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SUMS OF INDEPENDENT TRIANGULAR ARRAYS AND EXTREME ORDER STATISTICS

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Let $X_{n,i}$ denote an infinitesimal array of independent random variables with convergent partial sums $Z_n = \sum_{i=1}^{n} X_{n,i} - a_n \to_{\mathcal{D}} \xi$. Throughout, we find conditions for the convergence of the portion k_n of lower extremes $L_n(k_n) = \sum_{i=1}^{k_n} X_{i:n} - b_n$ given by order statistics $X_{i:n}$. Similarly, $W_n(r_n)$ denotes the sum of the r_n upper extremes and $M_n = Z_n - L_n - W_n$ stands for the middle part of the sum. It is shown that $(L_n, M_n, W_n) \to_{\mathcal{D}} (\xi_1, \xi_2, \xi_3)$ jointly converges for various sequences $k_n, r_n \to \infty$, where the components of the limit law are independent such that $\xi_1 + \xi_2 + \xi_3 =_{\mathcal{D}} \xi$. The limit of the middle part ξ_2 is asymptotically normal and ξ_1 (ξ_3) gives the negative (positive) spectral Poisson part of ξ. In the case of a compound Poisson limit distribution we obtain rates of convergence that can be used for applications to insurance mathematics.

1. Introduction. The convergence of triangular arrays of independent random variables is one of the most interesting and important classical fields of probability theory. The class of limit distributions is classified by the familiar Lévy–Hinčin formula for infinitely divisible distributions (2.4) given by a normal part and a Poisson part, which is determined by a Lévy measure η. Lévy (1935) first gave a representation of a stable nonnormal law as an infinite sum of an extreme value process. This work was continued by LePage, Woodroofe and Zinn (1981), Csörgő, Csörgő, Horváth and Mason (1986) and Csörgő, Häusler and Mason (1988). The last paper gives an integral representation of infinitely divisible distributions in terms of Poisson processes. They applied the quantile approach via empirical processes to show which part of a normalized partial sum of i.i.d. variables contributes to the Poisson part and which part is asymptotically normal. This approach seems to be restricted to the case of rowwise i.i.d. random variables. The role of a finite number of extremes was studied earlier by Loève (1956). He showed that in the general case the extreme order statistics are always convergent and their distribution is determined by the Lévy measure η.

It is the purpose of this paper to answer the question of which portion of order statistics of the sum is responsible for the Poisson part and the normal component for a general convergent array. Our Theorems 2.1, 2.2 and 3.1 establish conditions for the convergence of sums of extremes. It turns out that the lower, middle and upper parts of the partial sum become independent as $n \to \infty$. We combine the quantile approach of extreme value theory and the classical

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approach via characteristic functions. Earlier considerations of this type appear in Janssen (1989) for i.i.d. arrays with stable limit laws. In a special situation of nonnegative random variables and compound Poisson distributions, we give an upper bound for the amount of upper extreme observations that are needed to find an approximation at a given level of accuracy. This result is motivated by problems coming from insurance mathematics. Rates of convergence similar to Propositions 2.1 and 2.2 were obtained by Janssen and Mason (1990) for i.i.d. partial sums with stable nonnormal limit laws. Csörgő, (1989a, b) introduced rates of convergence for series representation of infinitely divisible laws without normal components.

The paper is organized as follows. Section 2 contains the main results, whereas Section 3 gives auxiliary limit theorems for sums of extremes without centering constants. The series representation of infinitely divisible laws is introduced in Section 4 and the proofs of the main theorems are given in Section 5. The results are based on nonnegative correlation of order statistics. This result is contained in the Appendix.

We will use the following notation. Let 1_A denote the indicator function of a set A and let A^c be its complement. Introduce $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Let $\mathcal{L}(\psi \mid \mu)$ denote the image measure of μ under ψ. Convergence in distribution and in probability is indicated by $\to_{\mathcal{D}}$ and $\to_P$, respectively.

2. Main results. Throughout let $(X_{n,i})_{i=1,\ldots,n}$ be an infinitesimal triangular array of rowwise independent, real-valued random variables (r.v.'s) with

$$\max_{1 \le i \le n} P(|X_{n,i}| > \varepsilon) \to 0 \tag{2.1}$$

as $n \to \infty$ for each $\varepsilon > 0$. Suppose that for suitable centering constants a_n their sums converge in distribution to some infinitely divisible r.v. ξ:

$$\sum_{i=1}^{n} X_{n,i} - a_n \to_{\mathcal{D}} \xi. \tag{2.2}$$

It is often convenient to center the partial sum at a restricted mean. According to Gnedenko and Kolmogorov [(1968), pages 84 and 116ff], there exists a shift $a = a(\tau)$ such that

$$\sum_{i=1}^{n} \Big(X_{n,1} - E\big(X_{n,i} 1_{(-\tau,\tau)}(X_{n,i})\big)\Big) \to_{\mathcal{D}} \xi + a \tag{2.3}$$

in distribution for all continuity points $\pm\tau \neq 0$ that receive 0 mass from the Lévy measure η given in (2.4) (briefly continuity points of η). Then the law $\mu := \mathcal{L}(\xi + a)$ has characteristic function

$$\hat{\mu}(t) = \exp\bigg(-\sigma^2 t^2/2 + \int_{\mathbb{R}\setminus\{0\}} \Big\{\big(\exp(iut) - 1 - iut\big) 1_{(-\tau,\tau)}(u) + \big(\exp(iut) - 1\big) 1_{(-\tau,\tau)^c}(u)\Big\}\, d\eta(u)\bigg) \tag{2.4}$$

based on the possibly unbounded Lévy measure η on $\mathbb{R}\backslash\{0\}$. Recall that

(2.5) $$\int_{\mathbb{R}\backslash\{0\}} (x^2 \wedge 1)\, d\eta(x) < \infty.$$

Observe also that

(2.6) $$\delta^2\big(\eta(-\infty, -\delta] + \eta[\delta, \infty)\big) \le \int (x^2 \wedge \delta^2)\, d\eta(x) \to 0 \quad \text{as } \delta \downarrow 0.$$

Let μ_1 denote the infinitely divisible law with characteristic function (2.4) with $\sigma^2 = 0$ and $\eta_1 := 1_{(-\infty,\,0)}\eta$ (the negative spectral part or negative Poisson part of μ). Similarly μ_2 is by definition the law with $\sigma^2 = 0$ and $\eta_2 := 1_{(0,\,\infty)}\eta$. Obviously, μ is given by the product

(2.7) $$\mu = \mu_1 * N(0, \sigma^2) * \mu_2$$

where $*$ indicates convolution of distributions and $N(b, \sigma^2)$ denotes the normal law with expectation b and variance σ^2. Let now

(2.8) $$X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$$

be the order statistics based on the nth row $X_{n,i}$, $i = 1, \dots, n$. Then it is well known that (2.2) implies

(2.9) $$X_{1:n} \to_{\mathcal{D}} Z$$

in distribution, where $Z \le 0$ is a random variable with distribution function $\exp(-\eta(-\infty, x])$, $x < 0$ [cf. Loève (1956)]. Notice that ξ is normally distributed or degenerate, iff

(2.10) $$X_{1:n} \to_{\mathcal{D}} 0 \quad \text{and} \quad X_{n:n} \to_{\mathcal{D}} 0;$$

see also Gnedenko and Kolmogorov [(1968), page 126].

In this section we will show that a sufficiently large but finite number of extreme order statistics give approximately the entire Poisson part $\mu_1 * \mu_2$ of the limit law μ. The rest of the sum is approximately normal.

For the sequel we introduce the inverse (or quantile function) $\psi_1\colon (0, \infty) \to (-\infty, 0]$ of the Lévy measure η_1 by

(2.11) $$\psi_1(y) = \inf\{t\colon\ \eta_1(-\infty, t] \ge y\} \wedge 0;$$

see Reiss [(1989), Appendix I] as reference for inverse functions. Notice that (i) ψ_1 is nonincreasing and left-sided continuous, (ii) $\psi_1(x) \to 0$ as $x \to \infty$ and (iii) $\int_0^\infty (\psi_1^2(x) \wedge 1)\, dx < \infty$. Property (iii) is based on (2.5) and the quantile representation of Lévy measures given by (iv) $\eta_1 = \mathcal{L}(\psi_1 \mid \lambda_{|(0,\,\infty)})_{|(-\infty,\,0)}$, where $\lambda_{|(0,\,\infty)}$ denotes Lebesgue measure on $(0, \infty)$.

In an analogous manner we introduce the right-sided continuous inverse ψ_2: $(0, \infty) \to [0, \infty)$ of $x \to \eta_2[x, \infty)$, $x > 0$, by

(2.12) $$\psi_2(y) = \sup\{t\colon\ \eta_2[t, \infty) \ge y\} \vee 0.$$

In order to motivate our result we turn to Theorem 4 of Loève's (1956) early paper and recall that, more generally than in (2.9), the joint distribution of the r lowest extremes

$$(X_{1:n},\dots,X_{r:n}) \to_{\mathcal{D}} \mathcal{L}(\psi_1(S_1),\dots,\psi_1(S_r)) \tag{2.13}$$

converges in distribution for fixed $r \in \mathbb{N}$, where throughout

$$S_n := \sum_{i=1}^{n} Y_i \tag{2.14}$$

denotes partial sums of a sequence of i.i.d. exponential distributed r.v.'s Y_i with mean 1. A modern approach to (2.13) can be given in terms of point processes; see, for instance, Resnick [(1987), page 222]. Since the infinite sum

$$\Delta^- := \sum_{i=1}^{\infty} \Big(\psi_1(S_i) - E\Big(\psi_1(S_i)1_{(-\tau,0]}(\psi_1(S_i))\Big)\Big) \tag{2.15}$$

has distribution μ_1 (cf. Lemma 4.1), one may hope that similarly centered sums of extremes

$$\sum_{i=1}^{r} \Big(X_{i:n} - E\big(X_{i:n}1_{(-\tau,\tau)}(X_{i:n})\big)\Big) \tag{2.16}$$

approximate μ_1 well for r large enough. These arguments are made precise in the main theorems, (cf. Remark 2.2). To give further motivation, consider rowwise i.i.d. r.v.s $(X_{n,i})_{i\le n}$ with joint distribution function (d.f.) F_n. It is well known that

$$(X_{1:n},\dots,X_{n:n}) =_{\mathcal{D}} \Big(F_n^{-1}(S_1/S_{n+1}),\dots,F_n^{-1}(S_n/S_{n+1})\Big); \tag{2.17}$$

see Breiman [(1968), Section 13.6] or Reiss [(1989), page 41]. Since the quantile function $\psi_{1,n}(x) = F_n^{-1}(x/n) \wedge 0$ of $nF_{n\,|\,(-\infty,0)}$ converges by (5.4) pointwise to ψ_1, except on a Lebesgue null set of $(0,\infty)$, (2.13) follows easily by the strong law of large numbers.

