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ON THE ROTATIONAL DIMENSION OF STOCHASTIC MATRICES

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Let $(S_i,\ i = 1, 2, \dots, n)$, $n > 1$, be a partition of the circle into sets S_i each consisting of union of $\delta(i) < \infty$ arcs A_{kl}. Let f_t be a rotation of length t of the circle and denote Lebesgue measure by λ. Then every recurrent stochastic matrix P on $S = \{1, \dots, n\}$ is given according to a theorem of Cohen ($n = 2$), Alpern and Kalpazidou ($n \geq 2$) by $p_{ij} = \lambda(S_i \cap f_t^{-1}(S_j))/\lambda(S_i)$ for some choice of rotation f_t and partition $\mathscr{S} = \{S_i\}$. The number $\delta(\mathscr{S}) = \max_i \delta(i)$ is called the length of description of the partition $\mathscr{S}$. Then it turns out that the minimal value of $\delta(\mathscr{S})$, when $\mathscr{S}$ varies, characterizes the matrix P. We call this value the rotational dimension of P. We prove that for certain recurrent $n \times n$ stochastic matrices the rotational dimension is provided by the number of Betti circuits of the graph of P. One preliminary result shows that there are recurrent $n \times n$ stochastic matrices which admit minimal positive circuit decompositions in terms of the Betti circuits of their graph. Finally, a generalization of the rotational dimension for the transition matrix functions is also given.

1. Background and notation. Let $n > 1$, $S = \{1, \dots, n\}$ and $P = (p_{ij},\ i, j \in S)$ be a stochastic matrix which defines an irreducible S-state Markov chain $\xi = (\xi_m)_{m \geq 0}$. Consider $([0,1], B, \lambda)$ the canonical probability space on $[0,1]$; that is, B and λ are the σ-algebra of all Borel subsets of $[0,1]$ and Lebesgue measure, respectively. Then a theorem of Cohen [3] ($n = 2$), Alpern and Kalpazidou ($n \geq 2$) asserts that there exist a shift transformation f_t of length $t = 1/M$ with M the least common multiple of $(1, 2, \dots, n)$ [for short lcm$(1, 2, \dots, n)$] defined as

$$f_t(x) = (x + t) \pmod 1, \qquad x \in [0, 1), \tag{1}$$

and a partition $\mathscr{S} = (S_i,\ i = 1, \dots, n)$ of $[0, 1)$ into sets S_i each consisting of a finite union of subintervals such that

$$p_{ij} = \lambda\big(S_i \cap f_t^{-1}(S_j)\big)/\lambda(S_i), \qquad i, j = 1, \dots, n \tag{2}$$

(see also Alpern [1]). Furthermore, if $\pi = (\pi_i,\ i = 1, \dots, n)$ denotes the invariant probability distribution of P, then $\pi_i = \lambda(S_i)$, $i = 1, \dots, n$. When (2) holds, the stochastic matrix P is said to have a rotational representation symbolized by $(t, \mathscr{S})$.

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The structure of each partitioning set S_i, $i = 1, \dots, n$, is described by a finite union of circle arcs A_{kl} whose indices (k, l) can be given by different ways of labeling. One may find such labelings in Alpern [2], Haigh [4] and Rodríguez del Tío and Valsero Blanco [10].

We shall now give a general labeling which reveals the intrinsic rotational device occurring in representation (2). To this end, a preliminary step will be to find a transformation of the edge distribution $E = (\pi_i p_{ij},\ i, j = 1, \dots, n)$ into a circuit distribution $(\omega_c,\ c \in \mathscr{C})$, where $\mathscr{C}$ denotes a collection of directed circuits in S. A directed circuit in S is any ordered sequence $c = (i_1, \dots, i_p, i_1)$, with $p \geq 1$, where $i_1, \dots, i_p$ are all distinct when $p > 1$. Each circuit c is assigned to a sequence $\hat{c} = (i_1, \dots, i_p)$ called a cycle. The above positive integer $p = p(c)$ is called the period of c (for more details, see [5]).

