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FÖLLMER-SCHWEIZER DECOMPOSITION AND MEAN-VARIANCE HEDGING FOR GENERAL CLAIMS

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Let X be an $\mathbb{R}^d$-valued special semimartingale on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, P)$ with decomposition $X = X_0 + M + A$ and Θ the space of all predictable, X-integrable processes θ such that $\int \theta\, dX$ is in the space $\mathscr{S}^2$ of semimartingales. If H is a random variable in $\mathscr{L}^2$, we prove, under additional assumptions on the process X, that H can be written as the sum of an $\mathscr{F}_0$-measurable random variable H_0, a stochastic integral of X and a martingale part orthogonal to M. Moreover, this decomposition is unique and the function mapping H with its decomposition is continuous with respect to the $\mathscr{L}^2$-norm. Finally, we deduce from this continuity that the subspace of $\mathscr{L}^2$ generated by $\int \theta\, dX$, where $\theta \in \Theta$, is closed in $\mathscr{L}^2$, and we give some applications of this result to financial mathematics.

1. Introduction. In this paper, we deal with a very important problem which arises in financial mathematics: we look for a solution to the problem of hedging a contingent claim and since, in an incomplete market, such a strategy does not always exist, we try to find a strategy which minimizes the expected value of the squared difference of the contingent claim and the portfolio value.

To express this problem in mathematical terms, consider an $\mathbb{R}^d$-valued special semimartingale $X = (X_t)_{0 \le t \le T}$ in $\mathscr{S}^2_{\text{loc}}$, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T}, P)$. Suppose that X admits the decomposition

$$X = X_0 + M + A.$$

In the market, X represents the discounted price of d risky assets at time t. The contingent claim is a random variable H, $\mathscr{F}_T$-measurable, which we assume to be in $\mathscr{L}^2$. For example, if $d = 1$ and the contingent claim is a European call option on X with expiration date T and strike price K, then $H = (X_T - K)^+$.

A trading strategy is described by an X-integrable process θ such that $\int \theta\, dX$ is in the space $\mathscr{S}^2$ of semimartingales. Then θ_t is the number of shares of X that the agent holds at time t. We suppose that θ is predictable, which

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squares with the obvious fact that the way of trading at t only depends on the past before t and not on what happens at t. If we assume that there is a riskless asset of price 1 at each time and that the agent's initial capital is c, then

$$c + \int_0^t \theta_s \, dX_s$$

is the value of the agent's portfolio at time t if the strategy followed is θ. For more details, we refer to Harrison and Kreps (1979) and Harrison and Pliska (1981).

Let us call Θ the set of all strategies. Our optimization problem is to define a strategy ξ such that

$$E\left(\left(H - c - \int_0^T \xi_s \, dX_s\right)^2\right) = \min_{\theta \in \Theta} E\left(\left(H - c - \int_0^T \theta_s \, dX_s\right)^2\right).$$

This problem has been previously studied in Duffie and Richardson (1991), Schäl (1994), Hipp (1993) and Schweizer (1992, 1993a, 1994).

In order to trade in a viable market, further assumptions must be added. First of all, a usual hypothesis is a "no arbitrage" condition, which roughly means that one cannot take a positive gain if one did not invest a positive sum at the beginning. This means that A, the finite variation part of X, is absolutely continuous with respect to $\langle M \rangle$ [Ansel and Stricker (1992)].

In a sense, this can also mean that there is a probability Q, equivalent to P, such that X is a local martingale under Q. These properties have been previously studied in Harrison and Pliska (1981). It was then a conjecture that the authors proved if Ω is finite. These problems have since been studied in Dalang, Morton and Willinger (1990), Stricker (1990), Ansel and Stricker (1992), Delbaen (1992) and Delbaen and Schachermayer (1994).

In a complete market, such an equivalent martingale measure always exists and is unique, so every contingent claim is attainable. However, in reality, we can rarely deal with a complete market, since the number of causes for uncertainty is greater than the number of assets held by the agent.

In an incomplete market, such an equivalent martingale measure is not unique. Hence, we need to look for a minimal martingale measure which was introduced in Föllmer and Schweizer (1991). If X is continuous, a necessary condition for the existence of such a probability is that, if B is an increasing predictable integrable process null at 0 such that

$$A^i = \gamma^i \cdot B \quad \text{and} \quad \langle M^i, M^j \rangle = \sigma^{ij} \cdot B \quad \text{for } i, j = 1, \dots, d,$$

then there exists a predictable process $\hat{\lambda}$ such that $\sigma\hat{\lambda} = \gamma$ [Ansel and Stricker (1992)].

So we assume that $\hat{\lambda}$ exists and we assume moreover that the mean-variance tradeoff process of X, $\hat{K}$ defined by

$$\hat{K} := \int \hat{\lambda}^* \gamma \, dB,$$

where the asterisk $(*)$ denotes the transposition, exists and is uniformly bounded in t and ω.

To solve our optimization problem, we can try to prove that the subspace $G_T(\Theta)$ of $\mathscr{L}^2$, generated by all the stochastic integrals $\int \theta \, dX$, where $\theta \in \Theta$, is closed. Then we only have to project $H - c$ on $G_T(\Theta)$ to find the solution.

The closedness of $G_T(\Theta)$ is an old problem. If X is a local martingale, then $G_T(\Theta)$ is closed because in this case, the stochastic integral is an isometry. In fact, there is a stronger result since Yor (1978) [a more convenient reference is Protter (1990), page 153] proved that if Y^n and Y are uniformly integrable martingales such that Y^n_∞ converges weakly to Y_∞ in $\mathscr{L}^1$, and if $Y^n_t = \int_0^t \varphi^n_s \, dX_s$, then there exists a predictable process φ satisfying $Y_t = \int_0^t \varphi_s \, dX_s$. When X is only a semimartingale, Schäl (1994) and Schweizer (1993a) recently proved that, in discrete time, if X satisfies the previous conditions and $\hat{K}$ is uniformly bounded, $G_T(\Theta)$ is closed. In continuous time, Schweizer (1994) proved that, under the assumptions that $\hat{K}$ exists, is deterministic and that the jumps of $\hat{K}$ are bounded by a constant $\delta \in (0, 1)$, $G_T(\Theta)$ is closed. In both cases, he has a better result for the optimization problem since he manages to determine a stochastic differential equation satisfied by the solution of the optimization problem. In this paper, we prove that $G_T(\Theta)$ is closed under the assumption that $\hat{K}$ is uniformly bounded.

To prove that $G_T(\Theta)$ is closed, a natural idea is to introduce the Föllmer–Schweizer decomposition (denoted by F-S decomposition in what follows) of a random variable $H \in \mathscr{L}^2(\Omega, \mathscr{F}_T, P)$, that is, to write H as

$$H = H_0 + \int_0^T \xi^H_s \, dX_s + L^H_T,$$

where H_0 is a random variable, ξ^H is a strategy and L^H is a martingale in $\mathscr{M}^2_0$, strongly orthogonal to $\int \theta \, dM$ for all processes $\theta \in L^2(M)$.

When X is a square-integrable martingale, this decomposition always exists and is known as the Galtchouk–Kunita–Watanabe decomposition [Kunita and Watanabe (1967), Galtchouk (1975) and Jacod (1979)].

In the general case, this decomposition does not always exist. It was introduced in Föllmer and Schweizer (1991) and was studied in Schweizer (1991) and in a slightly different way, in Ansel and Stricker (1992) in the case $d = 1$ and in the multidimensional case in Schweizer (1993a, c). Finally, assuming that H_0 is only an $\mathscr{F}_0$-measurable random variable, we prove the existence and the uniqueness of this decomposition, under the unique assumption that $\hat{K}$ is uniformly bounded. Moreover, we show that the function which associates H with its decomposition is continuous. A first version without proofs was published in Monat and Stricker (1994a).

After we submitted this paper, Schweizer (1994b) gave a direct proof for the closedness of $G_T(\Theta)$ when $\hat{K}$ is bounded and the jumps of $\hat{K}$ are less than 1. In Monat and Stricker (1994b), we extended his proof in the case where $\hat{K}$ is bounded, without further conditions on the jumps of $\hat{K}$.

In the case where $\hat{K}$ is not uniformly bounded, $G_T(\Theta)$ may not be closed and the F-S decomposition does not always exist. For example, if $d = 1$, then at a jump of $\hat{K}$, the F-S decomposition exists if and only if the jump of $\hat{K}$ is uniformly bounded. For the closedness of $G_T(\Theta)$, we will give a counterexample in the last section of this paper.

