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LARGE DEVIATIONS FROM A HYDRODYNAMIC SCALING LIMIT FOR A NONGRADIENT SYSTEM

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The hydrodynamic limit appears as a law of large numbers for rescaled density profiles of a large stochastic system. We study the large deviations from this scaling limit for a particular nongradient system, the nongradient version of the Ginzburg–Landau model.

1. Introduction. This paper is an extension of the basic paper of Donsker and Varadhan [2], on the large deviations from the hydrodynamic scaling limit of the Ginzburg–Landau model, to the nongradient version of that model considered in [5]. The setup in [5] is as follows: Let S denote the unit circle $0 \le \theta \le 1$ with $1 \equiv 0$ and, for an integer N, let S_N denote the periodic lattice $\{i/N\}_{i=1,\dots,N}$. The neighbours of a site i/N are $i + 1/N$ and $i - 1/N$ with addition modulo 1. To each site i/N is assigned a variable $x_i \in \mathbb{R}$ which is thought of as the charge at i/N. The vector $x \in \mathbb{R}^N$ evolves according to the system of stochastic differential equations

$$
\begin{aligned}
dx_i &= dz_{i-1,i} - dz_{i,i+1}, \\
dz_{i,i+1} &= \frac{N^2}{2} W_{i,i+1}\, dt + N\sqrt{a(x_i, x_{i+1})}\, d\beta_i,
\end{aligned}
\tag{1.1}
$$

where

$$
\begin{aligned}
W_{i,i+1} = {}& a(x_i, x_{i+1})(\varphi'(x_i) - \varphi'(x_{i+1})) \\
& - \frac{\partial a}{\partial x}(x_i, x_{i+1}) + \frac{\partial a}{\partial y}(x_i, x_{i+1}),
\end{aligned}
\tag{1.2}
$$

$a(x, y)$ is a function of $\mathbb{R}^2$ which is assumed to have bounded continuous first derivatives and satisfy

$$1/a^* < a(x, y) < a^* \tag{1.3}$$

for some $a^* < \infty$, and $\varphi \in C^2$ is assumed to satisfy

$$\int_{-\infty}^{\infty} e^{-\varphi(x)}\, dx = 1, \tag{1.4}$$

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$$\int_{-\infty}^{\infty} \exp(\lambda x - \varphi(x))\, dx = M(\lambda) < \infty \quad \text{for all } \lambda \in \mathbb{R}, \tag{1.5}$$

$$\int_{-\infty}^{\infty} \exp(\sigma|\varphi'(x)| - \varphi(x))\, dx < \infty \quad \text{for all } \sigma > 0 \tag{1.6}$$

and $1/K < \varphi'' < K$ for some $K < \infty$. The β_i are independent Brownian motions and the N^2 represents the diffusion scaling of time, corresponding to the lattice spacing of $1/N$.

When $a \equiv 1$, the system is the standard Ginzburg–Landau model (see [3], [2]) which is a *gradient system*, meaning that the microscopic current $W_{i,i+1}$ can be expressed as the gradient $\tau^{i+1}g - \tau^i g$ of the local function $g(x_{-l}, \dots, x_l)$. For such systems there is no net effect of fluctuations on the diffusion coefficient. While this simplifies the proofs considerably, it is not a natural condition. Many models, and in particular most stochastic lattice gases, do not satisfy the gradient condition. The Onsager coefficient a is introduced into the model to break the gradient structure.

The dynamics (1.1) can alternatively be described by the generator

$$\mathscr{L}_N = \frac{N^2}{2} \sum_{i=1}^{N} \Phi_N^{-1} D^{ii+1} \Phi_N a(x_i, x_{i+1}) D^{ii+1}, \tag{1.7}$$

where $D^{ii+1} = \partial/\partial x_i - \partial/\partial x_{i+1}$, or, since $\mathscr{L}_N$ is symmetric with respect to $\Phi_N(x) = \exp - \sum_{i=1}^N \varphi(x_i)$, by the Dirichlet form

$$\mathscr{D}_N(f) = \tfrac{1}{2} \int_{\mathbb{R}^N} \sum_{i=1}^{N} a(x_i, x_{i+1}) \big(D^{ii+1} f\big)^2 \Phi_N(x)\, dx. \tag{1.8}$$

As initial distribution for x we take $f_N^0 \Phi_N$, where

$$f_N^0(x) = \exp \sum_{i=1}^{N} h'(m_0(i/N)) x_i - \log M\big(h'(m_0(i/N))\big), \tag{1.9}$$

where $m_0(\theta)$ is a given smooth function on S and

$$h(m) = \sup_{\lambda} \{\lambda_m - \log M(\lambda)\}. \tag{1.10}$$

Corresponding to the initial distribution (1.9), the evolution (1.1) and $T < \infty$, we have a measure Q_N on $C([0,T] \to \mathbb{R}^N)$. For each $x \in \mathbb{R}^N$ define a measure on S by

$$\mu_M(d\theta) = \frac{1}{N} \sum_{i=1}^{N} x_i \delta_{i/N}, \tag{1.11}$$

where δ_θ represents the measure giving mass 1 to θ in S. The probability measure Q_N is carried by this mapping into a probability measure $\hat{Q}_N$ on $X = C([0,T] \to M[S])$, where $M[S]$ denotes the set of signed measures on S. In [5] it is shown that under these conditions,

$$\hat{Q}_N \Rightarrow \delta_{m(t,\theta)\, d\theta}, \tag{1.12}$$

where $\delta_{m(t,\theta)\,d\theta}$ denotes the probability measure on X whose mass is concentrated on the unique weak solution of the nonlinear diffusion equation

$$(1.13)\qquad \frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial}{\partial \theta}\left(\hat{a}(m(t,\theta))\frac{\partial}{\partial \theta}h'(m(t,\theta))\right), \qquad m(0,\theta) = m_0(\theta).$$

The function $\hat{a}(m)$ is given by a variational formula which we now describe. Consider $\Omega = \prod_{-\infty}^{\infty}\mathbb{R}$ with product measure $\pi_m = \prod_{i=-\infty}^{\infty}\exp\{h'(m)x_i - \varphi(x_i) - \log M(h'(m))\}\,dx_i$. By a *local function* we mean a function on Ω which depends on only finitely many coordinates. Let T be the shift operator on Ω defined by $(Tx)_i = x_{i+1}$, and for functions g on Ω, let $\tau^i g(x) = g(T^i x)$. For a local function g we can write the formal sum

$$(1.14)\qquad \xi_g = \sum_{i=-\infty}^{\infty} \tau^i g.$$

Although this is purely formal, the derivative $D^{12}\xi_g$ is well defined. The variational formula for $\hat{a}(m)$ is

$$(1.15)\qquad \hat{a}(m) = \inf_g E^{\pi_m}\left[a(x_1,x_2)\left(1 - D^{12}\xi_g\right)^2\right],$$

where the infimum is taken over all smooth local functions g.

