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# DIFFERENTIAL SUBORDINATION AND STRONG DIFFERENTIAL SUBORDINATION FOR CONTINUOUS-TIME MARTINGALES AND RELATED SHARP INEQUALITIES

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Let $X$ and $Y$ be two continuous-time martingales. If quadratic variation of X minus that of Y is a nondecreasing and nonnegative function of time, we say that $Y$ is differentially subordinate to $X$ and prove that $||Y||_p \le (p^* - 1)||X||_p$ for $1 < p < \infty$, where $p^* = p \vee q$ and $q$ is the conjugate of $p$. This inequality contains Burkholder's $L^p$-inequality for stochastic integrals, which implies that the above inequality is sharp. We also extend his concept of strong differential subordination and several other of his inequalities, and sharpen an inequality of Bañuelos.

**Introduction.** Let $(\Omega, \mathscr{F}_\infty, P)$ be a probability space and $\mathscr{F} = \{\mathscr{F}_n\}_{n\ge 0}$ be a nondecreasing sequence of sub-$\sigma$-fields of $\mathscr{F}_\infty$. Let $\mathbb{H}$ be a Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. Consider two martingales with respect to $\mathscr{F}$ taking values in $\mathbb{H}$: $f = \{f_n\}_{n\ge 0}$ and $g = \{g_n\}_{n\ge 0}$. Then $g$ is said to be *differentially subordinate* to $f$ if $|e_n| \le |d_n|$ for $n \ge 0$, where $d_0 = f_0, d_n = f_n - f_{n-1}$ for $n \ge 1$ and $e_n$ is defined similarly. Let $||f||_p = \sup_{n\ge 0} ||f_n||_p$. Burkholder (see Section 12 of [5] for the case $\mathbb{H} = \mathbb{R}$ and [7] for any real or complex Hilbert space $\mathbb{H}$) proved the following sharp martingale inequality:

THEOREM A. *Let $p^* = \max\{p, p/(p-1)\}$, where $1 < p < \infty$. If $f$ and $g$ are martingales relative to the same filtration as above and $g$ is differentially subordinate to $f$, then*

$$||g||_p \le (p^* - 1)||f||_p$$

*and the constant $(p^* - 1)$ is best possible. If $0 < ||f||_p < \infty$ and $p \ne 2$, the above inequality is a strict inequality.*

The proof is based on the existence of a special function satisfying appropriate conditions. Using approximation (see Bichteler [3]), Burkholder [5] extended the above theorem to stochastic integrals.

To state the theorem, we let $(\Omega, \mathscr{F}_\infty, P)$ be a complete probability space and $\mathscr{F} = \{\mathscr{F}_t\}_{t\ge 0}$ be a filtration that is continuous on the right. Assume $\mathscr{F}_0$ contains all $\mathscr{F}_\infty$ null sets. Consider an adapted martingale $M = \{M_t\}_{t\ge 0}$ with respect to $\mathscr{F}$ which is continuous on the right with limits from the left (r.c.l.l.) and two predictable processes $U = \{U_t\}_{t\ge 0}$ and $V = \{V_t\}_{t\ge 0}$. Let $||M||_p =$

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$\sup_{t\geq 0} ||M_t||_p$ and denote the stochastic integral between a predictable process $U$ and a martingale $M$ by $U \cdot M$:

$$(U \cdot M)_t = \int_0^t U_s \, dM_s.$$

Under each of the two conditions (i) the processes $U$ and $V$ are $\mathbb{H}$-valued and $M$ is scalar-valued and (ii) the processes $U$ and $V$ are scalar-valued and $M$ is $\mathbb{H}$-valued, Burkholder (see Theorem 5.1 of [7]; for $\mathbb{H} = \mathbb{R}$, see [5]) proved the following theorem.

THEOREM B. *Let $1 < p < \infty$. If either condition* (i) *or* (ii) *holds and if for all $t \geq 0$,*

$$|V_t| \leq |U_t|,$$

*then*

$$||V \cdot M||_p \leq (p^* - 1)||U \cdot M||_p.$$

*The constant $p^* - 1$ is the best possible.*

Using a different method, Bañuelos [1, 2] obtained a result similar to Theorem B when $M_t = B_t$, the $d$-dimensional Brownian motion. Let $H$ be a $d$-dimensional predictable process and let $\mathbf{A} = (A_1, A_2, \ldots)$ be a vector of $d \times d$ matrices. Let $||\mathbf{A}||^2$ be the square of the largest eigenvalue of $\sum_{i=1}^{\infty} A_i A_i^T$, which is assumed to be finite, where $A_i^T$ is the transpose of $A_i$. Let $X = H \cdot B$ and $Y = ((HA_1) \cdot B, (HA_2) \cdot B, \ldots) = (Y_1, Y_2, \ldots)$. Then Bañuelos proved ([2], Theorem 2.2) the following theorem.

THEOREM C. *For every $1 < p < \infty$, there exists a constant $c_p$ depending only on $p$ such that*

$$||Y||_p \leq c_p ||\mathbf{A}|| \; ||X||_p.$$

*Moreover, $c_p \leq cp$ for $p \geq 2$ and $c_p \leq c/(p-1)$ for $1 < p < 2$, where $c$ is an absolute constant.*

Theorem C has applications to singular integrals. See a forthcoming paper of Bañuelos and Wang.

Theorems A and B give examples of

$$||Y||_p \leq (p^* - 1)||X||_p, \tag{I.1}$$

where $X$ and $Y$ are adapted r.c.l.l. martingales. Burkholder asked whether this inequality always holds under the quadratic-variation condition

$$[X, X]_t - [Y, Y]_t \geq 0 \quad \text{for } t \geq 0, \tag{I.2}$$

a condition which is satisfied in both Theorem A and Theorem B. Although Burkholder's question remains open, we prove here, among other things, that

(I.1) does hold if the left-hand side of (I.2) is both nonnegative and nondecreasing in $t$. This allows us to show that the best constant $c_p$ in Theorem C is also $p^* - 1$.

Let $(\Omega, \mathscr{F}_\infty, P)$ and filtration $\mathscr{F}$ be defined as before. For two adapted r.c.l.l. semimartingales $X = \{X_t\}_{t\geq 0}$ and $Y = \{Y_t\}_{t\geq 0}$ taking values in a separable Hilbert space $\mathbb{H}$ with respect to $\mathscr{F}$, we say $Y$ is *differentially subordinate* (by quadratic variation) to $X$ if $[X,X]_t - [Y,Y]_t$ is a nondecreasing and nonnegative function of $t$, where $[X,X]_t$ and $[Y,Y]_t$ are the quadratic variation processes of $X$ and $Y$, respectively. [Likewise, we say $Y$ is *subordinate* (by quadratic variation) to $X$ if $[X,X]_t - [Y,Y]_t \geq 0$ for all $t \geq 0$. See the remark at the end of the paper.]

This definition is consistent with the analogue of discrete-time semimartingales introduced by Burkholder [7] defined before. In fact, if $f = \{f_n\}_{n\geq 0} = \{\sum_{i=0}^n d_i\}_{n\geq 0}$ and $g = \{g_n\}_{n\geq 0} = \{\sum_{i=0}^n e_i\}_{n\geq 0}$ are two discrete-time martingales, $g$ is differentially subordinate to $f$ if $[f,f]_n - [g,g]_n = \sum_{i=0}^n (|d_i|^2 - |e_i|^2)$ is a nonnegative and nondecreasing function of $n$ or $|e_n| \leq |d_n|$ for $n \geq 0$.

Similarly, suppose $M_t$ is a martingale and $U_t$ and $V_t$ are predictable processes given in (i) or (ii). Let $X_t = (U \cdot M)_t$ and $Y_t = (V \cdot M)_t$. Then $Y$ is differentially subordinate to $X$ if $|V_t| \leq |U_t|$ for all $t \geq 0$ since $[X,X]_t - [Y,Y]_t = \int_0^t (|U_s|^2 - |V_s|^2)\, d[M,M]_s$.

We shall prove the following theorem.

THEOREM 1. *Let $X$ and $Y$ be two adapted r.c.l.l. martingales in $\mathbb{H}$ such that $Y$ is differentially subordinate to $X$. For $1 < p < \infty$,*

$$||Y||_p \leq (p^* - 1)||X||_p$$

*and the constant $p^* - 1$ is best possible. Strict inequality holds when $0 < ||X||_p < \infty$ and $p \neq 2$.*

This theorem holds true for local martingales also provided the norm $||\cdot||_p$ is replaced by the norm $|||\cdot|||_p$ defined by $|||X|||_p = \sup_\tau ||X^\tau||_p$, where $\tau$ is any bounded stopping time and the process $X^\tau = \{X_t^\tau\}_{t\geq 0} = \{X_{\tau\wedge t}\}_{t\geq 0}$. In fact, if $p > 1$ and $|||X|||_p < \infty$, then the local martingale $X$ is an $L^p$-bounded martingale and $\lim_{t\to\infty} X_t = X_\infty$ exists almost surely. Moreover, $|||X|||_p = ||X||_p = ||X_\infty||_p$. Therefore, the inequality does not change.

Strict inequality is new even under the condition considered in Theorem B provided $\mathbb{H} \neq \mathbb{R}$. The case $\mathbb{H} = \mathbb{R}$ is contained in [6] by Burkholder.

Clearly, Theorems A, B and C are special cases of Theorem 1. In particular, the best constant in Theorem C is $p^* - 1$. This answers a question raised by Bañuelos.

The proof of Theorem 1 is based on the special function given in the proof of Theorem A by Burkholder and Itô's formula.

In the next section, we first give the proof of Theorem 1 for continuous-time local martingales under differential subordination when the dimension of $\mathbb{H}$ is finite. The proof of Theorem 1 when the dimension of $\mathbb{H}$ is infinite follows from Proposition 1.

In Section 2, we extend the concept of strong differential subordination introduced by Burkholder [9]. Theorem 4 is the main result of this section. It contains both (2.1) and (3.1) of [9]. The extension of the concepts of differential subordination and strong differential subordination makes possible the extension of some of the other sharp inequalities proved in [5] and [7]–[9].

At the end of this paper, we attempt to reduce the differential subordination condition in Theorem 1 to a weaker condition. In particular, we show the following theorem as a special case of Theorem 5.

THEOREM 2. *If $X$ and $Y$ are continuous path local martingales such that $|Y_0| \le |X_0|$ and $[X,X]-[Y,Y]$ is a submartingale, then*

$$||Y||_p \le (p^*-1)||X||_p$$

*and $p^*-1$ is best possible. Strict inequality holds when $p \neq 2$ and $0<||X||_p<\infty$.*

Throughout the rest of the paper, $\mathbb{H}$ denotes a separable Hilbert space, which in the proof can be taken to be $l^2$, and all semimartingales are r.c.l.l. and $\mathbb{H}$-valued unless otherwise stated.

**1. Continuous-time semimartingales and differential subordination.** Consider two $\mathbb{H}$-valued semimartingales $X=(X_1,X_2,\dots)$ and $Y=(Y_1,Y_2,\dots)$.

For example, let $B$ be $d$-dimensional Brownian motion and $H=(H_1, H_2,\dots)$, $K=(K_1,K_2,\dots)$ be two sequences of $d$-dimensional predictable processes. Then under proper boundedness assumptions on $H$ and $K$,

$$X=(H_1\cdot B, H_2\cdot B,\dots) \quad \text{and} \quad Y=(K_1\cdot B, K_2\cdot B,\dots)$$

may be considered as martingales taking values in $\mathbb{H}$. In particular, if $H=(H_1,0,\dots)$, where $H_1$ is a $d$-dimensional $L^2$ bounded predictable process, $K_i=H_1A_i$, where $\mathbf{A}=(A_1,A_2,\dots)$ is a sequence of $d\times d$ matrices, then $X$ and $Y$ defined above are the martingales considered in Theorem C by Bañuelos.

Let $|X|=(\sum_{i=1}^\infty |X_i|^2)^{1/2}$ and $[X,X]=\sum_{i=1}^\infty [X_i,X_i]$, where $[X_i,X_i]$ is the quadratic variation process (see [10], for example) of martingale $X_i$, $i\ge 1$. The processes $|Y|$ and $[Y,Y]$ are defined similarly. Recall $Y$ is differentially subordinate to $X$ if $[X,X]_t-[Y,Y]_t$ is a nonnegative and nondecreasing function of $t$. We now study some basic properties of differential subordination.