This paper is organized as follows: Section 2 describes the model and Section 3 contains all the results on the F-S decomposition, that is, some of Schweizer's results, the existence and the uniqueness when $\hat{K}$ is uniformly bounded and the continuity of the corresponding function. Section 4 deals with some applications to the properties of F-S decomposition: closedness of $G_T(\Theta)$ and $\{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$, approximation of a random variable by a stochastic integral, applications to financial mathematics and the case where the martingale part of X has the predictable representation property. Section 5 deals with the case where $\hat{K}$ is no longer uniformly bounded.

2. Model. We use the same notations as Schweizer (1994a). We recall them here. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $T > 0$ a fixed finite horizon. We suppose that we have a filtration $(\mathscr{F}_t)_{0 \le t \le T}$ on $(\Omega, \mathscr{F}, P)$ satisfying the usual conditions, that is, $(\mathscr{F}_t)_{0 \le t \le T}$ is right-continuous and complete, and we assume moreover that $\mathscr{F} = \mathscr{F}_T$. Let $X = (X_t)_{0 \le t \le T}$ be an $\mathbb{R}^d$-valued semimartingale in $\mathscr{S}^2_{\text{loc}}$. This means that if

$$X = X_0 + M + A$$

is the canonical decomposition of X, then $M \in \mathscr{M}^2_{0,\text{loc}}$ and the variation $|A^i|$ of the predictable finite variation process of X^i is locally square-integrable for each $i = 1, \dots, d$. For all unexplained notations, we refer to Jacod (1979). If $\langle M^i \rangle$ denotes the sharp bracket process of M^i for each $i = 1, \dots, d$, we suppose that

$$A^i \ll \langle M^i \rangle \text{ with predictable density } \alpha^i = \left(\alpha^i_t\right)_{0 \le t \le T} \quad \text{for } i = 1, \dots, d. \tag{2.1}$$

Let B be a fixed predictable integrable increasing RCLL process null at 0 such that $\langle M^i \rangle \ll B$ for each $i = 1, \dots, d$ (e.g., $B = \Sigma_i \langle M^i \rangle$). Using the Kunita–Watanabe inequality, this implies that

$$\langle M^i, M^j \rangle \ll B \text{ with predictable density } \sigma^{ij} = \left(\sigma^{ij}_t\right)_{0 \le t \le T} \quad \text{for } i = 1, \dots, d. \tag{2.2}$$

The process σ^{ij} is therefore a symmetric, nonnegative definite $d \times d$ matrix for each $t \in [0, T]$.

From (2.1) and (2.2), we deduce that

$$A^i \ll B \text{ with predictable density } \gamma^i = \alpha^i \sigma^{ii} \text{ for } i = 1, \dots, d. \tag{2.3}$$

Using these notations, we get

$$\langle M^i, M^j \rangle_t = \int_0^t \sigma^{ij}_s \, dB_s, \qquad P\text{-a.s. for } i = 1, \dots, d \text{ and } t \in [0, T] \tag{2.4}$$

and

$$A^i_t = \int_0^t \gamma^i_s \, dB_s, \qquad P\text{-a.s. for } i = 1, \dots, d \text{ and } t \in [0, T]. \tag{2.5}$$

We recall a definition introduced in Schweizer (1994a).

DEFINITION 2.1. We say that X satisfies the structure condition (SC) if there exists a predictable $\mathbb{R}^d$-valued process $\hat{\lambda} = (\hat{\lambda}_t)_{0 \le t \le T}$ such that

$$\sigma_t \hat{\lambda}_t = \gamma_t, \qquad P\text{-a.s. for all } t \in [0,T] \tag{2.6}$$

and

$$\hat{K}_t := \int_0^t \hat{\lambda}_s^* \gamma_s \, dB_s < +\infty, \qquad P\text{-a.s. for all } t \in [0,T]. \tag{2.7}$$

We then choose an RCLL version of $\hat{K}$ and we call it the mean-variance tradeoff (MVT) process of X.

REMARK 2.2. For the interpretation of the process $\hat{K}$, we refer to Schweizer (1994a).

DEFINITION 2.3. A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0 \le t \le T}$ belongs to $L^2_{(\text{loc})}(M)$ if the process

$$\left(\int_0^t \theta_s^* \sigma_s \theta_s \, dB_s \right)_{0 \le t \le T} \text{ is (locally) integrable.}$$

A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0 \le t \le T}$ belongs to $L^2_{(\text{loc})}(A)$ if the process

$$\left(\int_0^t |\theta_s^* \gamma_s| \, dB_s \right)_{0 \le t \le T} \text{ is (locally) square-integrable.}$$

Finally, Θ is the space defined by $\Theta := L^2(M) \cap L^2(A)$; $\theta \in \Theta$ is called a strategy.

REMARK 2.4. If $\theta \in \Theta$, we can define the stochastic integral process

$$G_t(\theta) := \int_0^t \theta_s \, dX_s$$

for all $t \in [0,T]$. Then $G(\theta)$ is a semimartingale in $\mathscr{S}^2$ if and only if $\theta \in \Theta$ and its canonical decomposition is given by $G(\theta) := \int \theta \, dM + \int \theta \, dA$.

DEFINITION 2.5. A random variable $H \in \mathscr{L}^2(\Omega, \mathscr{F}, P)$ admits a Föllmer–Schweizer decomposition if it can be written as

$$H = H_0 + \int_0^T \xi_s \, dX_s + L_T, \qquad P\text{-a.s.}, \tag{2.8}$$

where H_0 is an $\mathscr{F}_0$-measurable random variable, $\xi \in \Theta$ and $L = (L_t)_{0 \le t \le T}$ is a martingale in $\mathscr{M}_0^2$, strongly orthogonal to $\int \theta \, dM$ for all $\theta \in L^2(M)$.

We now recall a lemma of Schweizer (1994a).

LEMMA 2.6. *Suppose that X satisfies* (2.1) *and* (*SC*). *If $\hat{K}$ is bounded, then* $\Theta = L^2(M)$.

PROOF. If $\hat{K}$ is bounded, then the Cauchy–Schwarz inequality yields, for every $\theta \in L^2(M)$,

$$\begin{aligned}\int_0^T |\theta_s^* \gamma_s| \, dB_s &= \int_0^T |\theta_s^* \sigma_s \hat{\lambda}_s| \, dB_s \\ &\le \int_0^T (\theta_s^* \sigma_s \theta_s)^{1/2} \big(\hat{\lambda}_s^* \sigma_s \hat{\lambda}_s\big)^{1/2} \, dB_s \\ &\le \big(\hat{K}_T\big)^{1/2} \left(\int_0^T \theta_s^* \sigma_s \theta_s \, dB_s \right)^{1/2}.\end{aligned}$$

Therefore, $L^2(M) \subset L^2(A)$, which completes the proof of Lemma 2.6. □

In what follows, unless otherwise noted, we assume that X satisfies (2.1) and (SC) and that the MVT process $\hat{K}$ is uniformly bounded in t and ω.

3. Föllmer–Schweizer decomposition.

3.1. *Existence and uniqueness*. The beginning of this subsection is strongly inspired by Schweizer (1994a). Nevertheless, we do not work under the same hypothesis. Indeed, Schweizer has proved the existence of F-S decomposition in the case where the jumps of $\hat{K}$ are uniformly bounded by a constant less than 1. We assume only that $\hat{K}$ is uniformly bounded. That is why we recall the proof of the first result.

DEFINITION 3.1. Let $0 \le T_1 \le T_2 \le T$ be predictable stopping times and suppose that $H \in \mathscr{L}^2(\Omega, \mathscr{F}_{T_2-}, P)$. The mapping $\Psi_H \colon L^2(M) \to L^2(M)$ is defined by $\Psi_H(\theta) = \hat{\theta} := \mathbf{1}_{]\!]T_1;T_2[\![} \, \nu$, where ν is the integrand in the Galtchouk–Kunita–Watanabe decomposition of the random variable $H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s) \theta_s^* \, dA_s$ with respect to the martingale M.

Hence, the definition of Ψ_H yields

$$\begin{aligned}&H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^* \, dA_s \\ &\quad = E\left(H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^* \, dA_s \Big| \mathscr{F}_0 \right) + \int_0^T \nu_s \, dM_s + \hat{L}_T,\end{aligned} \tag{3.1}$$

where $\hat{L}$ is a martingale in $\mathscr{M}_0^2$, strongly orthogonal to M.