In the present article we are concerned with the large deviations from the scaling limit described above. Let $\langle f, g\rangle$ be the usual inner product on $L^2(S, d\theta)$. Define, for functions $m(\theta)$, $f(\theta)$ on S,

$$(1.16)\qquad \begin{aligned} I_{\mathrm{static}}(m(\cdot)) = \int_S \big(&h(m(\theta)) - h(m_0(\theta)) \\ &-h'(m_0(\theta))(m(\theta) - m_0(\theta))\big)\,d\theta, \end{aligned}$$

$$(1.17)\qquad \|f\|^2_{-1,\hat{a}(m(\cdot))} = \sup_{J\in H_1(S)}\left\{2\langle J, f\rangle - \int_S \hat{a}(m(\theta))\left(\frac{\partial J}{\partial \theta}(\theta)\right)^2 d\theta\right\}$$

and for a function $m(t,\theta)$ on $[0,T]\times S$,

$$(1.18)\qquad \begin{aligned} I_{\mathrm{dynamic}}(m(\cdot,\cdot))\frac{1}{2}\int_0^T \Bigg\| &\frac{\partial m}{\partial t} - \frac{1}{2}\frac{\partial}{\partial \theta} \\ &\times\left(\hat{a}(m(t,\theta))\frac{\partial}{\partial \theta}h'(m(t,\theta))\right)\Bigg\|^2_{-1,\hat{a}(m(t,\cdot))} dt. \end{aligned}$$

We will need as well the form $I_0(\mu(\cdot,\cdot))$, which is defined as follows: $I_0(\mu(\cdot,\cdot)) = \infty$ unless $\mu(t, d\theta) = m(t,\theta)\,d\theta$ for each $t\in[0,T]$, in which case

$$(1.19)\qquad I_0(\mu(\cdot,\cdot)) = \int_0^T\int_S \hat{a}(m)\left(\frac{\partial}{\partial\theta}(h'(m))\right)^2 d\theta\,dt.$$

We can now describe the rate function I and X. If $I_0(\mu(\cdot,\cdot)) = \infty$, then $I(\mu(\cdot,\cdot)) = \infty$. Otherwise, $\mu(t,d\theta) = m(t,\theta)\,d\theta$ for each $t\in[0,T]$ and

$$(1.20)\qquad I(\mu(\cdot,\cdot)) = I_{\mathrm{static}}(m(0,\cdot)) + I_{\mathrm{dynamic}}(m(\cdot,\cdot)).$$

The main result of this paper is that Q_N has the large deviations property with rate function I. In particular, let $M^l[S]$ be the set of signed measures on S with total variation at most l. The space $M^l[S]$ with the topology of weak convergence is metrizable. Therefore $X^l = C([0,T] \to M^l[S])$ can be given the topology of uniform convergence. Finally, $X = \bigcup X^l$ is given the direct limit topology. For any closed set $C \subset X$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \hat{Q}_N(C) \le - \inf_{u(\cdot,\cdot)\in C} I(\mu(\cdot,\cdot)) \tag{1.21}$$

and, for any open set $U \subset X$,

$$\liminf_{N\to\infty} \frac{1}{N} \log \hat{Q}_N(U) \ge - \inf_{\mu(\cdot,\cdot)\in U} I(\mu(\cdot,\cdot)). \tag{1.22}$$

2. Upper bound. To prove the upper bound (1.21) we will produce a family of functionals $\hat{F}_{J,G,\varepsilon_1,\varepsilon_2}(\mu(\cdot,\cdot))$, depending on certain parameters $J, G, \varepsilon_1, \varepsilon_2$, for which we can prove the following lemmas.

LEMMA 2.1. *For each fixed J and G,*

$$\lim_{\varepsilon_2\to 0} \limsup_{\varepsilon_1\to 0} \limsup_{N\to\infty} \frac{1}{N} \log E^{\hat{Q}_N}\left[\exp N\hat{F}_{J,G,\varepsilon_1,\varepsilon_2}\right] \le 0. \tag{2.1}$$

LEMMA 2.2. *For each $\mu(\cdot,\cdot) \in X$ for which $I_0(\mu(\cdot,\cdot)) < \infty$*

$$I(\mu(\cdot,\cdot)) \le \lim_{\varepsilon_2\to 0} \limsup_{\varepsilon_1\to 0} \sup_{J,G} \hat{F}_{J,G,\varepsilon_1,\varepsilon_2}(\mu(\cdot,\cdot)). \tag{2.2}$$

In addition we will show the following lemma.

LEMMA 2.3. *If $I_0(\mu(\cdot,\cdot)) \equiv \infty$ on a compact $K \subset X$, then*

$$\limsup_{N\to\infty} \frac{1}{N} \log \hat{Q}_N(K) = -\infty. \tag{2.3}$$

The upper bound for compact sets will follow by the exponential Chebyshev inequality. These are essentially standard arguments in large deviations (see [4], [2]). To extend this to arbitrary closed sets, we need the following exponential tightness estimates whose proofs can be found in [5].

LEMMA 2.4. *For each $T < \infty$, $\varepsilon > 0$ and smooth function J on S,*

$$\lim_{l\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log \hat{Q}_N\left(\sup_{0\le t\le T} \|\mu(t)\| \ge l\right) = -\infty, \tag{2.4}$$

$$\lim_{\delta\downarrow 0} \limsup_{N\to\infty} \frac{1}{N} \log \hat{Q}_N\left(\sup_{\substack{0\le s<t\le T\\ |s-t|<\delta}} |\langle J, \mu(t)\rangle - \langle J, \mu(s)\rangle| \ge \varepsilon\right) = -\infty. \tag{2.5}$$

We now describe the functionals $\hat{F}_{J,G,\varepsilon_1,\varepsilon_2}(\mu(\cdot,\cdot))$. Let $J(t,\theta)$ be a smooth test function on $[0,T]\times S$ and let $G(\theta)$ be a smooth test function on S. For $\varepsilon_1>0$, let $m_\theta^{\varepsilon_1}$ be the average density in an ε_1 neighbourhood of θ: $m_\theta^{\varepsilon_1}=(2\varepsilon_1)^{-1}\mu([\theta-\varepsilon_1,\theta+\varepsilon_1])$. For $\varepsilon_1,\varepsilon_2>0$, let

$$\begin{aligned}\hat{F}_{J,G,\varepsilon_1,\varepsilon_2}(\mu(\cdot,\cdot)) = &\int_S \log M(h'(m_0(\theta))) - \log M(G(\theta)+h'(m_0(\theta)))\,d\theta\\ &+\int_S J(T,\theta)\mu(T,d\theta)\\ &-\int_S (J(0,\theta)-G(\theta))\mu(0,d\theta)\\ &-\int_0^T\int_S \frac{\partial J}{\partial t}(t,\theta)\mu(t,d\theta)\,dt\\ &-\frac{1}{2}\int_0^T\int_S \frac{\partial J}{\partial\theta}(t,\theta)\hat{a}(m_\theta^{\varepsilon_1})\\ &\times\left(\frac{h'(m_{\theta-\varepsilon_2}^{\varepsilon_1})-h'(m_{\theta+\varepsilon_s}^{\varepsilon_1})}{2\varepsilon_2}\right)d\theta\,dt\\ &-\frac{1}{2}\int_0^T\int_S\left(\frac{\partial J}{\partial\theta}(t,\theta)\right)^2\hat{a}(m_\theta^{\varepsilon_1})\,d\theta\,dt.\end{aligned}\tag{2.6}$$

We will now begin the proof of Lemma 2.1, which depends on several preliminary lemmas. Before we start, it is worth making a remark. Let P_N be the *equilibrium process*, by which we mean the measure on $C([0,T]\to\mathbb{R}^N)$ corresponding to taking the invariant measure $\Phi_N(x)\,dx$ as initial distribution. Suppose that for some sequence of functionals F_N we can prove a superexponential estimate which says that for any $\alpha>0$,

$$\limsup_{N\to\infty}\frac{1}{N}\log E^{P_N}[\exp\alpha NF_N]\le 0.\tag{2.7}$$

Then, by Hölder's inequality, this estimate extends to Q_N:

