ be a one-dimensional Wiener process. Chung (1948) proved that with probability 1,

$$\liminf_{t \to \infty} \left(\log\log t/t\right)^{1/2} \sup_{0 \le s \le t} |B_s| = \pi/\sqrt{8}. \tag{1.1}$$

Motivated by recent results concerning processes of the form

$$L_t = \int_0^t \langle AW_u, dW_u\rangle,$$

we prove that the analog of (1.1) also holds for these processes. More precisely we will prove the following theorem.

THEOREM 1. *Suppose that $\{W_t\}_{t \ge 0}$ is a d-dimensional Wiener process and suppose that A is a skew-symmetric $d \times d$ matrix [i.e., $A^* = -A$, where the asterisk $(*)$ stands for the transpose] and $A \neq 0$. Let $L_t = \int_0^t \langle AW_u, dW_u\rangle$, $t \ge 0$. Then*

$$\lim_{t \to \infty} \frac{1}{t} \log P\left(\sup_{0 \le s \le t} |L_s| < 1 \right) = -a(A), \tag{1.2}$$

where $a(A) \in (0, \infty)$ and

$$P\left(\liminf_{t \to \infty} \left(\log\log t/t\right) \sup_{0 \le s \le t} |L(s)| = a(A) \right) = 1. \tag{1.3}$$

The rest of the paper is organized as follows: in Section 2 we prove (1.2) and we prove (1.3) in Section 3.

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2. Existence of $a(A)$. Suppose that (Ω, F, P) is a complete probability space and let $\{W_t, F_t\}_{t\geq 0}$ be a d-dimensional Wiener process on (Ω, F, P), when F_t is its standard filtration and $P(W_0 = 0) = 1$. From now on, A is a fixed $d \times d$ matrix such that $A \neq 0$ and $A^* = -A$.

Let L_t be a continuous version of the stochastic integral $\int_0^t \langle AW_u, dW_u\rangle$, where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product. Since for every $c > 0$, $\{W_{ct}/\sqrt{c}\}_{t\geq 0}$ is also a Wiener process, we see that $\{L_{ct}/c\}_{t\geq 0}$ has the same law as $\{L_t\}_{t\geq 0}$. Moreover, if we define $P_{(x,y)}$ as the probability measure induced by the process

$$\left(W_t^x, L_t^y\right) = \left(x + W_t, \langle Ax, W_t\rangle + y + L_t\right), \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R},$$

then $P_{(x,y)}$ is the solution of the martingale problem starting at (x,y) with generator $L = (1/2)\Sigma_{i=1}^d X_i^2$, where the vector fields X_i are given by $X_i = \partial x_i + (Ax)_i\, \partial y$, $1 \leq i \leq d$. Since $[X_i, X_j] = -2A_{ij}\,\partial y$ and $A \neq 0$, the diffusion is hypoelliptic and the process has a C^∞ density $P(t; (x,y), (w,l))$; that is, for every Borel subset C of $\mathbb{R}^{d+1}$,

$$P_{(x,y)}\big((W_t, L_t) \in C\big) = \int_C P\big(t; (x,y), (w,l)\big)\, dw\, dl.$$

For an open set $D \subset \mathbb{R}^{d+1}$, set $T_D = \inf\{t > 0;\ (W_t, L_t) \in D^c\}$. Using this notation, (1.2) can be written as

$$\lim_{t\to\infty} \frac{1}{t} \log P_{(0,0)}(T_G > t) = -a(A), \tag{2.1}$$
$$\text{where } G = \mathbb{R}^d \times (-1,1) \text{ and } 0 < a(A) < \infty.$$

We will now prove (2.1). Let $C_0^\infty(D)$ be the set of all infinitely differentiable functions with compact support contained in D.

Further let $|\ \ |_2$ be the L^2 norm on $L^2(\mathbb{R}^{d+1})$. Next define L_D to be the unique self-adjoint operator on $H_D = L^2(D)$ whose quadratic form is the closure of the form

$$Q(f,g) = (-Lf, g) = \frac{1}{2} \sum_{i:1}^{d} \int_D (X_i f)(X_i g)\, dz, \qquad f, g \in C_0^\infty(D).$$

Set $l(f) = Q(f,f)$.