Since $[X,X]_0=|X_0|^2$, the assumption that $Y$ is differentially subordinate to $X$ implies $|Y_0|\le|X_0|$. Similarly, $\Delta[X,X]_t=|\Delta X_t|^2$ implies that $|\Delta Y_t|\le|\Delta X_t|$ for all $t>0$, where $\Delta X_t=X_t-X_{t-}$. It is well known (see [10] or [12], for example) that for every semimartingale $X$, there exists a unique continuous local martingale part $X^c$ of $X$ such that for every $t\ge 0$,

$$[X,X]_t=|X_0|^2+[X^c,X^c]_t+\sum_{0<s\le t}|\Delta X_s|^2.$$

In fact, $[X^c,X^c]_t=[X,X]_t^c$, the pathwise continuous part of $[X,X]_t$. Therefore, $Y$ is differentially subordinate to $X$ if and only if $[X^c,X^c]_t-[Y^c,Y^c]_t$ and

$\sum_{0<s\le t}|\Delta X_s|^2 - \sum_{0<s\le t}|\Delta Y_s|^2$ are nonnegative and nondecreasing functions of $t$. The latter is equivalent to $|\Delta Y_t| \le |\Delta X_t|$ for all $t > 0$.

We summarize the above statement as a lemma:

LEMMA 1. *If $X$ and $Y$ are semimartingales, then $Y$ is differentially subordinate to $X$ if and only if $[X^c, X^c]_t - [Y^c, Y^c]_t$ is a nonnegative and nondecreasing function of $t$, the inequality $|\Delta Y_t| \le |\Delta X_t|$ holds for all $t > 0$ and $|Y_0| \le |X_0|$.*

We now state Theorem 1 again.

THEOREM 1. *Let $1 < p < \infty$. If the local martingale $X$ is differentially subordinate to the local martingale $Y$, then*

$$|||Y|||_p \le (p^* - 1)|||X|||_p. \tag{1.1}$$

*The constant $p^* - 1$ is best possible. Strict inequality holds if $0 < |||X|||_p < \infty$ and $p \ne 2$.*

Notice if $|||X|||_p < \infty$, then $X$ is a martingale and $|||X|||_p = ||X||_p$.

We consider $\mathbb{H} = \mathbb{R}^d$ for some integer $d$ first. In this case, the proof is an application of Itô's formula and Burkholder's special function defined in [7], page 76.

PROOF OF THEOREM 1 WHEN THE DIMENSION OF $\mathbb{H}$ IS FINITE. Let $x' = x/|x|$ when $|x| \ne 0$. It is enough to consider the case $|||X|||_p < \infty$. We start the proof by giving some standard reductions. First, by adding one more dimension to $\mathbb{H}$ if necessary, we can find a point $a$ in $\mathbb{H}$ which is orthogonal to the range of $X$ and $Y$. Let $\hat{X} = X + a$ and $\hat{Y} = Y + a$. Then $|\hat{X}_t|$ and $|\hat{Y}_t|$ are bounded away from 0 for all $t \ge 0$. If we can show (1.1) for $\hat{X}$ and $\hat{Y}$, then by letting $a \to 0$, we have (1.1) for $X$ and $Y$ as well. Thus we may assume at the beginning that $|X_t|$ and $|Y_t|$ are positive for all $t$. Second, since stochastic integrals preserve the local martingale property, for the function $U$ defined below we can find a sequence of nondecreasing bounded stopping times $T = \{T_n\}_{n\ge 1}$ going to $\infty$ almost surely such that for each $n$, $(U_x(X_-, Y_-)\cdot X)^{T_n}$ and $(U_y(X_-, Y_-)\cdot Y)^{T_n}$ are martingales, where $X_- = \{X_{t-}\}_{t>0}$ and $Y_- = \{Y_{t-}\}_{t>0}$. Because for any bounded stopping time and continuous function $g$, $[X^T, Y^T] = [X, Y]^T$ and $(g(X_-, Y_-)\cdot X)^T = g(X_-^T, Y_-^T)\cdot X^T$, we may assume at the beginning that $X$, $Y$, $U_x(X_-, Y_-)\cdot X$, $U_y(X_-, Y_-)\cdot Y$ are martingales, $Y$ is differentially subordinate to $X$ and $|X|$, $|Y|$ are bounded away from 0. We need only to prove for any $t \ge 0$,

$$||Y_t||_p \le (p^* - 1)||X_t||_p. \tag{1.2}$$

The inequality (1.1) then follows by Fatou's lemma. Third, by Theorem B, the constant $p^* - 1$ is sharp if we can prove (1.2), since it is already sharp in the $\mathbb{H} = \mathbb{R}$ case.

Recall Burkholder's special function $U(x,y)$ defined on $\mathbb{H}\times\mathbb{H}$ to $\mathbb{R}$:

$$U(x,y) = p(1-1/p^*)^{p-1}(|y|-(p^*-1)|x|)(|x|+|y|)^{p-1}.$$

Burkholder shows that function $U$ satisfies the following properties [7]:

(a) $|y|^p - (p^*-1)^p|x|^p \le U(x,y)$ and strict inequality holds if $|y| \neq (p^*-1)|x|$ and $p \neq 2$.

(b) For all $x, y, h, k \in \mathbb{H}$ such that $|k| \le |h|$ and $|x||y| \neq 0$,

$$U(x+h, y+k) \le U(x,y) + \langle U_x(x,y), h\rangle + \langle U_y(x,y), k\rangle.$$

(c) For all $x, y, h, k \in \mathbb{H}$ with $|x||y| \neq 0$,

$$\begin{aligned}&\langle hU_{xx}(x,y), h\rangle + 2\langle hU_{xy}(x,y), k\rangle + \langle kU_{yy}(x,y), k\rangle \\ &\quad = -c_p(A+B+C+D),\end{aligned}$$

where

$$c_p = p(1-1/p^*)^{p-1}$$

and $A$, $B$, $C$, $D$ are defined by

$$\begin{aligned}A &= p^*(p-1)(|h|^2-|k|^2)(|x|+|y|)^{p-2},\\ B &= (p^*-p)(|h|^2-\langle h, x'\rangle^2)|x|^{-1}(|x|+|y|)^{p-1},\\ C &= \{(p-1)p^*-p\}(|k|^2-\langle k, y'\rangle^2)|y|^{-1}(|x|+|y|)^{p-1},\\ D &= (p-1)[(p^*-p)|y| + \{(p-1)p^*-p\}|x|]\\ &\quad\times(\langle h, x'\rangle + \langle k, y'\rangle)^2(|x|+|y|)^{p-3}.\end{aligned}$$

(d) $U(x,y) \le 0$ if $|y| \le |x|$,

where $U_x$ and $U_y$ are first-order partial derivative vectors, $U_{xx}$ stands for the second order partial derivative matrix $\{U_{x_ix_j}\}_{1\le i,j\le d}$ and $U_{xy}, U_{yy}$ are defined similarly.

Since (1.2) is equivalent to $E(|Y_t|^p - (p^*-1)^p|X_t|^p) \le 0$, by (a), it suffices to show $EU(X_t, Y_t) \le 0$. Because the function $U$ has continuous second-order derivatives when $|x| > 0, |y| > 0$ and by assumption, $|X| > 0$, $|Y| > 0$, we may apply Itô's formula to $U(X,Y)$. We have

$$\begin{aligned}U(X_t, Y_t) = U(X_0, Y_0) &+ \int_{0+}^t \langle U_x(X_{s-}, Y_{s-}), dX_s\rangle\\ &+ \int_{0+}^t \langle U_y(X_{s-}, Y_{s-}), dY_s\rangle + I_1/2 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 = \int_{0+}^t \sum_{i=1}^d\sum_{j=1}^d &(U_{x_ix_j}(X_{s-}, Y_{s-})d[X_i^c, X_j^c]_s\\ &+ 2U_{x_iy_j}(X_{s-}, Y_{s-})d[X_i^c, Y_j^c]_s + U_{y_iy_j}(X_{s-}, Y_{s-})d[Y_i^c, Y_j^c]_s),\end{aligned}$$

$$I_2 = \sum_{0<s\le t} (U(X_s,Y_s) - U(X_{s-},Y_{s-}) \\ - \langle U_x(X_{s-},Y_{s-}),\Delta X_s\rangle - \langle U_y(X_{s-},Y_{s-}),\Delta Y_s\rangle).$$

Here we have used the fact that $[X,Y]^c = [X^c,Y^c]$ for any pair of semimartingales $X$ and $Y$.

Let $x = X_{s-}, y = Y_{s-}, h = \Delta X_s$ and $k = \Delta Y_s$. It follows from (b) that each term in $I_2$ is nonpositive since $|k|^2 \le |h|^2$ by Lemma 1. By (d) and the martingale property, it suffices to prove $I_1 \le 0$.

A easy calculation shows

$$I_1 = -c_p(E+F+G+H),$$

where

$$E = p^*(p-1)\int_{0+}^t (|X_{s-}|+|Y_{s-}|)^{p-2}\, d([X^c,X^c]_s - [Y^c,Y^c]_s),$$

$$F = (p^*-p)\int_{0+}^t |X_{s-}|^{-1}(|X_{s-}|+|Y_{s-}|)^{p-1}\, d([X^c,X^c]_s - [M,M]_s),$$

$$G = \{(p-1)p^*-p\}\int_{0+}^t |Y_{s-}|^{-1}(|X_{s-}|+|Y_{s-}|)^{p-1}\, d([Y^c,Y^c]_s - [N,N]_s),$$

$$H = (p-1)\int_{0+}^t [(p^*-p)|Y_{s-}| + \{(p-1)p^*-p\}|X_{s-}|] \\ \times (|X_{s-}|+|Y_{s-}|)^{p-3}\, d[W,W]_s$$

and

$$M_t = \int_0^t \sum_{i=1}^d X_{i,s-}/|X_{s-}|\, dX_{i,s}^c = \int_0^t \langle X_{s-}/|X_{s-}|, dX_s^c\rangle,$$

$$N_t = \int_0^t \sum_{i=1}^d Y_{i,s-}/|Y_{s-}|\, dY_{i,s}^c = \int_0^t \langle Y_{s-}/|Y_{s-}|, dY_s^c\rangle,$$

$$W_t = \int_0^t \sum_{i=1}^d \left(X_{i,s-}/|X_{s-}|\, dX_{i,s}^c + Y_{i,s-}/|Y_{s-}|\, dY_{i,s}^c\right) \\ = M_t + N_t.$$

Lemma 2 below shows that $[X^c,X^c]_t - [M,M]_t$ and $[Y^c,Y^c]_t - [N,N]_t$ are nondecreasing functions of $t$ or, in other words, $M$ and $N$ are differentially subordinate to $X^c$ and $Y^c$, respectively. Therefore $F$ and $G$ are nonnegative. Since $[W,W]_t$ is a nondecreasing function of $t$, $H$ is nonnegative too. By assumption, $E$ is also nonnegative. Therefore, $I_1 \le 0$ and

$$EU(X_t,Y_t) \le 0.$$

This proves (1.2) and hence (1.1). Note that by replacing time 0 above by time $s \le t$, we have

$$E([U(X_t,Y_t) - U(X_s,Y_s)] \mid \mathscr{F}_s) \le 0.$$

This shows that $U(X_t,Y_t)$ is a supermartingale.

*Strictness.* Our proof is a modification of the strictness proof given in Theorem 3.1 of [9] by Burkholder. Moreover, the proof works for the general Hilbert space case provided $U(X_t, Y_t)$ is a supermartingale.

Assume $p \neq 2$ and $0 < |||X|||_p < \infty$. Therefore, $||X||_p = |||X|||_p < \infty$ and $||Y||_p = |||Y|||_p < \infty$.

Let $V(x,y) = |y|^p - (p^* - 1)^p |x|^p$, $U_t = U(X_t, Y_t)$, $u(t) = EU_t$ and $v(t) = EV(X_t, Y_t)$. Then the above argument shows $v(t) \leq u(t)$. By the martingale convergence theorem, both $X$ and $Y$ have limits $X_\infty$ and $Y_\infty$ at $\infty$, respectively. Using Doob's maximal function inequality, we see that $v$ and $u$ are r.c.l.l. and have limits at infinity. Thus, strictness follows if we can show $v(\infty) < 0$.

If $E|X_0|^p \neq 0$, then $u(0) \leq (1 - 1/p^*)^p (1 - (p^* - 1)^p) E|X_0|^p < 0$. Thus $v(t) \leq u(t) \leq u(0) < 0$ for any $t \geq 0$. Therefore, without loss of generality, we may assume that $X_0 = Y_0 = 0$ and $||X_t||_p > 0$ for all $t > 0$.

It is enough to show $P(|Y_\infty| = (p^* - 1)|X_\infty|) < 1$ since otherwise by (a), $v(\infty) < u(\infty) \leq 0$.