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N}\log E^{Q_N}[\exp\alpha NF_N]\\ &\quad\le\limsup_{\gamma\downarrow 0}\limsup_{N\to\infty}\frac{1}{(1+\gamma)N}\log E^{P_N}\left[\left(f_N^0\right)^{1+\gamma}\right]\\ &\quad\quad+\limsup_{\gamma\downarrow 0}\limsup_{N\to\infty}\frac{\gamma}{(1+\gamma)N}\log E^{P_N}\left[\exp\frac{(1+\gamma)}{\gamma}\alpha NF_N\right]\\ &\quad\le 0.\end{aligned}\tag{2.8}$$

Now let $g(x_{-l},\dots,x_l,m)$ be a smooth function on $\mathbb{R}^{2l+2}$ with bounded first derivatives and for $\xi_g = \sum_{i=-\infty}^{\infty}\tau^i g$, let

$$A(g,) = E^{\pi_m}\Big[a(x_1,x_2)\big(1 - D^{12}\xi_g\big)^2\Big] - \hat{a}(m). \tag{2.9}$$

For $k > l$, let

$$\begin{aligned} V^1 = {} & \frac{1}{2}\sum_{i=1}^{N} J\Big(t,\frac{i}{N}\Big)\big[W_{i,i+1} - (\hat{\mathscr{L}}g)\big(x_{i-l},\dots,x_{i+l},m_{i/N}^{k/N}\big)\big] \\ & + \frac{1}{2N}\sum_{i=1}^{N} J\Big(t,\frac{i}{N}\Big)\hat{a}\big(m_{i/N}^{\varepsilon_1}\big)\left(\frac{h'\big(m_{i/N+\varepsilon_2}^{\varepsilon_1}\big) - h'\big(m_{i/N-\varepsilon_2}^{\varepsilon_1}\big)}{2\varepsilon_2}\right) \\ & - \frac{\alpha}{4N}\sum_{i=1}^{N}\Big(J\Big(t,\frac{i}{N}\Big)\Big)^2 A\big(g,m_{i/N}^{\varepsilon_1}\big). \end{aligned} \tag{2.10}$$

Here $\hat{\mathscr{L}}_g$ represents the generator, without the N^2 scaling, acting only on the variables x_{-l} through x_l.

The main point in our proof is the following estimate which is due to Varadhan.

Lemma 2.5. *For each $\alpha > 0$, g and J,*

$$\lim_{\varepsilon_2\to 0}\limsup_{\varepsilon_1\to 0}\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log E^{P_N}\Big[\exp \alpha N\int_0^T V^1(t,x(t))\,dt\Big] \le 0. \tag{2.11}$$

Proof. This is Theorem 7.1 of [5]. □

Lemma 2.6. *For any $\varepsilon > 0$ and $C < \infty$, we can find a smooth function $g(x_{-1},\dots,x_l,m)$ on $\mathbb{R}^{2l+2}$ with bounded first derivatives such that*

$$\sup_{|m|\le C} A(g,m) \le \varepsilon \tag{2.12}$$

and

$$\sup_m A(g,m) \le a^*. \tag{2.13}$$

Proof. Let $\mathscr{G}_{l,B}$ be the set of smooth g of the form $g(x_{-l},\dots,x_l)$ with $\|g\|_\infty \le B$ and $\|\partial g/\partial x_i\|_\infty \le B$ *for* $i = -l,\dots,l$, and let

$$A_{l,B}(m) = \inf_{g\in\mathscr{G}_{l,B}} A(g,m). \tag{2.14}$$

The quantity $A_{l,B}(m)$ is upper semicontinuous in m, nonincreasing in l and B and, for each m, $\lim_{l\to\infty}\lim_{B\to\infty} A_{l,B}(m) = 0$. Therefore,

$$\lim_{l\to\infty}\lim_{B\to\infty}\sup_{|m|\le C+1} A_{l,B}(m) = 0.$$

Thus we can find l, B and, for each $|m| \le C + 1$, $g(m) \in \mathscr{G}_{l,B}$ so that $A(g(m), m) \le \varepsilon/4$. For $|m| > C + 1$ take $g(m) = 0$. Of course this g is not regular in m, so we will smooth it out. Let $\mathscr{B}_M = \{x^2_{-2l+1} + \cdots + x^2_{2l+3} \le M\}$ and let M be large enough so that for all $g \in \mathscr{G}_{l,B}$,

$$\sup_{|m|\le C} E^{\pi_m}\Big[a(x_1, x_2)\big(1 - D^{12}\xi_g\big)^2 1_{\mathscr{B}_M^C}\Big] \le \frac{\varepsilon}{4}, \tag{2.15}$$

and let $1 > \delta > 0$ be small enough that if $|m| \le C + 1$ and $|m' - m| \le \delta$, then $|\hat{a}(m') - \hat{a}(m)| \le \varepsilon/4$ and

$$\sup_{\mathscr{B}_M}\left\{\frac{d\pi_{m'}}{d\pi_m}\bigg|_{\mathscr{F}_{-2l+1,\dots,2l+3}}\right\} \le 1 + \frac{\varepsilon}{4a^*}. \tag{2.16}$$

Then for $|m| \le C$ and $|m' - m| \le \delta$,

$$\begin{aligned} &A(g(m), m')\\ &\quad = E^{\pi_{m'}}\Big[a(x_1, x_2)\big(1 - D^{12}\xi_{g(m)}\big)^2\Big] - \hat{a}(m')\\ &\quad \le E^{\pi_{m'}}\Big[a(x_1, x_2)\big(1 - D^{12}\xi_g(m)\big)^2 1_{\mathscr{B}_M}\Big] + \frac{\varepsilon}{4} - \hat{a}(m') \qquad (2.17)\\ &\quad \le \left(1 + \frac{\varepsilon}{4a^*}\right)E^{\pi_m}\Big[a(x_1, x_2)\big(1 - D^{12}\xi_{g(m)}\big)^2 1_{\mathscr{B}_M}\Big] + \frac{\varepsilon}{4} - \hat{a}(m')\\ &\quad \le \varepsilon. \end{aligned}$$

Let φ be a smoothing kernel with support in $(-\delta, \delta)$ and let

$$\tilde{g}(x_{-l}, \dots, x_l, m) = \int \varphi(m' - m) g(x_{-l}, \dots, x_l, m')\, dm'. \tag{2.18}$$

Clearly $\partial\tilde{g}/\partial m$ is bounded and by (2.17) and convexity,

$$A(\tilde{g}(m), m) \le \left(\int \varphi(m' - m)\big(A(g(m'), m)\big)^{1/2}\, dm'\right)^2 \le \varepsilon \tag{2.19}$$

for $|m| \le C$. Finally, if $g = 0$, $A(g, m) = E^{\pi_m}[a(x_1, x_2)] - \hat{a}(m) \le a^*$, so by (2.19), for $|m| > C$, $A(\tilde{g}(m), m) \le a^*$. □

The next lemma allows us to control the difference between the corrector term which appears in (2.10),

$$\begin{aligned} &\sum_{i=1}^N J(t, i/N)\big(\hat{\mathscr{L}}g\big)\big(x_{i-1}, \dots, x_{i+1}, m^{k/N}_{i/N}\big)\\ &\qquad = \sum_{i=1}^N J(t, i/N) \sum_{j=i-k}^{i+k-1} \mathscr{L}^{j,j+1}g\big(x_{i-l}, \dots, x_{i+l}, m^{k/N}_{i/N}\big), \end{aligned} \tag{2.20}$$

where $\mathscr{L}^{i,i+1} = \Phi_N^{-1}D^{ii+1}\Phi_N a(x_i, x_{i+1})C^{ii+1}$, and the "derivative" term,

$$\sum_{i=1}^N J\left(t, \frac{i}{N}\right)\left(\frac{\mathscr{L}_N}{N^2}g\right)\big(x_{i-l}, \dots, x_{i+l}, m^{k/N}_{i/N}\big). \tag{2.21}$$

