The Annals of Probability
1995, Vol. 23, No. 3, 1061–1101

ENLARGEMENT OF OBSTACLES FOR THE SIMPLE RANDOM WALK

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We consider a continuous time simple random walk moving among obstacles, which are sites (resp., bonds) of the lattice Z^d. We derive in this context a version of the technique of enlargement of obstacles developed by Sznitman in the Brownian case. This method gives controls on exponential moments of certain death times as well as good lower bounds for certain principal eigenvalues. We give an application to recover an asymptotic result of Donsker and Varadhan on the number of sites visited by the random walk and another application to the number of bonds visited by the random walk.

Introduction. In this article we will derive the method of enlargement of obstacles for the d-dimensional continuous time simple random walk $(S_t)_{t\geq 0}$. This technique, developed by Sznitman [11, 13] in a Brownian motion context, enables us to derive controls on exponential moments of certain killing times as well as good lower bounds for certain principal Dirichlet eigenvalues of the Laplacian. Here we develop and give some applications of the method in a discrete setting. We consider for $\varepsilon > 0$ the lattice εZ^d embedded in R^d and $\mathscr{T}$, an arbitrary open subset of R^d. We look at the rescaled continuous time simple random walk $\varepsilon S_{t/\varepsilon^2}$ which is killed when it enters a certain obstacle set or when it leaves $\mathscr{T}$. We shall study two types of obstacles. In the first case, the obstacles are sites of the lattice and the process will be killed when it visits such a site. In the second case, the obstacles are bonds which kill the process when it crosses one of them. We denote the obstacle set by $\mathscr{H}^\varepsilon$ (site case) [resp., by $\mathscr{H}^\varepsilon_*$ (bond case)].

We are interested in certain estimates on exponential moments involving the above-mentioned killing time. These estimates enable us as well to derive interesting lower bounds for the lowest eigenvalue of the generator of the contraction semigroup associated with the killed process. We denote this eigenvalue by $\lambda^\varepsilon(\mathscr{H}^\varepsilon, \mathscr{T})$ in the site case [resp., by $\lambda^\varepsilon_*(\mathscr{H}^\varepsilon_*, \mathscr{T})$ in the bond case].

The idea of the method of enlargement of obstacles is to replace our true obstacles by obstacles of a much larger size in order to produce a coarse-grained picture of the obstacles which is easier to describe, and at the same time to derive a lower bound for the principal Dirichlet eigenvalue appearing in the true obstacle problem in terms of the principal Dirichlet eigenvalue in the enlarged obstacle problem. The enlargement methods for the two situations

Received November 1993; revised July 1994.
AMS 1991 *subject classifications*. 60J15, 82D30.
Key words and phrases. Random walk, killing traps, principal eigenvalues.

[sites (resp., bonds)] differ only in some technical details and we can treat both cases simultaneously.

Let us now briefly explain how the enlargement method works. We need three parameters b, δ and r to develop the method. First we introduce a parameter $b > 2$ and associate with each obstacle a cube of side $2b\varepsilon$ centered at the obstacle (in the bond case, at one end vertex of the obstacle). We call these cubes the enlarged obstacles.

We then chop R^d into cubes of unit side. Observe that ε will be chosen so small that $2b\varepsilon$, that is, the size of the enlarged obstacles, is much smaller than 1. In each cube we have "good" and "bad" obstacles. In order to derive our spectral estimates, we have to discard certain bad obstacles, where an enlargement would produce an increase of the corresponding principal eigenvalue on the relevant scale. So for the construction of our coarse-grained picture we shall only use the enlarged obstacles centered at the good true obstacles. Roughly speaking, an obstacle is good if it is well surrounded by other obstacles. That is, there is a nonvanishing fraction of volume of the enlarged obstacles in successive concentric cubes with geometrically increasing sides going from the size of one enlarged obstacle up to size 1. The parameter δ shall measure if an obstacle is good or bad.

Similarly as in [13] we also introduce the notion of clearing and forest boxes. This is done by picking a small number $r > 0$ and asking that in a box of forest type the total volume left unoccupied by the enlarged obstacles centered at the good true obstacles is smaller than the volume of a ball of radius r. The other boxes are said to be clearings. The main tool to estimate probabilities that a box is a clearing when obstacles are random is the covering Lemma 1.1. This lemma enables us to compare the volume left unoccupied by the enlarged obstacles centered at the good true obstacles to the volume left unoccupied by all enlarged obstacles. In fact, the set left unoccupied by all enlarged obstacles is probabilistically easier to handle and thanks to the covering lemma, a nondegenerate value of the volume left unoccupied by enlarged good obstacles will imply a nondegenerate value of the volume left unoccupied by all enlarged obstacles. This will be important for the application of the method to random trapping problems treated in Section 2.

We denote by $\mathscr{A}_1$ the 1-neighborhood of the clearing boxes. Our coarse-grained picture is the set Θ_b, defined as the open set complement in $\mathscr{T} \cap \mathscr{A}_1$ of the enlarged good obstacles. We look at the random walk killed when it leaves Θ_b and denote the lowest eigenvalue of the generator of the associated semigroup by $\lambda^\varepsilon(\Theta_b)$. We let $\tilde{T}$ (resp., $\tilde{T}_*$) stand for the killing time in the true obstacle problem. Then the crucial estimate coming out of the enlargement technique in Theorem 1.4 is that, for arbitrary $M > 0$ and $\rho > 0$, we have

$$\limsup_{r\to 0} \sup_{\substack{b>2\\0<\delta<1}} \limsup_{\varepsilon\to 0} \sup_{z,\mathscr{H}^\varepsilon,\mathscr{T}} E_z^\varepsilon[\exp\{(\lambda^\varepsilon(\Theta_b)\wedge M-\rho)\tilde{T}\}] \tag{I.1}$$
$$\le \mathrm{const}(d,M,\rho) < \infty,$$

$$\text{(I.2)}\qquad \limsup_{r\to 0} \sup_{\substack{b>2\\ 0<\delta<1}} \limsup_{\varepsilon\to 0} \sup_{z,\mathscr{H}^\varepsilon_*,\mathscr{T}} E^\varepsilon_z[\exp\{(\lambda^\varepsilon(\Theta_b)\wedge M-\rho)\tilde{T}_*\}] \le \operatorname{const}(d,M,\rho)<\infty.$$