For a chosen ordering in $\mathscr{C}$, say $\mathscr{C} = (c_1, \dots, c_s)$, the above transformation is defined by

$$(3) \qquad \pi_i p_{ij} = \sum_{k=1}^{s} w_{c_k} J_{c_k}(i, j), \qquad i, j \in S, w_{c_k} > 0, k = 1, \dots, s,$$

where

$$(4) \qquad C = \left(p(c_k) w_{c_k}, k = 1, \dots, s\right)$$

is a circuit distribution given either by deterministic algorithms as in [7] and [2] or by a probabilistic algorithm as in [5] and [6], and $J_{c_k}(i, j) = 1$ or 0 according as (i, j) is or is not an edge of c_k. The probabilistic algorithm uniquely determines both $\mathscr{C}$ and $\{w_{c_k}\}$. Let $J_{c_k}(i) = \sum_j J_{c_k}(i, j)$. When $J_{c_k}(i, j) = 1$ [$J_{c_k}(i) = 1$], we say that c_k passes through (i, j) (and i).

Notation. Throughout this paper the ingredients n, S, M, p and the function J_c will have the meanings above.

Once the circuit decomposition (3) and the starting points of the circuits are chosen, we may find a transformation A: $\{\mathscr{C}, C\} \to \{\{A_{kl}\}, \{\lambda(A_{kl})\}\}$ from the weighted circuits of $\mathscr{C}$ onto the weighted circle arcs A_{kl} defined as follows:

$$(5)(\mathrm{i}) \qquad A^{-1}(A_{kl}) = c_k,$$

that is, each index k of A_{kl} is assigned to a circuit c_k which occurs in the decomposition (3) and which passes a preassigned point i of S, and

$$(5)(\mathrm{ii}) \qquad \lambda(A_{kl}) = (1/M) p(c_k) w_{c_k},$$

for all $k = 1, \dots, s$, and all $l = 1, \dots, M$. Define the sets S_i by

$$(6) \qquad S_i = \bigcup_{(k,l)} A_{kl}, \qquad i = 1, \dots, n,$$

where the union is taken over all pairs $(k, l) = (k_i, l_i)$ defined as follows:

(7)(i) k_i is the index of a circuit c_k, $k \in \{1, \dots, s\}$, which passes through the preassigned point i and which occurs in the decomposition (3);

(7)(ii) l_i denotes those ranks $n \in \{1, \dots, M\}$ of all the points $\hat{c}_k(n)$ which are identical to i in the $M/p(c_k)$ repetitions of the cycle $\hat{c}_k = (\hat{c}_k(1), \dots, \hat{c}_k(p(c_k)))$ associated with the circuit c_k chosen at (i) above; that is, if for some $s \in \{1, \dots, p(c_k)\}$ we have $\hat{c}_k(s) = \hat{c}_k(s + p(c_k)) = \cdots = \hat{c}_k(s + (M/p(c_k) - 1)p(c_k)) = i$, then $l_i \in \{s, s + p(c_k), \dots, s + (M/p(c_k) - 1)p(c_k)\}$.

The lth repetition of the cycle $\hat{c}_k$ above is given by the sequence

$$\big(\hat{c}_k(1 + (l - 1)p(c_k)), \dots, \hat{c}_k(p(c_k) + (l - 1)p(c_k))\big).$$

Then $\mathscr{S} = \{S_1, \dots, S_n\}$ is a rotational partition of $[0, 1)$ associated to P with respect to the shift f_t, with $t = 1/M$.

The transformation A can be viewed as a pair (A_1, A_2) of transformations: The first component $A_1\colon \mathscr{C} \to \{A_{kl}\}$ is a topological (geometrical) transformation of a circuit to circle arcs given by (7)(i) and (ii); that is, it depends only on the connectivity relations of the graph $G(P)$ of P. The second component $A_2\colon C \to \{\lambda(A_{kl})\}$ is an algebraic transformation between the weights assigned to the circuits and arcs given by (5)(ii) above.

Let $\delta(i)$ denote the number of the components A_{kl} of S_i, $i = 1, \dots, n$, defined by (6). Then $\delta(i)$ depends only on A_1; that is, $\delta(i)$ is a topological feature of $\mathscr{S}$ (which depends neither on the ordering of $\mathscr{C}$ nor on the starting points of the circuits).

For instance, if i belongs to a single circuit c of period $p(c)$, then $\delta(i) = M/p(c)$, but if there is more than one circuit c containing i, then $\delta(i) = \sum_c (M/p(c))$. Hence $\delta(i)$ depends on the number s of the representative circuits in the decomposition (3) and on the connectivity of $\mathscr{C}$.