The difference is in the action of the generator on the local averages. It is given by

$$V^2 = \sum_{i=1}^{N} J(t, i/N)(\mathscr{L}^{i+k,i+k+1} + \mathscr{L}^{i-k-1,i-k}) \times g\left(x_{i-l}, \dots, x_{i+l}, m_{i/N}^{k/N}\right). \tag{2.22}$$

LEMMA 2.7. *For any* $\alpha > 0$,

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log E^{P_N}\left[\exp \alpha N \int_0^T V^2(t, x(t))\, dt\right] \le 0. \tag{2.23}$$

PROOF. Let g^i denote $g(x_{i-l}, \dots, x_{i+l}, m_{i/N}^{k/N})$. By Schwarz's inequality, for any f

$$\begin{aligned} &\int \sum_{i=1}^{N} \mathscr{L}^{i+k,i+k+1} g^i f \Phi_N\, dx \\ &= -\int \sum_{i=1}^{N} a(x_{i+k}, x_{i+k+1}) D^{i+k,i+k+1} g^i D^{i+k,i+k+1} f \Phi_N\, dx \\ &= -\int \sum_{i=1}^{N} a(x_{i+k}, x_{i+k+1}) D^{i+k,i+k+1} g^i \sqrt{f} \frac{D^{i+k,i+k+1} f}{\sqrt{f}} \Phi_n\, dx \\ &\le \left(\int \sum_{i=1}^{N} a(x_{i+k}, x_{i+k+1}) \left(D^{i+k,i+k+1} g^i\right)^2 f \Phi_N\, dx\right)^{1/2} \left(2\mathscr{D}_N\left(\sqrt{f}\right)\right)^{1/2}. \end{aligned} \tag{2.24}$$

There is of course an analogous estimate for $\int \sum_i \mathscr{L}^{i-k-1,i-k} g^i f \Phi_N\, dx$. Also,

$$D^{i+k,i+k+1} g^i = -D^{i-k-1,i-k} g^i = \frac{1}{2k} \frac{\partial g}{\partial m}\left(x_{i-l}, \dots, x_{i+l}, m_{i/N}^{k/N}\right). \tag{2.25}$$

By the spectral theorem and (2.24) and (2.25),

$$\begin{aligned} &\frac{1}{N} \log E^{P_N}\left[\exp \alpha N \int_0^T \sum_{i=1}^{N} J\left(t, \frac{i}{N}\right) \times (\mathscr{L}^{i+k,i+k+1} + \mathscr{L}^{i-k-1,i-k}) g^i\, dt\right] \\ &\le \int_0^T \frac{1}{N} \sup_f \left\{\alpha N \int \sum_{i=1}^{N} J\left(t, \frac{i}{N}\right) (\mathscr{L}^{i+k,i+k+1} + \mathscr{L}^{i-k-1,i-k}) \times g^i f \Phi_N\, dx - N^2 \mathscr{D}_N\left(\sqrt{f}\right)\right\} dt \\ &\le \frac{C}{N} \sup_f \left\{\alpha^2 \int \sum_{i=1}^{N} a(x_{i+k}, x_{i+k+1}) \times \left[\left(D^{i+k,i+k+1} g^i\right)^2 + \left(D^{i-k-1,i-k} g^i\right)^2\right] f \Phi_N\, dx\right\} \end{aligned} \tag{2.26}$$

$$\leq C'\alpha^2 k^{-2}\left\|\frac{\partial g}{\partial m}\right\|_\infty^2.$$

This completes the proof of Lemma 2.7. □

PROOF OF LEMMA 2.1. For a smooth function $g(x_{-l},\dots,x_l,m)$ on $\mathbb{R}^{2l+2}$ with bounded first derivatives, let

$$\xi^N(t) = \sum_{i=1}^N \frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right) g\left(x_{i-l}(t),\dots,x_{i+l}(t), m^{k/N}_{i/N}(t)\right) \tag{2.27}$$

and let

$$\begin{aligned}
F^1(x(\cdot)) = &\int_S \log M(h'(m_0(\theta))) - \log M(G(\theta)+h'(m_0(\theta)))\,d\theta\\
&+\frac{1}{N}\sum_{i=1}^N J\left(T,\frac{i}{N}\right)x_i(T) - \frac{1}{N^2}\xi^N(T)\\
&-\frac{1}{N}\sum_{i=1}^N\left(J\left(0,\frac{i}{N}\right)-G\left(\frac{i}{N}\right)\right)x_i(0)+\frac{1}{N^2}\xi^N(0)\\
&-\int_0^T\frac{1}{N}\sum_{i=1}^N\left(\frac{\partial J}{\partial t}\left(t,\frac{i}{N}\right)x_i(t)\right)dt\\
&+\int_0^T\frac{1}{N^2}\sum_{i=1}^N\left(\frac{\partial^2 J}{\partial t\,\partial\theta}\left(t,\frac{i}{N}\right)g\left(x_{i-l}(t),\dots,x_{i+l}(t),m^{k/N}_{i/N}(t)\right)\right)dt\\
&-\int_0^T\frac{N}{2}\sum_{i=1}^N\left(J\left(t,i+\frac{1}{N}\right)-J\left(t,\frac{i}{N}\right)\right)W_{i.i+1}-\frac{\mathscr{L}_N}{N^2}\xi^N(t)\,dt\\
&-\int_0^T\frac{1}{2N}\sum_{i=1}^N a(x_i,x_{i+1})\\
&\times\left(N\left(J\left(t,i+\frac{1}{N}\right)-J\left(t,\frac{i}{N}\right)\right)+D^{ii+1}\xi^N\right)^2 dt.
\end{aligned}\tag{2.28}$$

The functional $\exp NF^1(T)$ is a martingale under Q_N, so

$$\begin{aligned}
\limsup_{N\to\infty}&\frac{1}{N}\log E^{Q_N}\left[\exp NF^1(T)\right]\\
&=\limsup_{N\to\infty}\frac{1}{N}\log E^{Q_N}\left[\exp NF^1(0)\right]\\
&=-\int_S\log\frac{M(G(\theta)+h'(m_0(\theta)))}{M(h'(m_0(\theta)))}\,d\theta\\
&\quad+\limsup_{N\to\infty}\frac{1}{N}\log E^{f_N^0\Phi_N}\left[\exp\sum_{i=1}^N G\left(\frac{i}{N}\right)x_i\right]\\
&=0.
\end{aligned}\tag{2.29}$$

By bounding the last term in (2.28) in terms of the bound on the derivatives of g, it is not difficult to show that

$$\limsup_{\gamma\downarrow 0}\limsup_{N\to\infty}\frac{1}{N}\log E^{Q_N}\big[\exp(1+\gamma)NF^1\big]\le 0. \tag{2.30}$$

Consider the functional $\hat{F}(\mu(\cdot,\cdot))$ defined in (2.6). For each N a mapping $x\mapsto\mu$ is given by (1.11) and $\hat{F}(\mu(\cdot,\cdot))$ pulls back through this map to a function $F(x(\cdot))$ on $C([0,T]\to\mathbb{R}^N)$. By Hölder's inequality, for any $\gamma>0$,

$$\begin{aligned}\frac{1}{N}&\log E^{Q_N}[\exp NF]\\ &\le\frac{1}{(1+\gamma)N}\log E^{Q_N}\big[\exp(1+\gamma)NF^1\big]\\ &\quad+\frac{\gamma}{(1+\gamma)N}\log E^{Q_N}\left[\exp\frac{1+\gamma}{\gamma}N(F-F^1)\right].\end{aligned}\tag{2.31}$$