As a consequence of these exponential estimates, we get the promised estimates for the principal eigenvalues, namely, $\forall\ M>0$,

$$\text{(I.3)}\qquad \limsup_{r\to 0} \sup_{\substack{b>2\\ 0<\delta<1}} \limsup_{\varepsilon\to 0} \sup_{\mathscr{H}^\varepsilon,\mathscr{T}} (\lambda^\varepsilon(\Theta_b)\wedge M-\lambda^\varepsilon(\mathscr{H}^\varepsilon,\mathscr{T})\wedge M)_+ = 0,$$

$$\text{(I.4)}\qquad \limsup_{r\to 0} \sup_{\substack{b>2\\ 0<\delta<1}} \limsup_{\varepsilon\to 0} \sup_{\mathscr{H}^\varepsilon_*,\mathscr{T}} (\lambda^\varepsilon(\Theta_b)\wedge M-\lambda^\varepsilon_*(\mathscr{H}^\varepsilon_*,\mathscr{T})\wedge M)_+ = 0,$$

so with a suitable choice of parameters, $\lambda^\varepsilon(\Theta_b)\wedge M$ is not really bigger than $\lambda^\varepsilon(\mathscr{H}^\varepsilon,\mathscr{T})\wedge M$ [resp. $\lambda^\varepsilon_*(\mathscr{H}^\varepsilon_*,\mathscr{T})\wedge M$]; this yields the promised lower bound on $\lambda^\varepsilon(\mathscr{H}^\varepsilon,\mathscr{T})$ [resp. $\lambda^\varepsilon_*(\mathscr{H}^\varepsilon_*,\mathscr{T})$].

In Section 2 we give an application of the method (see also [4]). We consider a random configuration of traps (bonds or sites) in Z^d; more precisely we assume that the obstacles are i.i.d. Bernoulli distributed sites or bonds with parameter $p:=1-e^{-\nu}$. We denote by P the law of $S_\bullet$ and by $\mathbb{P}$ the law of the obstacle configurations; T (resp., T^* in the bond case) stands for the entrance time in the obstacle set. We also introduce R_t and R^*_t, the number of distinct sites (resp., bonds) visited by the random walk up to time t. Our main result in Theorem 2.1 is

$$\text{(I.5)}\qquad \lim_{t\to\infty} t^{-d/(d+2)}\log \mathbb{P}_*\otimes P_0[T_*>t] = \lim_{t\to\infty} t^{-d/(d+2)}\log E_0[\exp(-\nu R^*_t)] = -c_*(d,\nu),$$

$$\text{(I.6)}\qquad \lim_{t\to\infty} t^{-d/(d+2)}\log \mathbb{P}\otimes P_0[T_*>t] = \lim_{t\to\infty} t^{-d/(d+2)}\log E_0[\exp(-\nu R_t)] = -c_*(d,\nu),$$

where

$$\text{(I.7)}\qquad c(d,\nu)=(\nu\omega_d)^{2/(d+2)}\left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)},$$

$$\text{(I.8)}\qquad c_*(d,\nu)=c(d,\nu d).$$

Here λ_d and ω_d stand, respectively, for the principal Dirichlet eigenvalue of $-(1/2d)\triangle$ in the unit ball of R^d and the volume of this ball. In fact, the constant $c(d,\nu)$ comes out of the minimization problem

$$\text{(I.9)}\qquad c(d,\nu)=\inf_U(\nu|U|+\lambda(U)),$$

$$\text{(I.10)}\qquad c_*(d,\nu)=\inf_U(d\nu|U|+\lambda(U)),$$

where U runs over all bounded open subsets of R^d with negligible boundary.

Equation (I.6) recovers with a different approach a result originally due to Donsker and Varadhan (see also [3]). One might think at first that (I.5) is an easy consequence of (I.6); however, this is not the case. In fact, the ratio R_t^*/R_t converges almost surely to a constant $\zeta(d)$ for $t \to \infty$ and $\zeta(d)$ tends to 1 for large d. One could naively expect to obtain $c_*(d, \nu)$ from $c(d, \nu)$ by replacing ν by $\zeta\nu$, but as (I.8) shows, this is not the case.

We mention at this point that the problem studied in Section 2 gives only one possible application of the technique developed in Section 1. The method is, in fact, more general and for instance it could also be applied to derive almost sure estimates about the survival probability of a random walk among random obstacles.

Let us finally give some ideas of the proof of (I.5). The lower bound part is classical. One simply uses that the survival probability of the process until time t is bigger than the probability that the process remains in a large open ball centered at the origin and no obstacle falls in this ball.

The upper bound uses the enlargement technique of Section 1. First we adopt $t^{1/(d+2)}$ and $t^{2/(d+2)}$ as new space and time units, that is, we study a random walk on $t^{-1/(d+2)}Z^d$ until time $s = t^{d/(d+2)}$. We choose $\mathscr{T} := [-N[s], N[s]]$, where N is a large number. Our strategy to derive an upper bound on $\mathbb{P} \otimes P_0[T > s]$ is to write

$$\begin{aligned}\mathbb{P} \otimes P_0[T > s] \le P_0[T_{\mathscr{T}} \le s] + \mathbb{P}[\#(\text{clearings in } \mathscr{T}) > n_0]\\ + \mathbb{P} \otimes P_0[\#(\text{clearings in } \mathscr{T}) \le n_0 \text{ and the}\\ \text{process survives in } \mathscr{T} \text{ up to time } s].\end{aligned}$$

Here n_0 is an arbitrary large number which we eventually let tend to infinity. It is easy to see that the first term has exponential time decay and therefore it is negligible for our purpose. The second term can be estimated with our covering lemma and seen to be negligible for suitable choices of parameters. In fact, this is one point where the already mentioned comparison of volumes comes in. To estimate the third term, the enlargement technique is used. In fact, with $\varepsilon := t^{-1/(d+2)}$, the exponential estimates (I.1) [resp., (I.2)] give good controls on the probability that the process survives in $\mathscr{T}$ up to time s among the obstacles. Together with a lemma, which enables us to compare the principal Dirichlet eigenvalue of the discrete Laplacian with those of the usual Laplacian, this will imply the desired bound on the third term.

1. The enlargement technique.

1.1. *Notation.* In this section we derive uniform estimates which enable us to replace the true obstacles (which are single bonds or sites) in an open set of R^d by obstacles of a larger size without really raising the principal Dirichlet eigenvalue of the part of the open set left unoccupied by the obstacles, provided the principal Dirichlet eigenvalue for the initial configuration has a reasonable value. Our methods are inspired by the work of Sznitman [13], where the enlargement technique is applied to get estimates about the long time survival probability of d-dimensional Brownian motion among random obstacles. Let us introduce some notation.