Since $H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^*\,dA_s$ is an $\mathscr{F}_{T_2}$-measurable variable, we can rewrite (3.1) as

$$
\begin{aligned}
&H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^*\,dA_s \\
(3.2)\qquad &= E\left(H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^*\,dA_s \middle| \mathscr{F}_0\right) + \int_0^T \mathbf{1}_{]\!]0;T_2[\![}(s)\nu_s\,dM_s \\
&\quad + \int_0^T \mathbf{1}_{]\!]0;T_2[\![}(s)\,d\hat{L}_s.
\end{aligned}
$$

If $\hat{H}$ is defined by

$$
\begin{aligned}
\hat{H} := E\left(H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^*\,dA_s \middle| \mathscr{F}_0\right) &+ \int_0^T \mathbf{1}_{]\!]0;T_1]\!]}(s)\nu_s\,dM_s \\
&+ \int_0^T \mathbf{1}_{]\!]0;T_1]\!]}(s)\,d\hat{L}_s,
\end{aligned}
$$

then $\hat{H}$ is $\mathscr{F}_{T_1}$-measurable and

$$
\begin{aligned}
&H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^*\,dA_s \\
(3.3)\qquad &= \hat{H} + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\hat{\theta}_s\,dM_s + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\,d\hat{L}_s.
\end{aligned}
$$

Now let us characterize the fixed points of Ψ_H. If $\theta \in \Theta$ is a fixed point of Ψ_H, then (3.3) yields

$$
(3.4)\qquad H = \hat{H} + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}L(s)\theta_s\,dX_s + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}L(s)\,d\hat{L}_s.
$$

Conversely, if $\theta \in \Theta$ satisfies (3.4), then the Galtchouk–Kunita–Watanabe decomposition of $\hat{H}$ with respect to M implies that

$$
\begin{aligned}
&H - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s^*\,dA_s \\
(3.5)\qquad &= E(\hat{H}|\mathscr{F}_0) + \int_0^T \left(\mathbf{1}_{]\!]0;T_1]\!]}(s)\xi_s + \mathbf{1}_{]\!]T_1;T_2[\![}(s)\theta_s\right)dM_s \\
&\quad + \left(N_{T_1} + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\,d\hat{L}_s\right).
\end{aligned}
$$

From the uniqueness of the Galtchouk–Kunita–Watanabe decomposition, we conclude that $\nu = \mathbf{1}_{]\!]0;T_1]\!]}\,\xi + \mathbf{1}_{]\!]T_1;T_2[\![}\,\theta$ and therefore $\Psi_H(\theta) = \theta$.

We need some auxiliary results to prove the existence and the uniqueness of the F-S decomposition. Indeed, the idea of the proof is to deal with the jumps of $\hat{K}$ which are greater than $3/4$, on one hand, and, on the other hand, to deal with what happens between two such jumps. On these intervals, we can apply the same method as in Buckdahn (1993) and Schweizer (1994a), that is, to consider subintervals on which the growth of $\hat{K}$ is less than or equal to a constant $\delta \in (0;1)$. For this proof, we need two lemmas.

LEMMA 3.2. *Let $0 \le T_1 \le T_2 \le T$ be predictable stopping times and suppose that $H \in \mathscr{L}^2(\Omega, \mathscr{F}_{T_2-}, P)$. If there exists a constant $\delta \in (0;1)$ such that $\hat{K}_{T_2-} - \hat{K}_{T_1} \le \delta$ P-a.s., then Ψ_H admits a unique fixed point.*

PROOF. If $\|\cdot\|_{L^2(M)}$ denotes the norm defined by $\|\theta\|_{L^2(M)} := \|\int_0^T \theta_s\, dM_s\|_2$, then $(L^2(M), \|\cdot\|_{L^2(M)})$ is a Banach space. Hence, to prove that Ψ_H admits a unique fixed point, it suffices to prove that Ψ_H is a contraction on $L^2(M)$. Let $\theta, \theta' \in L^2(M)$. Then

$$\begin{aligned}
\|\hat{\theta} - \hat{\theta}'\|^2_{L^2(M)} &= \left\| \int_0^T \left(\hat{\theta}_s - \hat{\theta}'_s\right) dM_s \right\|_2^2 \\
&\le \left\| \int_0^T \left(\hat{\theta}_s - \hat{\theta}'_s\right) dM_s \right\|_2^2 + \|\hat{H} - \hat{H}'\|_2^2 \\
&\quad + \left\| \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\, d\hat{L}_s - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\, d\hat{L}'_s \right\|_2^2 \\
&= \left\| \left(\int_0^T \left(\hat{\theta}_s - \hat{\theta}'_s\right) dM_s \right) + (\hat{H} - \hat{H}') \right. \\
&\quad \left. + \left(\int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\, d\hat{L}_s - \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\, d\hat{L}'_s \right) \right\|_2^2 \\
&\qquad\qquad \text{(since these three terms are orthogonal)} \\
&= \left\| \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)(\theta_s - \theta'_s)^* \, dA_s \right\|_2^2 \\
&= \left\| \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)(\theta_s - \theta'_s)^* \sigma_s \hat{\lambda}_s \, dB_s \right\|_2^2 \quad \text{(from SC)} \\
&\le E\left(\int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\, \hat{\lambda}^*_s \sigma_s \hat{\lambda}_s \, dB_s \int_0^T (\theta_s - \theta'_s)^* \sigma_s (\theta_s - \theta'_s)\, dB_s \right) \\
&\qquad\qquad \text{(since } \sigma \text{ is a symmetric nonnegative matrix)} \\
&\le \|\hat{K}_{T_2-} - \hat{K}_{T_1}\|_\infty E\left(\int_0^T (\theta_s - \theta'_s)^* \sigma_s (\theta_s - \theta'_s)\, dB_s \right) \\
&\le \delta \|\theta - \theta'\|^2_{L^2(M)}.
\end{aligned}$$

Hence, Ψ_H is a contraction, which completes the proof. □

LEMMA 3.3. *Let T_0 be a predictable stopping time and suppose that H is a random variable in $\mathscr{L}^2(\Omega, \mathscr{F}_{T_0}, P)$. Then*

$$H = \tilde{H} + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\, \tilde{\xi}_s \, dX_s + \tilde{L}_{T_0}, \qquad P\text{-a.s.}, \tag{3.6}$$

where $\tilde{H} \in \mathscr{L}^2(\Omega, \mathscr{F}_{T_0-}, P)$, $\tilde{\xi} \in \Theta$ *and* $\tilde{L}$ *is a martingale in* $\mathscr{M}_0^2$ *that vanishes on* $[\![0; T_0 [\![$ *and is strongly orthogonal to* $\int \theta \, dM$ *for every* $\theta \in L^2(M)$. *Moreover, this decomposition is unique in the following sense: If*

$$H = \tilde{H} + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s \, dX_s + \tilde{L}_{T_0} = \tilde{H}' + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}'_s \, dX_s + \tilde{L}'_{T_0}$$

with $(\tilde{H}, \tilde{\xi}, \tilde{L})$ *and* $(\tilde{H}', \tilde{\xi}', \tilde{L}')$ *satisfying the previous conditions, then*

$$\tilde{H} = \tilde{H}', \qquad P\text{-}a.s.,$$

$$\mathbf{1}_{[\![T_0]\!]} \tilde{\xi} = \mathbf{1}_{[\![T_0]\!]} \tilde{\xi}' \quad in\ L^2(M)$$

and

$$\tilde{L}_{T_0} = \tilde{L}'_{T_0}, \qquad P\text{-}a.s.$$

PROOF. $(E(H|\mathscr{F}_t))_{0 \le t \le T}$ is a square-integrable martingale; hence, the Galtchouk–Kunita–Watanabe decomposition yields

$$(3.7) \qquad H = E(H|\mathscr{F}_T) = E(H|\mathscr{F}_0) + \int_0^T \tilde{\xi}_s \, dM_s + L_T, \qquad P\text{-a.s.}$$

with $\tilde{\xi} \in L^2(M)$ and $L \in \mathscr{M}_0^2$, strongly orthogonal to M. However, H is $\mathscr{F}_{T_0}$-measurable; hence, $\tilde{\xi}$ vanishes on $]\!] T_0; T]\!]$ and $L_{T_0} = L_T$ P-a.s. On the other hand,

$$(3.8) \quad E\big(H|\mathscr{F}_{T_0-}\big) = E(H) + \int_0^T \mathbf{1}_{]\!] 0; T_0 [\![}(s) \tilde{\xi}_s \, dM_s + L_{T_0-}, \qquad P\text{-a.s.}$$

Hence, subtracting (3.8) from (3.7) implies that

$$(3.9) \qquad H = E\big(H|\mathscr{F}_{T_0-}\big) + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s \, dM_s + L_{T_0} - L_{T_0-}.$$