To prove Lemma 2.1 it therefore suffices to show that for any $\alpha>0$,

$$\inf_{g}\lim_{\varepsilon_2\to 0}\limsup_{\varepsilon_1\to 0}\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log E^{Q_N}\big[\exp\alpha N(F-F^1)\big]\le 0. \tag{2.32}$$

We break it up as follows: $F-F^1=\int_0^T(V^3+V^4+V^5)\,dt+o(1)$, where

$$\begin{aligned}V^3&=\frac{1}{2}\sum_{i=1}^{N}\left(N\left(J\left(t,i+\frac{1}{N}\right)-J\left(t,\frac{i}{N}\right)\right)-\frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)\right)W_{i,i+1},\\ V^4&=\frac{1}{2}\sum_{i=1}^{N}\frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)W_{i,i+1}-\frac{\mathscr{L}_N}{N^2}\xi^N\\ &\quad+\frac{1}{2N}\sum_{i=1}^{N}\frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)\\ &\quad\times\hat{a}\big(m^{\varepsilon_1}_{i/N}\big)\left(\frac{h'\big(m^{\varepsilon_1}_{i/N+\varepsilon_2}\big)-h'\big(m^{\varepsilon_1}_{i/N-\varepsilon_2}\big)}{2\varepsilon_2}\right),\\ V^5&=\frac{1}{2N}\sum_{i=1}^{N}a(x_i,x_{i+1})\left(\frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)+D^{ii+1}\xi^N\right)^2\\ &\quad-\frac{1}{2N}\sum_{i=1}^{N}\hat{a}\big(m^{\varepsilon_1}_{i/N}\big)\left(\frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)\right)^2.\end{aligned}\tag{2.33}$$

By (3.8) of [5] there is a constant $C<\infty$ such that, for any $\alpha>0$,

$$\begin{aligned}\frac{1}{N}&\log E^{P_N}\left[\exp\alpha N\int_0^T V^3(x(t),t)\,dt\right]\\ &\le C\alpha^2\int_0^T\frac{1}{N}\sum_{i=1}^{N}\left(N\left(J\left(t,\frac{i+1}{N}\right)-J\left(t,\frac{i}{N}\right)\right)\right.\\ &\qquad\left.-\frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)\right)^2dt,\end{aligned}\tag{2.34}$$

the right hand side being $o(1)$. Now let $\alpha > 0$ and $\varepsilon > 0$ and choose $C < \infty$ so that

$$\limsup_{\varepsilon_1 \to 0} \limsup_{N \to \infty} \frac{1}{N} \log E^{P_N}\left[\exp 2\alpha^2 \int_0^T \sum_{i=1}^N \left(\frac{\partial J}{\partial \theta}\left(t, \frac{i}{N}\right)\right)^2 \times a^* 1_{\{|m_{i/N}^{\varepsilon_1}| > C\}}\, dt\right] \le \varepsilon \tag{2.35}$$

and, for this C, choose $g(m)$ as in Lemma 2.6 so that $A(g(m), m) \le a^*$ and

$$\sup_{|m| \le C} A(g(m), m) \le \frac{1}{4T} \varepsilon \alpha^{-2} \left\| \frac{\partial J}{\partial \theta} \right\|_\infty^{-2}. \tag{2.36}$$

By Schwarz's inequality,

$$\begin{aligned} &\frac{1}{N} \log E^{P_N}\left[\exp \alpha^2 \int_0^T \sum_{i=1}^N \left(\frac{\partial J}{\partial \theta}\left(t, \frac{i}{N}\right)\right)^2 A\left(g(m_{i/N}^{\varepsilon_1}), m_{i/N}^{\varepsilon_1}\right) dt\right] \\ &\le \frac{1}{2N} \log E^{P_N}\left[\exp \frac{\alpha^2}{2T} \int_0^T \sum_{i=1}^N \left(\frac{\partial J}{\partial \theta}\left(t, \frac{i}{N}\right)\right)^2 \varepsilon \alpha^{-2} \left\| \frac{\partial J}{\partial \theta} \right\|_\infty^{-2} dt\right] \\ &\quad + \frac{1}{2N} \log E^{P_N}\left[\exp 2\alpha^2 \int_0^T \sum_{i=1}^N \left(\frac{\partial J}{\partial \theta}\left(t, \frac{i}{N}\right)\right)^2 a^* 1_{\{|m_i^{\varepsilon_1}/N| > C\}}\, dt\right], \end{aligned} \tag{2.37}$$

the latter terms each being less than $\varepsilon/2$. Together with Lemmas 2.5 and 2.7 this shows that

$$\lim_{\varepsilon_2 \to 0} \limsup_{\varepsilon_1 \to 0} \limsup_{k \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log E^{P_N}\left[\exp \alpha N \int_0^T V^4(x(t), t)\, dt\right] \le \varepsilon. \tag{2.38}$$

To deal with V^5 we use Theorem 4.1 of [3], which says that if $\nu(x_{-k}, \dots, x_k)$ is bounded and continuous and $J(\theta)$ is smooth, then, for any $\alpha > 0$,

$$\limsup_{\varepsilon_1 \to 0} \limsup_{k \to \infty} \limsup_{N \to \infty} \frac{1}{N} \times \log E^{P_N}\left[\exp \alpha \int_0^T \left| \sum_{i=1}^N J\left(\frac{i}{N}\right)\left(\nu^i - E^{\pi_{m_{i/n}^{\varepsilon_1}}}[\nu^i]\right) \right| dt\right] \le 0, \tag{2.39}$$

where $\nu^i = \nu(x_{i-k}, \dots, x_{i+k})$. It is true that the theorem was only proved in [3] for the case $a \equiv 1$, but the proof extends to the general case by means of the inequality

$$\mathscr{D}_N(f) \ge \frac{1}{a^*} \int \sum_{i=1}^N (D^{i i+1} f)^2 \Phi_N\, dx. \tag{2.40}$$

From (2.37) and (2.39) it is easy to see that

$$\lim_{\varepsilon_2\to 0}\limsup_{\varepsilon_1\to 0}\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N}\log E^{P_N}\left[\exp \alpha N\int_0^T V^5(x(t),t)\,dt\right] \le \varepsilon. \tag{2.41}$$

To complete the proof of Lemma 2.1, we remark that (2.32) follows from (2.34), (2.38) and (2.41) by another application of Hölder's inequality. □

PROOF OF LEMMA 2.2. First note that

$$\begin{aligned} I_{\text{static}}(m(0,\cdot)) &= \sup_G\left\{\int_S G(\theta)m(0,\theta)\,d\theta + \int_S \log M(h'(m_0(\theta)))\right.\\ &\qquad\left. -\log M(G(\theta)+h'(m_0(\theta)))\,d\theta\right\}. \end{aligned}\tag{2.42}$$