Let $\widetilde{Y}_i$ denote independent copies of Y_i and set

$$\widetilde{S}_n = \sum_{i=1}^{n} \widetilde{Y}_i, \tag{2.18}$$

$$\Delta^+ := \sum_{i=1}^{\infty} \Big(\psi_2(\widetilde{S}_i) - E\Big(\psi_2(\widetilde{S}_i)1_{[0,\tau)}(\psi_2(\widetilde{S}_i))\Big)\Big). \tag{2.19}$$

We arrange Δ^+ to be independent of Δ^- for the joint limit in (2.41). Notice that $\mathcal{L}(\Delta^+) = \mu_2$.

Consider sequences k_n, m_n, s_n and $r_n \in \{0, 1, \dots, n\}$ and introduce centered sums of order statistics given by

$$L_n(k_n) := \sum_{i=1}^{k_n} \left(X_{i:n} - E\left(X_{i:n} 1_{(-\tau,\tau)}(X_{i:n})\right)\right), \tag{2.20}$$

$$M_n(m_n, s_n) := \sum_{i=m_n+1}^{n-s_n} \left(X_{i:n} - E\left(X_{i:n} 1_{(-\tau,\tau)}(X_{i:n})\right)\right), \tag{2.21}$$

$$W_n(r_n) := \sum_{i=n+1-r_n}^{n} \left(X_{i:n} - E\left(X_{i:n} 1_{(-\tau,\tau)}(X_{i:n})\right)\right), \tag{2.22}$$

which are labelled as the lower, middle and upper parts of the partial sum. Notice that $L_n(k_n) + M_n(k_n, r_n) + W_n(r_n)$ yields a decomposition of (2.3).

We want to find conditions concerning the sequences k_n, m_n, s_n and r_n such that L_n, M_n and W_n converge. The results are split in two cases depending on whether the normal part of μ is trivial or not. If $\sigma^2 = 0$, then almost no restrictions are required. Define

$$q := \begin{cases} 1, & \text{if } \eta(-\infty, 0) > 0, \\ 0, & \text{if } \eta(-\infty, 0) = 0, \end{cases} \qquad p := \begin{cases} 1, & \text{if } \eta(0, -\infty) > 0, \\ 0, & \text{if } \eta(0, -\infty) = 0. \end{cases} \tag{2.23}$$

THEOREM 2.1. *Suppose that the triangular array* (2.3) *converges in distribution and let* $\sigma^2 = 0$ *for the limit distribution* (2.7). *Let each sequence* $j_n = k_n, m_n, n - s_n, n - r_n$ *satisfy the condition*

$$q/(1 + j_n) + p/(n + 1 - j_n) \to 0 \quad \text{as } n \to \infty. \tag{2.24}$$

Then (a) $M_n(m_n, s_n) \to_P 0$ *and* (b) $(L_n(k_n), W_n(r_n)) \to_{\mathcal{D}} (\Delta^-, \Delta^+)$ *as* $n \to \infty$.

If the normal part of μ is not trivial, then obviously additional conditions for k_n and r_n are required. In the case of i.i.d. variables, limit theorems for the middle part can be found in Stigler (1973).

Let $F_{ni}(x) := P(X_{n,i} \le x)$ denote the d.f. of $X_{n,i}$ and introduce

$$G_n(x) := \sum_{i=1}^{n} F_{ni}(x) \tag{2.25}$$

with inverse G_n^{-1}. Also introduce for $0 < y < n$,

$$\overline{G}_n^{-1}(y) := \sup\{t: n - G_n(t-) \ge y\}, \tag{2.26}$$

where $-\overline{G}_n^{-1}$ is the inverse of $y \mapsto \sum_{i=1}^{n} P(X_{n,i} \ge -y)$.

Throughout assume that

$$0 < \liminf_{n\to\infty} \frac{G_n(0)}{n}, \qquad \limsup_{n\to\infty} \frac{G_n(0-)}{n} < 1 \tag{2.27}$$

hold. If (2.27) is violated, G_n may be centered at its "median" and an obvious modification of Theorem 2.2 can be applied.

THEOREM 2.2. *If* (2.3) *converges, consider for* $\sigma^2 > 0$ *sequences such that*

$$k_n + m_n + s_n + r_n = o(n) \tag{2.28}$$

and conditions (2.24) *and* (2.27) *hold. Moreover, define* $j_n := k_n \vee m_n$ *and* $i_n := r_n \vee s_n$ *and let*

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\left\{\int_{(-\infty, G_n^{-1}(j_n))}(x^2\wedge\delta^2)\,dG_n(x) + G_n^{-1}(j_n)^2\Big(j_n - G_n\big(G_n^{-1}(j_n)-\big)\Big)\right\} = 0, \tag{2.29}$$

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\left\{\int_{(\bar{G}_n^{-1}(i_n),\,\infty)}(x^2\wedge\delta^2)\,dG_n(x) + \overline{G}_n^{-1}(i_n-)^2\Big(i_n - \Big(n - G_n\big(\overline{G}_n^{-1}(i_n)\big)\Big)\Big)\right\} = 0, \tag{2.30}$$

be satisfied. Then

$$\big(L_n(k_n), M_n(m_n, s_n), W_n(r_n)\big) \to_{\mathcal{D}} (\Delta^-, Z, \Delta^+) \tag{2.31}$$

as $n \to \infty$ *where* Δ^- *and* Δ^+ *are as in* (2.15) *and* (2.19), *and* Z *is* $N(0,\sigma^2)$ *distributed such that* Δ^-, Z *and* Δ^+ *are independent.*

REMARKS 2.1. (a) It is easy to see that Theorem 2.2 remains true for smaller sequences $k_n' \le k_n$, $m_n' \le m_n$, $s_n' \le s_n$ and $r_n' \le r_n$ as long as (2.24) holds. In fact, condition (2.27) yields $\delta_n := G_n^{-1}(j_n) \to 0$ and thus condition (2.29) is based on the G_n integral of

$$(x^2\wedge\delta^2)\left(1_{(-\infty,\,\delta_n)}(x) + \frac{j_n - G_n(\delta_n-)}{G_n(\delta_n) - G_n(\delta_n-)}1_{\{\delta_n\}}(x)\right) \tag{2.32}$$

for $-\delta < \delta_n$. Thus j_n may be made smaller; see also Remark 3.1(b).

(b) The assumptions (2.29) and (2.30) are very natural conditions, which can be compared with condition (5.7). Observe that (at least in the symmetric case) expressions of the type

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\int(x^2\wedge\delta^2)\,dG_n(x)$$

indicate whether σ^2 vanishes or not.

(c) The rates $k_n, m_n, \ldots$ of Theorem 2.2 are sharp in the sense that they cannot be improved in general; see Example 2.1. However, one cannot speak of a "largest rate" k_n such that, for instance,

$$L_n(k_n) \to_{\mathcal{D}} \Delta^-, \qquad k_n \to \infty \tag{2.33}$$

holds. We may argue as follows. If $\mu \neq \mu_1$, then condition (2.24) implies $n - k_n \to \infty$. Now statement (2.13) implies

$$X_{k_n + r:n} \to_P 0 \tag{2.34}$$

for each $r > 0$. By a subsequence selection principle we may find by (2.34) a sequence $s_n \to \infty$ with $M_n(k_n, k_n + s_n) \to_P 0$, which shows that (2.33) remains valid for $k_n + s_n$.

(d) Under the conditions of Theorems 2.1 and 2.2 we have the joint convergence

$$\big(L_n(k_n), X_{1:n}, \dots, X_{r:n}\big) \to_{\mathcal{D}} \big(\Delta^-, \psi_1(S_1), \dots, \psi(S_r)\big)$$

for fixed $r \in \mathbb{N}$. The proof uses the Cramér–Wold device and the lines of the proofs of the theorems. As an application we have convergence of the slightly trimmed sums:

$$\sum_{i=r+1}^{k_n} \Big\{X_{i:n} - E\big(X_{i:n} 1_{(-\tau,\tau)}(X_{i:n})\big)\Big\}$$
$$\to_{\mathcal{D}} \sum_{i=r+1}^{\infty} \Big\{\psi_1(S_i) - E\Big(\psi_1(S_i) 1_{(-\tau,0)}\big(\psi_1(S_i)\big)\Big)\Big\}.$$

EXAMPLE 2.1. Consider i.i.d. r.v.'s X_i with d.f. F, $E(X_1) = 0$ and $0 < \mathrm{Var}(X_1) < \infty$. If we defined $X_{n,i} = X_i/n^{1/2}$, it is well known that

$$M_n(m_n, s_n) \to_{\mathcal{D}} N\big(0, \mathrm{Var}(X_1)\big) \quad \text{iff } m_n + s_n = o(n). \tag{2.35}$$

It is easy to see that assumptions (2.29) and (2.30) hold for m_n (and s_n, respectively) if (2.35) is valid. Notice that $G_n^{-1}(y) = F^{-1}(y/n)/n^{1/2}$.
Then

$$\int x^2 1_{(-\infty, F^{-1}(m_n/n)/n^{1/2})}(x)\, dG_n(x) = \int x^2 1_{(-\infty, F^{-1}(m_n/n))}(x)\, dF(x) \to 0.$$

The additional term can be written as

$$F^{-1}(m_n/n)^2 \Big\{m_n/n - F\Big(F^{-1}(m_n/n)-\Big)\Big\},$$

which vanishes for $n \to \infty$. This follows from (2.28) whenever the support of F is bounded below. Otherwise $F^{-1}(m_n/n) \to -\infty$ and

$$\int x^2 1_{(-\infty, F^{-1}(m_n/n))}(x)\, dF(x) \to 0$$

gives an upper bound of the term under consideration.