We denote by Z^d the d-dimensional lattice and by E^d the set of all edges between points of Z^d with distance 1. We will think of the graph (Z^d, E^d) as a graph embedded in R^d, the edges being straight line segments between their end vertices. E^d can be identified with the set of the middle points of the edges, that is, with $\bigcup_{i=1}^d \frac{1}{2}e_i + Z^d$, where e_i is the ith basis vector of the canonical basis of R^d. We also introduce a convenient metric on R^d and the balls w.r.t. this metric. We set, for $x, y \in R^d$ and $r > 0$,

$$d(x,y) := \max_{1\le i\le d} |x_i - y_i|, \tag{1.1}$$

$$B(x,r) := \{z \in R^d \mid d(x,z) < r\}, \qquad \overline{B}(x,r) := \{z \in R^d \mid d(x,z) \le r\}. \tag{1.2}$$

We let Ω stand for the set of all piecewise constant right continuous functions on R^+ with finitely many jumps on finite intervals and values in R^d, $\mathscr{F}$ the canonical σ-field on Ω, $(\mathscr{F}_t)$ the canonical right continuous filtration and P_x the probability measure on $\mathscr{F}$, under which the canonical coordinate process $(S_t)_{t\ge 0}$ is a simple (nearest neighbor) random walk starting at $x \in Z^d$ and having jump intensity 1.

We will also choose a small number $\varepsilon > 0$ and study the rescaled process $X_t^\varepsilon := \varepsilon S_{t/\varepsilon^2}$, whose law we denote by P_x^ε. The obstacles we want to study in this first section are deterministic subsets of εZ^d (resp., of εE^d), that is, sites (resp., bonds) of the scaled lattice. We denote the obstacle set by $\mathscr{H}^\varepsilon$ in the site case and by $\mathscr{H}_*^\varepsilon$ in the edge case.

If y is an edge in εE^d, then there exist a well defined site $y_1 \in \varepsilon Z^d$ and e_i, an element of the canonical basis of R^d, such that y_1 and $y_2 := y_1 + \varepsilon e_i$ are the two end vertices of y. We define the "vertex of the obstacle h" by

$$v(h) := \begin{cases} h & \text{(site case)}, \\ h_1 & \text{(edge case)}. \end{cases} \tag{1.3}$$

For a set $A \subset R^d$ we write A^ε for $A \cap \varepsilon Z^d$ and A_*^ε for the set of edges of E^d with middle point in A. The counting measure on εZ^d is denoted by $\#^\varepsilon$; that on εE^d by $\#_*^\varepsilon$. For $A \subset R^d$, $\#^\varepsilon A$ stands for the cardinality of A^ε and $\#_*^\varepsilon A$ for the cardinality of A_*^ε.

Finally we define for each multiindex $m \in Z^d$, the cube

$$C_m := \{z \in R^d,\ m_i \le z_i < m_i + 1,\ i = 1,\dots,d\}. \tag{1.4}$$

1.2. *Enlarged obstacles.* Let us now describe the enlarged obstacles. They are defined in the following fashion:

Let $\varepsilon > 0$ and $b > 2$ be fixed. For each obstacle h (site or bond) we define the corresponding enlarged obstacle as the ball $\overline{B}(v(h), \varepsilon b)$. In the bond situation, two distinct obstacles can give rise to the same enlarged obstacle.

In order to derive interesting bounds on eigenvalues dealing with the true obstacles in terms of enlarged obstacles, we shall have to discard certain "bad obstacles" which are poorly surrounded. We choose a number δ in $(0,1)$ which will measure if an obstacle is good or bad. Indeed, we say that an obstacle $h \in \mathscr{H}^\varepsilon \cap C_m$ is good if for all balls $C := \overline{B}(v(h), (10\sqrt{d})^{l+1}\varepsilon b)$, $l \geq 0$ and $(10\sqrt{d})^{l+1}\varepsilon b < \frac{1}{2}$,

$$\#^\varepsilon\Bigl(C_m \cap C \cap \bigcup_{\tilde{h} \in \mathscr{H}^\varepsilon \cap C_m} \overline{B}(v(\tilde{h}), \varepsilon b)\Bigr) \geq (\delta/12^d)\,\#^\varepsilon(C_m \cap C). \tag{1.5}$$

In the bond case we define similarly an obstacle with middle point in C_m to be good if (1.5) holds with the obvious modification that $\tilde{h}$ runs over all bond obstacles with middle point in C_m. In both cases we use the counting measure on εZ^d. We also chop each segment $[k, k+1]$ into at most $[1/((b-1)\varepsilon)] + 1$ intervals of length $(b-1)\varepsilon$ each, except perhaps the "last one." This yields closed subboxes of side less than $(b-1)\varepsilon$, with union $\overline{C}_m$. The crucial point of this construction is that if an obstacle h falls in a certain subbox (the meaning of this in the bond case is that the middle point falls in the box), then the box is entirely contained in the corresponding enlarged obstacle, since $v(h)$ has at most distance $\varepsilon/2$ from the box.

We now set U_m to be the open (in R^d) subset of C_m obtained by taking the complement in the interior of C_m of the closed boxes where an obstacle of C_m falls, and $\tilde{U}_m$ to be the complement of the boxes where a good obstacle of C_m falls.

1.3. *Two comparison lemmas.* For our application of the method to random trapping problems, a comparison between the volumes of U_m and $\tilde{U}_m \supset U_m$ is crucial. Indeed the natural probabilistic estimates are expressed in terms of U_m, whereas our spectral controls involve $\tilde{U}_m$. We are able to do this thanks to the following covering argument, which says that we have a good control on the volume (w.r.t. counting measure) of the enlarged "bad obstacles," that is, the union of balls of radius $b\varepsilon$ centered at the bad obstacles cover a small fraction of the volume of C_m^ε. So, for instance, a "sizable $\tilde{U}_m$" implies the occurrence of a "sizable U_m."

LEMMA 1.1. *Let $\mathscr{H}^\varepsilon_{\text{bad}}$ be the set of bad obstacles. Then we have, for $m \in Z^d$,*

$$\#^\varepsilon\Bigl(C_m \cap \Bigl(\bigcup_{h \in \mathscr{H}^\varepsilon_{\text{bad}} \cap C_m} \overline{B}(v(h), \varepsilon b)\Bigr)\Bigr) < \delta \#^\varepsilon C_m. \tag{1.6}$$

PROOF. We treat the two cases (sites, resp. bonds) simultaneously. Let h be a bad obstacle (with no loss of generality, we can assume that $\mathscr{H}^\varepsilon_{\text{bad}}$ is not empty). Then there is $l \geq 0$, $\varepsilon b(10\sqrt{d})^{l+1} < \frac{1}{2}$, such that

$$\#^\varepsilon\Bigl(C_m \cap D \cap \Bigl(\bigcup_{\substack{\text{obstacles}\\ \tilde{h} \text{ in } C_m}} \overline{B}(v(\tilde{h}), \varepsilon b)\Bigr)\Bigr) < (\delta/12^d)\,\#^\varepsilon(C_m \cap D),$$

where $D := \overline{B}(v(h), (10\sqrt{d})^{l+1}\varepsilon b)$ and C_m is the cube containing h (site case) [resp., the middle point of h (bond case)]. Observe that in the bond case, $v(h)$ is not necessarily an element of C_m^ε. Take $h_1 \in \mathscr{H}_{\text{bad}}^\varepsilon$ with maximal $l := l_1$ and set

$$D_1 := \overline{B}(v(h), (10\sqrt{d})^{l_1+1}\varepsilon b),$$

$$G_1 := \{\tilde{h} \in \mathscr{H}_{\text{bad}}^\varepsilon, \overline{B}^\varepsilon(v(\tilde{h}), \varepsilon b) \not\subset D_1^\varepsilon\}.$$