However, $L^2(M) = \Theta$ and $\int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s^* \, dA_s$ is $\mathscr{F}_{T_0-}$-measurable because A is predictable. Therefore,

$$(3.10) \qquad H = \tilde{H} + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s^* \, dX_s + \tilde{L}_{T_0}$$

with

$$\tilde{H} = E\big(H|\mathscr{F}_{T_0-}\big) - \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s^* \, dA_s$$

and

$$\tilde{L}_{T_0} = L_{T_0} - L_{T_0-}.$$

Let us show now the uniqueness of this decomposition. We can first assume that $H = 0$. If

$$0 = \tilde{H} + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s \, dX_s + \tilde{L}_{T_0},$$

then, taking the conditional expectation with respect to $\mathscr{F}_{T_0-}$ and subtracting, yields

$$0 = \int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s\, dM_s + \tilde{L}_{T_0}.$$

Now, the Galtchouk–Kunita–Watanabe decomposition is unique, so

$$\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s\, dM_s = 0 \quad \text{and} \quad \tilde{L}_{T_0} = 0, \qquad P\text{-a.s.}$$

Finally,

$$\begin{aligned}
0 &\le \left|\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s^*\, dA_s\right| = \left|\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s^*\gamma_s\, dB_s\right| \\
&= \left|\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s^*\sigma_s\hat{\lambda}_s\, dB_s\right| \quad \text{(from SC)} \\
&\le \left(\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s^*\sigma_s\tilde{\xi}_s\, dB_s\right)^{1/2}\left(\int_0^T \hat{\lambda}_s^*\sigma_s\hat{\lambda}_s\, dB_s\right)^{1/2} \\
&\le \left(\left\langle \int_0^{\cdot} \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s\, dM_s\right\rangle_T\right)^{1/2}\left(\hat{K}_T\right)^{1/2} \\
&\le 0,
\end{aligned}$$

because $\hat{K}_T$ is finite and $\langle \int_0^{\cdot}\mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s\, dM_s\rangle_T = 0$; hence, the decomposition is unique. □

THEOREM 3.4. *Every random variable $H \in \mathscr{L}^2(\Omega,\mathscr{F},P)$ admits a F-S decomposition. Moreover, this decomposition is unique in the following sense: If*

$$H = H_0 + \int_0^T \xi_s^H\, dX_s + L_T^H = H_0' + \int_0^T \xi_s'^H\, dX_s + L_T'^H,$$

where (H_0, ξ^H, L^H) and (H_0', ξ'^H, L'^H) satisfy the conditions of the F-S decomposition, then

$$H_0 = H_0', \qquad P\text{-a.s.},$$
$$\xi^H = \xi'^H \quad \text{in } L^2(M)$$

and

$$L_T^H = L_T'^H, \qquad P\text{-a.s.}$$

PROOF. Since $\hat{K}$ is uniformly bounded in t and ω, there exist a constant $\delta \in (0,1)$ and a finite sequence of predictable stopping times $(T_i)_{0\le i\le n}$ such that

$$0 = T_0 \le T_1 \le \cdots \le T_n = T \text{ and } \hat{K}_{T_i-} - \hat{K}_{T_{i-1}} \le \delta, \quad P\text{-a.s. for } i = 1,\dots,n.$$

To construct such a sequence, we can take $\varepsilon > 0$ such that $3/4 + \varepsilon < 1$ and set

$$T_0 = 0,$$

$$T_{i+1} = \begin{cases} \inf\left\{T_i < t \le T \mid \hat{K}_t - \hat{K}_{T_i} \ge 3/4 + \varepsilon\right\}, \\ T, \quad \text{if the previous set is empty,} \end{cases}$$

$$T_n = T \quad \text{with } n \text{ large enough to ensure that } \hat{K}_{T-} - \hat{K}_{T_{n-1}} \le 3/4 + \varepsilon \; P\text{-a.s.}$$

All stopping times T_i are predictable since they are "debuts" of right-closed, predictable sets.

Let $H \in \mathscr{L}^2(\Omega, \mathscr{F}, P)$. Remembering that $\mathscr{F} = \mathscr{F}_T = \mathscr{F}_{T_n}$, we can apply Lemma 3.3 to the stopping times T_n to decompose H as

$$H = \tilde{H}^n + \int_0^T \mathbf{1}_{[\![T_n]\!]}(s) \tilde{\xi}_s^n \, dX_s + \tilde{L}_{T_n}^n.$$

Applying Lemma 3.2 between T_{n-1} and T_n allows us to write $\tilde{H}^n$ as

$$\tilde{H}^n = \hat{H}^{n-1} + \int_0^T \mathbf{1}_{]\!] T_{n-1}; T_n [\![}(s) \hat{\xi}_s^{n-1} \, dX_s + \int_0^T \mathbf{1}_{]\!] T_{n-1}; T_n [\![}(s) \, d\hat{L}_s^{n-1}.$$

Therefore,

$$H = \hat{H}^{n-1} + \int_0^T \left(\mathbf{1}_{]\!] T_{n-1}; T_n [\![}(s) \hat{\xi}_s^{n-1} + \mathbf{1}_{[\![T_n]\!]}(s) \tilde{\xi}_s^n\right) dX_s$$
$$+ \tilde{L}_{T_n}^n + \int_0^T \mathbf{1}_{]\!] T_{n-1}; T_n [\![}(s) \, d\hat{L}_s^{n-1}.$$

By induction, by using successively Lemmas 3.2 and 3.3, H is decomposable as follows:

$$H = \hat{H}^0 + \int_0^T \sum_{i=1}^n \left(\mathbf{1}_{]\!] T_{i-1}; T_i [\![}(s) \hat{\xi}_s^{i-1} + \mathbf{1}_{[\![T_i]\!]}(s) \tilde{\xi}_s^i\right) dX_s$$
$$+ \sum_{i=1}^n \left(\tilde{L}_{T_i}^i + \int_0^T \mathbf{1}_{]\!] T_{i-1}; T_i [\![}(s) \, d\hat{L}_s^{i-1}\right).$$

Let

$$H_0 = \hat{H}^0,$$
$$\xi = \sum_{i=1}^n \left(\mathbf{1}_{]\!] T_{i-1}; T_i [\![} \hat{\xi}^{i-1} + \mathbf{1}_{[\![T_i]\!]} \tilde{\xi}^i\right)$$

and

$$L = \sum_{i=1}^n \left(\tilde{L}_{T_i}^i + \int_0^T \mathbf{1}_{]\!] T_{i-1}; T_i [\![}(s) \, d\hat{L}_s^{i-1}\right).$$

Then H_0 is $\mathscr{F}_0$-measurable, $\xi \in \Theta$ and $L \in \mathscr{M}_0^2$ is strongly orthogonal to $\int \theta \, dM$ for $\theta \in L^2(M)$, which completes the proof of the existence of the F-S decomposition.

Let us prove now the uniqueness of the decomposition. By subtracting, we can assume that $H = 0$ P-a.s. If

$$H_0 + \int_0^T \xi_s \, dX_s + L_T = 0,$$

then

$$\left(H_0 + \int_0^T \mathbf{1}_{]\!]0;T_n[\![}(s) \xi_s \, dX_s + L_{T_n-} \right) + \int_0^T \mathbf{1}_{[\![T_n]\!]}(s) \xi_s \, dX_s + L_T - L_{T_n-} = 0.$$

Now $(H_0 + \int_0^T \mathbf{1}_{]\!]0;T_n[\![}(s) \xi_s \, dX_s + L_{T_n-})$ is $\mathscr{F}_{T_n-}$-measurable so, by Lemma 3.3,

$$H_0 + \int_0^T \mathbf{1}_{]\!]0;T_n[\![}(s) \xi_s \, dX_s + L_{T_n-} = 0,$$

$$\int_0^T \mathbf{1}_{[\![T_n]\!]}(s) \xi_s \, dM_s = 0$$

and

$$L_T = L_{T_n-}, \qquad P\text{-a.s.}$$

Consequently,

$$\left(H_0 + \int_0^T \mathbf{1}_{]\!]0;T_{n-1}]\!]}(s) \xi_s \, dX_s + L_{T_{n-1}} \right) + \int_0^T \mathbf{1}_{]\!]T_{n-1};T_n[\![}(s) \xi_s \, dX_s$$
$$+ \int_0^T \mathbf{1}_{]\!]T_{n-1};T_n[\![}(s) \, dL_s = 0.$$

Using the uniqueness of decomposition (3.4) yields

$$H_0 + \int_0^T \mathbf{1}_{]\!]0;T_{n-1}]\!]}(s) \xi_s \, dX_s + L_{T_{n-1}} = 0,$$

$$\int_0^T \mathbf{1}_{]\!]T_{n-1};T_n]\!]}(s) \xi_s \, dM_s = 0$$

and

$$L_T = L_{T_{n-1}}.$$

By induction, we prove that

$$H_0 = 0,$$

$$\int_0^T \mathbf{1}_{]\!]T_0;T_n]\!]}(s) \xi_s \, dM_s = 0$$

and

$$L_T = L_{T_0} = L_0 = 0, \qquad P\text{-a.s.},$$

which completes the proof of the uniqueness of F-S decomposition. □

3.2. *Continuity.* To prove the continuity of F-S decomposition, we argue as in the proof of Theorem 3.4. That is, we want to see what happens at a high jump of $\hat{K}$ and outside a high jump. So again we need two auxiliary results.