Then it is easy to see that for every $f \in C_0^\infty(D)$, $L_D f = Lf$. Since the process (W_t, L_t) is continuous and D is open, we have $P_x(T = 0) = 0$, for any $x \in D$. Moreover, $f(W_{t\wedge T_D}, L_{t\wedge T_D}) - \int_0^{t\wedge T_D} Lf(Wu, Lu)\, du$ is a martingale for $f \in C_0^\infty(D)$. It follows that

$$\lim_{t\downarrow 0} \frac{1}{t}\Big(E_{(x,y)}\big(f(W_t, L_t) 1_{\{T_D > t\}}\big) - f(x,y)\Big) = Lf(x,y) = L_D f(x,y)$$

for $(x,y) \in D$. Therefore, the semigroup T_t, defined on all $H_D = L^2(D)$ by

$$T_t f(x,y) = E_{(x,y)}\big(f(W_t, L_t) 1_{\{T_D > t\}}\big),$$

has L_D as its generator. Since $C_0^\infty(D)$ is a core for L_D, it follows that $T_t = e^{tL_D}$.

Combining results of Azencott (1981) and Léandre [(1987), e.g., Théorème 11.3], we can prove that for every convex open set D with compact closure, $\inf_{(w,l)\in K} P_D(t_0;(x,y),(w,l)) > 0$ for every compact $K \subset D$ and for some $t_0 > 0$ [$t_0 = t_0(K)$].

Recall that $G = \mathbb{R}^d \times (-1,1)$.

LEMMA 2.1. *Suppose $f \in C_0^\infty(G)$ and $|f|_2 = 1$. Then*

$$\liminf_{t\to\infty} \frac{1}{t} \log P(T_G > t) \geq -l(f). \tag{2.2}$$

PROOF. Let support $(f) = K \subset G$. Then we can find a convex open set D with compact closure such that $K \subset D \subset \overline{D} \subset G$. Moreover there exists $t_0 > 0$ such that

$$\inf_{(w,l)\in K} P_D\big(t_0;\ (0,0),(w,l)\big) = c > 0.$$

Now $P_{(0,0)}(T_G > t) \geq P_{(0,0)}(T_D > t) = \int_D P_D(t_0;(0,0),(x,y))P_{(x,y)}(T_D > t-t_0)\,dx\,dy$ by the Markov property if $t > t_0$. Therefore,

$$P_{(0,0)}(T_G > t) \geq c\int_K P_{(x,y)}(T_D > t-t_0)\,dx\,dy.$$

Next if $|f|_\infty = \sup_{(x,y)}|f(x,y)|$, then

$$\begin{aligned}\int_K P_{(x,y)}(T_D > t)\,dx\,dy &\geq \int f\big((x,y)\big)E_{(x,y)}\big(f(W_t,L_t)1_{\{T_D>t\}}\big)dx\,dy/|f|_\infty^2\\ &= \int f(x,y)e^{tL_D}f(x,y)\,dx\,dy/|f|_\infty^2 = \big(e^{tL_D}f,f\big)_H/|f|_\infty^2.\end{aligned}$$

Since $|f|_2 = 1$, $E_f(d\lambda)$ is a probability measure. Therefore, using Jensen's inequality and the spectral theorem, we get

$$\big(\exp(tL_D)f,f\big)_H = \int_{-\infty}^0 \exp(\lambda t)E_f(d\lambda) \geq \exp\bigg(t\int_{-\infty}^0 \lambda\,E_f(d\lambda)\bigg).$$

However, $\int_{-\infty}^0 \lambda E_f(d\lambda) = (L_Df,f) = -l(f)$. Hence $(e^{tL_D}f,f) \geq e^{-tl(f)}$ and we can conclude that

$$P_{(0,0)}(T_G > t) \geq c\exp\big(-(t-t_0)l(f)\big)/|f|_\infty^2.$$

Therefore, $\liminf_{t\to\infty}(1/t)\log P_{(0,0)}(T_G > t) \geq -l(f)$. □

Since (2.2) holds for every $f \in C_0^\infty(G)$ such that $|f|_2 = 1$, we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\log P_{(0,0)}(T_G > t) \geq -\inf_{\substack{f\in C_0^\infty(G)\\ |f|_2=1}} l(f). \tag{2.3}$$

We now set

$$a(A) = \inf_{\substack{f \in C_0^\infty(G) \\ |f|_2 = 1}} l(f).$$

REMARK. We can see that $a(A)$ is the infimum of the spectrum of the self-adjoint operator $-L_G$.