Define h_2 to be an element of G_1 with maximal $l := l_2$ and set

$$G_2 := \{\tilde{h} \in \mathscr{H}_{\text{bad}}^\varepsilon, \overline{B}^\varepsilon(v(\tilde{h}), \varepsilon b) \not\subset D_1^\varepsilon \cup D_2^\varepsilon\}.$$

Continue like this with $D_1, D_2, \dots, D_L$, with $G_L = \varnothing$. So we have

$$\begin{aligned}
&\#^\varepsilon\Bigg(C_m \cap \bigg(\bigcup_{\tilde{h} \in \mathscr{H}_{\text{bad}}^\varepsilon \cap C_m} \overline{B}(v(\tilde{h}), \varepsilon b)\bigg)\Bigg) \\
&\quad \le \sum_{k=1}^{L} \#^\varepsilon\Bigg(C_m \cap D_k \cap \bigg(\bigcup_{\tilde{h} \in \mathscr{H}_{\text{bad}}^\varepsilon \cap C_m} \overline{B}(v(\tilde{h}), \varepsilon b)\bigg)\Bigg) \\
&\quad < (\delta/12^d) \sum_{k=1}^{L} \#^\varepsilon(D_k \cap C_m).
\end{aligned}$$

We now define the set $\tilde{D}_k := \overline{B}(v(h_k), \frac{1}{3}(10\sqrt{d})^{l_k+1}\varepsilon b)$. Our claim will follow if we show that the balls $\tilde{D}_k$, $1 \le k \le L$, are disjoint and

$$\#^\varepsilon(D_k \cap C_m) \le 12^d \#^\varepsilon(\tilde{D}_k \cap C_m). \tag{1.7}$$

First we show (1.7). We have $\tilde{D}_k \cap C_m = \prod_{i=1}^d \tilde{I}_i$, where the $\tilde{I}_i$ are closed or semiopen intervals in R. For the (Euclidean) length of these intervals we have

$$|\tilde{I}_i| \ge \frac{10}{3}\varepsilon - \frac{\varepsilon}{2}$$

and, therefore, there exist $\tilde{n}_i \in N$, $\tilde{n}_i \ge 2$ and $0 \le \tilde{r}_i < \varepsilon$ such that $|\tilde{I}_i| = \tilde{n}_i\varepsilon + \tilde{r}_i$. We also have $D_k \cap C_m = \prod_{i=1}^d I_i$ with $|I_i| = n_i\varepsilon + r_i$ and $n_i \le 3\tilde{n}_i + 5$, since $|I_i| \le 3(|\tilde{I}_i| + \varepsilon/2)$.

We now give a counting argument in a more general form than needed in the present proof, since we shall use it at several points in the sequel.

For any interval I (open, semiopen or closed) with $|I| = n\varepsilon + r$, $0 \le r < \varepsilon$, we have

$$n - 1 \le \#^\varepsilon I \le n + 1. \tag{1.8}$$

This implies

$$\#^\varepsilon(\tilde{D}_k \cap C_m) \ge \prod_{i=1}^{d}(\tilde{n}_i - 1),$$

$$\#^\varepsilon(D_k \cap C_m) \le \prod_{i=1}^{d}(n_i + 1) \le \prod_{i=1}^{d}(3\tilde{n}_i + 6).$$

Therefore, we have

$$\frac{\#^{\varepsilon}(D_k \cap C_m)}{\#^{\varepsilon}\tilde{D}_k \cap C_m} \le \frac{\prod_{i=1}^{d}(3\tilde{n}_i + 6)}{\prod_{i=1}^{d}(\tilde{n}_i - 1)} \le 12^d \tag{1.9}$$

since $\tilde{n}_i \ge 2$.

The last step is to show that the balls $\tilde{D}_k$, $1 \le k \le L$, are disjoint. For this we pick $1 \le k < k' \le L$. Then $\overline{B}^{\varepsilon}(h_{k'}, \varepsilon b) \not\subset D_k$ and so

$$d(h_k, h_{k'}) \ge \left(10\sqrt{d}\right)^{l_k+1}\varepsilon b - \varepsilon b \ge 9\varepsilon b\sqrt{d}\left(10\sqrt{d}\right)^{l_k}.$$

Since $(10/3)\varepsilon b\sqrt{d}(10\sqrt{d})^{l_k} + (10/3)\varepsilon b\sqrt{d}(10\sqrt{d})^{l_{k'}} \le (20/3)\varepsilon b\sqrt{d}(10\sqrt{d})^{l_k} < d(h_k, h_{k'})$, it follows that the balls with radius $(1/3)(10\sqrt{d})^{l_k+1}\varepsilon b$ centered at h_k and $h_{k'}$, respectively, are in fact disjoint and this proves our claim. □

The following lemma is useful for the comparison of the volumes of certain balls which we shall need later.

LEMMA 1.2. *Let a be an arbitrary real number such that $10\varepsilon < a$, let I_i, $1 \le i \le d$, be a collection of closed or semiopen intervals in R of length a and let J_i, $1 \le i \le d$, be a collection of such intervals of length $2a$. Then we have*

$$4^{-d}\#^{\varepsilon}\left(\prod_{i=1}^{d} J_i\right) < \#^{\varepsilon}\left(\prod_{i=1}^{d} I_i\right) < (2/3)^d\#^{\varepsilon}\left(\prod_{i=1}^{d} J_i\right). \tag{1.10}$$

PROOF. As in the proof of the preceding lemma, we set $a = n\varepsilon + r$ with $n \in N, 0 \le r < \varepsilon$, and therefore we have $2a = \tilde{n}\varepsilon + \tilde{r}$ with $2n \le \tilde{n} \le 2(n+1)$. Using (1.8) we obtain

$$(n-1)^d \le \#^{\varepsilon}\left(\prod_{i=1}^{d} I_i\right) \le (n+1)^d,$$

$$(2n-1)^d \le \#^{\varepsilon}\left(\prod_{i=1}^{d} J_i\right) \le (2n+3)^d.$$

Since $n \ge 10$, we have

$$\left(\frac{n-1}{2n+3}\right)^d \ge \left(\frac{9}{23}\right)^d > 4^{-d} \quad \text{and} \quad \left(\frac{n+1}{2n-1}\right)^d \le \left(\frac{11}{19}\right)^d < \left(\frac{2}{3}\right)^d$$

and this yields our claim. □

COROLLARY 1.3. *Let $10\varepsilon < a < \frac{1}{2}$ and $m \in Z^d$. Then we have, for any obstacle h in C_m,*

$$\#^{\varepsilon}(\overline{B}(v(h), a) \cap C_m) \ge 4^{-d}\#^{\varepsilon}\overline{B}(v(h), a). \tag{1.11}$$