REMARK 2.2. As a further conclusion the motivation (2.13)–(2.16) can be made precise. Let d denote the Lévy metric [(A1) of the Appendix] that gives us convergence in distribution. For each $\varepsilon > 0$ there exist r, $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d\big(L_n(r), \Delta^-\big) \leq \varepsilon \tag{2.36}$$

holds. We may combine (2.13) and Lemma 4.1.

In the special case of nonnegative r.v.'s we will now estimate how many upper extremes are really needed to find an accurate approximation of the total partial sum. The following propositions are motivated by applications to insurance mathematics, where

$$X_{n,i} \geq 0 \tag{2.37}$$

may be interpreted as the total claim of the ith person within a period and n denotes the total number of contracts under consideration. In this example the probability

$$p_{ni} := P(X_{n,i} > 0) \tag{2.38}$$

for the occurrence of a claim is typically small. Define

$$p_n := \sum_{i=1}^{n} p_{ni}. \tag{2.39}$$

Then the aggregate claims $\sum_{i=1}^n X_{ni}$ of all policy holders is approximately given by a sparse sum of extremes.

PROPOSITION 2.1. *Suppose that* (2.37)–(2.39) *hold and let* $m > p_n$. *Then*

$$\begin{aligned} &\sup_{x \geq 0} \left| P\left(\sum_{i=n+2-m}^{n} X_{i:n} \leq x \right) - P\left(\sum_{i=1}^{n} X_{n,i} \leq x \right) \right| \\ &\quad \leq 2 \exp\left(-(m-p_n)^2 \Big/ \left(2m/3 + 4p_n/3 - 2\sum_{i=1}^{n} p_{ni}^2 \right) \right). \end{aligned} \tag{2.40}$$

Notice that the choice $m = [\rho p_n]$ for some $\rho > 1$, where $[\cdot]$ indicates the entire function, gives for large p_n the approximate upper bound

$$2 \exp\Big(-p_n (1-\rho)^2 / (2\rho/3 + 4/3) \Big) \tag{2.41}$$

of (2.40). That result also provides information about the accuracy of the series representation (2.13) and (4.3) of compound Poisson distributions obtained by finite partial sums. Assume that μ has characteristic function

$$\widehat{\mu}(t) = \exp\left(\int_{(0,\infty)} \big(\exp(iut) - 1\big)\, d\eta(u) \right) \tag{2.42}$$

with $\lambda := \eta(0,\infty) < \infty$.

PROPOSITION 2.2. *For each $m > \lambda$ we have*

$$\sup_{x \geq 0} \left| P\left(\sum_{i=1}^{m-1} \psi_2(S_i) \leq x \right) - \mu(-\infty, x] \right| \leq 2\exp\Big(-(m-\lambda)^2 / \big(2m/3 + 4\lambda/3\big)\Big). \tag{2.43}$$

Further material concerning the accuracy of series approximation (2.15) in terms of the Lévy metric can be found in Csörgő (1989a, b).

3. Convergence of sums of extremes. In this section we are interested in the behavior of

$$\widetilde{L}_{k_n}(k_n) := \sum_{i=1}^{k_n} X_{i:n}, \tag{3.1}$$

where the centering constants are cancelled. Based on our motivation in Section 2, one may expect that for certain sequences k_n,

$$\widetilde{L}_n(k_n) \to_{\mathscr{D}} \widetilde{\Delta}^- := \sum_{i=1}^{\infty} \psi_1(S_i) \tag{3.2}$$

whenever the condition

$$\int_{(-1,0)} |x| \, d\eta(x) < \infty \tag{3.3}$$

holds. Then convergence of $\widetilde{\Delta}^-$ follows from Lemma 4.1.

THEOREM 3.1. *Suppose that* (2.3) *converges to an infinitely divisible law such that $\mu_1 \neq \varepsilon_0$ is nontrivial and* (3.3) *holds. Assume that the sequence k_n satisfies the conditions $k_n \to \infty$, $k_n = o(n)$ and*

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \int_{(-\infty, G_n^{-1}(k_n))} \big(|x| \wedge \delta\big)\, dG_n(x) + \left|G_n^{-1}(k_n)\right| \Big(k_n - G_n\big(G_n^{-1}(k_n)-\big)\Big) = 0. \tag{3.4}$$

Then assertion (3.2) *holds.*

REMARKS 3.1. (a) Notice that condition (2.29) implies (3.4) if we take (2.32) into account. Consequently, the portion of lower extremes k_n of Theorem 3.1 is smaller than in Theorem 2.2 provided $\liminf_{n\to\infty} G_n(0)/n > 0$. In the case of Example 2.1, we arrive at a portion $k_n = o(n^{1/2})$ such that (3.2) remains true.

That portion cannot be increased in general. For instance, consider a two point distribution with mean zero in connection with Example 2.1.

(b) As pointed out in Remark 2.1(a), condition (3.4) remains true if k_n is decreased.

(c) Under the conditions of Theorems 2.2 and 3.1 we have

$$\widetilde{L}_n(k_n) - L(k_n) \to \sum_{i=1}^{\infty} E\Big(\psi_1(S_i)1_{(-\tau,0)}\big(\psi_1(S_i)\big)\Big).$$

Thus $(\widetilde{L}_n(k_n), M_n(m_n, s_n), W_n(r_n))$ remain asymptotically independent. Upper extremes can be treated similarly.

4. Series representation of infinitely divisible laws. The following technical lemma gives a representation of "one-sided" infinitely divisible distributions of Poisson type that is related to the form of Csörgő, Häusler and Mason (1988). Special attention is paid to L_j-convergence for $j = 1, 2$.

LEMMA 4.1. *Suppose that* $\mu = \mu_1$ *is as in* (2.7) *with* $\sigma^2 = 0$ *and* $\eta_2 = 0$.

(a) *The infinite sum*

$$\sum_{i=1}^{\infty} \Big(\psi_1(S_i) - E\Big(\psi_1(S_i)1_{(-\tau,0]}\big(\psi_1(S_i)\big)\Big)\Big) \tag{4.1}$$

is convergent in probability and it is distributed according to μ.

(b) *The sum* (4.1) *is convergent in* $L_j(P)$ *whenever the condition*

$$\int_{(-\infty,-1]} |x|^j \, d\eta_1(x) < \infty \tag{4.2}$$

holds for $j \in \{1, 2\}$

(c) *Under condition* (3.3) *the infinite sum*

$$\sum_{i=1}^{\infty} \psi_1(S_i) \tag{4.3}$$

is almost surely finite and its characteristic function is given by

$$t \mapsto \exp\Big(\int_{(-\infty,0)} \big(\exp(iut) - 1\big) \, d\eta_1(u)\Big). \tag{4.4}$$

(d) *Under conditions* (3.3) *and* (4.2) *the series* (4.3) *converges in* $L_j(P)$.

REMARKS 4.1. The proof is based on the following elementary properties of quantile functions for Lévy measures. Let ψ_1 be as in (2.11) and choose $\delta < 0$.

(a) Then $\eta_\delta := \eta_{|(-\infty,\delta]}$ has the inverse

$$\psi_{1,\delta}(y) = \psi_1(y)1_{(-\infty,\delta]}\big(\psi_1(y)\big). \tag{4.5}$$

(b) The inverse of $\eta^{(\delta)} := \eta_{|(\delta,0)} + \eta(-\infty,\delta]\varepsilon_\delta$ is given by

$$\psi_1^{(\delta)}(y) = \psi_1(y)1_{(\delta,0)}(\psi_1(y)) + \delta 1_{(-\infty,\delta]}(\psi_1(y)). \tag{4.6}$$

(c) Let X denote a Poisson r.v. with mean $\lambda > 0$. Then δX and (4.3) are equal in distribution for $\delta < 0$ provided $\psi_1 = \delta 1_{(0,\lambda]}$.

PROOF OF LEMMA 4.1.

Step 1. Assume that $\eta_1(\delta, 0) = 0$ for some $\delta < 0$. Thus η_1 is bounded and $\psi_1(y) = 0$ whenever $y > \eta_1(-\infty, 0)$. Consequently, the series (4.3) converges almost surely. We now determine its limit law. Consider a sequence $(U_n)_n$ of i.i.d uniformly distributed r.v.s over (0, 1) and define

$$X_{n,i} := \psi_1(nU_i), \qquad i \le n. \tag{4.7}$$

As shown below, the partial sum $\sum_{i=1}^n X_{n,i}$ converges in distribution to a compound Poisson r.v. X with characteristic function (4.4). One easily verifies the sufficient conditions (5.2)–(5.5) for $\tau < |\delta|$. In order to do so, notice that for $x < 0$,

$$\psi_1(nU_1) \le x \quad \text{iff } nU_1 \le \eta(-\infty, x] \tag{4.8}$$

and thus

$$nP(X_{n,1} \le x) = nP(U_1 \le \eta(-\infty, x]/n) = \eta(-\infty, x] \wedge n.$$

Observe also that $X_{n,i}1_{(-\varepsilon,\varepsilon)}(X_{n,i}) = 0$ for $|\delta| > \varepsilon > 0$ [cf. (4.5)]. The proper choice of order statistics $U_{i:n}$ [see (2.17)] implies

$$\sum_{i=1}^n X_{n,i} =_{\mathcal{D}} \sum_{i=1}^n \psi_1(nS_i/S_{n+1}),$$

which converges almost surely to (4.3) by the strong law of large numbers. Thus $X =_{\mathcal{D}} \sum_{i=1}^\infty \psi_1(S_i)$ holds.