LEMMA 2.2.

$$\limsup_{t \to \infty} \frac{1}{t} \log P_{(0,0)}(T_G > t) \le -a(A). \tag{2.4}$$

PROOF. Let $D_t = B_{t^2} \times (-1, 1)$, when B_{t^2} is the open ball centered at 0 with radius t^2, $t > 0$. Clearly,

$$P_{(0,0)}\big(T_G > t, T_{D_t} < t\big) \le P\Big(\sup_{0 \le s \le t} |W_s| \ge t^2\Big) \le ke^{-\delta t^2}$$

for some positive constants k and δ. It follows that

$$\limsup_{t \to \infty} \frac{1}{t} \log P_{(0,0)}(T_G > t) \le \limsup_{t \to \infty} \frac{1}{t} \log P_{(0,0)}\big(T_{D_t} \ge t\big). \tag{2.5}$$

By the Markov property,

$$\begin{aligned} P_{(0,0)}\big(T_{D_t} > t\big) &= \int_{D_t} p_{D_t}\big(1; (0,0), (x,y)\big) P_{(x,y)}\big(T_{D_t} > t - 1\big) dx\,dy \\ &\le c_1 \int_{D_t} P_{(x,y)}\big(T_{D_t} > t - 1\big) dx\,dy, \end{aligned}$$

where $c_1 = \sup_{(x,y)} p(1; (0,0); (x,y)) = p(1; (0,0), (0,0))$ since the characteristic function of (W_1, L_1) is real and positive [see Helmes and Schwane (1983), Corollary 2]. The last inequality holds since

$$p_{D_t}\big(1, (0,0); (x,y)\big) \le p\big(1; (0,0), (x,y)\big) \le c_1.$$

Next set $V_t = \int_{D_t} dx\,dy$ and $\phi_t = 1_{D_t}/V_t^{1/2}$. Then $\phi_t \in L^2(D_t) = H_{D_t}$, $|\phi_t|_2 = 1$ and

$$\int_{D_t} P_{(x,y)}\big(T_{D_t} > t - 1\big) dx\,dy = V_t\Big(\exp\big((t-1)L_{D_t}\big)\phi_t, \phi_t\Big) \le V_t \exp\big((t-1)\lambda_t\big),$$

where

$$\lambda_t = \sup_{\substack{\phi \in \mathrm{Domain}(L_{D_t}) \\ |\phi|_{L^2(D_t)} = 1}} \big(L_{D_t}\phi, \phi\big) = -\inf_{\substack{f \in C_0^\infty(D_t) \\ |f|_2 = 1}} l(f),$$

which follows from the construction of L_{D_t}. Since

$$-\lambda_t \geq \inf_{\substack{f \in C_0^\infty(G) \\ |f|_2 = 1}} l(f) = a(1),$$

we obtain $P_{(0,0)}(T_{D_t} > t) \leq c_1 V_t \exp(-(t-1)a(A))$, $t > 1$. Combining the last inequality with (2.5) we get $\limsup_{t\to\infty} (1/t) \log P_{(0,0)}(T_G > t) \leq -a(A)$, proving the lemma. □

Using (2.3) and (2.4) we have

$$\lim_{t\to\infty} \frac{1}{t} \log P_{(0,0)}(T_G > t) = -a(A).$$

We will now find a representation for $a(A)$.

LEMMA 2.3. *Suppose that the set of nonnull eigenvalues of A is given by $\{\pm ia_1, \dots, \pm ia_{d_0}\}$, where $2d_0 \leq d$. Then $a(A) = (\pi/2)\Sigma_{k=1}^{d_0} |a_k|$.*

PROOF. Since A is skew-symmetric, there exists an orthogonal $d \times d$ matrix O such that $O^T A O = \Delta$, where $\Delta = \operatorname{diag}\{(a_k J)_{1 \leq k \leq d_0}\}$, that is, Δ is the $d \times d$ matrix with 2×2 matrices $a_k J = \begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix}$ along its diagonal and with zero entries elsewhere. It is easy to see that the process $(W_t, L_t^{(A)})_{t \geq 0}$ has the same law as the process $(OW_t, L_t^{(\Delta)})$, where $L_t^{(A)} = \int_0^t \langle AW_s, dW_s\rangle$, and $L_t^{(\Delta)} = \int_0^t \langle \Delta W_s, dW_s\rangle$. Therefore, $a(A) = a(\Delta)$. Moreover $L_t^{(\Delta)}$ is independent of the $(d - 2d_0)$ components of W_t; hence $a(\Delta) = a(\Delta_0)$, where Δ_0 is the $2d_0 \times 2d_0$ matrix defined by $\Delta = \begin{pmatrix} \Delta_0 & 0 \\ 0 & 0 \end{pmatrix}$. Our goal is to show that $a(\Delta_0)$ is the largest eigenvalue of L_{G_0}, where $G_0 = \mathbb{R}^{2d_0} \times (-1, 1)$ and

$$L_{G_0} f(x,y) = \tfrac{1}{2} \sum_{k=1}^{2d_0} \partial_{x_k}^2 f(x,y) + \tfrac{1}{2} |\Delta_0 x|^2 \, \partial_y^2 f(x,y) + \sum_{k=1}^{2d_0} (\Delta_0 x)_k \, \partial_y \partial_{x_k} f,$$