PROOF. Since $v(h)$ has at most distance $\varepsilon/2$ from C_m and $a < \frac{1}{2}$, there is a cubic box $\prod_{i=1}^d I_i$, where the I_i are semiopen intervals of length $a-\varepsilon$, such that $\prod_{i=1}^d I_i$ is contained in $C_m \cap \overline{B}(v(h), a)$. On the other hand, $\overline{B}(v(h), a)$ is the product of closed intervals of length $2a$, so we obtain by the same calculation as in the preceding lemma (with n replaced by $n-1$),

$$\#^\varepsilon \overline{B}(v(h), a) \leq (2n+1)^d,$$
$$\#^\varepsilon (\overline{B}(v(h), a) \cap C_m) \geq (n-2)^d.$$

Since $n \geq 10$, our claim follows from $((n-2)/(2n+1))^d \geq (8/21)^d > 4^{-d}$. □

1.4. *Clearings and forests.* When a cube C_m is of "clearing type" or "forest type" we define it in the following way: We introduce a number $r > 0$ and say that C_m is a clearing of size r if

$$\#^\varepsilon \tilde{U}_m \geq 4^{-d} \#^\varepsilon \overline{B}(0, r). \tag{1.12}$$

Observe that for the definition of clearings in both cases [site (resp., bond) obstacles] the counting measure on εZ^d is used. We then set $\mathscr{A}$ to be the closed union of all closed cubes $\overline{C}_m$ in R^d which are of clearing type of size r. We define $\mathscr{A}^1$ as the open set of points at distance less then 1 of $\mathscr{A}$. If $\mathscr{A}$ is empty, so is $\mathscr{A}^1$.

We can now define the operator semigroup which will be important in the sequel. Let $U \subset R^d$ be an arbitrary open set and T_U the entrance time in U^c. We set

$$\mathscr{L}_U := \{ f \in L^2(\varepsilon Z^d, \#^\varepsilon) \mid f(z) = 0 \ \forall \ z \in (U^c)^\varepsilon \} \tag{1.13}$$

and define

$$P_t^U \colon \mathscr{L}_U \to \mathscr{L}_U, \qquad f \mapsto P_t^U f := E_\bullet^\varepsilon [f(X_t^\varepsilon); \ t < T_U]. \tag{1.14}$$

If $U^\varepsilon \neq \varnothing$, $(P_t^U)_{t \geq 0}$ is a C_0 semigroup on $\mathscr{L}_U$. The generator L^ε of this semigroup [with the convention that $P_t^U = \exp(-tL^\varepsilon)$] is the bounded positive operator $-\Delta_\varepsilon^U$, where

$$\Delta_\varepsilon^U f(z) := \begin{cases} \dfrac{1}{2d\varepsilon^2} \displaystyle\sum_{\substack{e \in \varepsilon Z^d \\ |e| = \varepsilon}} (f(z+e) - f(z)), & \text{if } z \in U^\varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{1.15}$$

We denote by $\lambda^\varepsilon(U) \geq 0$ the bottom of the spectrum of $-\Delta_\varepsilon^U$ (Dirichlet discrete Laplacian in $U \cap \varepsilon Z^d$). We now introduce the entrance time into the obstacle set. In the site case we set

$$T_{\mathscr{H}^\varepsilon} := \inf\{ t \geq 0, \ X_t^\varepsilon \in \mathscr{H}^\varepsilon \}. \tag{1.16}$$

In the edge case we give the analogous definition as follows. For $t > 0$ we set $X_{t^-}^\varepsilon := \lim_{s \to t^-} X_s^\varepsilon$ and say that the process crosses the edge h at time $t > 0$, denoted by $X_t^\varepsilon \sim h$, if X_{t-}^ε and X_t^ε are the two different end vertices of h. So we define

$$T_{\mathscr{H}_*^\varepsilon} := \inf\{t > 0,\ X_t^\varepsilon \sim h \text{ for some } h \in \mathscr{H}_*^\varepsilon\}. \tag{1.17}$$

We shall investigate the effect of the obstacles in a nonempty open subset $\mathscr{T}$ of R^d on the principal eigenvalues. Neither the obstacles nor $\mathscr{T}$ should be viewed as fixed. In fact, our estimates will be uniform on $\mathscr{T}$ and on the obstacle sets. For the application of Section 2 we shall pick, for instance, $\mathscr{T} = (-N[t^{d/(d+2)}], N[t^{d/(d+2)}])^d$ and $\varepsilon = t^{-1/(d+2)}$, for t going to infinity.

Our coarse-grained picture Θ_b is now defined as the open set complement in $\mathscr{T} \cap \mathscr{A}^1$ of the enlarged good obstacles, that is,

$$\Theta_b := \mathscr{T} \cap \mathscr{A}^1 \setminus \bigcup_{h \in \mathscr{H}_{\text{good}}^\varepsilon} \overline{B}(v(h), b\varepsilon). \tag{1.18}$$

Finally we set $\tilde{T} := T_{\mathscr{T}} \wedge T_{\mathscr{H}^\varepsilon}$ and $\tilde{T}_* := T_{\mathscr{T}} \wedge T_{\mathscr{H}_*^\varepsilon}$, where $T_{\mathscr{T}}$ stands for the entrance time of X_t^ε in $\mathscr{T}^c$.

1.5. *The main result.* The main result coming out of the enlargement technique is the following exponential estimate for $\tilde{T}$ (resp., $\tilde{T}_*$).

THEOREM 1.4. *For $M > 0$ and $\rho > 0$ we have*

$$\begin{aligned} &\limsup_{r \to 0} \sup_{\substack{b > 2 \\ 0 < \delta < 1}} \limsup_{\varepsilon \to 0} \sup_{z, \mathscr{H}^\varepsilon, T} E_z^\varepsilon[\exp\{(\lambda^\varepsilon(\Theta_b) \wedge M - \rho)\, \tilde{T}\}] \\ &\qquad \le 1 + \frac{8}{3} C(d, M, \rho), \end{aligned} \tag{1.19}$$

$$\begin{aligned} &\limsup_{r \to 0} \sup_{\substack{b > 2 \\ 0 < \delta < 1}} \limsup_{\varepsilon \to 0} \sup_{z, \mathscr{H}_*^\varepsilon, T} E_z^\varepsilon[\exp\{(\lambda^\varepsilon(\Theta_b) \wedge M - \rho)\, \tilde{T}_*\}] \\ &\qquad \le 1 + \frac{8}{3} C(d, M, \rho), \end{aligned} \tag{1.20}$$

where $C(d, M, \rho)$ is defined below in (1.21).