LEMMA 3.5. *Let $0 \le T_1 \le T_2 \le T$ be predictable stopping times such that there exists a constant $\delta \in (0,1)$ satisfying $\hat{K}_{T_2} - \hat{K}_{T_1} \le \delta$ P-a.s. Suppose that H^p and H are random variables in $\mathscr{L}^2(\Omega, \mathscr{F}_{T_2-}, P)$. Let*

$$H^p = \hat{H}^p + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\xi_s^p\,dX_s + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\,d\hat{L}_s^p$$

and

$$H = \hat{H} + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\xi_s\,dX_s + \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\,d\hat{L}_s$$

be the decompositions of H^p and H defined in (3.4). *If*

$$H^p \xrightarrow[p\to\infty]{\mathscr{L}^2} H,$$

then

$$\hat{H}^p \xrightarrow[p\to\infty]{\mathscr{L}^2} \hat{H},$$

$$\mathbf{1}_{]\!]T_1;T_2[\![}\,\hat{\xi}^p \xrightarrow[p\to\infty]{L^2(M)} \mathbf{1}_{]\!]T_1;T_2[\![}\,\hat{\xi}$$

and

$$\int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\,d\hat{L}_s^p \xrightarrow[p\to\infty]{\mathscr{L}^2} \int_0^T \mathbf{1}_{]\!]T_1;T_2[\![}(s)\,d\hat{L}_s.$$

PROOF. Let $\hat{\xi}^p$ (resp. $\hat{\xi}$) be the unique fixed point of Ψ_{H^p} (resp. Ψ_H). Then

$$\begin{aligned}\|\hat{\xi}^p - \hat{\xi}\|_{L^2(M)} &= \|\Psi_{H^p}(\hat{\xi}^p) - \Psi_H(\hat{\xi})\|_{L^2(M)}\\ &\le \|\Psi_{H^p}(\hat{\xi}^p) - \Psi_{H^p}(\hat{\xi})\|_{L^2(M)} + \|\Psi_{H^p}(\hat{\xi}) - \Psi_H(\hat{\xi})\|_{L^2(M)}\\ &\le \sqrt{\delta}\|\hat{\xi}^p - \hat{\xi}\|_{L^2(M)} + \|\Psi_{H^p}(\hat{\xi}) - \Psi_H(\hat{\xi})\|_{L^2(M)}\end{aligned}$$

because Ψ_{H^p} is a contraction with parameter $\sqrt{\delta}$ for every p, and δ does not depend on p. Therefore,

$$\|\hat{\xi}^p - \hat{\xi}\|_{L^2(M)} \le \frac{1}{1-\sqrt{\delta}}\|\Psi_{H^p}(\hat{\xi}) - \Psi_H(\hat{\xi})\|_{L^2(M)}.$$

However, for every $\theta \in L^2(M)$, if $\hat{\theta}^p = \Psi_{H^p}(\theta)$ and $\hat{\theta} = \Psi_H(\theta)$, we have, from (3.3), using the orthogonality of the three terms,

$$\|H^p - H\|_2 \ge \left\|\int_0^T \left(\hat{\theta}_s^p - \hat{\theta}_s\right) dM_s\right\|_2.$$

Since H^p converges to H in $\mathscr{L}^2$,

$$\int_0^T \hat{\theta}_s^p\,dM_s \xrightarrow[p\to\infty]{\mathscr{L}^2} \int_0^T \hat{\theta}_s\,dM_s.$$

So,

$$\int_0^T \hat{\xi}_s^p\,dM_s \xrightarrow[p\to\infty]{\mathscr{L}^2} \int_0^T \hat{\xi}_s\,dM_s.$$

Using the fact that X satisfies (SC) and that $\hat{K}_{T_2-} - \hat{K}_{T_1} \le \delta$, we obtain

$$\begin{aligned}\left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s \right)^* dA_s \right\|_2^2 &= \left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s \right)^* \gamma_s \, dB_s \right\|_2^2 \\ &= \left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s \right)^* \sigma_s \hat{\lambda}_s \, dB_s \right\|_2^2 \\ &\le \| \hat{K}_{T_2-} - \hat{K}_{T_1} \|_\infty E\left(\int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s \right)^* \sigma_s \left(\hat{\xi}_s^p - \hat{\xi}_s \right) dB_s \right) \\ &\le \| \hat{K}_{T_2-} - \hat{K}_{T_1} \|_\infty \left\| \int_0^T \left(\hat{\xi}_s^p - \hat{\xi}_s \right) dM_s \right\|_2^2 .\end{aligned}$$

Hence,

$$\int_0^T \hat{\xi}_s^{p*} \, dA_s \xrightarrow[p \to \infty]{\mathscr{L}^2} \int_0^T \hat{\xi}_s^* \, dA_s ,$$

so

$$\left\| \left(H^p - \int_0^T \hat{\xi}_s^{p*} \, dA_s \right) - \left(H - \int_0^T \hat{\xi}_s^* \, dA_s \right) \right\|_2^2 \xrightarrow[p \to \infty]{\mathscr{L}^2} 0.$$

Now, from (3.4),

$$H - \int_0^T \hat{\xi}_s^* \, dA_s = \hat{H} + \int_0^T \hat{\xi}_s \, dM_s + \int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \, d\hat{L}_s$$

and

$$H^p - \int_0^T \hat{\xi}_s^{p*} \, dA_s = \hat{H}^p + \int_0^T \hat{\xi}_s^p \, dM_s + \int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \, d\hat{L}_s^p .$$

Therefore,

$$\begin{aligned}&\bigg\| \left(\hat{H}^p + \int_0^T \hat{\xi}_s^p \, dM_s + \int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \, d\hat{L}_s^p \right) \\ &\qquad - \left(\hat{H} + \int_0^T \hat{\xi}_s \, dM_s + \int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \, d\hat{L}_s \right) \bigg\|_2^2 \xrightarrow[p \to \infty]{\mathscr{L}^2} 0.\end{aligned}$$

However, the three terms $\hat{H}^p - \hat{H}$, $\int_0^T \hat{\xi}_s^p \, dM_s - \int_0^T \hat{\xi}_s \, dM_s$ and $\int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \times d\hat{L}_s^p - \int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \, d\hat{L}_s$ are orthogonal; hence, the previous convergence implies that

$$\hat{H}^p \xrightarrow[p \to \infty]{\mathscr{L}^2} \hat{H} \quad \text{and} \quad \int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \, d\hat{L}_s^p \xrightarrow[p \to \infty]{\mathscr{L}^2} \int_0^T \mathbf{1}_{]\!]T_1; T_2[\![} (s) \, d\hat{L}_s ,$$

which completes the proof of Lemma 3.5. □

REMARK 3.6. Since X satisfies (SC) and $\hat{K}$ is uniformly bounded, the convergence of ξ^p to ξ in $L^2(M)$ always implies the convergence of $\int_0^T \xi_s^p \, dX_s$ to $\int_0^T \xi_s \, dX_s$ in $\mathscr{L}^2$. Indeed

$$\begin{aligned}\left\| \int_0^T (\xi_s^p - \xi_s)\, dX_s \right\|_2 &\le \|\xi^p - \xi\|_{L^2(M)} + \left\| \int_0^T (\xi_s^p - \xi_s)^* \, dA_s \right\|_2 \\ &= \|\xi^p - \xi\|_{L^2(M)} + \left\| \int_0^T (\xi_s^p - \xi_s)^* \gamma_s \, dB_s \right\|_2 \\ &= \|\xi^p - \xi\|_{L^2(M)} + \left\| \int_0^T (\xi_s^p - \xi_s)^* \sigma_s \hat{\lambda}_s \, dB_s \right\|_2 \\ &\le \left(1 + \|\hat{K}\|_\infty^{1/2}\right) \|\xi^p - \xi\|_{L^2(M)}.\end{aligned}$$