Suppose $|Y_\infty| = (p^* - 1)|X_\infty|$ almost surely. Then $U_\infty = 0$. Since $\{U_t,\ t \geq 0\}$ is a uniformly integrable supermartingale starting from 0, this implies $P(U_t = 0, \text{ for all } t \geq 0) = 1$. Therefore, $|Y| = (p^* - 1)|X|$.

Let $T_n = \inf\{t > 0,\ |X_t| + |Y_t| \geq n\}$. Then $|X_{T_n-}| \leq n$ and $|Y_{T_n-}| \leq n$. Moreover, $(X_- \cdot X)^{T_n}$ and $(Y_- \cdot Y)^{T_n}$ are local martingales. In fact, Lemma 2 below shows that $\sum_{i=1}^d (X_{i-} \cdot X_i)^{T_n}$ is differentially subordinate to $n(X_1, \ldots, X_d)^{T_n} = nX^{T_n}$. Therefore, applying (1.1) to them, we see that $\sum_{i \geq 1} (X_{i-} \cdot X_i)^{T_n}$ converges in $L^p$ to $(X_- \cdot X)^{T_n}$. Thus, the latter is a local martingale. The same is true for $(Y_- \cdot Y)^{T_n}$. Using stopping times, we may further assume that they are martingales. By the definition of quadratic variation, we have, for any $t > 0$,

$$\begin{aligned} 0 &= |Y_{T_n \wedge t}|^2 - (p^* - 1)^2 |X_{T_n \wedge t}|^2 \\ &= 2((Y_- \cdot Y)_{T_n \wedge t} - (p^* - 1)^2 (X_- \cdot X)_{T_n \wedge t}) \\ &\quad + ([Y,Y]_{T_n \wedge t} - (p^* - 1)^2 [X,X]_{T_n \wedge t}) \\ &= 2J_1 + J_2. \end{aligned}$$

Observe that $J_1$ is a martingale, so $EJ_1 = 0$, and that $EJ_2$ is negative unless we have $E[X,X]_{T_n \wedge t} = 0$. Taking expectation on both sides, we must have $E[X,X]_{T_n \wedge t} = 0$. Therefore, $E|X_t|^2 = 0$. This contradicts that $E|X_t|^p > 0$ for $t > 0$ as we assumed at the beginning.

This completes the proof provided we can prove the following lemma.

LEMMA 2. *Let $M = (M_1, M_2, \ldots, M_d)$ be a semimartingale with finite quadratic variation for almost sure paths and $H$ and $K = (K_1, K_2, \ldots, K_d)$ be adapted r.c.l.l. processes such that $|K_t|^2 = \sum_{i=1}^d |K_{i,t}|^2 \leq H_t$ for all $t \geq 0$. If*

$$(K_- \cdot M)_t = \int_0^t \sum_{i=1}^d K_{i,s-}\, dM_{i,s} = \int_0^t \langle K_{s-}, dM_s \rangle,$$

*then for any nonnegative measurable function $f$ on $\mathbb{R}$,*

$$\int_0^\infty f(s)\,d[K_- \cdot M, K_- \cdot M]_s \le \int_0^\infty f(s) H_{s-}\,d[M,M]_s, \tag{1.3}$$

*almost surely.*

PROOF. First observe

$$[K_- \cdot M, K_- \cdot M]_t = \int_0^t \sum_{i=1}^d \sum_{j=1}^d K_{i,s-} K_{j,s-}\,d[M_i, M_j]_s.$$

Thus

$$\int_0^\infty f(s)\,d[K_- \cdot M, K_- \cdot M]_s = \int_0^\infty \sum_{i=1}^d \sum_{j=1}^d f(s) K_{i,s-} K_{j,s-}\,d[M_i, M_j]_s.$$

When $s < t$, let $[Z,Z]_s^t = [Z,Z]_t - [Z,Z]_s$ for any semimartingale $Z$. Then by the Kunita–Watanabe inequality,

$$|f(t) K_{i,t-} K_{j,t-}|[M_i, M_j]_s^t \le f(t)|K_{i,t-}||K_{j,t-}|([M_i, M_i]_s^t)^{1/2}([M_j, M_j]_s^t)^{1/2}.$$

Consequently, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \left|\sum_{i=1}^d \sum_{j=1}^d f(t) K_{i,t-} K_{j,t-}[M_i, M_j]_s^t\right| &\le f(t)\left(\sum_{i=1}^d |K_{i,t-}|([M_i, M_i]_s^t)^{1/2}\right)^2 \\ &\le f(t)|K_{t-}|^2 [M,M]_s^t \\ &\le f(t) H_{t-}[M,M]_s^t. \end{aligned}$$

By a density argument, the above inequality implies (1.3). This completes the proof. □

It is clear from the proof that the special function $U$ having continuous second-order derivatives on the range of $X$ and $Y$ plays an important role; that is, the requirement to apply Itô's formula to $U(X,Y)$ on $\mathbb{R}^d$. Although there is Itô's formula for $\mathbb{H}$-valued processes, the situation could become tricky (see, for example, [11]). To prove Theorem 1 for infinite-dimensional Hilbert space-valued martingales and to prove other sharp inequalities obtained by Burkholder [7, 8], we need to make a little modification of the above proof and consider the case when special functions are not $C^2$ in general. To this end, we give an approximation lemma. In the following lemma and Propositions 1–4 given later, we shall consider function $f(x,y)$ defined in $\mathbb{H} \times \mathbb{H}$ which satisfies the following condition:

$$f((0, x_1, x_2, x_3, \ldots), (0, y_1, y_2, y_3, \ldots)) = f((x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots))$$

for $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ in $\mathbb{H}$.

LEMMA 3. *Let $f(x, y)$ be a continuous function defined on $\mathbb{H} \times \mathbb{H}$ such that $f$ is bounded on bounded sets, is in $C^1$ on $\mathbb{H} \times \mathbb{H} \setminus (\{|x| = 0\} \cup \{|y| = 0\})$ and is in $C^2$ on $S_i$, $i \geq 1$, where $S_i$ is a sequence of open connected sets such that the union of the closures of $S_i$ is $\mathbb{H} \times \mathbb{H}$. Suppose for each $i \geq 1$, there exists a nonnegative measurable function $c_i(x, y)$ defined on $S_i$ satisfying the following condition: for $(x, y) \in S_i$ with $|x||y| \neq 0$,*

$$\langle hf_{xx}(x, y), h\rangle + 2\langle hf_{xy}(x, y), k\rangle + \langle kf_{yy}(x, y), k\rangle \leq -c_i(x, y)(|h|^2 - |k|^2) \tag{1.4}$$

*for all $h, k \in \mathbb{H}$. Assume further that for each $n \geq 1$, there exists a function $M_n$ which is nondecreasing on $n$ and*

$$\sup c_i(x, y) \leq M_n < \infty,$$

*where the supremum is taken over all $(x, y) \in S_i$ such that $1/n^2 \leq |x|^2 + |y|^2 \leq n^2$ and then over all $i \geq 1$. Let $X$ and $Y$ be two bounded martingales with bounded quadratic variations. If $Y$ is differentially subordinate to $X$, then for any $t \geq 0$,*

$$Ef(X_t, Y_t) \leq Ef(X_0, Y_0). \tag{1.5}$$

This lemma is essential to prove the following proposition.

PROPOSITION 1. *Suppose $f$ satisfies the conditions given in Lemma 3. Assume further that $f_x$ and $f_y$ are bounded on any bounded set that does not contain 0, the origin of $\mathbb{H} \times \mathbb{H}$, and for $h, k \in \mathbb{H}$,*

$$f(x + h, y + k) - f(x, y) - \langle f_x(x, y), h\rangle - \langle f_y(x, y), k\rangle \leq 0 \tag{1.6}$$

*when $|x||y| \neq 0$ and $|k| \leq |h|$. For any local martingale $Y$ that is differentially subordinate to local martingale $X$, if for $0 \leq s \leq t$ and any sequence of nondecreasing stopping times $\{T_n\}_{n\geq 1}$ going to $\infty$,*

$$\begin{aligned} &E(f(X_t, Y_t) \mid \mathscr{F}_s) \\ &\quad \leq \limsup_{a\to 0} \limsup_{n\to\infty} E(f((a, X_{T_n\wedge t}), (a, Y_{T_n\wedge t})) \mid \mathscr{F}_s) \quad \text{a.e.} \end{aligned} \tag{A1}$$

*then*

$$E(f(X_t, Y_t) \mid \mathscr{F}_s) \leq f(X_s, Y_s) \quad \text{a.e.}$$

*for $0 \leq s \leq t$.*

The proof of Lemma 3 is based on the convolution argument and Itô's formula.

PROOF OF LEMMA 3. Since $X$ and $Y$ are bounded martingales with bounded quadratic variations, there exists $\bar{n}_1$ such that $|X_t|^2 + |Y_t|^2 + [X, X]_t + [Y, Y]_t \leq \bar{n}_1{}^2$ for all $t$ and all paths. Given $0 < a < 1/2$, choose $\bar{n} \geq \bar{n}_1$ such that $1/(\bar{n} + 2) \leq a$. Let $\bar{X}_t = (a, X_t) \in \mathbb{R} \times \mathbb{H}$ and $\bar{Y}_t = (a, Y_t) \in \mathbb{R} \times \mathbb{H}$. For $\varepsilon > 0$, let $m_{a,\varepsilon}$ be an integer such that $\sum_{i\geq m_{a,\varepsilon}}(\|[X_i, X_i]\|_\infty + \|[Y_i, Y_i]\|_\infty) \leq \varepsilon/(2M_{\bar{n}+2})$. This is possible since $\sum_{i\geq 1}(\|[X_i, X_i]\|_\infty + \|[Y_i, Y_i]\|_\infty) \leq \|[X, X]\|_\infty + \|[Y, Y]\|_\infty \leq \bar{n}_1{}^2 < \infty$.

For $m > 1$, we denote

$$\bar{X}^m = (a, X_1, \ldots, X_{m-1}, 0, \ldots), \qquad \bar{Y}^m = (a, Y_1, \ldots, Y_{m-1}, 0, \ldots),$$

where

$$X = (X_1, X_2, \ldots), \qquad Y = (Y_1, Y_2, \ldots).$$

Now let $m > m_{a,\varepsilon}$ and let $g$ be a $C^\infty$ nonnegative function on $\mathbb{R}^m \times \mathbb{R}^m$ such that $g$ has support inside the unit ball and assume

$$\int_{R^m}\int_{R^m} g(x, y)\, dx\, dy = 1.$$

Such a $g$ exists. In fact, we can choose $g$ such that it is a radial function: $g(x, y) = g(|x|^2 + |y|^2)$.

Let $l$ be a positive integer such that $1/l < a$ and $1/l \le \sqrt{2}a - 1/(\bar{n} + 2)$. Define for $x, y \in \mathbb{R}^m$,

$$f^l(x, y) = \int_{R^m}\int_{R^m} f(x - u/l, y - v/l) g(u, v)\, du\, dv.$$

In the above, we use the notation $f(x, y) = f((x, 0, \ldots), (y, 0, \ldots))$ for $x, y \in \mathbb{R}^m$.

Since $X$ and $Y$ are bounded and $f$ is continuous and bounded on bounded sets, by the dominated convergence theorem we have

$$Ef(X_t, Y_t) = \lim_{a \to 0} \lim_{m \to \infty} \lim_{l \to \infty} Ef^l(\bar{X}_t^m, \bar{Y}_t^m). \tag{1.7}$$

When $|x| \ge a$ and $|y| \ge a$, because $f$ is continuous and in $C^1$ on $\mathbb{H} \times \mathbb{H} \backslash (\{|x| = 0\} \cup \{|y| = 0\})$, using integration by parts, we get

$$f^l_{xx}(x, y) = \int_{R^m}\int_{R^m} f_{xx}(x - u/l, y - v/l) g(u, v)\, du\, dv,$$

$$f^l_{xy}(x, y) = \int_{R^m}\int_{R^m} f_{xy}(x - u/l, y - v/l) g(u, v)\, du\, dv,$$

$$f^l_{yy}(x, y) = \int_{R^m}\int_{R^m} f_{yy}(x - u/l, y - v/l) g(u, v)\, du\, dv.$$

Therefore, by (1.4), when $|x| \ge a$ and $|y| \ge a$,

$$\langle hf^l_{xx}(x, y), h\rangle + 2\langle hf^l_{xy}(x, y), k\rangle + \langle kf^l_{yy}(x, y), k\rangle \le -c(x, y)(|h|^2 - |k|^2), \tag{1.8}$$

where

$$c(x, y) = \sum_i \iint_{R_i} c_i(x - u/l, y - v/l) g(u, v)\, du\, dv$$

and $R_i = \{(u, v)\colon\ (x, y) - (u, v)/l \in S_i\}$.