In order to define the constant appearing in the r.h.s. of these inequalities, we need the following result:

LEMMA 1.5. *For $M > 0$, $\rho > 0$ we have*

$$\sup_{0 < \varepsilon < 1} \sup_{\substack{U \subset R^d \text{ open} \\ z \in \varepsilon Z^d}} E_z^\varepsilon[\exp\{(M \wedge \lambda^\varepsilon(U) - \rho) T_U\}] \vee 1 := C(d, M, \rho) < \infty. \tag{1.21}$$

PROOF. Pick $M > 0$, $\rho > 0$, $\varepsilon \in (0, 1)$, U an arbitrary open subset of R^d and $z \in \varepsilon Z^d$. We set $\mu := M \wedge \lambda^\varepsilon(U) - \rho$. With no loss of generality we assume that $\mu > 0$, $U^\varepsilon \neq \varnothing$. Let $r_\varepsilon(t, z, \cdot)$ be the density of P_t^U w.r.t. the counting measure on εZ^d. Then we have

$$\begin{aligned} E_z^\varepsilon[\exp(\mu T_U)] &= 1 + \mu \int_0^\infty P_z^\varepsilon[T_U > s] \exp(\mu s)\, ds \\ &= 1 + \mu \int_0^\infty \sum_{y \in U^\varepsilon} \exp(\mu s) r_\varepsilon(s, z, y)\, ds \\ &\le 1 + M \int_0^2 \sum_{y \in U^\varepsilon} \exp(\mu s) r_\varepsilon(s, z, y)\, ds \\ &\quad + M \int_2^\infty \sum_{y \in U^\varepsilon} \exp(\mu s) r_\varepsilon(s, z, y)\, ds \\ &\le 1 + M \int_0^2 \exp(Ms)\, ds \\ &\quad + M \int_2^\infty \sum_{y \in U^\varepsilon} \exp(\mu s) r_\varepsilon(s, z, y)\, ds. \end{aligned}$$

Therefore our claim will follow once we show that

$$\int_2^\infty \sum_{y \in U^\varepsilon} e^{\mu s} r_\varepsilon(s, z, y)\, ds < K(d, M, \rho) < \infty. \tag{1.22}$$

The main tools of the proof of (1.22) are the following results:

PROPOSITION 1.6. *There is a constant $K_1(d)$ such that*

$$\forall\, t > 2,\ \forall\, x, y \in Z^d, \qquad r_\varepsilon(t, x, y) \le K_1 \varepsilon^d \exp(-\lambda^\varepsilon(U)(t-2)). \tag{1.23}$$

PROPOSITION 1.7. *There is a constant $a > 0$ such that*

$$\forall\, \varepsilon \in (0,1),\ \forall\, c > 0, \qquad P_0^\varepsilon[T_s \le s] \le 2d e^{s(a-c)}, \tag{1.24}$$

where T_s denotes the exit time from the box $(-cs, cs)^d$.

Let us first show how these propositions imply (1.22). For this we define $c := M + a + 1$, where a is the constant appearing in Proposition 1.7. We also introduce the set

$$V_s := \{y \in \varepsilon Z^d | d(z, y) < sc\}.$$

Then the l.h.s. of (1.22) is equal to

$$\int_2^\infty \sum_{y \in V_s} e^{\mu s} r_\varepsilon(s, z, y)\, ds + \int_2^\infty \sum_{y \in V_s^c} e^{\mu s} r_\varepsilon(s, z, y)\, ds = I_1 + I_2. \tag{1.25}$$

Using (1.23) we see that

$$\begin{aligned}
I_1 &\le K_1 \int_2^\infty \exp(\mu s) \sum_{y\in V_s} \varepsilon^d \exp(-\lambda^\varepsilon(U)(s-2))\,ds \\
&\le K_1 \int_2^\infty \exp(\mu s)\exp(-\lambda^\varepsilon(U)(s-2))(2sc+1)^d\,ds \\
&= K_1 \int_0^\infty \exp(\mu(t+2))\exp(-\lambda^\varepsilon(U)t)(2ct+4c+1)^d\,dt \\
&\le K_1 e^{2M} \int_0^\infty \exp(t(\mu-\lambda^\varepsilon(U)))(2ct+4c+1)^d\,dt \\
&\le K_1 e^{2M} \int_0^\infty \exp(-\rho t)(2ct+4c+1)^d\,dt \\
&:= K_2(M,\rho,a) < \infty.
\end{aligned}$$

It now remains to estimate I_2. We have

$$\begin{aligned}
I_2 &\le \int_2^\infty e^{\mu s} P_z^\varepsilon[d(X_s^\varepsilon, z) \ge cs]\,ds \\
&\le \int_2^\infty e^{Ms} P_0^\varepsilon[T_s \le s]\,ds \\
&\le 2d \int_0^\infty e^{-s}\,ds = 2d,
\end{aligned}$$

where we have used Proposition 1.7 and the definition of c.

We still have to give the proofs of the two propositions we have used. We begin with Proposition 1.6. We denote by $(\cdot,\cdot)_\varepsilon$ the inner product on $L^2(\varepsilon Z^d, \#^\varepsilon)$ and by $\|\cdot\|_{2;\varepsilon}$ the corresponding norm. Then we have, for all $t > 2$ and $x \in \varepsilon Z^d$,

$$\begin{aligned}
r_\varepsilon(t,x,x) &= \left(P_{t-2}^U(r_\varepsilon(1,x,\cdot)),\ r_\varepsilon(1,x,\cdot)\right)_\varepsilon \\
&\le \|r_\varepsilon(1,x,\cdot)\|_{2;\varepsilon}^2\ \|P_{t-2}^U\| \\
&\le \exp(-\lambda^\varepsilon(U)(t-2))\|r_\varepsilon(1,x,\cdot)\|_{2;\varepsilon}^2 \\
&\le \exp(-\lambda^\varepsilon(U)(t-2))P_0^\varepsilon[X_2^\varepsilon = 0],
\end{aligned} \tag{1.26}$$

where we have used that $\|r_\varepsilon(1,x,\cdot)\|_{2;\varepsilon}^2 = r_\varepsilon(2,x,x) = P_0^\varepsilon[X_2^\varepsilon = 0]$. The asymptotic behavior of $P_0[S_t = 0]$ can be obtained classically (e.g., [10], page 78, Example 2) by integrating the characteristic function of S_t, which is given by

$$\phi_t(\theta) = \exp\left\{-t + \frac{t}{d}\sum_{i=1}^d \cos(\theta_i)\right\} \qquad (\theta \in R^d) \tag{1.27}$$

and consequently

$$P_0[S_t = 0] = (2\pi)^{-d}\int_0^{2\pi}\cdots\int_0^{2\pi} \phi_t(\theta)\,d\theta_1\cdots d\theta_d \sim \text{const}\cdot t^{-d/2}, \qquad \text{as } t\to\infty.$$