Suppose now in addition that condition (4.2) holds. Routine differentiation of the characteristic function (4.4) proves that $E(X)$ exists for $j = 1$ and $\mathrm{Var}(X)$ exists for $j = 2$ with

$$E(X) = \int_{(-\infty,0)} x\, d\eta_1(x), \qquad \mathrm{Var}(X) = \int_{(-\infty,0)} x^2\, d\eta_1(x). \tag{4.9}$$

For these reasons the monotone convergence theorem proves L_1-convergence (and L_2-convergence, respectively) of the partial sums $\sum_{i=1}^n \psi_1(S_i)$ and

$$\sum_{i=1}^n E(\psi_1(S_i)) \to E(X) = \int_{(-\infty,0)} x\, d\eta_1(x) \tag{4.10}$$

as $n \to \infty$. Hence assertions (b)–(d) are proved for the special type of Lévy measures of Step 1. Next the influence of the centering constants given in (4.1) is studied. For this reason consider (4.6) for $-\tau$, which implies

$$\sum_{i=1}^{\infty} E\left(\psi_1^{(-\tau)}(S_i)\right) = \int_{[-\tau,\,0)} x\, d\eta^{(-\tau)}(x). \tag{4.11}$$

Similarly, consideration of $\tilde{\eta} = \eta(-\infty, -\tau]\varepsilon_{-\tau}$ yields the related $\tilde{\psi}_1$ function

$$\tilde{\psi}_1(y) = -\tau \mathbf{1}_{(0,\,\eta(-\infty,\,-\tau]]}(y) = -\tau \mathbf{1}_{(-\infty,\,-\tau]}\big(\psi_1(y)\big)$$

by considerations similar to (4.8). Thus

$$\sum_{i=1}^{\infty} E\left(-\tau \mathbf{1}_{(-\tau,0)}(S_i)\right) = \int_{(-\infty,\,0)} x\, d\tilde{\eta}(x). \tag{4.12}$$

Combination of (4.11) and (4.12) proves

$$\sum_{i=1}^{\infty} E\Big(\psi_1(S_i)\mathbf{1}_{(-\tau,\,0)}\big(\psi_1(S_i)\big)\Big) = \int_{(-\tau,\,0)} x\, d\eta_1(x), \tag{4.13}$$

which gives the difference of (4.1) and (4.3). Altogether, (4.4) and (4.13) yield the desired form of the characteristic function of (4.1) in Step 1.

The general situation uses a truncation argument that is based upon (4.5). For $\delta < 0$, introduce

$$\Delta_\delta^- := \sum_{i=1}^{\infty} \left\{ \psi_1(S_i)\mathbf{1}_{(-\infty,\,\delta]}\big(\psi_1(S_i)\big) - E\Big(\psi_1(S_i)\mathbf{1}_{(-\tau,\,\delta]}\big(\psi_1(S_i)\big)\Big) \right\}. \tag{4.14}$$

LEMMA 4.2. *Suppose that* (4.2) *holds for* $j = 2$. *Then* Δ_δ^- *is an* L_2*-Cauchy sequence as* $\delta \uparrow 0$. *Its limit distribution has characteristic function* (2.4) *for* $\mu = \mu_1$.

PROOF. According to Step 1 of the proof of Lemma 4.1 and (4.5) the r.v. Δ_δ^- has characteristic function (2.4) with Lévy measure η_δ and $\sigma^2 = 0$. The continuity theorem for characteristic functions implies convergence in distribution of Δ_δ^- as $\delta \uparrow 0$.

The L_2-convergence is based on the variance formula (4.9). For fixed $\tau > 0$, consider $-\tau < \delta < \alpha < 0$. Then $E(\Delta_\alpha^- - \Delta_\delta^-) = 0$ and

$$\begin{aligned} \operatorname{Var}\big(\Delta_\alpha^- - \Delta_\delta^-\big) &= \operatorname{Var}\left(\sum_{i=1}^{\infty}\left\{\psi_1(S_i)\mathbf{1}_{(\delta,\,\alpha]}\big(\psi_1(S_i)\big)\right\}\right) \\ &\le 2\left\{\operatorname{Var}\left(\sum_{i=1}^{\infty}\left\{\psi_1(S_i)\mathbf{1}_{(\delta,\,\alpha]}\big(\psi_1(S_i)\big) + \delta\mathbf{1}_{(-\infty,\,\delta]}\big(\psi_1(S_i)\big)\right\}\right)\right. \\ &\qquad \left. + \operatorname{Var}\left(\sum_{i=1}^{\infty}\delta\mathbf{1}_{(-\infty,\,\delta]}\big(\psi_1(S_i)\big)\right)\right\} =: 2(V_1 + V_2). \end{aligned} \tag{4.15}$$

By (4.5), V_1 is the variance of a r.v. with Lévy measure $\eta_{|(\delta,\alpha]} + \eta(-\infty,\delta]\varepsilon_\delta$:

$$V_1 = \int_{(\delta,\alpha]} u^2\,d\eta(u) + \delta^2\eta(-\infty,\delta]. \tag{4.16}$$

Up to a constant, the second term is a Poisson r.v. with variance

$$V_2 = \delta^2\eta(-\infty,\delta]. \tag{4.17}$$

The integrability condition (2.6) of the Lévy measure ensures that $V_1 + V_2$ is small whenever δ is sufficiently small. Thus Δ_δ^- is an L_2-Cauchy sequence. □

Step 2 (of the proof of Lemma 4.1). Throughout we establish the proof for Lévy measures with bounded support, which gives the boundedness of ψ_1. Introduce partial sums

$$\Delta^-(n) := \sum_{i=1}^{n}\Big\{\psi_1(S_i) - E\Big(\psi_1(S_i)1_{(-\tau,0)}\big(\psi_1(S_i)\big)\Big)\Big\} \tag{4.18}$$

and similarly $\Delta_\delta^-(n)$ for $\psi_1(y)1_{(-\infty,\delta]}(\psi_1(y))$ instead of ψ_1 for $\delta < 0$. Also define remainder terms $R_\delta(n)$ by

$$\Delta^-(n) = \Delta_\delta^-(n) + R_\delta(n). \tag{4.19}$$

Employing Step 1 and Lemma 4.2, we have $\Delta_\delta^-(n) \to \Delta_\delta^-$ and $\Delta_\delta^- \to \Delta^-$ as $\delta \uparrow 0$ in L_2, where Δ^- denotes the limit r.v. of Lemma 4.2. By Lemma B of the Appendix it remains to check that the remainders become uniformly small in L_2. To be explicit it suffices to show

$$\lim_{\delta\uparrow 0}\limsup_{n\to\infty}\operatorname{Var}\big(R_\delta(n)\big) = 0 \tag{4.20}$$

since $E(R_\delta(n)) = 0$ for $\delta > -\tau$. Obviously

$$\operatorname{Var}\big(R_\delta(n)\big) = \lim_{\alpha\uparrow 0}\operatorname{Var}\big(\Delta_\alpha^-(n) - \Delta_\delta^-(n)\big), \tag{4.21}$$

and analogously to (4.15),

$$\begin{aligned}\operatorname{Var}&\big(\Delta_\alpha^-(n) - \Delta_\delta^-(n)\big)\\ &\le 2\Bigg\{\operatorname{Var}\Bigg(\sum_{i=1}^{n}\Big\{\psi_1(S_i)1_{(\delta,\alpha]}\big(\psi_1(S_i)\big) + \delta 1_{(-\infty,\delta]}\big(\psi_1(S_i)\big)\Big\}\Bigg)\\ &\qquad + \operatorname{Var}\Bigg(\sum_{i=1}^{n}\delta 1_{(-\infty,\delta]}\big(\psi_1(S_i)\big)\Bigg)\Bigg\} =: a_n\end{aligned} \tag{4.22}$$

for $\delta < \alpha < 0$. Next let φ denote any bounded nondecreasing function. Employing Lemma A and (2.17), we obtain

$$\operatorname{Cov}\big(\varphi(S_i),\varphi(S_j)\big) = \lim_{n\to\infty}\operatorname{Cov}\left(\varphi\left(\frac{nS_i}{S_{n+1}}\right),\varphi\left(\frac{nS_j}{S_{n+1}}\right)\right) \ge 0. \tag{4.23}$$

This result is applied to (4.22), which gives

$$a_n \le a_{n+m}, \qquad m \in \mathbb{N}. \tag{4.24}$$

On the other hand, we may make use of the L_2-convergence (Step 1) of the partial sums of the right-hand side of (4.22) showing $a_n \uparrow 2(V_1 + V_2)$, where V_1 and V_2 are taken from (4.15). The choice of $\alpha \uparrow 0$ yields

$$\operatorname{Var}(R_\delta(n)) \le 2\int_{(\delta,0)} u^2\,d\eta(u) + 4\delta^2\eta(-\infty,\delta], \tag{4.25}$$

which converges to zero as $\delta \uparrow 0$.