Let $\delta = \delta(s, \mathscr{C}) = \max_j \delta(j)$. Then the number of components A_{kl} of each S_i, according to labeling (7), is less than or equal to δ. We call δ the *length of description of the partition* $\mathscr{S} = \{S_i,\ i = 1, \dots, n\}$ *associated with* $\mathscr{C}$. Then there exists a pair $(s_0, \mathscr{C}_0)$ which provides the minimal value of δ when $(\mathscr{C}, C)$ varies in (3).

Let $D = D(P) = \delta(s_0, \mathscr{C}_0) = \min_{s, \mathscr{C}} \delta(s, \mathscr{C})$. We call D the *rotational dimension of* P. When either n or s is a large number, the corresponding rotational partition $\mathscr{S}$ will comprise a vast number of components and the construction (6) will become very complicated. This motivates our interest in rotational partitions with a minimal length of description. An immediate probabilistic implication of these investigations will then consist in improving the solution (of Theorem 1 of [6]) to the well-known coding problem arising in dynamical systems, which in our context has the following formulation: find a one-to-one correspondence from the space of $n \times n$ irreducible stochastic matrices into rotational n partitions.

In the present paper we prove that for certain recurrent $n \times n$ stochastic matrices P the number s_0 in the definition of D is equal to or less than the Betti number B of the graph $G(P)$ of P, while $\mathscr{C}_0$ is a collection $\tilde{\Gamma} \subseteq \{\gamma_1, \dots, \gamma_B\}$ of Betti circuits of $G(P)$. The Betti number B is the least number of independent circuits of $G(P)$; a rigorous presentation is given below.

A preliminary result (Theorem 1) shows that there exists a circuit distribution $C_{\min} = (p(\gamma_k)\omega_{\gamma_k}, \gamma_k \in \tilde{\Gamma})$ of minimal length. Then, in Theorem 2 it is proved that transformation A with labeling (7) on $C_{\min}$ determines a rotational partition of $[0, 1)$ whose length is D and is given by $\delta(\tilde{B}, \tilde{\Gamma})$ for some $\tilde{B} \leq B$.

When the algorithm in the circuit decomposition (3) is chosen to be the probabilistic one, the w_{c_k}'s are uniquely determined: w_{c_k} is the mean number of occurrences of c_k on almost all the trajectories $(\xi_k(\omega))_k$ of ξ (see [5] and [6]). The probabilistic algorithm is the only algorithm which allows us to generalize the rotational dimension to continuous parameter Markov processes. Namely, if $\mathscr{S}(h) = \{S_i(h)\}$ is a rotational partition of a recurrent stochastic matrix function $P(h) = (p_{ij}(h),\ i, j \in S)$, $h \geq 0$, as given in [6], then $\mathscr{S}(h)$ can be analogously characterized by a length of description for all h, provided that the class of representative circuits does not depend on h. This happens only when we choose the probabilistic algorithm in the decomposition (3) (see [6]).

On the other hand, the sample-path description given by the probabilistic algorithm gives a natural connection to Kolmogorov-type descriptions (see Kolmogorov and Uspensky [8]) as follows. Let i, j be fixed states of S and let ω be a fixed trajectory of ξ. Consider $y = y^{\omega}_{(i,j)} = (y(0), y(1), \dots, y(k), \dots)$ a binary sequence whose coordinate $y(k), k = 0, 1, \dots,$ is 1 or 0 according as the pair (i, j) occurs or does not occur on $(\xi_m(\omega))_m$ at moment k. Then each finite subsequence $y_k = (y(0), \dots, y(k-1))$ admits two descriptions. One description is given in terms of the edges by the binary sequence $x_k = (x(0), \dots, x(k-1))$, where $x(l) = y(l)$, $l = 0, \dots, k-1$, and the other in terms of the circuits by $\eta_k = (\eta(0), \dots, \eta(k-1))$, where $\eta(l)$, $l = 0, \dots, k-1$, is 1 or 0 according as a circuit passing (i, j) occurs or does not occur on $(\xi_m(\omega))_m$ at time l.