LEMMA 3.7. *Let T_0 be a predictable stopping time. Suppose that H^p and H are random variables in $\mathscr{L}^2(\Omega, \mathscr{F}_{T_0}, P)$. Let*

$$H^p = \tilde{H}^p + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s^p \, dX_s + \tilde{L}_{T_0}^p$$

and

$$H = \tilde{H} + \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s \, dX_s + \tilde{L}_{T_0}$$

be the decomposition defined in (3.6). *If*

$$H^p \xrightarrow[p \to \infty]{\mathscr{L}^2} H,$$

then

$$\tilde{H}^p \xrightarrow[p \to \infty]{\mathscr{L}^2} \tilde{H},$$

$$\mathbf{1}_{[\![T_0]\!]} \tilde{\xi}^p \xrightarrow[p \to \infty]{L^2(M)} \mathbf{1}_{[\![T_0]\!]} \tilde{\xi}$$

and

$$\tilde{L}_{T_0}^p \xrightarrow[p \to \infty]{\mathscr{L}^2} \tilde{L}_{T_0}.$$

PROOF. $H^p \xrightarrow[p \to \infty]{\mathscr{L}^2} H$, so $E(H^p | \mathscr{F}_{T_0 -}) \xrightarrow[p \to \infty]{\mathscr{L}^2} E(H | \mathscr{F}_{T_0 -})$. Hence, by subtracting,

$$H^p - E\left(H^p | \mathscr{F}_{T_0 -}\right) \xrightarrow[p \to \infty]{\mathscr{L}^2} H - E\left(H | \mathscr{F}_{T_0 -}\right).$$

Replacing H^p and H by their decompositions implies that

$$\int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s^p \, dM_s + \tilde{L}_{T_0}^p \xrightarrow[p \to \infty]{\mathscr{L}^2} \int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s \, dM_s + \tilde{L}_{T_0}.$$

However, $\int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s^p \, dM_s + \tilde{L}_{T_0}^p$ is the Galtchouk–Kunita–Watanabe decomposition of $H^p - E(H^p | \mathscr{F}_{T_0 -})$ and $\int_0^T \mathbf{1}_{[\![T_0]\!]}(s) \tilde{\xi}_s \, dM_s + \tilde{L}_{T_0}$ is the Galtchouk–Kunita–Watanabe decomposition of $H - E(H | \mathscr{F}_{T_0 -})$. The

Galtchouk–Kunita–Watanabe decomposition is continuous with respect to the $\mathscr{L}^2$-norm, since it is a projection on a closed subspace of $\mathscr{L}^2$. Therefore

$$\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s^p\, dM_s \xrightarrow[p\to\infty]{\mathscr{L}^2} \int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s\, dM_s \quad \text{and} \quad \tilde{L}_{T_0}^p \xrightarrow[p\to\infty]{\mathscr{L}^2} \tilde{L}_{T_0}.$$

By Remark 3.6, we also have the convergence of $\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s^p\, dX_s$ to $\int_0^T \mathbf{1}_{[\![T_0]\!]}(s)\tilde{\xi}_s\, dX_s$ in $\mathscr{L}^2$. Consequently, $\tilde{H}^p$ converges to $\tilde{H}$ in $\mathscr{L}^2$, which completes the proof. □

THEOREM 3.8. *Let H^p and H be random variables in $\mathscr{L}^2(\Omega, \mathscr{F}, P)$. Suppose that*

$$H^p = H_0^p + \int_0^T \xi_s^p\, dX_s + L_T^p$$

and

$$H = H_0 + \int_0^T \xi_s\, dX_s + L_T$$

are the F-S decompositions of H^p and H.
If

$$H^p \xrightarrow[p\to\infty]{\mathscr{L}^2} H,$$

then

$$H_0^p \xrightarrow[p\to\infty]{\mathscr{L}^2} H_0,$$

$$\xi^p \xrightarrow[p\to\infty]{L^2(M)} \xi$$

and

$$L_T^p \xrightarrow[p\to\infty]{\mathscr{L}^2} L_T.$$

PROOF. Using the notations of the proof of Theorem 3.4, we have the equalities

$$H_0 = \hat{H}^0, \qquad H_0^p = \hat{H}^{p,0},$$

$$\xi = \sum_{i=1}^n \left(\mathbf{1}_{]\!]T_{i-1};T_i[\![}\, \hat{\xi}^{i-1} + \mathbf{1}_{[\![T_i]\!]}\, \tilde{\xi}^i\right), \qquad \xi^p = \sum_{i=1}^n \left(\mathbf{1}_{]\!]T_{i-1};T_i[\![}\, \hat{\xi}^{p,i-1} + \mathbf{1}_{[\![T_i]\!]}\, \tilde{\xi}^{p,i}\right)$$

and

$$L = \sum_{i=1}^n \left(\tilde{L}_{T_i}^i + \int_0^T \mathbf{1}_{]\!]T_{i-1};T_i[\![}(s)\, d\hat{L}_s^{i-1}\right),$$

$$L^p = \sum_{i=1}^n \left(\tilde{L}_{T_i}^{p,i} + \int_0^T \mathbf{1}_{]\!]T_{i-1};T_i[\![}(s)\, d\hat{L}_s^{p,i-1}\right).$$

Hence, applying Lemma 3.7 to stopping times T_i and Lemma 3.5 between T_{i-1} and T_i, we prove Theorem 3.8. □

4. Applications.

4.1. *Closedness of* $G_T(\Theta)$ *and* $\{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$. As an application of the existence, the uniqueness and the continuity of F-S decomposition, we have the following result:

THEOREM 4.1. *The subspaces* $G_T(\Theta)$ *and* $\{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$ *are closed subspaces of* $\mathscr{L}^2$.

PROOF. Let $(H^p)_{p \ge 0}$ be a sequence of $G_T(\Theta)$ which converges to H in $\mathscr{L}^2$. For every p, H^p is in $G_T(\Theta)$. Hence there exists a process $\theta^p \in \Theta$ such that

$$H^p = \int_0^T \theta_s^p \, dX_s$$

and since the F-S decomposition is unique, this is the F-S decomposition of H^p. The random variable H is in $\mathscr{L}^2$. Hence, from the existence of the F-S decomposition, we can write

$$H = H_0 + \int_0^T \xi_s^H \, dX_s + L_T^H$$

as in Theorem 3.4.

Now, H^p converges to H in $\mathscr{L}^2$, so by Theorem 3.8, $H_0 = 0$ P-a.s. and $L_T^H = 0$ P-a.s. Hence, H is in $G_T(\Theta)$, so $G_T(\Theta)$ is closed in $\mathscr{L}^2$.

Let $(H^p)_{p \ge 0}$ be a sequence of $\{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$ which converges to H in $\mathscr{L}^2$. For every p, H^p is in $\{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$. Hence, there exist a random variable $H_0^p \in \mathscr{L}^2(\mathscr{F}_0)$ and a process $\theta^p \in \Theta$ such that

$$H^p = H_0^p + \int_0^T \theta_s^p \, dX_s.$$

and since the F-S decomposition is unique, this is the F-S decomposition of H^p. The random variable H is in $\mathscr{L}^2$. Hence, from the existence of the F-S decomposition, we can write

$$H = H_0 + \int_0^T \xi_s^H \, dX_s + L_T^H$$

as in Theorem 3.4.

Now, H^p converges to H in $\mathscr{L}^2$, so by Theorem 3.8, the sequence $(H_0^p)_{p \ge 0}$ converges to H_0 and $L_T^H = 0$ P-a.s. Hence, $H \in \{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$, so $\{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$ is closed. □

We thank M. Schweizer for the following remark.

REMARK 4.2. Since $G_T(\Theta)$ is closed, so is any sum of $G_T(\Theta)$ and a finite-dimensional subspace of $\mathscr{L}^2$. In particular, $\{c + G_T(\Theta) | c \in \mathbb{R}\}$ is closed.