By assumption, $I_0(\mu(\cdot,\cdot))<\infty$ and therefore $m_\theta^{\varepsilon_1}(t)\to m(t,\theta)$ for almost every t and θ. Suppose that $I_{\text{dynamic}}(\mu(\cdot,\cdot))\ge M$. Let $\varepsilon>0$ and choose $J(t,\theta)$ so that

$$\begin{aligned} M-\varepsilon \le{}& \int_S J(T,\theta)\mu(T,d\theta) - \int_S J(0,\theta)\mu(0,d\theta)\\ &-\int_0^T\int_S \frac{\partial J}{\partial t}(t,\theta)\mu(t,d\theta)\,dt\\ &+\frac{1}{2}\int_0^T\int_S\frac{\partial J}{\partial\theta}(t,\theta)\hat a(m(t,\theta))\frac{\partial}{\partial\theta}h'(m(t,\theta))\,d\theta\,dt\\ &-\frac{1}{2}\int_0^T\int_S\left(\frac{\partial J}{\partial\theta}(t,\theta)\right)^2\hat a(m(t,\theta))\,d\theta\,dt. \end{aligned}\tag{2.43}$$

By the bounded convergence theorem, since $\hat a$ is bounded and continuous,

$$\begin{aligned} &\int_0^T\int_S\left(\frac{\partial J}{\partial\theta}(t,\theta)\right)^2\hat a(m_\theta^{\varepsilon_1})\,d\theta\,dt\\ &\qquad\to\int_0^T\int_S\left(\frac{\partial J}{\partial\theta}(t,\theta)\right)^2\hat a(m(t,\theta))\,d\theta\,dt. \end{aligned}\tag{2.44}$$

Since $I_0(\mu(\cdot,\cdot))<\infty$, we have that

$$\int_0^T\int_S\frac{\partial J}{\partial\theta}\hat a(m)\left(\frac{\partial}{\partial\theta}h'(m)-\frac{h'(m_{\theta+\varepsilon_2}^{\varepsilon_1})-h'(m_{\theta-\varepsilon_2}^{\varepsilon_1})}{2\varepsilon_2}\right)d\theta\,dt\to 0 \tag{2.45}$$

as ε_1 is sent to zero, followed by ε_2. By a straightforward application of Schwarz's inequality and the fact that $\hat{a}(m)$ is continuous and bounded away from both 0 and ∞, we see that as ε_1 and then ε_2 go to zero,

$$\int_0^T \int_S \frac{\partial J}{\partial \theta}(\hat{a}(m) - \hat{a}(m_\theta^{\varepsilon_1}))\left(\frac{h'(m_{\theta+\varepsilon_2}^{\varepsilon_1}) - h'(m_{\theta-\varepsilon_2}^{\varepsilon_1})}{2\varepsilon_2}\right) d\theta\, dt \to 0. \tag{2.46}$$

Therefore, for ε_1 and ε_2 sufficiently small,

$$\begin{aligned} M - 2\varepsilon \le &\int_S J(T,\theta)\mu(T,d\theta) - \int_S J(0,\theta)\mu(0,d\theta) \\ &- \int_0^T \int_S \frac{\partial J}{\partial t}(t,\theta)\mu(t,d\theta)\,dt \\ &+ \frac{1}{2}\int_0^T \int_S \frac{\partial J}{\partial \theta}(t,\theta)\hat{a}(m_\theta^{\varepsilon_1})\left(\frac{h'(m_{\theta+\varepsilon_2}^{\varepsilon_1}) - h'(m_{\theta-\varepsilon_2}^{\varepsilon_1})}{2\varepsilon_2}\right) d\theta\, dt \\ &- \frac{1}{2}\int_0^T \int_S \left(\frac{\partial J}{\partial \theta}(t,\theta)\right)^2 \hat{a}(m_\theta^{\varepsilon_1})\, d\theta\, dt. \end{aligned} \tag{2.47}$$

Comparing with (2.6) we see that Lemma 2.2 follows. □

PROOF OF LEMMA 2.3. For smooth test functions $J(t,\theta)$ on $[0,T] \times S$ and ε_1, $\varepsilon_2 > 0$, consider the family of functionals given by

$$\begin{aligned} &\hat{F}_{J,\varepsilon_1,\varepsilon_2}(\mu(\cdot,\cdot)) \\ &\quad = 2\int_0^T \int_S J(t,\theta)\left(\frac{h'(m_{\theta+\varepsilon_2}^{\varepsilon_1}) - h'(m_{\theta-\varepsilon_2}^{\varepsilon_1})}{2\varepsilon_2}\right) d\theta\, dt \\ &\qquad - \int_0^T \int_S J^2(t,\theta)\hat{a}^{-1}(m_\theta^{\varepsilon_1})\, d\theta\, dt, \end{aligned} \tag{2.48}$$

so that for each $\mu(\cdot,\cdot)$,

$$\lim_{\varepsilon_2 \to 0} \limsup_{\varepsilon_1 \to 0} \sup_J \hat{F}_{J,\varepsilon_1,\varepsilon_2}(\mu(\cdot,\cdot)) = I_0(\mu(\cdot,\cdot)), \tag{2.49}$$

in the sense that if the right-hand side is infinite, then the left-hand side is as well. Let

$$\begin{aligned} &F_{J,\varepsilon_1,\varepsilon_2,N}(x(\cdot)) \\ &\quad = 2\int_0^T \frac{1}{N}\sum_{i=1}^N J\left(t,\frac{i}{N}\right)\left(\frac{h'(m_{i/N+\varepsilon_2}^{\varepsilon_1}) - h'(m_{i/N-\varepsilon_2}^{\varepsilon_1})}{2\varepsilon_2}\right) dt \\ &\qquad - \int_0^T \frac{1}{N}\sum_{i=1}^N J^2\left(t,\frac{i}{N}\right)\hat{a}^{-1}(m_{i/N}^{\varepsilon_1})\, dt, \end{aligned} \tag{2.50}$$

so that $F_{J,\varepsilon_1,\varepsilon_2,N}$ differs from the image of $\hat{F}_{J,\varepsilon_1,\varepsilon_2}$ under the mapping $x \mapsto \mu$ by $o(1)$. To prove the lemma it is therefore enough to show that for each fixed J,

$$\lim_{\varepsilon_2\to 0}\limsup_{\varepsilon_1\to 0}\limsup_{N\to\infty}\frac{1}{N}\log E^{P_N}\left[\exp NF_{j,\varepsilon_1,\varepsilon_2,N}\right] \le 0. \tag{2.51}$$

This follows from Theorems 4.1 and 6.7 and Lemmas 7.2, 7.3 and 7.4 of [5]. □

3. Hydrodynamic limit for system with weak driving forces. We introduce a configuration dependent weak driving force into the system by changing the generator to

$$\mathscr{L}_{N,H,f} = \mathscr{L}_N + N/2\sum_{i=1}^{N} H(t,i/N)a(x_i,x_{i+1})(1 + D^{i,i+1}\xi_f)D^{i,i+1}, \tag{3.1}$$

where $H(t,\theta)$ is a continuous function on $[0,T]\times S$, $f(t,\theta,x_{-r},\dots,x_r)$ is a smooth function on $[0,T]\times S\times\mathbb{R}^{2r+1}$ with bounded first derivatives and $\xi_f = \Sigma_{i=1}^N f(t,\theta,x_{i-r},\dots,x_{i+r})$. The ξ_f term does not affect the hydrodynamics, but it allows us to obtain the correct lower bound in Section 4. Let $m(\theta)$ be a smooth function on S and let $f_N = \exp\Sigma_{i=1}^N h'(m(i/N))x_i - \log M(h'(m(i/N)))$. Corresponding to the generator $\mathscr{L}_{N,H,f}$ and initial distribution $f_N\Phi_N$, we have probability measures $Q_{N,H,f}$ on $C([0,T]\to\mathbb{R}^N)$ and $\hat{Q}_{N,H,f}$ on X.