Furthermore, one may characterize an irreducible stochastic matrix P as "chaotic" in the spirit of Kolmogorov if the connectivity relations of the graph $G(P)$ of P are complex enough. Then the Betti number of the graph $G(P)$ should be the maximal one. It turns out that for a given $n \geq 1$ the largest Betti number of all connected oriented graphs on $\{1, \dots, n\}$ is $n^2 - n + 1$. Then there is an irreducible stochastic matrix on $\{1, \dots, n\}$ whose graph has the Betti number $n^2 - n + 1$.

Let us now consider the maximal rotational dimension of P when P varies in the set of all $n \times n$ recurrent stochastic matrices. Another way to approach this concept was initiated by Alpern [2] and extended by the author [6] as follows. We say that a rotational partition $\mathscr{S} = \{S_1, \dots, S_n\}$ has the type L if the number of components of each S_i is less than or equal to L, $i = 1, \dots, n$. Let $D(n)$ be the least integer such that every $n \times n$ recurrent matrix has a rotational representation of type $D(n)$; that is, a representation $(t, \mathscr{S})$, where $\mathscr{S}$ is of type $D(n)$. Then $D(n)$ may be connected with the maximal rotational dimension over all $n \times n$ recurrent stochastic matrices. Alpern [2] proved that $D(n)$ can be estimated by an interval $(\exp(\alpha n^{1/2}), \exp(\beta n))$ for some positive constants α and β.

Let $P = (p_{ij}, i, j = 1, \dots, n)$ be an irreducible stochastic matrix. As is well known, P may be assigned to a graph $G = G(P)$ as follows: the set of points is given by $S = \{1, \dots, n\}$ and the set of directed branches consists of all pairs (i, j) for which $p_{ij} > 0$. In general, one may dissociate the graph from any matrix, in which case the concepts below are related to the graph alone.

We are concerned here with a circuit decomposition which holds in any finite connected directed graph $G = (\mathscr{B}_0, \mathscr{B}_1)$, where $\mathscr{B}_0 = \{n_1, \dots, n_{\nu_0}\}$ and $\mathscr{B}_1 = \{b_1, \dots, b_{\nu_1}\}$ denote, respectively, the set of nodes and the set of directed branches. This approach comes from algebraic topology.

Let us consider that $\mathscr{B}_0$ and $\mathscr{B}_1$ are the bases of two real vector spaces $\mathbf{C}_0$ and $\mathbf{C}_1$. Then any two elements $\mathbf{c}_0 \in \mathbf{C}_0$ and $\mathbf{c}_1 \in \mathbf{C}_1$ have the formal expressions

$$\mathbf{c}_0 = \sum_{h=1}^{\nu_0} x_h n_h = \mathbf{x}'\mathbf{n}, \qquad x_h \in R,$$

$$\mathbf{c}_1 = \sum_{k=1}^{\nu_1} y_k b_k = \mathbf{y}'\mathbf{b}, \qquad y_k \in R,$$

where by convention $y_k(-b_k) = -y_k b_k$ for all $(-b_k)$ which do not belong to $\mathscr{B}_1$, and R denotes the set of reals.

The linear map Δ: $C_1 \to C_0$ defined by $\Delta b_j = n_k - n_h$ (where n_h and n_k are the initial point and end point of b_j) describes the orientation of G. Any circuit $\mathbf{c}$ in G can be written as $\mathbf{c} = b_1 + \cdots + b_k$ (say), and plainly $\Delta \mathbf{c} = 0$, that is, $\mathbf{c} \in \ker(\Delta)$. Conversely, if $\mathbf{c}_1$ is any such sum, and $\mathbf{c}_1 \in \ker(\Delta)$, then either $\mathbf{c}_1$ is a circuit or $\mathbf{c}_1$ contains a subgraph that is a circuit. Let η be the $\mathscr{B}_1 \times \mathscr{B}_0$ matrix that defines Δ with respect to the given bases.

Let T be any spanning tree of G and let $\mathscr{B}(T)$ be its set of branches. Although T may not be unique, the number B of branches not in $\mathscr{B}(T)$ is a characteristic of G, called its Betti number (see [9]). It is given by $B = \nu_1 - \operatorname{rank}(\eta)$. Let β_k be any branch not in T. Since G is connected there exists a sequence σ_k of connected branches in $\mathscr{B}(T)$ such that $\beta_k + \boldsymbol{\sigma}_k = \boldsymbol{\gamma}_k$ belongs to $\ker(\Delta)$. We call β_k and $\boldsymbol{\gamma}_k$, $k = 1, \dots, B$, Betti branches and Betti one-cycles of G, respectively.