REMARK 4.3. When X is continuous, we only need the existence of the F-S decomposition to prove that the previous subspaces are closed. In fact, in that case, since $\hat{K}$ is bounded, the process $\hat{Z}$ defined by

$$\hat{Z}_t := \mathscr{E}\left(-\int \hat{\lambda}\, dM\right)_t$$

is a positive square-integrable martingale. Therefore, we can define a new probability law $\hat{P}$ equivalent to P by setting

$$\frac{d\hat{P}}{dP} := \hat{Z}_T.$$

Then $\hat{Z}G(\theta) \in \mathscr{M}_0^1(P)$ for every $\theta \in \Theta$, so $G(\theta)$ is a $\hat{P}$-martingale and

$$\begin{aligned}\hat{E}\left(\sup_{0\le t\le T} |G_t(\theta)|\right) &= E\left(\hat{Z}_T \sup_{0\le t\le T} |G_t(\theta)|\right)\\ &\le \left(E\left(\hat{Z}_T^2\right)\right)^{1/2}\left(E\left(\sup_{0\le t\le T} |G_t(\theta)|\right)^2\right)^{1/2}\\ &\le C\left(E\left(\hat{Z}_T^2\right)\right)^{1/2} \sup_{0\le t\le T}\left\|\int_0^T \theta_s\, dM_s\right\|_2\\ &< +\infty.\end{aligned}$$

Therefore, $G(\theta) \in \mathscr{M}_0^1(\hat{P})$.

Let $(G_T(\theta^n))_{n\in\mathbb{N}}$ be a sequence in $G_T(\Theta)$ which converges to H in $\mathscr{L}^2(P)$. Then $G_T(\theta^n)$ converges to H in $\mathscr{L}^1(\hat{P})$. Under the probability law $\hat{P}$, $G(\theta^n)$ is a u.i. martingale, so by Yor's lemma, there exists a predictable process ξ such that

$$H = \int_0^T \xi_s\, dX_s.$$

Let $H_0 + \int_0^T \xi_s^H\, dX_s + L_T^H$ be the F-S decomposition of H. Since $\int \xi\, dX$ is a $\hat{P}$-martingale,

$$\begin{aligned}\hat{E}(H|\mathscr{F}_t) &= \int_0^t \xi_s\, dX_s\\ &= \hat{E}\left(H_0 + \int_0^T \xi_s^H\, dX_s + L_T^H|\mathscr{F}_t\right)\\ &= H_0 + \int_0^t \xi_s^H\, dX_s + \hat{E}\left(L_T^H|\mathscr{F}_t\right).\end{aligned}$$

However, $\hat{Z}L \in \mathscr{M}_0^1(P)$, so $L \in \mathscr{M}_0^1(\hat{P})$. Hence

$$\int_0^t \xi_s\, dX_s = H_0 + \int_0^t \xi_s^H\, dX_s + L_t^H, \qquad P\text{-a.s. for all } t \in [0;T].$$

If $t = 0$, we conclude that $H_0 = 0$ P-a.s. Therefore, taking the sharp bracket between L^H and $\int \xi\, dX$ yields $\langle L^H \rangle = 0$; that is, $L_t^H = 0$ P-a.s. for all $t \in [0;T]$, which completes the proof. □

REMARK 4.4. Using the second part of Theorem 4.1, it is easy to prove that if $\mathscr{G} \subset \mathscr{F}_0$ is a subfiltration of $\mathscr{F}_0$, then $\{\mathscr{L}^2(\mathscr{G}) + G_T(\Theta)\}$ is closed.

THEOREM 4.5. *If $\hat{K}$ is uniformly bounded, the norms $\|\cdot\|_{L^2(M)}$, $\|G_T(\cdot)\|_2$ and $\|G_T(\cdot)\|_{\mathscr{S}^2}$ are equivalent.*

PROOF. It is obvious that $\|G_T(\cdot)\|_2 \le \|G_T(\cdot)\|_{\mathscr{S}^2}$. On the other hand, for $\theta \in \Theta$,

$$\begin{aligned}\|G_T(\theta)\|_{\mathscr{S}^2} &= \left\|\int_0^T \theta_s\, dM_s\right\|_2 + \left\|\int_0^T |\theta_s^* \gamma_s|\, dB_s\right\|_2 \\ &= \|\theta\|_{L^2(M)} + \left\|\int_0^T |\theta_s^* \sigma_s \hat{\lambda}_s|\, dB_s\right\|_2 \\ &\le \|\theta\|_{L^2(M)} + \left(E\left(\int_0^T \theta_s^* \sigma_s \theta_s\, dB_s \int_0^T \hat{\lambda}_s^* \sigma_s \hat{\lambda}_s\, dB_s\right)\right)^{1/2} \\ &\le \left(1 + \|\hat{K}\|_\infty^{1/2}\right)\|\theta\|_{L^2(M)}.\end{aligned}$$

Finally, if $G_T(\theta^n) \to_{n\to\infty}^{\mathscr{S}^2} G_T(\theta)$, then by Theorem 4.1, there exists $\theta' \in \Theta$ such that $G_T(\theta^n) \to_{n\to\infty}^{\mathscr{S}^2} G_T(\theta')$. However, $H^n = G_T(\theta^n)$ and $H = G_T(\theta)$ are the F-S decompositions of H^n and H. So by the continuity of this decomposition, $\theta^n \to_{n\to\infty}^{L^2(M)} \theta'$. Therefore, there exists a positive constant α such that $\|\cdot\|_{L^2(M)} \le \alpha \|G_T(\cdot)\|_2$, which completes the proof of Theorem 4.5. □

4.2. *Approximation of random variables by stochastic integrals.* By Theorem 4.1, we can now project any random variable $H \in \mathscr{L}^2$ on $G_T(\Theta)$ and $\{\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)\}$. So we prove the existence and the uniqueness of a solution of the optimization problem.

THEOREM 4.6. *For every $H \in \mathscr{L}^2(\Omega, \mathscr{F}, P)$ and every $c \in \mathscr{L}^2(\mathscr{F}_0)$, there exists a unique strategy $\xi^{(c)} \in \Theta$ such that*

$$E\left(\left(H - c - \int_0^T \xi_s^{(c)}\, dX_s\right)^2\right) = \min_{\theta\in\Theta} E\left(\left(H - c - \int_0^T \theta_s\, dX_s\right)^2\right).$$

Similarly, there exists a unique $(c^H, \xi^H) \in \mathscr{L}^2(\mathscr{F}_0) \times \Theta$ such that

$$E\left(\left(H - c^H - \int_0^T \xi_s^H\, dX_s\right)^2\right) = \min_{(c,\theta)\in\mathscr{L}^2(\mathscr{F}_0)\times\Theta} E\left(\left(H - c - \int_0^T \theta_s\, dX_s\right)^2\right).$$

4.3. *Applications to financial mathematics.* The two previous optimization problems have a natural interpretation in financial mathematics. Indeed, if H is interpreted as a contingent claim, the first optimization problem amounts to finding the best strategy to minimize the expected net quadratic loss at time T, given an initial capital c. As for the second optimization problem, it amounts to finding an initial capital c^H and a strategy ξ^H so as to minimize the expected net quadratic loss at time T.

4.4. *A special case.* If M, the martingale part of X, has the predictable representation property (denoted by PRP in what follows), then we prove that $G_T(\Theta)$ is equal to $\mathscr{L}^2$ up to constants.

THEOREM 4.7. *If M has the PRP and if $\hat{K}$ is bounded, then*

$$G_T(\Theta) = \left\{H \in \mathscr{L}^2(\Omega, \mathscr{F}, P) | E\left(\hat{Z}_T H\right) = 0\right\},$$

where the process $\hat{Z}$ is defined by

$$\hat{Z}_t := \mathscr{E}\left(-\int \hat{\lambda}\, dM\right)_t.$$

PROOF. Let H be a random variable in $\mathscr{L}^2(\Omega, \mathscr{F}, P)$. From the existence of the F-S decomposition, we can write H as

$$H = H_0 + \int_0^T \xi_s^H \, dX_s + L_T^H.$$

Since the martingale L^H is strongly orthogonal to M and vanishes at 0, and since M has the PRP, $L^H = 0$ P-a.s. Hence, every random variable $H \in \mathscr{L}^2(\Omega, \mathscr{F}, P)$ can be written as

$$H = H_0 + \int_0^T \xi_s^H \, dX_s.$$

However, from the uniqueness of the F-S decomposition, H belongs to $G_T(\Theta)$ if and only if $H_0 = 0$. If $\hat{Z}$ is the process defined by

$$\hat{Z}_t := \mathscr{E}\left(-\int \hat{\lambda}\, dM\right)_t$$

and if we assume that $\hat{K}$ is bounded, then Theorem II2 of Lépingle and Mémin (1978) implies that $\hat{Z}$ is in $\mathscr{M}^2$ and for every $\theta \in \Theta$, $\hat{Z}G(\theta)$ is in $\mathscr{M}_0^1(P)$. Hence $H_0 = E(\hat{Z}_T H)$, which completes the proof of Theorem 4.7. □

5. When $\hat{K}$ is no longer uniformly bounded.

If $\hat{K}$ is no longer uniformly bounded, the F-S decomposition does not always exist and $G_T(\Theta)$ may be closed or not, as we show in the three following examples.