$f \in C_0^\infty(G_0)$, $(x,y) \in G_0$. Using a limiting argument, it can be shown that the last formula also holds for infinitely differentiable functions f that are continuous on $\overline{G}_0$ and such that $f|_{\partial G_0} \equiv 0$. It is the case for the following function f_0 defined by

$$f_0(x,y) = \left(\prod_{k=1}^{d_0} \frac{|a_k|}{2}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\frac{\pi}{2}\right) \sum_{k=1}^{d_0} |a_k| \left(x_{2k-1}^2 + x_{2k}^2\right)\right) \cos \frac{\pi}{2} y,$$
$$x \in \mathbb{R}^{2d_0},\ y \in [-1, 1].$$

Then it is easy to see that $f_0 > 0$ on G_0, $\int_{G_0} f^2(z)\,dz = 1$ and $L_{G_0} f_0 = -\lambda_0 f_0$ on G_0, where $\lambda_0 = (\pi/2)\Sigma_{k=1}^{d_0} |a_k|$. Therefore, $a(\Delta_0) \leq \lambda_0$. We will now prove that $a(\Delta_0) \geq \lambda_0$. So suppose that $f \in C_0^\infty(G_0)$ and $|f|_2 = 1$. Since the support of f is compact and contained in G_0, we see that $K_f = \sup_{x \in G_0} |f(x)|/f_0(x)$ is finite.

Therefore, if $Z_t = (W_t, L_t^{(\Delta_0)})$, we have

$$\begin{aligned}\left(\exp(tL_{G_0})f, f\right) &= \int_{G_0} f(z) E_z\left(f(z_t) 1_{\{T_{G_0} > t\}}\right) dz \\ &\le K_f^2 \int_{G_0} f_0(z) E_z\left(f_0(z_t) 1_{\{T_{G_0} > t\}}\right) dz = K_f^2 \exp(-\lambda_0 t), \qquad t > 0.\end{aligned}$$

On the other hand, using Jensen's inequality, we get

$$\left(\exp(tL_{G_0})f, f\right) \ge \exp\left(t(L_{G_0}f, f)\right) = \exp(-tl(f)), \qquad t > 0.$$

Hence $-\lambda_0 \ge -I(f)$, that is, $\lambda_0 \le l(f)$. Since $a(\Delta_0)$ is the infimum of $I(f)$ over all $f \in C_0^\infty(G_0)$, $|f|_2 = 1$, we obtain $a(\Delta_0) \ge \lambda_0$, completing the proof. □

REMARK. For Lévy's area process $L_t = L_t^{(J)}$, we obtain $a(J) = \pi/2$.

3. Chung's LIL. Set $\phi(t) = \log\log t/t$, $t \ge 3$. The proof of (1.3) is based on the following lemmas.

LEMMA 3.1.

$$P\left(\liminf_{t\to\infty} \phi(t) \sup_{0\le s\le t} |L(s)| \ge a(A)\right) = 1.$$

PROOF. Let r be such that $0 < r < a(A)$. Then we can find $c > 1$ such that $rc < a(A)$. Then

$$\begin{aligned}P\left(\liminf_{t\to\infty} \phi(t) \sup_{0\le s\le t} |L(s)| < r\right) &\le P\left(\inf_{c^n \le t \le c^{n+1}} \phi(t) \sup_{0\le s\le t} |L(s)| < r \quad \text{i.o.}\right) \\ &\le P\left(\frac{\phi(c^n)}{rc} \sup_{0\le s\le c^n} |L(s)| < 1 \quad \text{i.o.}\right).\end{aligned}$$

Using the scaling property of L,

$$P\left(\frac{\phi(c^n)}{rc} \sup_{0\le s\le c^n} |L(s)| < 1\right) = P_{(0,0)}\left(T_G > \frac{c^n\phi(c^n)}{rc}\right).$$