Step 3. *The general case.* Let η be an arbitrary Lévy measure on $(-\infty, 0)$. For $-\tau/2 < \delta < 0$ let us split the Lévy measure and consider η_δ, $\eta^{(\delta)}$ and their ψ-functions (4.5) and (4.6). Thus

$$\psi_1^{(\delta)} + \psi_{1,\delta} = \psi_1 + \overline{\psi}, \tag{4.26}$$

where $\overline{\psi} := \delta 1_{(-\infty,\delta]}(\psi_1(\cdot))$ is the ψ-function of a Poisson r.v. Note that the series (4.1) and (4.3) with ψ replaced by $\psi_1^{(\delta)}$, $\psi_{1,\delta}$ and $\overline{\psi}$ are also convergent in L_j under the assumptions of (b)–(d). Then (4.26) implies the convergence of (4.1) and (4.3). If we now consider the $\psi_1^{(\delta)}$ series for $\delta \to -\infty$, the continuity theorem for characteristic functions determines their limit distribution. This completes the proof of Lemma 4.1. □

REMARK 4.2. The series representation can be compared with the integral representation of μ_1 of Csörgő, Häusler and Mason (1988) given by improper Lebesgue–Stieltjes integrals. Let $X_t = -\sum_{i=1}^{\infty} 1_{(0,t)}(S_i)$ denote a left-continuous Poisson process on $(-\infty, 0)$. Under the conditions of Lemma 4.1., the r.v. (4.3) and

$$\int_{(0,\infty)} X_s\,d\psi_1(s) \tag{4.27}$$

are equal in distribution. Moreover, it can be shown that (4.1) and

$$\int_{(0,\infty)} \big(X_s - s1_{[\eta(-\infty,-\tau],\infty)}(s)\big)\,d\psi_1(s) - \psi_1\big(\eta(-\infty,-\tau]\big)\eta(-\infty,-\tau] \tag{4.28}$$

have the same distribution. That proof must be obtained first for η with $\eta(\delta, 0) = 0$, $\delta < 0$. Here one uses integration by parts and (4.13).

5. Technical results and the proofs. We begin with the well-known inequality of Bernstein [(5.1); cf. Bennett (1962)]. Let $X_1, \dots, X_n$ be independent r.v.'s such that $\sigma_i^2 = \operatorname{Var}(X_i)$ and $|X_i - E(X_i)| \le 1$ almost surely for $i = 1, \dots, n$. Then for each $\varepsilon > 0$,

$$P\left(\left|\sum_{i=1}^{n}(X_i - E(X_i))\right| \ge \varepsilon\right) \le 2\exp\left(-\varepsilon^2 \Big/ \left(2\sum_{i=1}^{n}\sigma_i^2 + 2\varepsilon/3\right)\right). \tag{5.1}$$

Throughout we use a few technical tools concerning the convergence of triangular arrays quoted from Gnedenko and Kolmogrov [(1968), page 116]. The partial sums (2.3) have the limit distribution (2.4) iff the following conditions (5.2)–(5.5) hold:

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \operatorname{Var}\left(\sum_{i=1}^{n} X_{n,i} 1_{(-\delta,\delta)}(X_{n,i}) \right) = \sigma^2, \tag{5.2}$$

$$\lim_{\delta \downarrow 0} \liminf_{n \to \infty} \operatorname{Var}\left(\sum_{i=1}^{n} X_{n,i} 1_{(-\delta,\delta)}(X_{n,i}) \right) = \sigma^2. \tag{5.3}$$

Restricted to continuity points x of η, one has

$$\sum_{i=1}^{n} P(X_{n,i} \le x) \to \eta(-\infty, x], \qquad x < 0, \tag{5.4}$$

and

$$\sum_{i=1}^{n} P(X_{n,i} \ge x) \to \eta[x, \infty), \qquad x > 0. \tag{5.5}$$

Further equivalent conditions are needed.

LEMMA 5.1. *Assume that the array is infinitesimal* [*see* (2.1)] *and let* (5.4) *and* (5.5) *be satisfied. Define* $\alpha_{ni} := E(X_{n,i} 1_{(-\tau,\tau)}(X_{n,i}))$. *Then condition* (5.2) *and either of the following are equivalent*:

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sum_{i=1}^{n} \int_{|x| < \delta} x^2 \, dF_{ni}(x + \alpha_{ni}) = \sigma^2, \tag{5.6}$$

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sum_{i=1}^{n} \int (x^2 \wedge \delta^2) \, dF_{ni}(x + \alpha_{ni}) = \sigma^2. \tag{5.7}$$

The equivalence also holds if $\limsup_{n \to \infty}$ *is replaced by* $\liminf_{n \to \infty}$.

PROOF. The equivalence of (5.2) and (5.6) is already contained in Gnedenko and Kolmogorov [(1968), page 119ff]. On the other hand, (5.4) and (5.5) together with (2.6) yield

$$\delta^2 \sum_{i=1}^{n} P(|X_{n,i}| \ge \delta) \to \delta^2 \int_{|x| \ge \delta} x^2 \, d\eta(x), \tag{5.8}$$

which becomes arbitrarily small as $\delta \downarrow 0$ restricted to continuity points δ. □

The proof of Theorem 2.1 requires the following lemma.

LEMMA 5.2. *Let X denote a Poisson r.v. with mean $\lambda = \eta(-\infty, -\delta]$, where $-\delta < 0$ denotes a continuity point of η. Then for each $k \in \mathbb{N}$,*

$$\lim_{n \to \infty} \sum_{i=k+1}^{n} P(X_{i:n} \le -\delta) = E\big(X 1_{(k,\infty)}(X)\big). \tag{5.9}$$

PROOF. By Poisson's limit theorem one gets

$$\xi_n := \sum_{i=1}^{n} 1_{(-\infty,-\delta]}(X_{n,i}) \to_{\mathcal{D}} X$$

and

$$E(\xi_n \wedge k) \to E(X \wedge k) \quad \text{as } n \to \infty.$$

Notice that

$$E(\xi_n) = \sum_{i=1}^{n} P(X_{i:n} \le -\delta) = E(X)$$

by (5.4) and

$$E(\xi_n \wedge k) = \sum_{i=1}^{k} P(X_{i:n} \le -\delta),$$

which proves (5.9). □

PROOF OF THEOREM 2.1. (a) Condition (2.24) together with (2.13) implies

$$X_{m_n:n} \to_P 0 \quad \text{and} \quad X_{n-s_n:n} \to_P 0. \tag{5.10}$$

In the next step we will use the identity

$$x = (-\delta) \vee x \wedge \delta + (x-\delta) 1_{[\delta,\infty)}(x) + (x+\delta) 1_{(-\infty,-\delta]}(x)$$

for $\delta > 0$, which gives us the relation

$$M_n(m_n, s_n) = Z_n^\delta + X_n^\delta + Y_n^\delta. \tag{5.11}$$

The r.v.'s on the right-hand side are defined for continuity points $\pm\delta$ of η, $0 < \delta < \tau$, by

$$Z_n^\delta := \sum_{i=m_n+1}^{n-s_n} \Big\{(-\delta) \vee X_{i:n} \wedge \delta - E\big((-\delta) \vee X_{i:n} \wedge \delta\big)\Big\}, \tag{5.12}$$

$$X_n^\delta := \sum_{i=m_n+1}^{n-s_n} \Big\{(X_{i:n} - \delta) 1_{[\delta,\infty)}(X_{i:n}) - E\big((X_{i:n} - \delta) 1_{[\delta,\tau)}(X_{i:n})\big)\Big\}, \tag{5.13}$$

$$Y_n^\delta := \sum_{i=m_n+1}^{n-s_n} \Big\{(X_{i:n} + \delta) 1_{(-\infty,-\delta]}(X_{i:n}) - E\big((X_{i:n} + \delta) 1_{(-\tau,-\delta]}(X_{i:n})\big)\Big\}. \tag{5.14}$$

By Lemma 5.2, the exceptional part of X_n^δ asymptotically vanishes since

$$\left|\sum_{i=m_n+1}^{n-s_n} E\big((X_{i:n}+\delta)1_{(-\tau,-\delta]}(X_{i:n})\big)\right| \le \tau \sum_{i=m_n+1}^{n-s_n} P(X_{i:n} \le -\delta) \to 0 \tag{5.15}$$

as $n \to \infty$. A similar result holds for Y_n^δ. If we now combine (5.10) and (5.15), then we obtain

$$X_n^\delta + Y_n^\delta \to_P 0 \tag{5.16}$$

for fixed δ. According to Lemma B of the Appendix it remains to show

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \operatorname{Var}(Z_n^\delta) = 0. \tag{5.17}$$

The variances can be bounded by Lemma A of the Appendix, which is applied to $\varphi(x) = (-\delta) \vee x \wedge \delta$. Since order statistics are nonnegatively correlated, we have

$$\begin{aligned}\operatorname{Var}(Z_n^\delta) &\le \operatorname{Var}\left(\sum_{i=1}^n (-\delta) \vee X_{i:n} \wedge \delta\right) = \sum_{i=1}^n \operatorname{Var}\big((-\delta)\vee X_{n,i} \wedge \delta\big)\\ &\le 2\left\{\sum_{i=1}^n \operatorname{Var}\big(X_{n,i}1_{(-\delta,\delta)}(X_{n,i})\big)\right.\\ &\quad \left. + \sum_{i=1}^n \operatorname{Var}\big(-\delta 1_{(-\infty,-\delta]}(X_{n,i}) + \delta 1_{[\delta,\infty)}(X_{n,i})\big)\right\}.\end{aligned} \tag{5.18}$$

Since $\sigma^2 = 0$, the convergence condition (5.2) proves that the first sum of the right-hand side of (5.18) converges to zero if first $n \to \infty$ and then $\delta \downarrow 0$. On the other hand, observe that the second sum is bounded above by

$$\delta^2 \sum_{i=1}^n P(|X_{n,i}| \ge \delta) \to \delta^2 \int_{|x|\ge\delta} x^2\, d\eta(x), \tag{5.19}$$

which becomes arbitrary small for $\delta \downarrow 0$ [cf. (2.6)].