Let $\{h_i^j\}$ and $\{k_i^j\}$ be two triangular sequences elements of $\mathbb{R}^m$ such that

$$\sup_j \sum_i |k_i^j|^2 < \infty \quad \text{and} \quad \sup_j \sum_i |h_i^j|^2 < \infty.$$

Using (1.8), we then have

$$\begin{aligned}\lim_{j\to\infty} \sum_i (\langle h_i^j f_{xx}^l(x,y), h_i^j\rangle + 2\langle h_i^j f_{xy}^l(x,y), k_i^j\rangle \\ + \langle k_i^j f_{yy}^l(x,y), k_i^j\rangle) \le -c(x,y) \lim_{j\to\infty} \sum_i (|h_i^j|^2 - |k_i^j|^2)\end{aligned} \tag{1.9}$$

for all $|x| \ge a$ and $|y| \ge a$.

Now let $s_0 < s_1 \le t$. Replace $x, y$ by $\bar{X}_{s-}^m, \bar{Y}_{s-}^m$ and $h_i^j, k_i^j$ by $\bar{X}_{T_{i+1}^j}^{mc} - \bar{X}_{T_i^j}^{mc}, \bar{Y}_{T_{i+1}^j}^{mc} - \bar{Y}_{T_i^j}^{mc}$, respectively, where for each $j$, $\{T_i^j\}_{1\le i\le i_j}$ is a sequence of nondecreasing finite stopping times with $T_1^j = s_0$ and $T_{i_j}^j = s_1$ such that $\lim_{j\to\infty} \max_{1\le i\le i_j-1} |T_{i+1}^j - T_i^j| = 0$. Then the above inequality (1.9) gives

$$\begin{aligned}\sum_{i=1}^m \sum_{j=1}^m (f_{x_i x_j}^l(\bar{X}_{s_0-}^m, \bar{Y}_{s_0-}^m)[\bar{X}_i^{mc}, \bar{X}_j^{mc}]_{s_0}^{s_1} + 2 f_{x_i y_j}^l(\bar{X}_{s_0-}^m, \bar{Y}_{s_0-}^m)[\bar{X}_i^{mc}, \bar{Y}_j^{mc}]_{s_0}^{s_1} \\ + f_{y_i y_j}^l(\bar{X}_{s_0-}^m, \bar{Y}_{s_0-}^m)[\bar{Y}_i^{mc}, \bar{Y}_j^{mc}]_{s_0}^{s_1}) \\ \le -c(\bar{X}_{s_0-}^m, \bar{Y}_{s_0-}^m) \sum_{i=1}^m ([\bar{X}_i^{mc}, \bar{X}_i^{mc}]_{s_0}^{s_1} - [\bar{Y}_i^{mc}, \bar{Y}_i^{mc}]_{s_0}^{s_1}).\end{aligned}$$

This implies, by differential subordination,

$$\begin{aligned}\sum_{i=1}^m \sum_{j=1}^m \int_{0+}^t (f_{x_i x_j}^l(\bar{X}_{s-}^m, \bar{Y}_{s-}^m)\, d[\bar{X}_i^{mc}, \bar{X}_j^{mc}]_s \\ + 2 f_{x_i y_j}^l(\bar{X}_{s-}^m, \bar{Y}_{s-}^m)\, d[\bar{X}_i^{mc}, \bar{Y}_j^{cm}]_s \\ + f_{y_i y_j}^l(\bar{X}_{s-}^m, \bar{Y}_{s-}^m)\, d[\bar{Y}_i^{mc}, \bar{Y}_j^{mc}]_s) \\ \le -\sum_{i=1}^m \int_{0+}^t c(\bar{X}_{s-}^m, \bar{Y}_{s-}^m)\, d([\bar{X}_i^{mc}, \bar{X}_i^{mc}]_s - [\bar{Y}_i^{mc}, \bar{Y}_i^{mc}]_s) \\ \le M_{\bar{n}+2} \sum_{i\ge m} ([X_i^c, X_i^c]_t + [Y_i^c, Y_i^c]_t)\end{aligned} \tag{1.10}$$

since $0 \le c(\bar{X}_{s-}^m, \bar{Y}_{s-}^m) \le M_{\bar{n}+2}$ because $\sqrt{2}a \le \sqrt{|\bar{X}_{s-}^m|^2 + |\bar{Y}_{s-}^m|^2} \le \bar{n}_1 + 1$.

For $x, y, h, k \in \mathbb{R}^m$, where $|x| \ge a$ and $|y| \ge a$, and $t \in [0,1]$, define

$$G(t) = f^l(x + th, y + tk).$$

Then by the mean value theorem and (1.8), there exists $t_0 \in (0,1)$ such that for $x_0 = x + t_0 h$ and $y_0 = y + t_0 k$,

$$G(1) - G(0) - G'(0) = G''(t_0) \le -c(x_0, y_0)(|h|^2 - |k|^2).$$

Let $x = \bar{X}^m_{s-}$, $y = \bar{Y}^m_{s-}$, $h = \Delta\bar{X}^m_s$ and $k = \Delta\bar{Y}^m_s$. Then we have

$$
\begin{aligned}
&f^l(\bar{X}^m_s, \bar{Y}^m_s) - f^l(\bar{X}^m_{s-}, \bar{Y}^m_{s-}) \\
&\qquad - \langle f^l_x(\bar{X}^m_{s-}, \bar{Y}^m_{s-}), \Delta\bar{X}^m_s\rangle - \langle f^l_y(\bar{X}^m_{s-}, \bar{Y}^m_{s-}), \Delta\bar{Y}^m_s\rangle \\
&\qquad \le -c(\bar{X}^m_{s-} + t_0\Delta\bar{X}^m_s, \bar{Y}^m_{s-} + t_0\Delta\bar{Y}^m_s)(|\Delta\bar{X}^m_s|^2 - |\Delta\bar{Y}^m_s|^2) \\
&\qquad \le M_{\bar{n}+2}\sum_{i\ge m}(\Delta[X_i, X_i]_s + \Delta[Y_i, Y_i]_s).
\end{aligned}
\tag{1.11}
$$

The last inequality is from differential subordination and $c(\bar{X}^m_{s-} + t_0\,\Delta\bar{X}^m_s, \bar{Y}^m_{s-} + t_0\,\Delta\bar{Y}^m_s) \le M_{\bar{n}+2}$ since $\sqrt{2}a \le \sqrt{|\bar{X}^m_{s-} + t_0\,\Delta\bar{X}^m_s|^2 + |\bar{Y}^m_{s-} + t_0\,\Delta\bar{Y}^m_s|^2} \le \bar{n}_1 + 1$.

Applying Itô's formula to $f^l(\bar{X}^m, \bar{Y}^m)$, by (1.10) and (1.11), we have

$$
Ef^l(\bar{X}^m_t, \bar{Y}^m_t) - Ef^l(\bar{X}^m_0, \bar{Y}^m_0) \le M_{\bar{n}+2}E\sum_{i\ge m}([X_i, X_i]_t + [Y_i, Y_i]_t) \le \varepsilon.
$$

Thus, by (1.7), we get $Ef(X_t, Y_t) \le Ef(X_0, Y_0) + \varepsilon$. This completes the proof by letting $\varepsilon \to 0$. □

PROOF OF PROPOSITION 1. It is sufficient to prove $E(f(X_t, Y_t) \mid \mathscr{F}_0) \le f(X_0, Y_0)$. For notational simplicity, we use $E$ to denote $E(\cdot \mid \mathscr{F}_0)$. Let $T_n = \inf\{t > 0,\ |X|^2_t + |Y|^2_t + [X, X]_t + [Y, Y]_t \le n\}$. Then $T_n$ is a stopping time and $X^{T_n}$ and $Y^{T_n}$ are local martingales. Moreover, $X^{T_n-}$, $Y^{T_n-}$, $[X, X]^{T_n-}$ and $[Y, Y]^{T_n-}$ are bounded. Checking the proof of Lemma 3 carefully, we have that given any $n$, for $\varepsilon$, $a$, $l$ and $m$ defined there, the following holds: For each $m$, there exists a sequence of finite stopping times $\{T_{n,i}\}_{i\ge1} = \{T^m_{n,i}\}_{i\ge1}$ such that $T_{n,i} \uparrow T_n$ as $i \uparrow \infty$, $(f^l_x(\bar{X}^m_-, \bar{Y}^m_-)\cdot\bar{X}^m)^{T_{n,i}}$, $(f^l_y(\bar{X}^m_-, \bar{Y}^m_-)\cdot\bar{Y}^m)^{T_{n,i}}$ are martingales and

$$
\begin{aligned}
&f^l(\bar{X}^m_{t\wedge T_{n,i}-}, \bar{Y}^m_{t\wedge T_{n,i}-}) \\
&\quad \le f^l(\bar{X}^m_0, \bar{Y}^m_0) + M_{\bar{n}+2}\sum_{i\ge m}([X_i, X_i]_{t\wedge T_n-} + [Y_i, Y_i]_{t\wedge T_n-}) \\
&\qquad + \int_{0+}^{t\wedge T_{n,i}-}\langle f^l_x(\bar{X}^m_{s-}, \bar{Y}^m_{s-}), d\bar{X}^m_s\rangle + \int_{0+}^{t\wedge T_{n,i}-}\langle f^l_y(\bar{X}^m_{s-}, \bar{Y}^m_{s-}), d\bar{Y}^m_s\rangle \\
&\quad \le f^l(\bar{X}^m_0, \bar{Y}^m_0) + \varepsilon + \int_{0+}^{t\wedge T_{n,i}}\langle f^l_x(\bar{X}^m_{s-}, \bar{Y}^m_{s-}), d\bar{X}^m_s\rangle \\
&\qquad + \int_{0+}^{t\wedge T_{n,i}}\langle f^l_y(\bar{X}^m_{s-}, \bar{Y}^m_{s-}), d\bar{Y}^m_s\rangle - \langle f^l_x(\bar{X}^m_{t\wedge T_{n,i}-}, \bar{Y}^m_{t\wedge T_{n,i}-}), \Delta\bar{X}^m_{t\wedge T_{n,i}}\rangle \\
&\qquad - \langle f^l_y(\bar{X}^m_{t\wedge T_{n,i}-}, \bar{Y}^m_{t\wedge T_{n,i}-}), \Delta\bar{Y}^m_{t\wedge T_{n,i}}\rangle.
\end{aligned}
$$

Taking the conditional expectation of the above and letting $i \to \infty$, we have, by the dominated convergence theorem,

$$
\begin{aligned}
Ef^l(\bar{X}^m_{t\wedge T_n-}, \bar{Y}^m_{t\wedge T_n-}) \le f^l(\bar{X}^m_0, \bar{Y}^m_0) &- E\langle f^l_x(\bar{X}^m_{t\wedge T_n-}, \bar{Y}^m_{t\wedge T_n-}), \Delta\bar{X}^m_{t\wedge T_n}\rangle \\
&- E\langle f^l_y(\bar{X}^m_{t\wedge T_n-}, \bar{Y}^m_{t\wedge T_n-}), \Delta\bar{Y}^m_{t\wedge T_n}\rangle + \varepsilon.
\end{aligned}
$$

Let $l \to \infty$, then $m \to \infty$ and then $\varepsilon \to 0$. By the dominated convergence theorem again, we have

$$Ef(\bar{X}_{t\wedge T_n-}, \bar{Y}_{t\wedge T_n-}) + E\langle f_x(\bar{X}_{t\wedge T_n-}, \bar{Y}_{t\wedge T_n-}), \Delta \bar{X}_{t\wedge T_n}\rangle$$
$$+ E\langle f_y(\bar{X}_{t\wedge T_n-}, \bar{Y}_{t\wedge T_n-}), \Delta \bar{Y}_{t\wedge T_n}\rangle \le f(\bar{X}_0, \bar{Y}_0).$$

From (1.6) and differential subordination,

$$f(\bar{X}_{t\wedge T_n}, \bar{Y}_{t\wedge T_n}) - f(\bar{X}_{t\wedge T_n-}, \bar{Y}_{t\wedge T_n-}) - \langle f_x(\bar{X}_{t\wedge T_n-}, \bar{Y}_{t\wedge T_n-}), \Delta \bar{X}_{t\wedge T_n}\rangle$$
$$- \langle f_y(\bar{X}_{t\wedge T_n-}, \bar{Y}_{t\wedge T_n-}), \Delta \bar{Y}_{t\wedge T_n}\rangle \le 0.$$

Therefore, combining the above two inequalities, we have

$$Ef(\bar{X}_{t\wedge T_n}, \bar{Y}_{t\wedge T_n}) \le f(\bar{X}_0, \bar{Y}_0). \tag{1.12}$$

Next letting $n \to \infty$ and $a \to 0$, condition (A1) and continuity of $f$ imply

$$Ef(X_t, Y_t) \le f(X_0, Y_0).$$

This completes the proof. □

Now we return to the proof of Theorem 1 when $\mathbb{H}$ is infinite dimensional. Note that (1.12) implies that

$$||Y_{t\wedge T_n}||_p \le (p^* - 1)||X_{t\wedge T_n}||_p \le (p^* - 1)||X_t||_p.$$

By Fatou's lemma, we have

$$||Y_t||_p \le (p^* - 1)||X_t||_p.$$

This proves (1.1). For the strictness, by Proposition 1, we need to show $U$ satisfies condition (A1). However, from the above inequality, Hölder's inequality and Doob's maximal function inequality, we have

$$\lim_{a\to 0}\lim_{n\to\infty} E|U(X_t, Y_t) - U((a, X_{T_n\wedge t}), (a, Y_{T_n\wedge t}))| = 0.$$

This implies (A1). The proof of Theorem 1 is then complete. □

Go back to Theorem C. Now we have the following corollary.