5.1. *Counterexample for the F-S decomposition.* Suppose that $d = 1$ and that we are at a jump of $\hat{K}$. Let $H = \theta_S \, \Delta M_S$ with $\theta \in L^2(M)$ and let S be the time when $\hat{K}$ jumps. If H admits a F-S decomposition, then

$$H = H_0 + \int_0^T \xi_s \, dX_s + L_T.$$

However, $E(H | \mathscr{F}_{S-}) = 0$, so

$$\begin{aligned} H &= \theta_S \, \Delta M_S \\ &= \xi_S \, \Delta X_S + \Delta L_S. \end{aligned}$$

Now L is a martingale strongly orthogonal to $\int \theta\, dM$ for every $\theta \in \Theta$. Hence, necessarily $\theta_S = \xi_S$.

However,

$$E\left(\theta_S^2 (\Delta A_S)^2\right) = E\left(\theta_S^2\, \Delta\langle M\rangle_S\, \Delta \hat{K}_S\right)$$

and $\theta_S^2\, \Delta\langle M\rangle_S$ spans $L^1(\mathscr{F}_{S-})$ when $\theta \in \Theta$, so $E(\theta_S^2(\Delta A_S)^2) < +\infty$ if and only if $\Delta \hat{K}_S$ is uniformly bounded. Consequently, H does not admit a F-S decomposition if $\Delta \hat{K}$ is not uniformly bounded.

5.2. *Counterexamples for the closedness of* $G_T(\Theta)$. While the assumption that $\hat{K}$ is bounded might seem excessively strong, there are counterexamples when it is relaxed. In discrete time, there is a counterexample due to Schachermayer which is explained in Schweizer (1993a).

We now give another counterexample in which we work in continuous time and we assume, moreover, that the process X is continuous. More precisely, consider a standard Brownian motion W and its natural filtration $(\mathscr{F}_t)_{t \geq 0}$. Let $\alpha = (\alpha_t)_{1 \leq t \leq 2}$ be an $\mathbb{R}_+ \times \mathscr{F}_1$-measurable process such that

$$E\left(\exp\left(\int_1^2 \alpha_s^2\, ds\right)\right) < +\infty,$$

$$\forall\, \varepsilon \in (0;1), \qquad \int_1^{2-\varepsilon} \alpha_s^2\, ds \in \mathscr{L}^\infty$$

and

$$\int_1^2 \alpha_s^2\, ds \notin \mathscr{L}^\infty.$$

The last assumption implies that there exists a nonnegative random variable $H \in \mathscr{L}^1(\Omega, \mathscr{F}_1, P)$ such that

$$E\left(H \int_1^2 \alpha_s^2\, ds\right) = +\infty.$$

Define the process X by

$$X_t := \begin{cases} W_t, & \text{if } t \in [0,1], \\ W_t + \int_1^t \alpha_s\, ds, & \text{if } t \in [1,2]. \end{cases}$$

Finally, let us define the processes ψ, θ and θ^n by

$$\psi_t := \sqrt{E\left(\frac{H}{\hat{K}_2 + 1}\,\middle|\, \mathscr{F}_t\right)} \quad \text{for all } t \in [0,2],$$

$$\theta_t := \begin{cases} 0, & \text{if } t \in [0,1], \\ \psi_{t\text{-}}\alpha_t, & \text{if } t \in [1,2], \end{cases}$$

and $\theta_t^n := \mathbf{1}_{\{\hat{K}_t \leq n\}}\theta_t$, where $\hat{K}_t := \int_1^t \alpha_s^2\, ds$ for all $t \in [1,2]$. Then $\theta^n \in \Theta$ and the process θ is in $L^2(W)$, but it is not in $L^2(\int \alpha_s\, ds)$.

On the other hand, $\int_1^2 \theta_s^n \alpha_s \, ds$ is $\mathscr{F}_1$-measurable, so using the predictable representation property of the Brownian motion implies that there exists a process $\psi^n \in \Theta$ such that

$$\int_0^1 \psi_s^n \, dW_s = -\int_1^2 \theta_s^n \alpha_s \, ds, \qquad P\text{-a.s.}$$

For every $n \in \mathbb{N}$, if φ^n is the strategy defined by

$$\varphi^n := \begin{cases} \psi^n, & \text{on } [0,1], \\ \theta^n, & \text{on } [1,2], \end{cases}$$

then

$$\int_0^2 \varphi_s^n \, dX_s = \int_0^1 \psi_s^n \, dW_s + \int_1^2 \theta_s^n \, dW_s + \int_1^2 \theta_s^n \alpha_s \, ds$$
$$= \int_1^2 \theta_s^n \, dW_s.$$

Hence, the sequence $(G_2(\varphi^n))_{n \in \mathbb{N}}$ converges to $\int_1^2 \theta_s \, dW_s$ in $\mathscr{L}^2$.

Now $E(\exp(\int_1^2 \alpha_s^2 \, ds))$ is finite, so Novikov's criterion implies that we can define a new probability law $\hat{P}$ equivalent to P by setting

$$\frac{d\hat{P}}{dP} = \hat{Z}_2 := \exp\left(\int_1^2 \alpha_s \, dW_s - \frac{1}{2} \int_1^2 \alpha_s^2 \, ds \right).$$

Assuming that there exists a process $\varphi \in \Theta$ such that $(G_2(\varphi^n))_{n \in \mathbb{N}}$ converges to $G_2(\varphi)$ in $\mathscr{L}^2(P)$ and noting that, under $\hat{P}$, $G(\xi)$ is a martingale for all $\xi \in \Theta$, yields

$$\forall \varepsilon \in (0,1), \qquad \int_1^{2-\varepsilon} \varphi_s^n \, dX_s \xrightarrow[n \to \infty]{\mathscr{L}^1(\hat{P})} \int_1^{2-\varepsilon} \varphi_s \, dX_s,$$

since $\hat{Z}_2$ is square-integrable. However, the definition of φ^n implies that $\int_1^{2-\varepsilon} \varphi_s^n \, dX_s = \int_1^{2-\varepsilon} \theta_s^n \, dX_s$, so

$$\forall \varepsilon \in (0,1), \qquad \int_1^{2-\varepsilon} \theta_s^n \, dX_s \xrightarrow[n \to \infty]{\mathscr{L}^1(\hat{P})} \int_1^{2-\varepsilon} \varphi_s \, dX_s.$$

Now, for all $\varepsilon \in (0,1)$,

$$\int_1^{2-\varepsilon} \theta_s^n \, dW_s \xrightarrow[n \to \infty]{\mathscr{L}^2(P)} \int_1^{2-\varepsilon} \theta_s \, dW_s \quad \text{and} \quad \int_1^{2-\varepsilon} \alpha_s^2 \, ds \in \mathscr{L}^\infty.$$

Therefore, for all $\varepsilon \in (0,1)$,

$$\int_1^{2-\varepsilon} \theta_s^n \alpha_s \, ds \xrightarrow[n \to \infty]{\mathscr{L}^2(P)} \int_1^{2-\varepsilon} \theta_s \alpha_s \, ds,$$

which implies that

$$\forall \varepsilon \in (0,1), \qquad \int_1^{2-\varepsilon} \theta_s^n \, dX_s \xrightarrow[n \to \infty]{\mathscr{L}^1(\hat{P})} \int_1^{2-\varepsilon} \theta_s \, dX_s.$$

Consequently, for all $\varepsilon \in (0,1)$, $\int_1^{2-\varepsilon} \theta_s \, dX_s = \int_1^{2-\varepsilon} \varphi_s \, dX_s$, so

$$\forall \varepsilon \in (0,1), \qquad \hat{E}\left(\left(\int_1^{2-\varepsilon} (\theta_s - \varphi_s) \, dX_s \right)^2 \right) = 0.$$

Under $\hat{P}$, X is a standard Brownian motion, so taking the sharp bracket yields

$$\forall\, \varepsilon \in (0,1), \qquad \hat{E}\left(\int_1^{2-\varepsilon} (\theta_s - \varphi_s)^2\, ds\right) = 0,$$

which proves that $\theta = \varphi$. However, $\theta \notin \Theta$, so the limit of the sequence $(G_2(\theta^n))_{n\in\mathbb{N}}$ is not in $G_2(\Theta)$, which completes the counterexample.

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