(b) Throughout we will prove convergence of the marginal distributions of L_n and W_n. We restrict ourselves to L_n since the proof for the upper extremes is similar. Following (2.13) we have

$$L_n(m) \to_{\mathcal{D}} \Delta^-(m) := \sum_{i=1}^m \Big\{\psi_1(S_i) - E\Big(\psi_1(S_i)1_{(-\tau,0)}\big(\psi_1(S_i)\big)\Big)\Big\} \tag{5.20}$$

for each $m \in \mathbb{N}$. Let d be the Lévy metric of the topology of convergence in distribution (A1). We will now construct a sequence $m_n \to \infty$, $m_n \le n/2$, such that

$$d\big(L_n(m_n), \Delta^-(m_n)\big) \to 0 \quad \text{as } n \to \infty. \tag{5.21}$$

The sequence m_n is defined as follows. For fixed j we may choose $n_j > 2j$ and $n_j > n_{j-1}$ with

$$d\big(L_n(j), \Delta^-(j)\big) \le 1/j, \tag{5.22}$$

whenever $n > n_j$. If we set $n_0 = 0$, we have

$$\mathbb{N} = \bigcup_{j=0}^{\infty} (n_j, n_{j+1}]. \tag{5.23}$$

For each $n \in \mathbb{N}$ we have exactly one index j with $n \in (n_j, n_{j+1}]$. If we take $m_n = j$, we obtain

$$d\big(L_n(m_n), \Delta^-(m_n)\big) \le 1/m_n, \tag{5.24}$$

which implies (5.21). Since $\Delta^-(m_n) \to_{\mathfrak{D}} \Delta^-$ holds (see Lemma 4.1.), statement (5.21) proves

$$L_n(m_n) \to_{\mathfrak{D}} \Delta^-. \tag{5.25}$$

On the other hand, observe that for $m_n \le k_n$,

$$L_n(k_n) = L_n(m_n) + M_n(m_n, k_n), \tag{5.26}$$

where the middle part of (5.26) vanishes by part (a). An obvious modification of (5.26) also applies in the case $m_n > k_n$. Thus (5.25) proves the desired convergence of $L_n(k_n)$.

The asymptotic independence of the three parts of the Theorems 2.1 and 2.2 is based on the splitting lemma, which is proved first. □

LEMMA 5.3. *There exists a sequence* $0 \le \tau_n \to 0$ *such that*

$$Y_n := (R_n, S_n, T_n) \to_{\mathfrak{D}} Y := (R, S, T), \tag{5.27}$$

where $\mathfrak{L}(Y) = \mu_1 \otimes N(0, \sigma^2) \otimes \mu_2$ *and*

$$R_n := \sum_{i=1}^{n} \Big\{X_{n,i} 1_{(-\infty, -\tau_n)}(X_{n,i}) - E\big(X_{n,i} 1_{(-\tau, -\tau_n)}(X_{n,i})\big)\Big\},$$

$$S_n := \sum_{i=1}^{n} \Big\{X_{n,i} 1_{[-\tau_n, \tau_n]}(X_{n,i}) - E\big(X_{n,i} 1_{[-\tau_n, \tau_n]}(X_{n,i})\big)\Big\},$$

$$T_n := \sum_{i=1}^{n} \Big\{X_{n,i} 1_{(\tau_n, \infty)}(X_{n,i}) - E\big(X_{n,i} 1_{(\tau_n, \tau)}(X_{n,i})\big)\Big\}.$$

An additional result is needed:

LEMMA 5.4. *Let* $x_n \le 0,\ x_n \to 0$. *If* (2.3) *converges, then the condition*

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \int_{(-\infty, x_n)} \left(x^2 \wedge \delta^2\right) dG_n(x) = 0 \tag{5.28}$$

implies

$$\sum_{i=1}^{n} \left\{X_{n,i} 1_{(-\infty, x_n)}(X_{n,i}) - E\left(X_{n,i} 1_{(-\tau, x_n)}(X_{n,i})\right)\right\} \to_{\mathcal{D}} R, \tag{5.29}$$

where $\mathcal{L}(R) = \mu_1$.

PROOF. According to (5.2)–(5.7) it remains to check the condition

$$\lim_{\delta \uparrow 0} \limsup_{n \to \infty} \sum_{i=1}^{n} \int_{(\delta, x_n)} x^2 \, dF_{ni}(x + \beta_{ni}) = 0, \tag{5.30}$$

where $\beta_{ni} = \int_{(-\tau, x_n)} x \, dF_{ni}(x)$. Since $\beta_{ni} \le 0$, we have for $\delta < 0$,

$$\begin{aligned} \int_{(\delta, x_n)} x^2 \, dF_{ni}(x + \beta_{ni}) &\le \int_{(-\infty, x_n + \beta_{ni})} \left((x - \beta_{ni})^2 \wedge \delta^2\right) dF_{ni}(x) \\ &\le \int_{(-\infty, x_n)} \left(x^2 \wedge \delta^2\right) dF_{ni}(x). \end{aligned} \tag{5.31}$$

Thus assumption (5.28) implies (5.30). □

PROOF OF LEMMA 5.3. Choose a sequence $0 \le \tau_n \to 0$ such that

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \int_{[-\tau_n, \tau_n]^c} \left(x^2 \wedge \delta^2\right) dG_n(x) = 0 \tag{5.32}$$

holds. This sequence exists since (5.19) implies (5.32) for each sequence of positive constants. In the next step the convergence of the one-dimensional distributions of R_n, S_n and T_n will be established. By Lemma 5.4 we have

$$R_n \to_{\mathcal{D}} R, \qquad T_n \to_{\mathcal{D}} T. \tag{5.33}$$

Thus (5.2) applied to $X_{n,i} 1_{(-\infty, -\tau_n)}(X_{n,i})$ proves

$$\begin{aligned} &\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sum_{i=1}^{n} \operatorname{Var}\left(X_{n,i} 1_{(-\delta, \delta) - [-\tau_n, \tau_n]}(X_{n,i})\right) \\ &\quad \le \lim_{\delta \downarrow 0} \limsup_{n \to \infty} 2 \sum_{i=1}^{n} \left\{\operatorname{Var}\left(X_{n,i} 1_{(-\delta, -\tau_n)}(X_{n,i})\right)\right. \\ &\qquad\qquad \left. + \operatorname{Var}\left(X_{n,i} 1_{(\tau_n, \delta)}(X_{n,i})\right)\right\} = 0. \end{aligned} \tag{5.34}$$

The treatment of S_n requires the variance inequality

$$|\mathrm{Var}(X_n) - \mathrm{Var}(Z_n)| \le \mathrm{Var}(X_n - Z_n) + 2\big(\mathrm{Var}(X_n - Z_n)\mathrm{Var}(Z_n)\big)^{1/2}. \tag{5.35}$$

A suitable choice of X_n and Z_n together with (5.2) and (5.34) now proves

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \mathrm{Var}\big(X_{n,i} 1_{[-\tau_n,\tau_n]}(X_{n,i})\big) = \sigma^2 \tag{5.36}$$

and the same for $\liminf_{n\to\infty}$, which proves $S_n \to_{\mathcal{D}} S$ in view of the validity of the conditions (5.2)–(5.5).

The independence proof uses the following device. Let

$$\widetilde{Y}_n = (\widetilde{R}_n, \widetilde{S}_n, \widetilde{T}_n) \tag{5.37}$$

be copies of Y_n such that $\widetilde{R}_n$, $\widetilde{S}_n$ and $\widetilde{T}_n$ are independent. Due to the Cramér–Wold device it is enough to show that the limit distributions of λY_n^{T} and $\lambda \widetilde{Y}_N^{\mathsf{T}}$ coincide for each $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ since obviousl $\lambda \widetilde{Y}_n^{\mathsf{T}} \to_{\mathcal{D}} \lambda Y^{\mathsf{T}}$. For these reasons, define

$$Y_{n,i} := \lambda_1 X_{n,i} 1_{(-\infty,-\tau_n)}(X_{n,i}) + \lambda_2 X_{n,i} 1_{[-\tau_n,\tau_n]}(X_{n,i}) + \lambda_3 X_{n,i} 1_{(\tau_n,\infty)}(X_{n,i}), \tag{5.38}$$

whose partial sum up to a shift coincides with λY_n^{T}. By (5.34)–(5.36) one gets

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sum_{i=1}^{n} \mathrm{Var}\big(Y_{n,i} 1_{(-\delta,\delta)}(Y_{n,i})\big) = \lambda_2^2 \sigma^2 \tag{5.39}$$

and the same statement for $\liminf_{n\to\infty}$. On the other hand, we can calculate the Lévy measure η_λ at all continuity points $x < 0$ as

$$\begin{aligned} \sum_{i=1}^{n} P(Y_{n,i} \le x) &\to \eta\big(\{y\colon \lambda_1 y \le x,\ y < 0\} \cup \{y\colon \lambda_3 y \le x,\ y > 0\}\big) \\ &= \eta_\lambda\big((-\infty, x]\big) \end{aligned} \tag{5.40}$$

and similarly for $x > 0$. Thus η_λ coincides with the Lévy measure of λY^{T}. Moreover, the shift, which ensures convergence of λY_n^{T}, is only based on expectations [cf. Gnedenko and Kolmogorov (1968), page 117] and it is the same for λY_n^{T} and $\lambda \widetilde{Y}_n^{\mathsf{T}}$. All together we see that λY_n^{T} converges in distribution to the same limit distribution as $\lambda \widetilde{Y}_n^{\mathsf{T}}$. □

THE INDEPENDENCE PROOF FOR THEOREM 2.1. Without restriction we may assume $pq \neq 0$. Otherwise Δ^+ or Δ^- vanish. Since we may subtract middle parts, it suffices to give the proof for one pair of sequences k_n, r_n.