Denote

$$\Gamma = \{\boldsymbol{\gamma}_k, k = 1, \dots, B\}.$$

Then from algebraic topology we have the following (see Kalpazidou [7]).

LEMMA 1. *The set Γ of Betti one-cycles in G is a base of $\tilde{C}_1 = \ker \Delta$.*

When $\gamma_1, \dots, \gamma_B$ are certain directed circuits in the graph G such that the associated vectors $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_B$ in C_1 form a base of Betti one-cycles, then we call $\gamma_1, \dots, \gamma_B$ the Betti circuits of G and $\{\gamma_1, \dots, \gamma_B\}$ a base of Betti circuits.

2. A minimal circuit decomposition. Consider $P = (p_{ij}, i, j \in S)$, any stochastic matrix defining an S-state homogeneous irreducible Markov chain $\xi = (\xi_m, m \geq 0)$.

We show in this section that there exists a circuit decomposition of P in terms of a minimal number of directed circuits when the graph of P satisfies some topological condition.

Notation. Let $G = G(P) = (\mathscr{B}_0(P), \mathscr{B}_1(P))$, $\eta = \eta(P)$, $B = B(P)$ and $\Gamma = \Gamma(P) = \{\gamma_1, \dots, \gamma_B\}$ denote, respectively, the graph of P, the branch-point incidence matrix of this graph, the Betti number of G and any base of Betti circuits. Then we may prove the following theorem.

THEOREM 1. *Let $P = (p_{ij}, \ i, j = 1, \dots, n)$ be an irreducible stochastic matrix whose invariant probability distribution is $\pi = (\pi_1, \dots, \pi_n)$. Assume that the graph $G(P)$ contains a base $\Gamma = \{\gamma_1, \dots, \gamma_B\}$ of Betti circuits. Then P has a circuit decomposition in terms of the circuits of Γ, that is,*

$$\sum_{(i,j)} \pi_i p_{ij} b_{(i,j)} = \sum_{\kappa=1}^{B} \omega_{\gamma_\kappa} \boldsymbol{\gamma}_\kappa, \qquad b_{(i,j)} \in \mathscr{B}_1(P),\ \omega_{\gamma_\kappa} \in R, \tag{8}$$

or, in term of the (i,j) coordinates,

$$\pi_i p_{ij} = \sum_{\kappa=1}^{B} \omega_{\gamma_\kappa} J_{\gamma_\kappa}(i,j), \qquad i, j \in S, \tag{9}$$

where the corresponding circuit weights are defined as

$$\omega_{\gamma_\kappa} = \sum_{c \in \mathscr{C}} a(c, \gamma_\kappa) w_c, \qquad a(c, \gamma_\kappa) \in Z, w_c > 0,$$

with $\mathscr{C}$ and w_c given by random or nonrandom algorithms as in (3).

PROOF. If $p_{ij} > 0$, let $b_{(i,j)}$ be the branch in $\mathscr{B}_1(P)$ from i to j and write $\mathbf{w} = \sum_{i,j} \pi_i p_{ij} b_{(i,j)}$.

From (3), there is a class $\mathscr{C}$ of directed circuits so that

$$\mathbf{w} = \sum_{(i,j)} \sum_{c \in \mathscr{C}} w_c J_c(i,j) b_{(i,j)}, \tag{10}$$

where each $w_c > 0$. Then we have

$$\begin{aligned} \mathbf{w} &= \sum_{c \in \mathscr{C}} w_c \Bigg(\sum_{(i,j)} J_c(i,j) b_{(i,j)} \Bigg) \\ &= \sum_{c \in \mathscr{C}} w_c \mathbf{c} \end{aligned}$$

since $J_c(i,j)$ is the (i,j) coordinate of $\mathbf{c}$ with respect to the base $\mathscr{B}_1$ of C_1. Therefore $\mathbf{w} \in \tilde{C}_1$.