Using (2.4), we see that for any r_1 satisfying $rc < r_1 < a(A)$,

$$P_{(0,0)}\left(T_G > \frac{c^n\phi(c^n)}{rc}\right) \le \exp\left(-\left(\frac{r_1}{rc}\right)\log(n\log c)\right) = (n\log c)^{-r_1/(rc)}$$

if n_0 is large enough. Therefore, using the Borel–Cantelli lemma we obtain

$$P\left(\liminf_{t\to\infty} \phi(t) \sup_{0\le s\le t} |L(s)| < r\right) = 0 \qquad \forall\, 0 < r < a(A).$$

Hence $P(\liminf_{t\to\infty} \phi(t)\sup_{0\le s\le t}|L(s)|\ge a(A)) = 1$. □

LEMMA 3.2. *Set $t_n = n^n$. Then for every $\varepsilon > 0$,*

$$P\Bigl(\phi(t_n)\sup_{0\le s\le t_{n-1}}|L(s)| > \varepsilon \quad \text{i.o.}\Bigr) = 0 \tag{3.1}$$

and

$$P\Bigl(\phi(t_n)\sup_{t_{n-1}\le s\le t_n}\bigl|\langle AW_{t_{n-1}}, W_s - W_{t_{n-1}}\rangle\bigr| > \varepsilon \quad \text{i.o.}\Bigr) = 0. \tag{3.2}$$

PROOF. It follows from Baldi (1986), that there exists $c\in(0,\infty)$ such that with probability 1, $\sup_{0\le s\le t_{n-1}}|L(s)| \le ct_{n-1}^2\phi(t_{n-1})$ eventually. Hence $\phi(t_n)$ $\sup_{0\le s\le t_{n-1}}|L(s)| \le ct_{n-1}^2\phi(t_{n-1})\phi(t_n)\le\varepsilon$ eventually for any $\varepsilon>0$. Thus (3.1) holds true.

Next it is easy to see that

$$P\Bigl(\sup_{0\le s\le t}|\langle h, W_s\rangle| > a\Bigr) \le 2\exp\Bigl(-\lambda a + \frac{\lambda^2 t|h|^2}{2}\Bigr), \qquad \lambda > 0.$$

Therefore,

$$\begin{aligned}
&P\Bigl(\sup_{t_{n-1}\le s\le t_n}\bigl|\langle AW_{t_{n-1}}, W_s - W_{t_{n-1}}\rangle\bigr| > \varepsilon/\phi(t_n)\Bigr)\\
&\quad\le 2\exp\left(-\lambda\varepsilon/\phi(t_n)\right)E\Bigl(\exp\Bigl((\lambda^2/2)(t_n - t_{n-1})|AW_{t_{n-1}}|^2\Bigr)\Bigr)\\
&\quad\le 2\exp\bigl(-\lambda\varepsilon/\phi(t_n)\bigr)\left(1-\lambda^2c_1^2t_{n-1}(t_n-t_{n-1})\right)^{-d/2},
\end{aligned}$$

where c_1 is such that $|Ah|^2 \le c_1^2|h|^2$, $\forall\, h\in\mathbb{R}^d$, and λ is small enough. In particular, if $\lambda = (t_{n-1}(t_n - t_{n-1})/2)^{-1/2}r/c_1$, where $0<r<1$, we get

$$\begin{aligned}
&P\Bigl(\phi(t_n)\sup_{t_{n-1}\le s\le t_n}\bigl|\langle AW_{t_{n-1}}, W_s - W_{t_{n-1}}\rangle\bigr| > \varepsilon\Bigr)\\
&\quad\le 2\left(1-r^2\right)^{-d/2}\exp\Bigl\{-\Bigl((\varepsilon r/c_1)\phi(t_n)\Bigr)\left(t_{n-1}(t_n - t_{n-1})/2\right)^{-1/2}\Bigr\}\\
&\quad= k\exp(-a_n), \quad \text{say.}
\end{aligned}$$

Now $t_n^2/(t_{n-1}(t_n - t_{n-1})) \sim ne$ [here $b_n\sim c_n$ means $\lim_{n\to\infty}(b_n/c_n) = 1$]. Therefore, $a_n \sim (\varepsilon re^{1/2}n^{1/2})/(c_1\log\log n^n))$ and $\Sigma_{n\ge 2}\exp(-a_n) < \infty$.