Throughout let τ_n, R_n and T_n be as in Lemma 5.3. Choose $r_n = k_n \to \infty, k_n \le n/2$, such that

$$k_n \tau_n \to 0. \tag{5.41}$$

Thus it remains to prove

$$R_n - L_n(k_n) \to_P 0 \quad \text{and} \quad T_n - W_n(k_n) \to_P 0. \tag{5.42}$$

We restrict our attention to R_n. Consider r.v.'s $\overline{X}_{n,i} := X_{n,i}1_{(-\infty,-\tau_n)}(X_{n,i})$, their order statistics $X_{i:n}1_{(-\infty,-\tau_n)}(X_{i:n})$ and the related upper sum $\overline{W}_n(\cdot)$ of $\overline{X}_{n,i}$. The first part of the proof of Theorem 2.1 yields

$$\overline{W}_n(n-k_n) \to_P 0. \tag{5.43}$$

Consequently,

$$R_n - \sum_{i=1}^{k_n} \Big\{ X_{i:n}1_{(-\infty,-\tau_n)}(X_{i:n}) - E\big(X_{i:n}1_{(-\tau,-\tau_n)}(X_{i:n})\big)\Big\} \to_P 0. \tag{5.44}$$

The same type of argument shows

$$\sum_{i=1}^{k_n} \Big\{ X_{i:n}1_{(-\tau_n,\infty)}(X_{i:n}) - E\big(X_{i:n}1_{(\tau_n,\infty)}(X_{i:n})\big)\Big\} \to_P 0. \tag{5.45}$$

In addition, (5.41) yields

$$\left| \sum_{i=1}^{k_n} \Big\{ X_{i:n}1_{(-\tau_n,\tau_n)}(X_{i:n}) - E\big(X_{i:n}1_{(-\tau_n,\tau_n)}(X_{i:n})\big)\Big\} \right| \le 2k_n\tau_n \to 0. \tag{5.46}$$

The statements (5.44)–(5.46) prove the first assertion of (5.42). The proof of the second claim is similar. Thus the proof of Theorem 2.1 is complete. □

PROOF OF THEOREM 2.2. Let R_n, S_n and T_n denote the r.v.s of Lemma 5.3, where τ_n may fulfill condition (5.32). Introduce

$$\delta_n := G_n^{-1}(k_n) \quad \text{where } \delta_n \to 0. \tag{5.47}$$

By (2.27) we have $\delta_n < 0$ whenever n is large enough. The proof is based on assertions (I) and (II), which are proved below:

$$\text{(I)} \quad \xi_n := \sum_{i=1}^{n} \Big\{ X_{n,i}1_{(-\infty,\delta_n)}(X_{n,i}) - E\big(X_{n,i}1_{(-\tau,\delta_n)}(X_{n,i})\big)\Big\} - R_n \to_P 0,$$

$$\text{(II)} \quad \sum_{i=1}^{k_n} \big(X_{i:n} - E_{(-\tau,\tau)}(X_{i:n})\big) - \sum_{i=1}^{n} \Big\{ X_{n,i}1_{(-\infty,\delta_n)}(X_{n,i}) - E\big(X_{n,i}1_{(-\tau,\delta_n)}(X_{n,i})\big)\Big\} \to_P 0.$$

The claim can then be deduced as follows. Combining (I) and (II), we arrive at

$$L_n(k_n) - R_n \to_P 0. \tag{5.48}$$

An analogous consideration for upper extremes yields

$$W_n(r_n) - T_n \to_P 0. \tag{5.49}$$

The middle part can be dealt with by the following arguments. Notice that

$$L_n(m_n) + M_n(m_n, s_n) + W_n(s_n) = R_n + S_n + T_n. \tag{5.50}$$

Next we may substitute k_n by m_n and r_n by s_n. Then (5.48)–(5.50) yield

$$S_n - M_n(m_n, s_n) \to_P 0. \tag{5.51}$$

Thus Lemma 5.3 implies the result. It remains to prove (I) and (II).

(I) Let $\delta < 0$. Consider n large enough such that $-\tau < \delta_n < \tau$, $\tau_n < \tau$ and $\delta < -\tau_n \wedge \delta_n$ hold. Next define $B_n = [-\tau_n, \delta_n)$ if $-\tau_n < \delta_n$ and $B_n = [\delta_n, -\tau_n)$ otherwise. Then $E(\xi_n) = 0$ and

$$\begin{aligned}\operatorname{Var}(\xi_n) = \operatorname{Var}\left(\sum_{i=1}^{n} X_{n,i} 1_{B_n}(X_{n,i})\right) &\le \sum_{i=1}^{n} E\left(X_{n,i}^2 1_{B_n}(X_{n,i})\right) \\ &\le \int_{(-\infty, -\tau_n \vee \delta_n)} \left(x^2 \wedge \delta^2\right) dG_n(x)\end{aligned} \tag{5.52}$$

hold. By (2.29) and (5.32) the right-hand side of (5.52) converges to zero if we take first $\limsup_{n\to\infty}$ and then $\delta \downarrow 0$.

(II) Throughout we may assume $-\tau < \delta_n < 0$. Due to Lemma 5.4 it is transparent that

$$Z_n := \sum_{i=1}^{n} \left\{X_{n,i} 1_{(-\infty,\delta_n)}(X_{n,i}) - E\left(X_{n,i} 1_{(-\tau,\delta_n)}(X_{n,i})\right)\right\} \to_{\mathcal{D}} R \tag{5.53}$$

with $\mathcal{L}(R) = \mu_1$. Similarly to (5.42) one gets from Theorem 2.1,

$$\sum_{i=k_n+1}^{n} \left\{X_{i:n} 1_{(-\infty,\delta_n)}(X_{i:n}) - E\left(X_{i:n} 1_{(-\tau,\delta_n)}(X_{i:n})\right)\right\} \to_P 0 \tag{5.54}$$

and, consequently,

$$\sum_{i=1}^{k_n} \left\{X_{i:n} 1_{(-\infty,\delta_n)}(X_{i:n}) - E\left(X_{i:n} 1_{(-\tau,\delta_n)}(X_{i:n})\right)\right\} - Z_n \to_P 0. \tag{5.55}$$

Thus it remains to check

$$\sum_{i=1}^{k_n} \left\{X_{i:n} 1_{[\delta_n,\infty)}(X_{i:n}) - E\left(X_{i:n} 1_{[\delta_n,\tau)}(X_{i:n})\right)\right\} \to_P 0. \tag{5.56}$$

This assertion will be rewritten in an equivalent form. Consider r.v.'s $Y_n = \sum_{i=1}^{n} \delta_n 1_{(-\infty,\delta_n)}(X_{n,i})$. Notice that

$$\operatorname{Var}(Y_n) \to 0 \tag{5.57}$$

by assumption (2.29) since

$$\mathrm{Var}(Y_n) = \delta_n^2 G_n(\delta_n -) \le \int_{(-\infty,\,\delta_n)} (\delta^2 \wedge x^2)\, dG_n(x) \tag{5.58}$$

for each $\delta < \delta_n$. Lemma A of the Appendix applied to $\varphi(x) = \delta_n 1_{(-\infty,\,\delta_n)}(x)$ yields

$$\mathrm{Var}(\widetilde{Y}_n) \to 0, \tag{5.59}$$

where $\widetilde{Y}_n := \sum_{i=1}^{k_n} \delta_n 1_{(-\infty,\,\delta_n)}(X_{i:n})$. If we now add $\widetilde{Y}_n - E(\widetilde{Y}_n)$, then statement (5.56) is equivalent to

$$\sum_{i=1}^{k_n} \Big\{ \delta_n \vee X_{i:n} - E\big(\delta_n \vee X_{i:n} 1_{(-\tau,\,\tau)}(\delta_n \vee X_{i:n})\big) \Big\} \to_P 0. \tag{5.60}$$

Four different steps prove the validity of (5.60):

Step 1. The expectations of (5.60) can be treated partially by proving

$$a_n := \sum_{i=1}^{k_n} E(\delta_n \vee X_{i:n} \wedge 0) - \delta_n k_n \to 0. \tag{5.61}$$

Observe that

$$0 \le a_n \le |\delta_n| \sum_{i=1}^{k_n} P(X_{i:n} > \delta_n) =: b_n. \tag{5.62}$$

For each $i \le k_n$ and $\varepsilon = G_n(\delta_n) + 1 - i$ we can apply Bernstein's inequality (5.1), which shows

$$\begin{aligned} P(X_{i:n} > \delta_n) &= P\left(\sum_{j=1}^{n} 1_{(\delta_n,\,\infty)}(X_{n,j}) \ge n + 1 - i \right) \\ &\le P\left(\left| \sum_{j=1}^{n} 1_{(\delta_n,\,\infty)}(X_{n,j}) - \big(n - G_n(\delta_n)\big) \right| \ge G_n(\delta_n) + 1 - i \right) \\ &\le 2 \exp\left(- \frac{\big(G_n(\delta_n) + 1 - i\big)^2}{8 G_n(\delta_n)/3} \right), \end{aligned} \tag{5.63}$$

since the variances $\sum_{i=1}^{n} \sigma_i^2$ of (5.1) are bounded above by $G_n(\delta_n)$. The required convergence of b_n (5.62) is first established under the additional assumption

$$G_n(\delta_n) = k_n. \tag{5.64}$$

For $\overline{\sigma}_n^2 := 4k_n/3$ we get

$$\begin{aligned} \sum_{i=1}^{k_n} \exp\left(- \frac{(k_n + 1 - i)^2}{2\overline{\sigma}_n^2} \right) &\le \sqrt{2\pi}\,\overline{\sigma}_n \int_0^{k_n + 1} dN(k_n + 1, \overline{\sigma}_n^2) \\ &\le \sqrt{2\pi}\,\overline{\sigma}_n. \end{aligned} \tag{5.65}$$

Since $\delta_n^2 G_n(\delta_n -) \to 0$ holds by (5.58), the assumption (2.29) implies $\delta_n^2 k_n \to 0$. Consequently, b_n (5.62) converges to zero.