On the other hand, any circuit **c** can be written according to Lemma 1 as a linear combination of the Betti circuits of Γ, that is,

$$\mathbf{c} = \sum_{k=1}^{B} a(c, \gamma_k)\boldsymbol{\gamma}_k,$$

where $a(c, \gamma_k) \in \{-1, 0, 1\}$. Then the vector **w** has the expression

$$\mathbf{w} = \sum_{k=1}^{B} \left(\sum_{c \in \mathscr{C}} a(c, \gamma_k) w_c \right) \boldsymbol{\gamma}_k.$$

Hence the (i, j) coordinates of **w** are given by

$$w(i,j) = \pi_i p_{ij} = \sum_{k=1}^{B} \left(\sum_{c \in \mathscr{C}} a(c, \gamma_k) w_c \right) J_{\gamma_k}(i,j)$$

and the proof is complete. □

REMARKS. (i) Theorem 1 still remains valid if we consider any circuit to be modulo the cyclic permutations, that is, instead of a single sequence, the circuit c is understood to be the equivalence class

$$\hat{c} = \{(i_1, \dots, i_s, i_1), (i_2, \dots, i_s, i_1, i_2), \dots, (i_s, i_1, \dots, i_{s-1}, i_s)\},$$

where the equivalence relation is defined as follows: $\tilde{c} \sim c$ iff $\tilde{c} \in \hat{c}$ (see [5]). That the class circuit $\hat{c}$ can be viewed as a vector of $\tilde{C}_1$ follows from the fact that all the representatives of $\hat{c}$ are identical to the same vector $1 \cdot b_{(i_1, i_2)} + \cdots + 1 \cdot b_{(i_s, i_1)}$ in C_1.

(ii) The coefficients ω_{γ_k}, $k = 1, \dots, B$, of decompositions (8) and (9) can be negative numbers. When we can find a base $\Gamma = \{\gamma_1, \dots, \gamma_B\}$ of Betti circuits such that the circuit weights w_{γ_k} occurring in (3) are larger than or equal to w_c for all the circuits $c \notin \Gamma$, then the corresponding weights ω_{γ_k} of (9) will be nonnegative numbers.

For instance in Figure 1 the circuits $c_1 = (1, 2, 3, 1)$, $c_2 = (1, 3, 4, 5, 1)$, $c_3 = (1, 3, 1)$ and $c_4 = (1, 2, 3, 4, 5, 1)$ have the positive weights w_{c_1}, w_{c_2}, w_{c_3} and w_{c_4}. If $w_{c_3} \geq w_{c_4}$, then we may choose the Betti circuits to be $\gamma_1 = c_1$, $\gamma_2 = c_2$ and $\gamma_3 = c_3$, while $\mathbf{c}_4 = \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_3$.

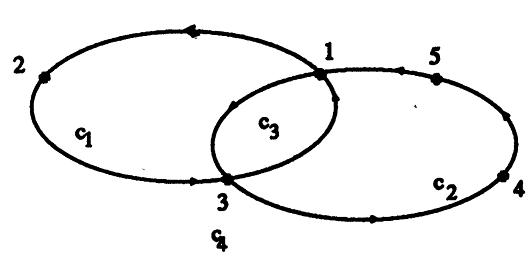

FIG. 1.

Note that for any branch (i, j) of c_4 we have

$$J_{c_4}(i,j) = J_{\gamma_1}(i,j) + J_{\gamma_2}(i,j) - J_{\gamma_3}(i,j)$$
$$= \begin{cases} J_{\gamma_1}(i,j), & \text{if } (i,j) \in \{(1,2),(2,3)\}, \\ J_{\gamma_2}(i,j), & \text{if } (i,j) \in \{(3,4),(4,5),(5,1)\}. \end{cases}$$

Therefore, the circuit c_4 passes through a branch iff a single Betti circuit does. Then the weights of decomposition (9) are as follows: $\omega_{c_1} = w_{c_1} + w_{c_4}$, $\omega_{c_2} = \mathrm{w}_{c_2} + \mathrm{w}_{c_4}$ *and* $\omega_{c_3} = \mathrm{w}_{c_3} - \mathrm{w}_{c_4} \geq 0$. *Hence decomposition* (9) *becomes*

$$\pi_i p_{ij} = (w_{c_1} + w_{c_4}) J_{c_1}(i,j) + (w_{c_2} + w_{c_4}) J_{c_2}(i,j) + (w_{c_3} - w_{c_4}) J_{c_3}(i,j).$$