Applying this result to X_t^ε we can find a constant $K_1(d)$ with $P_0^\varepsilon[X_2^\varepsilon = 0] \le K_1\varepsilon^d$ and so, thanks to (1.26),

$$r_\varepsilon(t,x,x) \le K_1\varepsilon^d \exp(-\lambda^\varepsilon(U)(t-2)). \tag{1.28}$$

This implies (1.23), indeed

$$\begin{aligned} r_\varepsilon(t,x,y) &= \left(r_\varepsilon\left(\frac{t}{2},x,\cdot\right),\ r_\varepsilon\left(\frac{t}{2},y,\cdot\right)\right)_\varepsilon \\ &\le (r_\varepsilon(t,x,x))^{1/2}(r_\varepsilon(t,y,y))^{1/2} \\ &\le \sup_{z\in U^\varepsilon} r_\varepsilon(t,z,z). \end{aligned}$$

It now remains to prove Proposition 1.7. First we observe that

$$P_0^\varepsilon[T_s \le s] \le 2\,dP_0^\varepsilon[T_s^1 \le s], \tag{1.29}$$

where $T_s^1 := \inf\{t > 0,\ Y_t^\varepsilon \ge cs\}$, Y_t^ε being the first coordinate of X_t^ε. The process $(Y_t^\varepsilon)_{t\ge 0}$ is a one-dimensional simple random walk with jump intensity $1/d$. Therefore we know that

$$M_t := \exp\left\{Y_t^\varepsilon - \int_0^t \frac{(1/d)\triangle_\varepsilon(e^x)}{e^x}(Y_u^\varepsilon)\,du\right\}$$

is a supermartingale (see [7]), where $\triangle_\varepsilon$ stands for the one-dimensional discrete Laplacian. For twice differentiable functions f we have

$$\begin{aligned} \triangle_\varepsilon f(x) &= \frac{1}{2\varepsilon^2}(f(x+\varepsilon) - f(x) + f(x-\varepsilon) - f(x)) \\ &= \frac{1}{2}\int_0^1\int_0^1 f''(x-\varepsilon+\varepsilon u+\varepsilon v)\,du\,dv \end{aligned} \tag{1.30}$$

and therefore

$$\sup_{\substack{0<\varepsilon<1\\ x\in R}} \frac{(1/d)\triangle_\varepsilon(e^x)}{e^x} \le \sup_{0<\varepsilon<1} \frac{1}{2}\int_0^1\int_0^1 \exp(-\varepsilon+\varepsilon u+\varepsilon v)\,du\,dv \ := a < \infty. \tag{1.31}$$

Using this and the optional sampling theorem, we obtain

$$\begin{aligned} 1 &\ge E_0^\varepsilon[M_{s\wedge T_s^1}] \\ &\ge E_0^\varepsilon\big[\exp\{Y^\varepsilon_{T_s^1\wedge s} - a(T_s^1\wedge s)\}\big] \\ &\ge E_0^\varepsilon\big[\exp\{Y^\varepsilon_{T_s^1\wedge s} - as\}\big] \\ &\ge e^{-as}e^{cs}P_0^\varepsilon\big[Y^\varepsilon_{T_s^1\wedge s} \ge cs\big] \\ &\ge e^{-s(a-c)}P_0^\varepsilon\big[T_s^1 \le s\big], \end{aligned}$$

and together with (1.29) this implies our claim. □

1.6. *More notation.* Let us now define some constants which will be needed in the proof of Theorem 1.4. In what follows, P_z stands for the law of the discrete time random walk S_n on Z^d with starting point $z \in Z^d$. For $c > 0$ we define

$$H_c := \inf\{n \geq 0,\ d(S_n, S_0) \geq c\},$$
$$H_c^\varepsilon := \inf\{t \geq 0,\ d(X_t^\varepsilon, X_0^\varepsilon) \geq c\}. \tag{1.32}$$

We set

$$\beta(d, M, \rho) := \frac{1}{24C(d, M, \rho) + 2}. \tag{1.33}$$

As follows from Lemma 1.12 [see (1.60)], we have a strictly positive constant

$$C_1(d) := \tfrac{1}{2} \inf_{l \geq 1} \inf_{A} P_0[H_A < H_{20l\sqrt{d}}], \tag{1.34}$$

where A runs over all closed subsets of $\overline{B}(0, 2l)$ with relative volume (w.r.t. counting measure in Z^d!) $\geq 4^{-d}(1-(2/3)^d)$ and H_A denotes the entrance time in A.

With the help of Lemma 1.13 [see (1.87)], we pick $r \in (0, \frac{1}{4})$ small enough so that

$$(1 - C_1)^{[1/(22\sqrt{dr})]} \leq \beta \quad \text{and} \quad \sup_{0<\varepsilon<1} E_0^\varepsilon[\exp(MH_{\sqrt{r}}^\varepsilon)] \leq 1 + \beta. \tag{1.35}$$

Let $b > 2$ and $0 < \delta < 1$ be fixed. We denote by $H_{\{0\}}$ the hitting time of zero and by $H_{e_1}^*$ the first time when the process passes through the edge having one end vertex in zero and the other one in the point $(1, 0, \ldots, 0)$. For $u > 0$ we denote by H_u^0 the exit time from the ball $B(0, u)$. Then by Lemma 1.12 we have strictly positive constants

$$\alpha(\delta, b, C, d) := \inf_{l \geq 1} \inf_{\substack{z \in \tilde{B}(0, l+1) \\ A}} P_z[H_A < H_{10l\sqrt{d}}^0] \times \inf_{z \in \overline{B}(0,b)} P_z[H_{\{0\}} < H_{B(0,3b)^c}] \quad \text{(site case)}, \tag{1.36}$$

$$\alpha_*(\delta, b, C, d) := \inf_{l \geq 1} \inf_{\substack{z \in \tilde{B}(0, l+1) \\ A}} P_z[H_A < H_{10l\sqrt{d}}^0] \times \inf_{z \in \overline{B}(0,b)} P_z[H_{e_1}^* < H_{B(0,3b)^c}] \quad \text{(edge case)}. \tag{1.37}$$

In both cases, A runs over all closed subsets of $\overline{B}(0, l)$ with relative volume (for the counting measure in Z^d) bigger than $\delta/48^d$ and z has integer coordinates. We also set

$$q = q(\delta, b, C, d) := \inf\{n \in N\colon\ (1 - \alpha)^n \leq 1/8C(d, M, \rho)\}. \tag{1.38}$$

In the case of bonds, α is of course replaced by α_*. Finally we pick $0 < \varepsilon < 1$, so that

$$(10\sqrt{d})^{q+1} \varepsilon b + \varepsilon b < r < \tfrac{1}{4}. \tag{1.39}$$

We recall that now we have fixed $b > 2$, $0 < \delta < 1$ and $r \in (0, \frac{1}{4})$, $0 < \varepsilon < 1$ satisfying (1.35) and (1.39), respectively.