COROLLARY 1. *Under the condition of Theorem C,*

$$\left\| \left( \sum_{i=1}^{\infty} Y_i^2 \right)^{1/2} \right\|_p \le (p^* - 1)||\mathbf{A}||\ ||X||_p.$$

*The constant $p^* - 1$ is best possible. Strict inequality holds if $p \neq 2$ and $0 < ||X||_p < \infty$.*

REMARK 1. Let $S$ be a closed set satisfying the following: (a) If $(x, y) \in S$ and $|x| = |\bar{x}|$, $|y| = |\bar{y}|$, then $(\bar{x}, \bar{y}) \in S$ and (b) for any $0 < r < 1$, there exists $d_r > 0$ such that $rS + d_r O \subset S^\circ$, where $O$ is the open unit ball and $S^\circ$ is the interior of $S$. If $S$ contains the range of the martingale $(X, Y)$, replacing $\mathbb{H} \times \mathbb{H}$ by $S$ and condition (A1) by

$$\tag{A2} \begin{aligned} &E(f(X_t, Y_t) \mid \mathscr{F}_s) \\ &\quad \le \limsup_{a\to 0} \limsup_{r\uparrow 1} \limsup_{n\to\infty} E(f(a, rX_{T_n\wedge t}), (a, rY_{T_n\wedge t})) \mid \mathscr{F}_s) \quad \text{a.e.} \end{aligned}$$

for any sequence of nondecreasing stopping times $\{T_n\}_{n\ge 1}$ going to $\infty$ and $0 \le s \le t$ in Proposition 1, then the same conclusion holds. The proof is the same except for one change in the argument: replace $\bar{X}^m$ by $(a, rX_1, \dots, rX_{m-1}, 0, \dots)$ and $\bar{Y}^m$ accordingly, where $a < d_r/2$.

REMARK 2. When function $f$ is nonnegative, condition (A1) follows from Fatou's lemma. It is easy to verify that for all the special functions used in the proofs of norm inequalities in [7, 8], conditions required in Proposition 1 or its modification given in Remark 1 above are met. Therefore, we can generalize them to the continuous-time differential subordination setting. For weak type inequalities (Theorems 3.5, 8.1 and 9.1 in [8]), the special function found there by Burkholder is not $C^1$, but is piecewise $C^1$. Therefore, Proposition 1 is not applicable in those situations. We give the following variation of Proposition 1. As before, for $S \subset \mathbb{H} \times \mathbb{H}$, denote $S^\circ$ to be interior of $S$ and $O$ the open unit ball of $\mathbb{H} \times \mathbb{H}$.

PROPOSITION 2. *Let $f$ be a function defined on $\mathbb{H}\times\mathbb{H}$ such that $f$ is bounded on bounded sets, is in $C^1$ on $S \setminus (\{|x| = 0\} \cup \{|y| = 0\})$, is in $C^2$ in $S_i$, where the union of the closures of $S_i$ is $S$, and $S$ is a bounded closed set of $\mathbb{H} \times \mathbb{H}$ satisfying* (a) *$(x, y) \in S$, $|x| = |\bar{x}|$, $|y| = |\bar{y}|$, then $(\bar{x}, \bar{y}) \in S$, and* (b) *for each $0 < r < 1$, there exists $d_r > 0$ such that $rS + d_r O \subset S^\circ$. Assume for $(x, y) \in S_i$, $|x||y| \neq 0$, $h, k \in \mathbb{H}$,*

$$\tag{1.13} \begin{aligned} &\langle hf_{xx}(x, y), h\rangle + 2\langle hf_{xy}(x, y), k\rangle + \langle kf_{yy}(x, y), k\rangle \\ &\quad \le -c_i(x, y)(|h|^2 - |k|^2), \end{aligned}$$

*where $c_i(x, y)$ is a measurable function nonnegative on $S_i$ satisfying for each $0 < r < 1$,*

$$\sup_i \sup_{\substack{(x,y)\in S_i\cap rS \\ |x||y|\neq 0}} c_i(x, y) \le M_r$$

*for some nondecreasing sequence $M_r$. Assume further*

$$\tag{1.14} f(x + h, y + k) - f(x, y) - \langle f_x(x, y), h\rangle - \langle f_y(x, y), k\rangle \le 0$$

*when $(x, y) \in S$, $|x||y| \neq 0$ and $h, k \in \mathbb{H}$. Let $X$ and $Y$ be two local martingales such that $Y$ is differentially subordinate to $X$. If for any sequence of* *nondecreasing stopping times* $\{T_n\}_{n\ge1}$ *going to* $\infty$,

$$\text{(A3)}\qquad \begin{aligned} &Ef(X_T, Y_T)\\ &\quad \le \limsup_{a\to0}\limsup_{r\uparrow1}\limsup_{n\to\infty} Ef((a, rX_{T_n\wedge T}), (a, rY_{T_n\wedge T}))\quad a.e., \end{aligned}$$

*then*

$$E(f(X_T, Y_T)\mid \mathscr{F}_0) \le f(X_0, Y_0)\quad a.e.,$$

*where* $T = \inf\{t\ge 0\colon\ (X_t, Y_t)\notin S\}$.

PROOF. By Remark 1 and the arguments given in Lemma 3 as well as Proposition 1, we may assume that $\mathbb{H}$ is finite dimensional, $f$ is of $C^2$ in $S^\circ$ and there exists a nonnegative continuous function $c$ such that

$$\langle hf_{xx}(x, y), h\rangle + 2\langle hf_{xy}(x, y), k\rangle + \langle kf_{yy}(x, y), k\rangle \le -c(x, y)(|h|^2 - |k|^2),$$

for $(x, y)\in S$. This implies, by differential subordination,

$$\text{(1.15)}\qquad \begin{aligned} \int_{0+}^{T-} \sum_{i,j}(&f_{x_ix_j}(X_{s-}, Y_{s-})\, d[X_i^c, X_j^c]_s\\ &+ 2f_{x_iy_j}(X_{s-}, Y_{s-})\, d[X_i^c, Y_j^c]_s\\ &+ f_{y_iy_j}(X_{s-}, Y_{s-})\, d[Y_i^c, Y_j^c]_s)\\ &\le -\int_{0+}^{T-} c(X_s, Y_s)\, d([X^c, X^c]_s - [Y^c, Y^c]_s)\\ &\le 0, \end{aligned}$$

since the range of $(X^{T-}, Y^{T-})$ is inside $S$. Similarly, by differential subordination and (1.14),

$$\text{(1.16)}\qquad \begin{aligned} f(X_T, Y_T) &\le f(X_{T-}, Y_{T-}) + \langle f_x(X_{T-}, Y_{T-}), \Delta X_T\rangle\\ &\qquad + \langle f_y(X_{T-}, Y_{T-}), \Delta Y_T\rangle\\ &= f(X_{T-}, Y_{T-}) + \int_{T-}^{T}\langle f_x(X_{s-}, Y_{s-}), dX_s\rangle\\ &\qquad + \int_{T-}^{T}\langle f_y(X_{s-}, Y_{s-}), dY_s\rangle. \end{aligned}$$

Applying Itô's formula to $f(X_{T-}, Y_{T-})$ and using (1.16), we have

$$\text{(1.17)}\qquad f(X_T, Y_T) \le f(X_0, Y_0) + I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{0+}^{T} \langle f_x(X_{s-}, Y_{s-}), dX_s\rangle + \int_{0+}^{T} \langle f_y(X_{s-}, Y_{s-}), dY_s\rangle,$$

$$2I_2 = \int_{0+}^{T-} \sum_{i,j} (f_{x_ix_j}(X_{s-}, Y_{s-})\, d[X_i^c, X_j^c]_s + 2f_{x_iy_j}(X_{s-}, Y_{s-})\, d[X_i^c, Y_j^c]_s$$
$$+ f_{y_iy_j}(X_{s-}, Y_{s-})\, d[Y_i^c, Y_j^c]_s),$$

$$I_3 = \sum_{0<s<T} (f(X_s, Y_s) - f(X_{s-}, Y_{s-}) - \langle f_x(X_{s-}, Y_{s-}), \Delta X_s\rangle$$
$$- \langle f_y(X_{s-}, Y_{s-}), \Delta Y_s\rangle).$$

Without loss of generality, $I_1$ is a martingale, so $E(I_1 \mid \mathscr{F}_0) = 0$. By (1.15), $I_2 \leq 0$. Finally, by (1.14) and differential subordination, each term of $I_3$ is nonpositive. Consequently, $E(f(X_T, Y_T) \mid \mathscr{F}_0) \leq f(X_0, Y_0)$. This finishes the proof. □

Checking the proof of Theorem 8.1 of [8], we see that Proposition 2 can be applied to $S_1 = D_0$, $S_2 = D_1$, $S_3 = D_2$ and $S_4 = D_3$, where $D_0, \ldots, D_3$ are defined there. We then have the following corollary.

COROLLARY 2. *Let $X$ and $Y$ be $\mathbb{H}$-valued local martingales such that $Y$ is differentially subordinate to $X$ and $||X||_\infty \leq 1$. Then for $\lambda > 2$,*

$$P\Big\{\sup_t(|Y|_t^2 - |X|_t^2) \geq \lambda^2 - 1\Big\} \leq e^2/4e^{-\lambda}$$

*and the constant $e^2/4$ is best possible.*

Since

$$P\{\sup |Y|_t \geq \lambda\} \leq P\Big\{\sup_t(|Y|_t^2 - |X|_t^2) \geq \lambda^2 - 1\Big\},$$

Corollary 2 shows, under the same condition,

$$P\{\sup |Y|_t \geq \lambda\} \leq e^{2-\lambda}/4.$$

This will lead to the proof of the analogue of Theorem 8.1 of [8] for the continuous-time setting.

The example given in the Theorem 8.1 of [8] shows the constant in Corollary 2 is best possible.

The proof of Theorem 9.1 is similar. This time, we need the following proposition.

PROPOSITION 3. *Let $f$ be a function defined on $\mathbb{H}\times\mathbb{R}$ such that $f$ is bounded on bounded set, of $C^1$ on $S \setminus \{|x| = 0\}$ and bounded on bounded set, of $C^2$ in $S_i$, where the union of the closure of $S_i$ is $S$, and $S$ is a bounded closed set of $\mathbb{R} \times \mathbb{H}$ satisfying* (a) *$(x, y) \in S, |x| = |\bar{x}|$, then $(\bar{x}, y) \in S$, and* (b) *for each*

$0 < r < 1$, *there exists* $d_r > 0$ *such that* $rS + d_r O \subset S^\circ$. *Assume for* $(x, y) \in S_i$, $|x| \neq 0$, $h \in \mathbb{H}$, $k \in \mathbb{R}$,

$$\langle hf_{xx}(x, y), h\rangle + 2\langle f_{xy}(x, y), h\rangle k + k^2 f_{yy}(x, y) \leq -c_i(x, y)(|h|^2 - k^2),$$

*where* $c_i(x, y)$ *is a measurable function nonnegative on* $S_i$ *satisfying for each* $0 < r < 1$,

$$\sup_i \sup_{\substack{(x,y)\in S_i \cap rS \\ |x|\neq 0}} c_i(x, y) \leq M_r$$

*for some nondecreasing sequence* $M_r$. *Assume further*

$$f(x + h, y + k) - f(x, y) - \langle f_x(x, y), h\rangle - kf_y(x, y) \leq 0$$

*when* $(x, y) \in S$, $|x| \neq 0$ *and* $h \in \mathbb{H}$, $k \in \mathbb{R}$. *If* $X$ *is an* $\mathbb{H}$*-valued local martingale and* $Y$ *is a* $\mathbb{R}$*-valued local martingale such that* $Y$ *is differentially subordinate to* $X$ *and* $f$ *satisfies* (A3), *then*

$$E(f(X_T, Y_T) \mid \mathscr{F}_0) \leq f(X_0, Y_0) \quad a.e.,$$

*where* $T = \inf\{t \geq 0\colon\ (X_t, Y_t) \notin S\}$.