If $\mathrm{w}_{c_4} \geq \mathrm{w}_{c_3}$, *we choose* $\gamma_1 = \mathrm{c}_1$, $\gamma_2 = \mathrm{c}_2$, $\gamma_3 = \mathrm{c}_4$, *while* $\mathbf{c}_3 = \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_3$, *and we may repeat the same reasoning above.*

The class of $n \times n$ stochastic matrices which admit a circuit representation (9) with positive weights is large. For instance, we may obtain such matrices by the following construction: Let $\mathscr{C}$ be any class of overlapping circuits in $S = \{1, 2, \dots, n\}$ and let G be the graph (S, arcset $\mathscr{C}$). If the cardinal number of $\mathscr{C}$ is less than or equal to the Betti number of G, then the circuits of $\mathscr{C}$ can be chosen as Betti circuits of G. Otherwise we may choose a base $\Gamma \subset \mathscr{C}$ of Betti circuits of G and assign a weight $w_c > 0$ to each circuit c of $\mathscr{C}$ such that $w_\gamma \geq \sum_{c \in \mathscr{C} \setminus \Gamma} w_c$ for all $\gamma \in \Gamma$. Then $p_{ij} \equiv \sum_{c \in \mathscr{C}} w_c J_c(i,j) / \sum_{c \in \mathscr{C}} w_c J_c(i)$, $i, j = 1, \dots, n$, define a stochastic matrix which admits a decomposition (9) with positive weights.

(iii) Let P be an irreducible $n \times n$ stochastic matrix as in Theorem 1, whose graph $G(P)$ is the complete graph on $\{1, 2, \dots, n\}$. Since each circuit matrix $C_c = (1/p(c)) J_c$ is defined by a circuit c, then the number of circuit matrices equals the number of circuits of $G(P)$. On the other hand, since each circuit of $G(P)$ is written in terms of $n^2 - n + 1$ independent (Betti) circuits γ_k, the decomposition (9) will comprise $n^2 - n + 1$ terms, that is,

$$\pi P = \sum_{k=1}^{n^2-n+1} \left(p(\gamma_k) \omega_{\gamma_k} \right) C_{\gamma_k}. \tag{11}$$

Then the Betti dimension equals the Carathéodory-type dimension of the convex hull on the circuit matrices. Namely, according to Alpern [2], the latter follows when πP, as an equisummed matrix, is considered to be a vector of an $(n^2 - n)$-dimensional Euclidean space. [A matrix $R = (r_{ij})$ is called equisummed iff $\sum_j r_{ij} = \sum_j r_{ji}$ and $\sum_{ij} r_{ij} = 1$.]

3. The rotational dimension. In Remark (ii) of the preceding section we have shown that there are connected oriented graphs where any B circuits are Betti circuits ($B = 3$ in Figure 1). From this standpoint one may obtain a method of construction of finite stochastic matrices admitting minimal positive decomposition in terms of Betti circuits. Then we may prove the following theorem.

THEOREM 2. *Let G be a connected oriented graph on S with Betti number B, where any B circuits are Betti circuits. Then, if the stochastic matrix P has G as its graph and decompositions* (3) *provide positive decompositions* (9), *each of the lengths of description of the rotational partitions is greater than or equal to the length of description on a collection $\{\gamma_1, \dots, \gamma_{\tilde{B}}\}$ of Betti circuits whose graph is G, where $\tilde{B} \le B$.*

PROOF. Let P be an irreducible stochastic matrix on S which has G as its graph and admits positive decompositions (9). Then we shall start labeling (7) with a decomposition of the form

$$\pi P = \sum_{k=1}^{B} \big(p(\gamma_k)\,\omega_{\gamma_k}\big) C^k, \qquad \omega_{\gamma_k} \ge 0,\ k = 1, \dots, B, \tag{12}$$

where $\Gamma = \{\gamma_1, \dots, \gamma_B\}$ is a base of Betti circuits of the graph $G = G(P)$ of P and $C^k \equiv (1/p(\gamma_k))J_{\gamma_k}$, $k = 1, \dots, B$.