If (5.64) is violated, the proof of (5.61) can be modified as follows. Define $p_n = (k_n - G_n(\delta_n -))(G_n(\delta_n) - G_n(\delta_n -))^{-1}$ and let Z_i denote i.i.d. $\mathcal{B}(1, p_n)$ binomial r.v.'s. Define new r.v.'s $\widetilde{X}_{n,i}$ by

$$\widetilde{X}_{n,i} = X_{n,i} \quad \text{if } X_{n,i} \neq \delta_n.$$

Otherwise set $\widetilde{X}_{n,i} = \delta_n$ if $Z_i = 1$ and $\widetilde{X}_{n,i} = 0$ if $Z_i = 0$. Thus $\widetilde{X}_{n,i} \geq X_{n,i}$ holds and

$$P(X_{i:n} > \delta_n) \leq P(\widetilde{X}_{i:n} > \delta_n). \tag{5.66}$$

The related d.f.'s of $\widetilde{X}_{n,i}$ satisfy

$$\widetilde{F}_{n,i}(\delta_n) = F_{n,i}(\delta_n -) + p_n\big(F_{n,i}(\delta_n) - F_{n,i}(\delta_n -)\big), \tag{5.67}$$

which implies $\widetilde{G}_n(\delta_n) = k_n$. Evidently, assumption (2.29) remains true for $\widetilde{G}_n$. Thus the previous proof gives the result for $\widetilde{X}_{i:n}$. Altogether, we see that $b_n \longrightarrow 0$ and the proof of Step 1 is finished.

Step 2. The proof of statement (5.60) depends on a second inequality for expectations. We would like to prove

$$\sum_{i=1}^{k_n} E\big(X_{i:n} 1_{[0,\tau)}(X_{i:n})\big) \leq \tau k_n P(X_{k_n:n} > 0) \to 0. \tag{5.68}$$

Inequality (5.63) together with $x \exp(-x) \leq 1$ for $x \geq 0$ yields

$$k_n P\big(X_{k_n:n} > 0\big) \leq 2k_n \frac{8G_n(0)/3}{\big(G_n(0) + 1 - k_n\big)^2}. \tag{5.69}$$

By our assumptions (2.27) and (2.28) we obtain $k_n/G_n(0) \to 0$. Thus (5.69) implies condition (5.68).

Step 3. Suppose that $k_n \to \infty$. Similarly to (5.63) and (5.69), we have

$$P\big(X_{k_n:n} > \delta_n\big) \leq \frac{6}{G_n(\delta_n)}\left(1 + \frac{1 - k_n}{G_n(\delta_n)}\right)^{-2} \to 0 \tag{5.70}$$

since $G(\delta_n) \geq k_n$.

Step 4. Now we are in the position to put everything together and to give the proof of (5.60). Two cases must be considered. If $q = 0$, then assumption (2.24) implies that k_n does not converge to infinity. Suppose that k_n remains bounded along a subsequence. Then (5.60) holds along that subsequence since $\psi_1 = 0$ and $X_{i:n} \to_P 0$ for each i.

Therefore we may assume $k_n \to \infty$. Then Step 3 implies

$$\sum_{i=1}^{k_n} \delta_n \vee X_{i:n} - k_n \delta_n \to_P 0. \tag{5.71}$$

If we combine (5.61), (5.68) and (5.71) we obtain the desired statement (5.60). This completes the proof of assertion (II). □

PROOF OF PROPOSITION 2.1. Note that

$$\left\{ \sum_{i=n+2-m}^{n} X_{i:m} \le x \right\} \Big\backslash \left\{ \sum_{i=1}^{n} X_{n,i} \le x \right\} \subset \{X_{n+1-m:n} > 0\}.$$

As a conclusion of Bernstein's inequality we have, as in (5.63),

$$\begin{aligned} P(X_{n+1-m:n} > 0) &= P\left(\left| \sum_{j=1}^{n} 1_{(0,\infty)}(X_{n,j}) - p_n \right| \ge m - p_n \right) \\ &\le 2\exp\left(-\frac{(m-p_n)^2}{2\sum_{i=1}^{n}(p_{ni} - p_{ni}^2) + 2(m-p_n)/3} \right). \end{aligned}$$ □

PROOF OF PROPOSITION 2.2. Consider rowwise independent r.v.'s $X_{n,i}$ with compound Poisson distribution given by the Lévy measure η/n and characteristic function (2.42). For each n the sum is equal in distribution to μ. On the other hand,

$$\sum_{i=1}^{m-1} X_{n+1-i:n} \to \sum_{i=1}^{m-1} \psi_2(S_i)$$

in distribution by (2.13). Notice that $p_n = n(i - \exp(-\lambda/n)) \to \lambda$ implies the result. □

PROOF OF THEOREM 3.1. Suppose that k_n fulfills the conditions of Theorem 3.1. Repeating the proof of (5.20)–(5.24) we can find a sequence $m_n \to \infty$, $m_n \le k_n$, with

$$d(\tilde{L}_n(m_n), \tilde{\Delta}^-) \to 0 \tag{5.72}$$

if $m_n := j \wedge k_n$ is chosen in connection with (5.24). To prove (3.2) it is sufficient to check that

$$\sum_{i=m_n+1}^{k_n} X_{i:n} \to_P 0. \tag{5.73}$$

We have $\delta_n \to 0$ for δ_n given in (5.47). Again Bernstein's inequality yields $P(X_{k_n:n} > \delta_n) \to 0$ as in (5.63) and (5.70). Check also that assumption (3.4) implies $\sum_{i=m_n+1}^{k_n} \delta_n 1_{\{\delta_n\}}(X_{i:n}) \to_P 0$ since

$$\delta_n G_n(\delta_n -) \to 0 \quad \text{and} \quad k_n \delta_n \to 0$$

as $n \to \infty$. Thus (5.73) is equivalent to

$$\xi_n := \sum_{i=m_n+1}^{k_n} X_{i:n} 1_{(-\infty,\, \delta_n)}(X_{i:n}) \to_P 0. \tag{5.74}$$

Condition (5.74) is now proved via Lemma B of the Appendix. Define

$$\xi_{1,n}^{(\delta)} := \sum_{i=m_n+1}^{k_n} X_{i:n} 1_{[\delta,\, \delta_n)}(X_{i:n}) \tag{5.75}$$

and $\xi_n =: \xi_{1,n}^{(\delta)} + \xi_{2,n}^{(\delta)}$. For each $\delta < 0$ we have $\xi_{1,n}^{(\delta)} \to_P 0$ since $P(X_{m_n:n} \le \delta) \to 0$ [cf. (2.13)]. Moreover,

$$E\left(\left|\xi_{2,n}^{(\delta)}\right|\right) \le \int_{[\delta,\, \delta_n)} |x|\, dG_n(x)$$

and condition (3.4) implies

$$\lim_{\delta \uparrow 0} \limsup_{n \to \infty} E\left(\left|\xi_{2,n}^{(\delta)}\right|\right) = 0.$$

Thus Lemma B proves (5.74). □

APPENDIX

This Appendix serves to show that order statistics are nonnegatively correlated. Lemma A is our main technical tool. That result was earlier proved by Bickel (1967) for i.i.d. random variables with densities.

LEMMA A. *Let $X_1, \dots, X_n$ denote independent random variables and let φ: $\mathbb{R} \to \mathbb{R}$ be nonincreasing or nondecreasing such that $|\varphi(X_i)|$ is square integrable for each $i = 1, \dots, n$. Then the order statistics have nonnegative covariance.*

A proof is given by Hàjek [(1968), Lemma 3.1]

Throughout, some facts concerning the Lévy metric are recalled. Let X, Y denote real r.v.'s with distribution functions F_X and F_Y. The topology of convergence in distribution is given by the Lévy metric

$$d(X,Y) := \inf\left\{\varepsilon > 0\colon F_Y(x-\varepsilon) - \varepsilon \le F_X(x) \le F_Y(x+\varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}\right\} \tag{A1}$$

Notice that the condition $P(|Z| \ge \varepsilon) \le \varepsilon$ implies

$$d(X+Z, Y) \le d(X,Y) + \varepsilon. \tag{A2}$$

Let $\|\cdot\|_2$ denote the L_2-norm.

LEMMA B. (a) *Consider sequences of r.v.'s* $Y_n = X_n^{(k)} + Z_n^{(k)}$. *Assume that for fixed* $k \in \mathbb{N}$,

$$X_n^{(k)} \to_{\mathcal{D}} X^{(k)} \tag{A3}$$

holds as $n \to \infty$ *and*

$$X^{(k)} \to_{\mathcal{D}} X \quad \text{as } k \to \infty. \tag{A4}$$

Suppose also that for each $\varepsilon > 0$,

$$\lim_{k \to \infty} \limsup_{n \to \infty} P\Big(\big|Z_n^{(k)}\big| \geq \varepsilon\Big) = 0. \tag{A5}$$

Then $Y_n \to_{\mathcal{D}} X$ *follows. If* (A3) *and* (A4) *converge in probability, then also* $Y_n \to_P X$ *holds.*

(b) *If* (A3) *and* (A4) *converge in* L_2, *then* $\|Y_n - X\|_2 \to 0$ *provided* (A5) *is substituted by condition*

$$\lim_{k \to \infty} \limsup_{n \to \infty} \big\|Z_n^{(k)}\big\|_2 = 0. \tag{A6}$$

PROOF. The distributional convergence part is Theorem 4.2 of Billingsley (1968). The other assertions are quite obvious and their proofs are similar. □

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