The proof is like that of Proposition 2. We may assume that $f$ is of $C^2$ and $\mathbb{H}$ is finite dimensional by using the argument given in the proofs of Lemma 3 and Proposition 1 with $\tilde{Y}^m$ being replaced by $Y$.

Similarly, when $X$ is real-valued and $Y$ is $\mathbb{H}$-valued, the analogue of the result of Proposition 3 holds too. We omit the details.

Back to Theorem 9.1 of [8]. Let $S_1 = D_1$ and $S_2 = D_2$, where $D_1$ and $D_2$ are defined there. We then have the following corollary.

COROLLARY 3. *Let* $X$ *and* $Y$ *be* $\mathbb{H}$*-valued and* $\mathbb{R}$*-valued local martingales, respectively, such that* $Y$ *is differentially subordinate to* $X$ *and* $\|X\|_\infty \leq 1$. *Then for* $\lambda > 1$,

$$P\Big\{\sup_t (Y_t - |X|_t) \geq \lambda - 1\Big\} \leq \begin{cases} (1 - \sqrt{\lambda - 1}/2)^2, & \text{if } 1 < \lambda \leq 2, \\ e^{2-\lambda}/4, & \text{if } \lambda > 2, \end{cases}$$

*and the bound on the right is sharp.*

This will lead to the analogue of Theorem 9.1 of [8] in the continuous parameter setting.

REMARK 3. The function used in the proof of Theorem 3.5 of [8] is so special that we can give the following theorem.

THEOREM 3. *Assume* $X$ *and* $Y$ *are local martingales such that for any stopping time* $T$, $|\Delta Y_T| \leq |\Delta X_T|$ *and*

$$E(X_{T-}^2 - Y_{T-}^2)\chi_{\{T>0\}} \geq 0,$$

*where $\chi_A$ is the indicator function of $A$. Then for any $\lambda > 0$,*

$$\lambda P\Big\{\sup_t(|X_t| + |Y_t|) \geq \lambda\Big\} \leq 2\|X\|_1.$$

*The constant 2 is best possible.*

Note if $Y$ is subordinate to $X$, then $[X,X]_{T-} - [Y,Y]_{T-} \geq X_0^2 - Y_0^2 \geq 0$ for any stopping time $T$ when $T > 0$. Consequently, the assumption of the theorem is satisfied. When $X$ and $Y$ are Brownian motion, Burkholder gave the proof of this theorem in [4].

PROOF OF THEOREM 3. It is enough to prove the inequality for $\lambda = 1$. Let $T = \inf\{t \geq 0\colon\ |X_t| + |Y_t| \geq 1\}$. Then $|X_T| + |Y_T| \geq 1$ and $|X_t| + |Y_t| < 1$ for $t < T$. Define for $x, y \in \mathbb{H}$,

$$V(x,y) = \begin{cases} -2|x|, & \text{when } |x| + |y| < 1, \\ 1 - 2|x|, & \text{when } |x| + |y| \geq 1, \end{cases}$$

and

$$U(x,y) = \begin{cases} |y|^2 - |x|^2, & \text{when } |x| + |y| < 1, \\ 1 - 2|x|, & \text{when } |x| + |y| \geq 1. \end{cases}$$

It is clear that $V(x,y) \leq U(x,y)$. Thus we need only to prove $EU(X_T, Y_T) \leq 0$.

Let $\delta_{ij}$ be 1 if $i = j$ and 0 otherwise. It is easy to check the following (see Burkholder [7]):

$$\begin{aligned} U_{x_ix_j}(x,y) &= -2\delta_{ij}, \\ U_{x_iy_j}(x,y) &= 0, \\ U_{y_iy_j}(x,y) &= 2\delta_{ij} \end{aligned} \tag{1.18}$$

for $|x| + |y| < 1$,

$$\begin{aligned} &U(x+h, y+k) - U(x,y) - \langle U_x(x,y), h\rangle - \langle U_y(x,y), k\rangle \\ &\quad = -(|h|^2 - |k|^2) \end{aligned} \tag{1.19}$$

when $|x| + |y| < 1$ and $|x+h| + |y+k| < 1$ and

$$\begin{aligned} &U(x+h, y+k) - U(x,y) - \langle U_x(x,y), h\rangle - \langle U_y(x,y), k\rangle \\ &\quad = -(|h|^2 - |k|^2) - (|y+k|^2 - (1 - |x+h|)^2) \\ &\quad \leq -(|h|^2 - |k|^2) \end{aligned} \tag{1.20}$$

when $|x| + |y| < 1$ and $|x+h| + |y+k| \geq 1$.

Therefore, using the idea in the proof of Proposition 2, we have on $\{T > 0\}$,

$$U(X_T, Y_T) = U(X_0, Y_0) + I_1 + I_2 + I_3,$$

where, in this context,

$$\begin{aligned} I_1 &= \int_{0+}^{T} \langle U_x(X_{s-}, Y_{s-}), dX_s \rangle + \int_{0+}^{T} \langle U_y(X_{s-}, Y_{s-}), dY_s \rangle, \\ I_2 &= [Y, Y]_{T-} - [X, X]_{T-}, \\ I_3 &= U(X_T, Y_T) - U(X_{T-}, Y_{T-}) - \langle U_x(X_{T-}, Y_{T-}), \Delta X_s \rangle \\ &\quad - \langle U_y(X_{T-}, Y_{T-}), \Delta Y_T \rangle, \end{aligned}$$

and $I_1$ is a local martingale. We may assume it is a martingale without loss of generality. Otherwise, use the stopping time argument. By (1.18)–(1.20),

$$I_2 + I_3 \le [Y, Y]_{T-} - [X, X]_{T-}.$$

Since $U(X_0, Y_0) = Y_0^2 - X_0^2 \le 0$ because $|X_0| + |Y_0| < 1$, we have

$$U(X_T, Y_T) \le I_1 + ([Y, Y]_{T-} - [X, X]_{T-}) \tag{1.21}$$

on $\{T > 0\}$. On $\{T = 0\}$, because $|X_0| + |Y_0| \ge 1$ and $|X_0| \ge |Y_0|$, we have

$$U(X_T, Y_T) = U(X_0, Y_0) = 1 - 2|X_0| \le 0.$$

Together with (1.21), this gives

$$U(X_T, Y_T) \le I_1 + ([Y, Y]_{T-} - [X, X]_{T-})\chi_{\{T>0\}}.$$

Now apply expectation to the above argument and we get the result. □

REMARK 4. In most cases (see [7]–[9]) inequality (1.4) is proved along with

$$f(x+h, y+k) - f(x, y) - \langle f_x(x, y), h \rangle - \langle f_y(x, y), k \rangle \le 0$$

when $|k|^2 \le |h|^2$. For discrete martingale or stochastic integrals satisfying condition (i) or (ii) in the Introduction, however, the above condition alone does guarantee the right result. We give the following analogue of Proposition 1 for purely jump martingales and stochastic integrals:

PROPOSITION 4. *Let $f$, $f_1$ and $f_2$ be three functions defined on $\mathbb{H} \times \mathbb{H}$ such that*

$$f(x+h, y+k) - f(x, y) - \langle f_1(x, y), h \rangle - \langle f_2(x, y), k \rangle \le 0 \tag{1.22}$$

*for $|k|^2 \le |h|^2$ and $|x||y| \ne 0$. Let $X$ and $Y$ be two purely jump martingales such that $Y$ is differentially subordinate to $X$ or let $X$ and $Y$ be stochastic integrals satisfying condition* (i) *or* (ii) *in the Introduction. Then for any $t \ge 0$,*

$$Ef(X_t, Y_t) \le Ef(X_0, Y_0)$$

*provided that $f$ satisfies condition* (A1).

The proof of Proposition 3 is simple. Let $\bar{X} = (a, X)$ and $\bar{Y} = (a, Y)$ for $a \neq 0$. Differential subordination and (1.22) imply

$$f(\bar{X}_s, \bar{Y}_s) - f(\bar{X}_{s-}, \bar{Y}_{s-}) - \langle f_1(\bar{X}_{s-}, \bar{Y}_{s-}), \Delta\bar{X}_s\rangle - \langle f_2(\bar{X}_{s-}, \bar{Y}_{s-}), \Delta\bar{Y}_s\rangle \leq 0.$$

Write

$$\begin{aligned} f(\bar{X}_t, \bar{Y}_t) = f(\bar{X}_0, \bar{Y}_0) &+ \int_{0+}^{t} f_1(\bar{X}_{s-}, \bar{Y}_{s-})\, d\bar{X}_s + \int_{0+}^{t} f_2(\bar{X}_{s-}, \bar{Y}_{s-})\, d\bar{Y}_s \\ &+ \sum_{0<s\leq t} (f(\bar{X}_s, \bar{Y}_s) - f(\bar{X}_{s-}, \bar{Y}_{s-}) - \langle f_1(\bar{X}_{s-}, \bar{Y}_{s-}), \Delta\bar{X}_s\rangle \\ &- \langle f_2(\bar{X}_{s-}, \bar{Y}_{s-}), \Delta\bar{Y}_s\rangle). \end{aligned}$$

By the martingale property, stopping time argument and condition (A1), we get the result for the purely jump martingale case. Using a theorem of Bichteler [3], we can approximate the stochastic integrals with discrete-time martingales such that the differential subordination structure is also preserved. We omit the details. When $f$ is $U$ defined in Theorem 1, the above argument can be found in [7] given by Burkholder.

REMARK 5. The condition

$$\sup c_i(x, y) \leq M_n < \infty$$

in Lemma 3 and Propositions 1, 2 and 3 are only needed when the dimension of Hilbert space $\mathbb{H}$ is infinite. When $\mathbb{H}$ is finite dimensional, it is easy to verify the argument given in Lemma 3; hence Propositions 1–3 will go through if we only require nonnegativeness of $c_i(x, y)$. In fact, if there exists a sequence of integers $m_n$ going to $\infty$, such that the subordination structure is preserved by $X^{m_n}$ and $Y^{m_n}$, then Lemma 3 and Proposition 1 hold true if $c_i(x, y) \geq 0$. This is the case when $Y$ and $X$ are stochastic integrals satisfying condition (i) or (ii).

**2. Continuous-time semimartingale and strong differential subordination.** When $\mathbb{H} = \mathbb{R}$, if $X$ is a sequence of integrable functions, then Doob's decomposition implies that there exist a unique martingale $M_n$ with $M_0 = 0$ and a predictable process $A_n$ with $A_0 = 0$ such that

$$X_n = X_0 + M_n + A_n.$$

In fact, $M_n = \sum_{i=1}^{n}(d_i - E(d_i \mid \mathscr{F}_{i-1}))$ and $A_n = \sum_{i=1}^{n} E(d_i \mid \mathscr{F}_{i-1})$, where $d = \{d_n\}_{n\geq 0}$ is the difference sequence of $X$: $d_0 = X_0$, $d_n = X_n - X_{n-1}$, $n \geq 1$. For continuous-time sub- or supermartingale $X$, the Doob–Meyer decomposition assures a similar result:

$$X_t = X_0 + M_t + A_t,$$

where $M$ is a local martingale with $M_0 = 0$ and $A$ is a predictable FV process with $A_0 = 0$ defined as follows: An adapted, r.c.l.l. process $A$ is FV if almost surely the path of $A$ is of finite variation on any compact interval of $[0, \infty)$.

We denote $|A|_t$ to be the total variation of $A$ on $[0, t]$. Such a decomposition is also unique.

For general semimartingale $X$, such decomposition is possible if $A$ is not required to be predictable, see [12], for example. That is, there exist a local martingale $M = \{M_t\}_{t\geq 0}$ with $M_0 = 0$ and an FV process $A = \{A_t\}_{t\geq 0}$ with $A_0 = 0$ such that

$$X_t = X_0 + M_t + A_t. \tag{2.1}$$

The above decomposition may not be unique in general.

All the above results can easily be generalized to $\mathbb{H}$-valued semimartingales by applying the $\mathbb{R}$-valued result coordinatewise.