Consider the shift f_t with $t = 1/M$ and all $\omega_{\gamma_k} > 0$. Then according to Alpern's procedure [2], let $(A_k,\ k = 1, 2, \dots, B)$ be a partition of $A = [0, 1/M)$ into B intervals $A_1, \dots, A_B$ such that the relative distribution $[\lambda(A_k)/\lambda(A),\ k = 1, 2, \dots, B]$ is given by the circuit distribution $[p(\gamma_k)\omega_{\gamma_k},\ k = 1, 2, \dots, B]$, that is,

$$\lambda(A_k) = (1/M)\,p(\gamma_k)\,\omega_{\gamma_k}, \qquad k = 1, \dots, B.$$

Define $A_{kl} = f_t^{l-1}(A_k)$ for $k = 1, \dots, B$; $l = 1, \dots, M$. Then for each choice of the starting points of γ_k, $k = 1, \dots, B$, the sets $S_i = \bigcup A_{kl}$, $i = 1, \dots, n$, labeled by (7), provide a rotational partition $(1/M, \mathscr{S}(P))$ of P.

On the other hand, each base Γ of Betti circuits of G determines a different length $\delta(B, \Gamma)$ of description of $\mathscr{S}(P)$. Then we may choose a base $\tilde{\Gamma}$ of Betti circuits of G which provides a minimal length $\delta(\tilde{B}, \tilde{\Gamma})$ of description with $\tilde{B} \le B$. We have $\tilde{B} < B$ when certain $\omega_{\gamma_k} = 0$ in equation (12). The proof is complete. □

Let us now consider a standard transition matrix function $P(h) = (p_{ij}(h),\ i, j \in S)$, $h \ge 0$, which defines a recurrent continuous parameter Markov process $\xi = (\xi_h)_{h \ge 0}$. Let $\pi = (\pi_i,\ i \in S)$ be a positive invariant probability distribution of $P = (P(h),\ h \ge 0)$, that is, $\pi_i > 0$, $i \in S$, and $\pi P(h) = \pi$, $h \ge 0$. Then according to Theorem 2 of [6], for each $h > 0$ there exists a rotational representation $(t, \mathscr{S}(h))$ of $P(h)$; that is,

$$p_{ij}(h) = \lambda\big(S_i \cap f_t^{-1}(S_j)\big)/\lambda(S_i), \qquad i, j \in S,\ h > 0,$$

where $t^{-1} = \operatorname{lcm}(1, 2, \dots, n)$ and $\mathscr{S}(h) = (S_i(h),\ i = 1, \dots, n)$ is an n-partition of $[0, 1)$ with $\lambda(S_i(h)) = \pi_i$, $i = 1, \dots, n$. Furthermore, the unique class $\mathscr{C}$ which provides the cycle distributions for all the matrices $P(h)$, $h > 0$, comprises all the directed cycles occurring along the sample paths of the discrete skeletons $\Xi_h = (\xi_{nh})_{n \geq 0}$ (see [6]). Accordingly, the circuit decomposition of each $P(h)$ is given on each recurrent class (except for a constant) by equations

$$\pi_i p_{ij}(h) = \sum_{\hat{c} \in \mathscr{C}} (p(c)\tilde{\omega}_c(h))C_c(i,j), \qquad i,j \in S, h > 0,$$

where $\tilde{\omega}_c(h)$, $\hat{c} \in \mathscr{C}$, are uniquely determined by the probabilistic criterion stated in Section 1.

Let σ be the cardinal number of $\mathscr{C}$. Then each partition $\mathscr{S}(h)$, $h > 0$, is characterized by a unique length $\delta = \delta(\sigma, \mathscr{C})$ of description which is independent of h. We call $\delta(\sigma, \mathscr{C})$ the *rotational dimension of the transition matrix function* of $P = (P(h), h \geq 0)$. Then we have the following theorem.

THEOREM 3. *The rotational dimension $\delta(\sigma, \mathscr{C})$ of all recurrent $n \times n$ transition matrix functions with the same graph G is provided by the collection $\mathscr{C}$ of all directed circuits of G where σ is the cardinal number of $\mathscr{C}$.*

PROOF. Let $\mathscr{C}$ be the collection of all the circuits of a graph G on $\{1, \dots, n\}$ and let $\sigma = \operatorname{card} \mathscr{C}$. Then the rotational dimension $\delta(\sigma, \mathscr{C})$ of a transition matrix function $P = (P(h), h \geq 0)$, whose graph is G, will remain invariant to the change of P (on G). The proof is complete. □

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