THEOREM 3.1. *For each continuous H and smooth local f,*

$$\hat{Q}_{N,H,f} \Rightarrow \delta_{m(t,\theta)\,d\theta}, \tag{3.2}$$

where $\delta_{m(t,\theta)\,d\theta}$ is the probability measure concentrated on the unique weak solution of

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial}{\partial\theta}\left(\hat{a}(m)\left(\frac{\partial}{\partial\theta}h'(m) + H\right)\right), \qquad m(0,\theta) = m(\theta). \tag{3.3}$$

PROOF. For $dQ_{N,H,f}/dP_N$ we have the following Cameron–Martin–Girsanov formula:

$$\begin{aligned}\frac{dQ_{N,H,f}}{dP_N} = f_N(x(0))\exp\Bigg\{&\sum_{i=1}^{N}\int_0^T H\left(t,\frac{i}{N}\right)\left(1 + D^{i,i+1}\xi_f\right)\sqrt{a(x_i,x_{i+1})}\,d\beta_i\\ &-\frac{1}{2}\sum_{i=1}^{N}\int_0^T H^2\left(t,\frac{i}{N}\right)a(x_i,x_{i+1})\left(1 + D^{i,i+1}\xi_f\right)^2 dt\Bigg\}.\end{aligned} \tag{3.4}$$

Using this formula it is quite easy to show that there exists $C < \infty$ so that for all $\gamma > 0$,

$$E^{P_N}\left[\left(\frac{dQ_{N,H,f}}{dP_N}\right)^{1+\gamma}\right] \le \exp(C\gamma(1+\gamma)N). \tag{3.5}$$

Therefore, as discussed in (2.8), by Hölder's inequality superexponential estimates for P_N extend to $Q_{N,H,f}$. In particular $\hat{Q}_{N,H,f}$ has a weak limit, $\hat{Q}_{H,f}$. Let

$$\hat{F}^H_{J,G,\varepsilon_1,\varepsilon_2} = \hat{F}_{J,G,\varepsilon_1,\varepsilon_2} + \frac{1}{2}\int_0^T\int_S \frac{\partial J}{\partial\theta}(t,\theta)\hat{a}(m_\theta^{\varepsilon_1})H(t,\theta)\,d\theta\,dt, \tag{3.6}$$

where $\hat{F}_{J,G,\varepsilon_1,\varepsilon_2}$ is defined in (2.6). We claim that for each fixed J and G,

$$\lim_{\varepsilon_2\to 0}\limsup_{\varepsilon_1\to 0}\limsup_{N\to\infty}\frac{1}{N}\log E^{\hat{Q}_{N,H,f}}\left[\exp N\hat{F}^H_{J,G,\varepsilon_1,\varepsilon_2}\right] \le 0. \tag{3.7}$$

To this end note that $\exp NF^2(T)$ is a martingale under $Q_{N,H,f}$, where

$$F^2(T) = F^1(T) + \int_0^T \frac{1}{2}\sum_{i=1}^N \frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)H\left(t,\frac{i}{N}\right)\nu^i(x(t))\,dt + o(1). \tag{3.8}$$

Here F^1 is defined in (2.28) and

$$\nu^i = a(x_i, x_{i+1})\left(1 + D^{i,i+1}\xi_f\right)\left(1 + D^{i,i+1}\xi_g\right), \tag{3.9}$$

where ξ_g is given by

$$\xi_g = \sum_{i=1}^N g\left(x_{i-l},\dots,x_{i+l}, m_{i/N}^{k/N}\right). \tag{3.10}$$

Let

$$V^6 = \frac{1}{N}\sum_{i=1}^N \frac{\partial J}{\partial\theta}\left(t,\frac{i}{N}\right)H\left(t,\frac{i}{N}\right)\left(\nu^i - \hat{a}(m_{i/N}^{\varepsilon_1})\right). \tag{3.11}$$

Let $\alpha > 0$, $\varepsilon > 0$ and choose $C < \infty$ as in (2.35) and $g(x_{-l},\dots,x_l,m)$ with $l > r$ such that $A(g(m),m) \le a^*$ and

$$\sup_{|m|\le C} A(g(m),m) \le \varepsilon\alpha^{-1}\left\|\frac{\partial J}{\partial\theta}\right\|_\infty^{-1}. \tag{3.12}$$

Let $g^*(x_{-l},\dots,x_l,m)$ by the minimizer of $A(g,m)$ among functions of x_{-l} through x_l. We can assume in addition that

$$\sup_{|m|\le C} E^{\pi_m}\left[a(x_1,x_2)\left(D^{12}\xi_{g^*} - D^{12}\xi_g\right)^2\right] \le \varepsilon^2\alpha^{-2}\left\|\frac{\partial J}{\partial\theta}\right\|_\infty^{-2} \tag{3.13}$$

and that $E^{\pi_m}[a(x_1, x_2)(D^{12}\xi_f)^2]$ and $E^{\pi_m}[a(x_1, x_2)(D^{12}\xi_g)^2]$ are bounded by some $M < \infty$. Since g^* is the minimizer, we have

$$E^{\pi_m}\Big[a(x_1, x_2)\big(D^{12}\xi_f - D^{12}\xi_g\big)\big(1 - D^{12}\xi_{g^*}\big)\Big] = 0. \tag{3.14}$$

By Schwarz's inequality, we have for $|m| \le C$,

$$E^{\pi_m}[\nu^i] - \hat{a}(m) \le \varepsilon\alpha^{-1}\left\|\frac{\partial J}{\partial\theta}\right\|_\infty^{-1}\big(1 + \sqrt{4M}\big) \tag{3.15}$$

and for $|m| > C$,

$$E^{\pi_m}[\nu^i] - \hat{a}(m) \le M. \tag{3.16}$$

Then

$$\begin{aligned} V^6 \le {}& \frac{1}{N}\sum_{i=1}^N \frac{\partial J}{\partial\theta}\left(t, \frac{i}{N}\right)H\left(t, \frac{i}{N}\right)\left(\nu^i - E^{\pi_{m_{i/N}^{\varepsilon_1}}}[\nu^i]\right) \\ & + \frac{1}{N}\sum_{i=1}^N \left|\frac{\partial J}{\partial\theta}\left(t, \frac{i}{N}\right)\right|\left|H\left(t, \frac{i}{N}\right)\right| M 1_{\{|m_{i/N}^{\varepsilon}| > C\}} \\ & + \frac{1}{2N}\sum_{i=1}^N \left|\frac{\partial J}{\partial\theta}\left(t, \frac{i}{N}\right)\right|\left|H\left(t, \frac{i}{N}\right)\right|\varepsilon\alpha^{-1}\left\|\frac{\partial J}{\partial\theta}\right\|_\infty^{-1}\big(1 + \sqrt{4M}\big). \end{aligned} \tag{3.17}$$

So, by Schwarz's inequality, if we take the limit superior as $N \to \infty$, followed by $k \to \infty$, then $\varepsilon_1 \to 0$, we get

$$\begin{aligned} & \limsup_{\substack{\varepsilon_1 \to 0 \\ k\to\infty}} \limsup_{N\to\infty} \frac{1}{N}\log E^{P_N}\left[\exp \alpha N\int_0^T V^6(x(t), t)\,dt\right] \\ & \le \limsup_{\substack{\varepsilon_1 \to 0 \\ k\to\infty}} \limsup_{N\to\infty} \frac{1}{2N}\log E^{P_N} \\ & \qquad \times\left[\exp 2\alpha N\int_0^T \sum_{i=1}^N \frac{\partial J}{\partial\theta}\left(t, \frac{i}{N}\right)H\left(t, \frac{i}{N}\right)\left(\nu^i - e^{\pi_{m_{i/N}^{\varepsilon_1}}}[\nu^i]\right)dt\right] \\ & \quad + \limsup_{\substack{\varepsilon_1 \to 0 \\ k\to\infty}} \limsup_{N\to\infty} \frac{1}{2N}\log E^{P_N} \\ & \qquad \times\left[\exp 2\alpha N\int_0^T \sum_{i=1}^N \left|\frac{\partial J}{\partial\theta}\left(t, \frac{i}{N}\right)\right|\left|H\left(t, \frac{i}{N}\right)\right| M 1_{\{|m_{i/N}^{\varepsilon}| > C\}}\,dt\right] \\ & \quad + \varepsilon\big(1 + \sqrt{4M}\big)\int_0^T\int_S |H(t, \theta)|\,d\theta\,dt \\ & \le 0\ [\text{by } (2.39)] + \varepsilon\ (\text{for sufficiently large } C) + \text{const. } \varepsilon. \end{aligned} \tag{3.18}$$