Burkholder [9] introduced the concept of *strong differential subordination* for a sequence of adapted integrable functions as follows. Consider two sequences of integrable functions $f = \{f_n\}_{n\geq 0}$ and $g = \{g_n\}_{n\geq 0}$. Let $d$ and $e$ be their difference sequences. Then $g$ is strongly differentially subordinate to $f$ if $|e_n| \leq |d_n|$ for all $n \geq 0$ and $|E(e_n \mid \mathscr{F}_{n-1})| \leq |E(d_n \mid \mathscr{F}_{n-1})|$ for all $n \geq 1$. The following two theorems are proved in [9]:

THEOREM D. *Let $1 < p < \infty$. If a sequence of adapted, $\mathbb{H}$-valued integrable functions $g$ is strongly differentially subordinate to a nonnegative submartingale $f$, then*

$$||g||_p \leq (p^{**} - 1)||f||_p,$$

*where $p^{**} = \max\{2p, p/(p-1)\}$. Strict inequality holds if $0 < ||f||_p < \infty$. The constant $p^{**} - 1$ is best possible.*

THEOREM E. *Let $1 < p < \infty$. If $X$ is a continuous-time nonnegative submartingale and $Y$ is the stochastic integral of $H$ with $X$ where $H$ is an adapted predictable $\mathbb{H}$-valued process bounded by 1, then*

$$||Y||_p \leq (p^{**} - 1)||X||_p.$$

*Strict inequality holds if $0 < ||X||_p < \infty$. The constant $p^{**} - 1$ is best possible.*

Theorems D and E give examples of

$$||Y||_p \leq (p^{**} - 1)||X||_p \tag{2.2}$$

when $X$ and $Y$ are semimartingales. We would like to broaden Burkholder's examples. To this end, we introduce the following definition. For two semimartingales $X$ and $Y$, we say $Y$ is *strongly differentially subordinate* to $X$ if (i) $Y$ is differentially subordinate to $X$ and (ii) there exist FV processes $A$ and $B$, where $A$ is in the decomposition (2.1) for $X$ and $B$ is similarly defined for $Y$, such that $|A|_t - |B|_t$ is a nondecreasing function of $t$. [$Y$ is *strongly subordinate* to $X$ if (i) $Y$ is subordinate to $X$ and (ii) $|A|_t \geq |B|_t$ for all $t \geq 0$.]

Since any adapted sequence of integrable functions can be thought of as a continuous-time semimartingale, our definition is consistent with that given by Burkholder in [9]. In fact, for two adapted sequences of integrable functions

$f$ and $g$, (i) is satisfied if $|e_n| \le |d_n|$ for all $n \ge 0$ since $[f,f]_n - [g,g]_n = \sum_{i=0}^n(|d_i|^2 - |e_i|^2)$. By using Doob's decomposition, we see (ii) is satisfied if for all $n \ge 1$, $|E(e_n|\mathscr{F}_{n-1})| \le |E(d_n|\mathscr{F}_{n-1})|$ since $|A|_n - |B|_n = \sum_{i=1}^n(|E(d_i|\mathscr{F}_{i-1})| - |E(e_i|\mathscr{F}_{i-1})|)$.

Similarly, $|H_t| \le 1$ assures that $Y = H \cdot X$ is strongly differentially subordinate to $X$ when $X$ is a nonnegative submartingale. To see this, note (i) is satisfied since $[X,X]_t - [Y,Y]_t = \int_0^t(1 - |H_s|^2)\,d[X,X]_s$. Let $A$ be the unique predictable FV process of $X$ in the Doob–Meyer decomposition. Then $B$ can be $H \cdot A$. Consequently, $|A|_t - |B|_t = A_t - |B|_t = \int_0^t(1 - |H_s|)\,dA_s$ is a nondecreasing function of $t$. So (ii) is satisfied.

We now prove the following extension of Theorems D and E.

THEOREM 4. *For $1 < p < \infty$, if $X$ is a nonnegative local submartingale and $Y$ is a semimartingale taking values in $\mathbb{H}$ and strongly differentially subordinate to $X$, then*

$$|||Y|||_p \le (p^{**} - 1)|||X|||_p \tag{2.3}$$

*and the constant $p^{**} - 1$ is best possible, where $p^{**} = \max\{2p, p/(p-1)\}$. The inequality*

$$||Y||_p \le (p^{**} - 1)||X||_p$$

*is strict when $0 < ||X||_p < \infty$.*

When $X$ is a nonnegative local submartingale and $p > 1$, $|||X|||_p < \infty$ implies that $X$ is a nonnegative submartingale and $||X||_p = |||X|||_p$.

The proof is again based on the special function found by Burkholder in [9] and Itô's formula. It follows the same line as that of Theorem 1.

PROOF OF THEOREM 4 WHEN THE DIMENSION OF $\mathbb{H}$ IS FINITE. Similar to the reduction given in Theorem 1, we assume without loss of generality that $X$ is a positive submartingale and $|Y|$ is a positive semimartingale. In addition, we may let $X_t = X_0 + M_t + A_t$ and $Y_t = Y_0 + N_t + B_t$, where $M$ and $N$ are martingales such that for the function $\tilde{U}$ defined below, $\tilde{U}_x(X_-, Y_-) \cdot M$ and $\tilde{U}_y(X_-, Y_-) \cdot N$ are also martingales and $A$ and $B$ are FV processes such that $A_t - |B|_t$ is a nondecreasing function of $t$. It suffices to show

$$||Y_t||_p \le (p^{**} - 1)||X_t||_p \tag{2.4}$$

for any $t > 0$.

Let $x' = x/|x|$ for the nonzero element $x$ of $\mathbb{H}$.

Recall Burkholder's special function $\tilde{U}(x,y)$ mapping $\mathbb{R}_+ \times \mathbb{H}$ to $\mathbb{R}$:

$$\tilde{U}(x,y) = p(1 - 1/p^{**})^{p-1}(|y| - (p^{**} - 1)x)(x + |y|)^{p-1}.$$

The function $\tilde{U}$ satisfies the following properties as shown by Burkholder [9]:

(a) $|y|^p - (p^{**} - 1)^p x^p \le \tilde{U}(x,y)$ and strict inequality holds if $|y| \ne (p^{**} - 1)x$.

(b) For all $x, h \in \mathbb{R}$ and $y, k \in \mathbb{H}$ such that $x \geq 0$, $x + h \geq 0$, $|y| \neq 0$ and $|k| \leq |h|$,

$$\tilde{U}(x+h, y+k) \leq \tilde{U}(x, y) + \tilde{U}_x(x, y)h + \langle \tilde{U}_y(x, y), k \rangle.$$

(c) For all $x, h \in \mathbb{R}$ and $y, k \in \mathbb{H}$ such that $x \geq 0$, $|y| \neq 0$ and $x + h \geq 0$,

$$\tilde{U}_{xx}(x, y)h^2 + 2\langle \tilde{U}_{xy}(x, y)h, k \rangle + \langle k\tilde{U}_{yy}(x, y), k \rangle = -c_p(A + B + C),$$

where $c_p > 0$ is a constant that depends only on $p$ and $A$, $B$ and $C$ are defined by

$$A = p^{**}(p-1)(|h|^2 - |k|^2)(x + |y|)^{p-2},$$
$$B = \{(p-1)p^{**} - p\}(|k|^2 - \langle k, y' \rangle^2)|y|^{-1}(x + |y|)^{p-1},$$
$$C = (p-1)[(p^{**} - p)|y| + \{(p-1)p^{**} - p\}x](h + \langle k, y' \rangle)^2(x + |y|)^{p-3}.$$

(d) $\tilde{U}(x, y) \leq 0$ if $|y| \leq x$.

(e) $|\tilde{U}_y(x, y)| \leq -\tilde{U}_x(x, y)$ for all $x \geq 0$ and $y \in \mathbb{H}$,

where $\tilde{U}_x$ and $\tilde{U}_y$ are first-order partial derivative vectors and $\tilde{U}_{xx}$, $\tilde{U}_{x,y}$ and $\tilde{U}_{yy}$ stand for the second-order partial derivative matrices.

Since (2.4) is equivalent to $E(|Y_t|^p - (p^{**}-1)^p|X_t|^p) \leq 0$, by (a) it is enough to show $E\tilde{U}(X_t, Y_t) \leq 0$. Itô's formula gives

$$\tilde{U}(X_t, Y_t) = \tilde{U}(X_0, Y_0) + \int_{0+}^{t} \tilde{U}_x(X_{s-}, Y_{s-})\, dX_s + \int_{0+}^{t} \langle \tilde{U}_y(X_{s-}, Y_{s-}), dY_s \rangle + I_1/2 + I_2,$$

where

$$I_1 = \int_{0+}^{t} \Big(\tilde{U}_{xx}(X_{s-}, Y_{s-})d[X^c, X^c]_s + 2\sum_{i=1}^{d} \tilde{U}_{xy_i}(X_{s-}, Y_{s-})d[X^c, Y_i^c]_s + \sum_{i=1}^{d}\sum_{j=1}^{d} \tilde{U}_{y_i y_j}(X_{s-}, Y_{s-})d[Y_i^c, Y_j^c]_s\Big),$$

$$I_2 = \sum_{0<s\leq t} \big(\tilde{U}(X_s, Y_s) - \tilde{U}(X_{s-}, Y_{s-}) - \tilde{U}_x(X_{s-}, Y_{s-})\Delta X_s - \langle \tilde{U}_y(X_{s-}, Y_{s-}), \Delta Y_s \rangle\big).$$

Let $x = X_{s-}$, $y = Y_{s-}$, $h = \Delta X_s$ and $k = \Delta Y_s$. It follows from (b) that each term in $I_2$ is nonpositive since $|k|^2 \leq |h|^2$ by Lemma 1. Similarly, following the exact argument as in the proof of Theorem 1, we can show that $I_1$ is nonpositive by using the assumption that $Y$ is differentially subordinate to $X$. By (d) and the martingale property, it suffices to prove

$$\int_{0+}^{t} \tilde{U}_x(X_{s-}, Y_{s-})\, dA_s + \int_{0+}^{t} \sum_{i=1}^{d} \tilde{U}_{y_i}(X_{s-}, Y_{s-})\, dB_{i,s} \leq 0.$$

This follows from the Cauchy–Schwarz inequality, property (e) and the fact that $A_t - |B|_t$ is a nondecreasing function of $t$:

$$\begin{aligned}\left|\int_{0+}^{t}\sum_{i=1}^{d}\tilde{U}_{y_i}(X_{s-},Y_{s-})\,dB_{i,s}\right| &+ \int_{0+}^{t}\tilde{U}_x(X_{s-},Y_{s-})\,dA_s\\ &\le \int_{0+}^{t}|\tilde{U}_y(X_{s-},Y_{s-})|\,d|B|_s + \int_{0+}^{t}\tilde{U}_x(X_{s-},Y_{s-})\,dA_s\\ &\le \int_{0+}^{t}\tilde{U}_x(X_{s-},Y_{s-})\,d(A_s - |B|_s)\\ &\le 0.\end{aligned}$$

This proves (2.4) and then (2.3).

*Strictness.* This is similar to that part of Theorem 1 and Theorem 3.1 of Burkholder [9] with slight changes. The argument given here also works when $\mathbb{H}$ is infinite dimensional provided that $\tilde{U}(X_t, Y_t)$ is a supermartingale. Assume $0 < ||X||_p < \infty$.

Let $\tilde{V}(x,y) = |y|^p - (p^{**}-1)^p|x|^p$, $\tilde{v}(t) = E\tilde{V}(X_t, Y_t)$, $\tilde{U}_t = \tilde{U}(X_t, Y_t)$, and $\tilde{u}(t) = E\tilde{U}_t$. Then the above argument shows that $\tilde{v}(t) \le \tilde{u}(t)$ and $\tilde{U}$ is a supermartingale. We want to show that both $X$ and $Y$ have limits $X_\infty$ and $Y_\infty$.

By the submartingale convergence theorem, it is clear that $X$ has limit $X_\infty$ since $||X||_p < \infty$. To see $Y$ has a limit too, we need a lemma given below. However, first we introduce some notation. Given $\varepsilon > 0$, let $\tau_0 = \inf\{t \ge 0\colon |Y_t| \ge \varepsilon\}$ and

$$\tau_{i+1} = \inf\{t \ge \tau_i\colon |Y_t - Y_{\tau_i}| \ge \varepsilon\}.$$

The number of $\varepsilon$-escapes of $Y$ as defined by Burkholder [8, 9] is

$$C_\varepsilon(Y) = \max\{i\colon \tau_i < \infty\}.$$

It is clear that $Y$ has limit $Y_\infty$ if and only if $P\{C_\varepsilon(Y) = \infty\} = 0$ for all $\varepsilon > 0$.