By the Borel–Cantelli lemma, we can conclude that (3.2) holds true for any $\varepsilon>0$. □

We have already proved that

$$P\Bigl(\liminf_{t\to\infty}\phi(t)\sup_{0\le s\le t}|L(s)|\ge a(A)\Bigr) = 1.$$

To prove our theorem, we only need to prove the following lemma.

LEMMA 3.3.

$$P\left(\liminf_{t\to\infty}\phi(t)\sup_{0\le s\le t}|L(s)|\le a(A)\right)=1.$$

PROOF. To prove the lemma, it is sufficient to show that for any $r > a(A)$,

$$P\left(\phi(t_n)\sup_{0\le s\le t_n}|L(s)|\le r \quad \text{i.o.}\right)=1. \tag{3.3}$$

Fix $r > a(A)$ and choose r_1 such that $r > r_1 > a(A)$.

Define the event B_n as

$$B_n=\left\{2\sup_{0\le s\le t_{n-1}}|L(s)|+\sup_{t_{n-1}\le s\le t_n}\left|\langle AW_{t_{n-1}},W_s-W_{t_{n-1}}\rangle\right|<(r-r_1)/\phi(t_n)\right\}.$$

By Lemma 3.2, $P(B_n^c \text{ i.o.}) = 0$. Hence,

$$\begin{aligned}P&\left(\phi(t_n)\sup_{0\le s\le t_n}|L(s)|\le r\text{ i.o.}\right)\\&\ge P\Bigg(B_n\cap\Bigg\{\phi(t_n)\sup_{t_{n-1}\le s\le t_n}\big|L(s)-L(t_{n-1})\\&\qquad\qquad -\langle AW_{t_{n-1}},W_s-W_{t_{n-1}}\rangle\big|<r_1\Bigg\}\text{ i.o.}\Bigg)\\&=P\left(B_n\cap\left\{\phi(t_n)\sup_{0\le s\le t_n-t_{n-1}}|L_n(s)|<r_1\right\}\text{ i.o.}\right),\end{aligned}$$

where

$$L_n(s)=L(s+t_{n-1})-L(t_{n-1})-\langle AW_{t_{n-1}},W_{s+t_{n-1}}-W_{t_{n-1}}\rangle,\qquad s\ge 0$$

and it is easy to see that $\{L_n(s)\}_{s\ge 0}$ is independent of $F_{t_{n-1}}$ and has the same law as $\{L(s)\}_{s\ge 0}$. Next if A_n and B_n are two sequences of events, then

$$P((A_n\text{ i.o.}))=P\left((A_n\text{ i.o.})\cap(\{B_n^c\text{ i.o.}\})^c\right)\le P(A_n\cap B_n\text{ i.o.})\quad\text{if } P(B_n^c\text{ i.o.})=0.$$

Taking $A_n = \{\phi(t_n)\sup_{0\le s\le t_n-t_{n-1}}|L_n(s)| < r_1\}$ we get $P(\phi(t_n)\sup_{0\le s\le t_n}|L(s)| \le r \text{ i.o.}) \ge P(A_n \text{ i.o.})$. Since $A_n \in F_{t_n}$ and A_n is independent of $F_{t_{n-1}}$, then $P(A_n \text{ i.o.}) = 1$, if $\sum_n P(A_n) = +\infty$. Now from (2.3) we know that if r_2 is chosen so that $r_1 > r_2 > a(A)$, then $P(T_G > t) \ge e^{-tr_2}$ if t is large enough.

Hence

$$
\begin{aligned}
P(A_n) &= P\Big(\phi(t_n) \sup_{0 \le s \le t_n - t_{n-1}} |L_n(s)| < r_1\Big) \\
&= P\Big(T_G > \frac{\phi(t_n)}{r_1}\Big) \ge \exp\Big(-\Big(\frac{t_n}{t_n - t_{n-1}}\Big)(\log\log t_n)\Big(\frac{r_2}{r_1}\Big)\Big) \\
&\ge (n \log n)^{-p}
\end{aligned}
$$

if n is large, where $p > 0$ is chosen so that $r_2/r_1 < p < 1$. Now the series $(n \log n)^{-p}$ diverges, proving that $\sum_{n \ge 2} P(A_n) = +\infty$. Therefore, $P(A_n \text{ i.o}) = 1$, which completes the proof of the theorem. □

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DÉPARTMENT DE MATHÉMATIQUES ET D'INFORMATIQUE
UNIVERSITÉ DU QUÉBEC À TROIS-RIVIÈRES
C.P. 500
TROIS-RIVIÈRES, QUÉBEC
CANADA G9A 5H7