If F^N is the image of $\hat{F}^H$ under $x \mapsto \mu$, then

$$F^H = \left(F^1(T) + \int_0^T \frac{1}{2}\sum_{i=1}^N \left(J\left(t, i + \frac{1}{N}\right) - J\left(t, \frac{i}{N}\right)\right) \times H\left(t, \frac{i}{N}\right)\nu^i(x(t))\,dt\right) + (F - F^1) - \left(\int_0^T V^6\,dt\right) + o(1). \tag{3.19}$$

So (3.7) follows by Hölder's inequality and (2.32). If $I_0(\mu(\cdot,\cdot)) < \infty$, then as in Lemma 2.2 we can calculate $\limsup_{\varepsilon_1, \varepsilon_2 \to 0} \sup_{J,G} \hat{F}^H_{J,G,\varepsilon_1,\varepsilon_2}(\mu(\cdot,\cdot)) > 0$ unless $m(t,\theta)$ is a weak solution of (3.3). Therefore, $\hat{Q}_{H,f}$ is concentrated on such solutions. It only remains to prove uniqueness for weak solutions of (3.3). It is shown in [5] that $\hat{a}$ is continuous in m. A general result for equations of the form (3.3), whose coefficients are only known to be continuous, can be found in [1]. □

4. Lower bound. For a smooth $m(\cdot,\cdot)$ we can find a continuous H so that m is a weak solution of (3.3). We can also assume that for each $0 \le t \le T$,

$$\int_S \hat{a}(m(t,\theta))H(t,\theta)\,d\theta = 0. \tag{4.1}$$

For each smooth local f we construct $\hat{Q}_{N,H,f}$ which by Theorem 3.1 converges weakly to the probability measure on X concentrated on $m(\cdot,\cdot)$. By the Cameron–Martin–Girsanov formula,

$$\begin{aligned}\frac{1}{N}\log\frac{dQ_N}{dQ_{N,H,f}} &= \frac{1}{N}\log\frac{f_N^0(x(0))}{f_N(x(0))} \\ &\quad + \frac{1}{2N}\sum_{i=1}^N \int_0^T H\left(t, \frac{i}{N}\right)\left(1 + D^{ii+1}\xi_f\right)\sqrt{a(x_i, x_{i+1})}\,d\beta_i \\ &\quad - \frac{1}{8N}\sum_{i=1}^N \int_0^T H^2\left(t, \frac{i}{N}\right)a(x_i, x_{i+1})\left(1 + D^{ii+1}\xi_f\right)^2 dt.\end{aligned} \tag{4.2}$$

Let

$$\hat{a}_f(m) = E^{\pi_m}\left[a(x_1, x_2)\left(1 + D^{12}\xi_f\right)^2\right]. \tag{4.3}$$

Then

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N}\log\frac{dQ_N}{dQ_{N,H,f}} &= -I_{\text{static}}(m(0,\cdot)) \\ &\quad - \frac{1}{8}\int_0^T\int_S \hat{a}_f(m(t,\theta))H^2(t,\theta)\,d\theta\,dt\end{aligned} \tag{4.4}$$

in $Q_{N,H,f}$-probability. Call the right-hand side $-I^f(\mu(\cdot,\cdot))$. This provides us with a family of lower bounds: For each open $U \subset X$,

$$\liminf_{N\to\infty} \frac{1}{N} \log \hat{Q}_N(U) \geq - \inf_{\mu(\cdot,\cdot)\in U} I^f(\mu(\cdot,\cdot)). \tag{4.5}$$

By the following lemma we obtain the lower bound (1.22) by optimizing (4.3) over f.

LEMMA 4.1. $I(\mu(\cdot,\cdot)) = \inf_f I^f(\mu(\cdot,\cdot))$.

PROOF. Let $\varepsilon > 0$. Since $m(t,\theta)$ is bounded, we can choose $f(t,\theta,x_{-r},\dots,x_r)$ so that

$$1 \geq \frac{\hat{a}(m(t,\theta))}{\hat{a}_f(m(t,\theta))} > 1 - \varepsilon. \tag{4.6}$$

Since H is continuous and $\int_S \hat{a} H \, d\theta = 0$,

$$\begin{aligned} &\frac{1}{8}\int_0^T \int_S H^2(t,\theta)\hat{a}_f(m(t,\theta))\, d\theta\, dt \\ &\quad = \sup_{\varphi} \left\{ \frac{1}{2}\int_0^T\int_S H(t,\theta)\hat{a}(m(t,\theta))\frac{\partial\varphi}{\partial\theta}(t,\theta)\, d\theta\, dt \right. \\ &\qquad \left. - \frac{1}{2}\int_0^T\int_S \left(\frac{\partial\varphi}{\partial\theta}(t,\theta)\right)^2 \hat{a}(m(t,\theta))\frac{\hat{a}(m(t,\theta))}{\hat{a}_f(m(t,\theta))}\, d\theta\, dt \right\}. \end{aligned} \tag{4.7}$$

Since m is a weak solution of (3.3),

$$\begin{aligned} &\frac{1}{2}\int_0^T\int_S \hat{a}(m) H \frac{\partial\varphi}{\partial\theta}\, d\theta\, dt \\ &\quad = -\int_S m\varphi\, d\theta \Big|_0^T + \int_0^T \int_S m \frac{\partial\varphi}{\partial\theta}\, d\theta\, dt \\ &\qquad - \frac{1}{2}\int_0^T \hat{a}(m)\frac{\partial h'}{\partial\theta}\frac{\partial\varphi}{\partial\theta}\, d\theta\, dt. \end{aligned} \tag{4.8}$$

Therefore,

$$\begin{aligned} I_{\text{dynamic}}(m(\cdot,\cdot)) &\leq \tfrac{1}{8}\int_0^T\int_S H^2(t,\theta)\hat{a}_f(m(t,\theta))\, d\theta\, dt \\ &< (1-\varepsilon)^{-1} I_{\text{dynamic}}(m(\cdot,\cdot)). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the lemma. □

To complete the argument it remains to show that for any $\mu(\cdot,\cdot)$ with $I(\mu(\cdot,\cdot)) < \infty$ we can find a sequence of smooth functions $\mu^k(\cdot,\cdot) \to \mu(\cdot,\cdot)$ with $I(\mu^k(\cdot,\cdot)) \to I(\mu(\cdot,\cdot))$. To do this we first approximate $m(t,\theta)$ by $m^C(t,\theta) = -C \vee m(t,\theta) \wedge C$. Then we convolute with a smoothing kernel $\varphi_\varepsilon(t,\theta)$ and use the fact that for $|m| \leq C$, $b(m) = \hat{a}(m)h''(m)$ is continuous and bounded away from both 0 and ∞.

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