LEMMA 4. *Suppose $X$ is a continuous-time nonnegative submartingale and $Y$ is an $\mathbb{H}$-valued continuous-time semimartingale such that $Y$ is strongly differentially subordinate to $X$. If $1 < p < \infty$ and $||X||_p < \infty$, then for all $i \ge 1$,*

$$P\{C_\varepsilon(Y) \ge i\} \le [(p^{**}-1)||X||_p/(\varepsilon i^{1/2})]^p.$$

Lemma 4 is proved in a similar way to Theorem 5.2 of [9]. The only difference is to use Chebyshev's inequality and (2.3). We omit the details. This shows $Y_\infty$ exists.

Observe that the maximum function of $Y$ is bounded even though Doob's maximal function inequality does not apply here. To see this, note that $X$ has limit $X_\infty$ implies that $M$ and $A$ have limits $M_\infty$ and $A_\infty$. Also $||M_\infty||_p$

and $||A_\infty||_p$ are finite since $||X||_p = ||X_\infty||_p$ is. Therefore, by strong differential subordination, $B$ has limit $B_\infty$ and $||B_\infty||_p \le ||A_\infty||_p < \infty$. Fatou's lemma and (2.3) imply $||Y_\infty||_p < \infty$. Therefore, $N$ has limit $N_\infty$ and $||N_\infty||_p \le ||Y_\infty||_p + ||B_\infty||_p < \infty$. Consequently, the maximum function $Y^* = \sup_{0\le t<\infty} |Y_t|$ has finite $L^p$ norm since both $N^*$ and $|B|$ do. Here we use Doob's maximal inequality for $N$, which is a martingale.

Next, by the dominated convergence theorem, both $\tilde{v}(t)$ and $\tilde{u}(t)$ are r.c.l.l. Strictness of (2.3) is then implied by

$$\tilde{v}(t-) < 0, \qquad \tilde{v}(t) < 0 \quad \text{and} \quad \tilde{v}(\infty) < 0. \tag{2.5}$$

In fact, by right continuity of $||Y_t||_p$ and $0 < ||X||_p < \infty$, there exists $\varepsilon > 0$ such that

$$||Y_t||_p \le (p^* - 1)||X||_p - \varepsilon$$

for $t < \varepsilon$. Let $\delta = -\sup_{t\ge\varepsilon} \tilde{v}(t)$. Then $\delta > 0$ by (2.5). Therefore, for $t \ge \varepsilon$,

$$||Y_t||_p^p = (p^* - 1)^p||X_t||_p^p + \tilde{v}(t) \le (p^* - 1)^p||X||_p^p - \delta.$$

We give a detailed proof of $\tilde{v}(\infty) < 0$. The proofs of $\tilde{v}(t-) < 0$ and $\tilde{v}(t) < 0$ for $t > 0$ are similar.

If $E|X_0|^p \ne 0$, then $\tilde{u}(0) \le (1 - 1/p^{**})^p(1 - (p^{**} - 1)^p)E|X_0|^p < 0$. Thus $\tilde{v}(t) \le \tilde{u}(t) \le \tilde{u}(0) < 0$ for any $t \ge 0$. Particularly, $\tilde{v}(\infty) \le \tilde{u}(0) < 0$. Therefore, we may assume without loss of generality that $X_0 = Y_0 = 0$ and $||X_t||_p > 0$ for all $t > 0$.

It is enough to show $P(|Y_\infty| = (p^{**} - 1)X_\infty) < 1$ since otherwise by (a), $\tilde{v}(\infty) < \tilde{u}(\infty) \le 0$.

Suppose $|Y_\infty| = (p^{**} - 1)X_\infty$ almost surely. Then $\tilde{U}_\infty = 0$. Since $\{\tilde{U}_t,\ t \ge 0\}$ is a uniformly integrable supermartingale starting from 0, this implies $P(\tilde{U}_t = 0, \text{ for all } t \ge 0) = 1$. Therefore, $|Y| = (p^{**} - 1)X$.

Without loss of generality, assume $X_- \cdot M$ and $Y_- \cdot N$ are martingales. By the definition of quadratic variation, we have for any $t > 0$,

$$\begin{aligned} 0 &= |Y_t|^2 - (p^{**} - 1)^2 X_t^2 \\ &= 2\int_{0+}^t (\langle Y_{s-}, dY_s\rangle - (p^{**} - 1)^2 X_{s-}\, dX_s) \\ &\quad + [Y, Y]_t - (p^{**} - 1)^2[X, X]_t \\ &= 2J_1 + J_2. \end{aligned} \tag{2.6}$$

Since $|Y| = (p^{**} - 1)X$ and $p^{**} > 2$, by strong differential subordination, we have

$$\begin{aligned} J_1 &= \int_{0+}^t (\langle Y_{s-}, dN_s\rangle - (p^{**} - 1)^2 X_{s-}\, dM_s) \\ &\quad + \int_{0+}^t (\langle Y_{s-}, dB_s\rangle - (p^{**} - 1)^2 X_{s-}\, dA_s) \end{aligned}$$

$$\le \text{martingale} + \int_{0+}^{t} (|Y_{s-}|\, d|B|_s - (p^{**}-1)^2 X_{s-}\, dA_s)$$
$$\le \text{martingale}.$$

Note that $EJ_1 \le 0$ and $EJ_2$ is negative unless $E[X,X]_t = 0$ because $p^{**} > 2$ and because of differential subordination. Taking expectation on both sides of (2.6), we must have $E[X,X]_t = 0$. Therefore, $E|X_t|^2 = 0$ for any $t > 0$. This contradicts that $E|X_t|^p > 0$ for $t > 0$ as we assumed at the beginning. This completes the proof. □

Note here that the strictness is only for $||Y||_p$. It is interesting to know if strictness holds for $|||Y|||_p$.

Like the proof of Theorem 1, that function $\tilde{U}$ is of $C^2$ and $\mathbb{H}$ has finite dimension play an important role. When the special function is not $C^2$, but is of $C^1$, continuous on $\mathbb{H} \times \mathbb{H}$, and of piecewise $C^2$, then the same proof can go through by using convolution and approximation. We give the analogue of Proposition 1 below. The proof is similar. One needs to modify $T_n$ in the proof of Proposition 1 to be $T_n = \inf\{t > 0\colon |X_t| + |Y_t| + |A|_t + |B|_t + [X,X]_t + [Y,Y]_t \le n\}$. We omit the details of the proof.

PROPOSITION 5. *Let $f(x,y)$ be a function continuous on $\mathbb{R}_+ \times \mathbb{H}$, bounded on bounded sets, in $C^1$ on $\mathbb{R}_+ \times \mathbb{H} \setminus (\{x=0\} \cup \{|y|=0\})$ and in $C^2$ on $S_i$, $i \ge 1$, where $\{S_i\}_{i\ge1}$ is a sequence of open connected sets such that the union of the closures of the $S_i$ is $\mathbb{R}_+ \times \mathbb{H}$. Assume that for $i \ge 1$, there exists a nonnegative function $c_i(x,y)$ on $S_i$ and*

$$f_{xx}(x,y)h^2 + 2\langle hf_{xy}(x,y), k\rangle + \langle hf_{yy}(x,y), k\rangle \le -c_i(x,y)(h^2 - |k|^2)$$

*for $h \in \mathbb{R}$, $k \in \mathbb{H}$, and $x|y| \ne 0$. Assume further that for $(x,y) \in S_i$, $i \ge 1$, and $x|y| \ne 0$,*

$$|f_y(x,y)| \le -f_x(x,y) \quad [\text{or } |f_y(x,y)| \le f_x(x,y)]$$

*and for each $n \ge 1$, there exists $M_n$ which is nondecreasing on $n$ satisfying*

$$\sup c_i(x,y) \le M_n < \infty,$$

*where supremum is taken over all $(x,y) \in S_i$ and $1/n^2 \le x^2 + |y|^2 \le n^2$ and all $i \ge 1$. Also assume that $f_x$ and $f_y$ are bounded on bounded set that does not contain 0 and for any $x|y| \ne 0$ and $|k| \le |h|$,*

$$f(x+h, y+k) - f(x,y) - f_x(x,y)h - \langle f_y(x,y), k\rangle \le 0.$$

*For any nonnegative submartingale (or supermartingale) $X$ and semimartingales $Y$ with $Y$ being strongly differentially subordinate to $X$, if $f$ satisfies condition* (A1)*, then for $0 \le s \le t$,*

$$E(f(X_t, Y_t) \mid \mathcal{F}_s) \le f(X_s, Y_s) \quad \text{a.e.}$$

Proposition 5 implies Theorem 4 when $\mathbb{H}$ is infinite dimensional.

It is easy to verify that conditions given in Proposition 5 are satisfied for the special functions found in [9] used in the other norm inequalities. For weak type inequalities proved in [9], we need to use the analogue of Proposition 3. In fact, all the remarks at the end of the previous section are also applicable in this section. We omit the details.

**3. Concluding Remarks.** The above arguments show that once the special function is found for the finite-dimensional discrete-time case, the same function can be used to produce the same result for the infinite-dimensional Hilbert space-valued continuous-time case with the help of Itô's formula and convolution arguments. Therefore, when working on sharp martingale inequalities, the finite-dimensional discrete-time case is more challenging and more interesting.

It is also clear from the above proofs that differential subordination and strong differential subordination play an important role. Motivated by Burkholder's conjecture given in the Introduction, we ask whether strong subordination implies (2.3). Although we cannot solve either of the above questions, we would like to make the following observation.

Conditions required in Theorems 1 and 4 can be weakened. For semimartingales $X$ and $Y$, we say that $Y$ is *in a weak sense differentially subordinate* to $X$ if (a) $[X^c, X^c] - [Y^c, Y^c]$ is a local submartingale and (b) $|Y_0| \le |X_0|$ and $|\Delta Y_t| \le |\Delta X_t|$ for all $t > 0$. For semimartingales $X$ and $Y$, $Y$ is *in a weak sense strongly differentially subordinate* to $Y$ if (a) $Y$ is in a weak sense differentially subordinate to $X$ and (b) $|A|_t - |B|_t$ is a local submartingale, where $X = X_0 + M + A$ and $Y = Y_0 + N + B$ are defined in (2.1). When $X$ and $Y$ have continuous paths, $Y$ is in a weak sense differentially subordinate to $X$ if $[X, X] - [Y, Y]$ is a local submartingale and $|Y_0| \le |X_0|$. This condition is very easy to verify.

By checking the previous proofs carefully, we can see that all the above theorems remain valid when (strong) differential subordination is replaced by, in a weak sense (strong), differential subordination. The only change is to decompose the integral

$$-\int_0^t c(X_{s-}, Y_{s-})\,(d[X^c, X^c]_s - [Y^c, Y^c]_s), \qquad \int_0^t f_x(X_{s-}, Y_{s-})\,d(A_s - |B|_s)$$

into

$$\begin{aligned} &-\int_{0+}^t c(X_{s-}, Y_{s-})\,dZ_s^1 - \int_{0+}^t c(X_{s-}, Y_{s-})\,dC_s^1, \\ &\int f_x(X_{s-}, Y_{s-})\,dZ_s^2 + \int_0^t f_x(X_{s-}, Y_{s-})\,dC_s^2, \end{aligned} \tag{3.1}$$

where $[X, X] - [Y, Y] = X_0^2 - Y_0^2 + Z^1 + C^1$, $A - |B| = Z^2 + C^2$, $Z^1, Z^2$ are local martingales and $C^1, C^2$ are predictable nondecreasing processes as given in the Doob–Meyer decomposition. Note in (3.1) that each of the two sums consists of a local martingale term and a nonnegative term. This ensures that the proofs of all our previous theorems go through. Therefore, all Burkholder's sharp inequalities hold for continuous-time semimartingale setting under, in a weak sense, differential subordination or strong differential subordination. As a special case, we have the following theorem.

THEOREM 5. *If $X$ and $Y$ are local martingales such that $Y$ is in a weak sense differential subordinate to $X$, then*

$$|||Y|||_p \leq (p^* - 1)|||X|||_p$$

*and constant $p^* - 1$ is best possible. Strict inequality holds if $p \neq 2$ and $0 < |||X|||_p < \infty$. Similarly, if $X$ is a nonnegative submartingale and $Y$ is a semimartingale such that $Y$ is in a weak sense strongly differentially subordinate to $X$, then*

$$|||Y|||_p \leq (p^{**} - 1)|||X|||_p$$

*and the constant $p^{**} - 1$ is best possible. When $0 < ||X||_p < \infty$,*

$$||Y||_p < (p^{**} - 1)||X||_p.$$

We note that the conditions of the theorem do not imply (I.2). Therefore, even if the above two conjectures are true, Theorem 5 is still a new result.

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