1.7. *More lemmas.* For the proof of Theorem 1.4, we need several lemmas which we are now going to prove. We formulate the statements for the site case; by consistent use of the notation * for bonds, we can see that the calculations in the bond case are exactly the same. In fact, there is only one point where the nature of the obstacles comes in, namely, the different definitions of the constants α and α^* in (1.36) [resp., (1.37)] which are used to estimate the probability appearing in (1.42) below.

LEMMA 1.8. *Let x be a good obstacle. Then we have for $z \in \overline{B}(v(x), \varepsilon b) \cap \mathscr{T}$,*

$$P_z^\varepsilon[H_r^\varepsilon > \tilde{T}] \geq \tfrac{1}{2}. \tag{1.40}$$

PROOF. Thanks to the choice of ε we have

$$\begin{aligned} P_z^\varepsilon[H_r^\varepsilon > \tilde{T}] &\geq P_z^\varepsilon[H^\varepsilon_{(10\sqrt{d})^{q+1}b\varepsilon + b\varepsilon} > \tilde{T}] \\ &\geq P_z^\varepsilon[H^{\varepsilon;x}_{(10\sqrt{d})^{q+1}b\varepsilon} > \tilde{T}] \end{aligned}$$

with $H_\rho^{\varepsilon;x} := \inf\{u \geq 0, d(X_u^\varepsilon, v(x)) \geq \rho\}$. Using the strong Markov property, we see that

$$\begin{aligned} &P_z^\varepsilon[H^{\varepsilon;x}_{(10\sqrt{d})^{q+1}b\varepsilon} < \tilde{T}] \\ &\quad \leq E_z^\varepsilon\left[I_{\{H^{\varepsilon;x}_{(10\sqrt{d})^q b\varepsilon} < T\}} P^\varepsilon_{X^\varepsilon_{H^{\varepsilon;x}_{(10\sqrt{d})^q b\varepsilon}}}\left[H^{\varepsilon;x}_{(10\sqrt{d})^{q+1}b\varepsilon} < T\right]\right]. \end{aligned} \tag{1.41}$$

We set $\tau_q := H^{\varepsilon;x}_{(10\sqrt{d})^q b\varepsilon}$ and denote by T_q the entrance time in $\mathscr{H}^\varepsilon \cap \overline{B}(v(x), (10\sqrt{d})^q b\varepsilon)$. We shall give an upper bound on $P^\varepsilon_{X^\varepsilon_{\tau_q}}[T > \tau_{q+1}]$ by getting a lower bound on

$$\begin{aligned} P^\varepsilon_{X^\varepsilon_{\tau_q}}[\tau_{q+1} > T] &\geq P^\varepsilon_{X^\varepsilon_{\tau_q}}[\tau_{q+1} > T_q] \\ &\geq E^\varepsilon_{X^\varepsilon_{\tau_q}}\left[I_{\{\tau_{q+1} > H_A\}} P^\varepsilon_{X^\varepsilon_{H_A}}[H_y < H_{B(y,3b)^c}]\right]. \end{aligned} \tag{1.42}$$

In the last term A denotes the set $\bigcup_{h \in \mathscr{H}^\varepsilon} \overline{B}(v(h), b\varepsilon) \cap \overline{B}(v(x), (10\sqrt{d})^q b\varepsilon)$ and $y := y(X^\varepsilon_{H_A})$ stands for a true obstacle such that $X^\varepsilon_{H_A} \in \overline{B}(v(y), b\varepsilon)$. Let C_m be the cube containing x. Since x is a good obstacle, we know that

$$\begin{aligned} \#^\varepsilon A &\geq \#^\varepsilon\left(\overline{B}(v(x), (10\sqrt{d})^q b\varepsilon) \cap \bigcup_{h \in C_m \cap \mathscr{H}^\varepsilon} \overline{B}(v(h), \varepsilon b) \cap C_m\right) \\ &\geq \frac{\delta}{12^d} \#^\varepsilon(\overline{B}(v(x), (10\sqrt{d})^q b\varepsilon) \cap C_m) \\ &\geq \frac{\delta}{48^d} \#^\varepsilon \overline{B}(v(x), (10\sqrt{d})^q b\varepsilon), \end{aligned}$$

where the last inequality follows by (1.11) and (1.39). Now using scaling, translation invariance and the definition of the constant α in (1.36), we see that the expression on the r.h.s. of (1.42) is bigger than α. Iterating this in (1.41) we have

$$P_z^\varepsilon[H^{\varepsilon;x}_{(10\sqrt{d})^{q+1}b\varepsilon} < \tilde{T}] \le (1-\alpha)^q \le \frac{1}{8C(d,M,\rho)} \quad \text{[by (1.38)]}$$

and therefore

$$P_z^\varepsilon[H^{\varepsilon;x}_{(10\sqrt{d})^{q+1}b\varepsilon} > \tilde{T}] \ge 1 - \frac{1}{8C(d,M,\rho)} > \frac{1}{2}$$

since $C < 1$ by definition [see (1.21)]. □

The next result we need is the following corollary.

COROLLARY 1.9. *For all $z \in \mathscr{A}^c \cap \varepsilon Z^d$ we have*

$$P_z^\varepsilon[\tilde{T} < H^\varepsilon_{21r\sqrt{d}}] \ge C_1. \tag{1.43}$$

PROOF. Pick $z \in C_m^\varepsilon \cap \mathscr{A}^c$ (forest). Then we have $\#^\varepsilon \tilde{U}_m \le 4^{-d}\,\#^\varepsilon \overline{B}(0,r)$. Set $V := \overline{B}(z,2r) \cap C_m \cap \tilde{U}_m^c$. Using (1.10), (1.11), (1.12) and $10\varepsilon < r < \frac{1}{4}$, we obtain

$$\begin{aligned}\#^\varepsilon V &\ge 4^{-d}\#^\varepsilon \overline{B}(z,2r) - 4^{-d}\#^\varepsilon \overline{B}(0,r)\\ &\ge 4^{-d}(1-(2/3)^d)\#^\varepsilon \overline{B}(z,2r).\end{aligned}$$

So by the definition of C_1 in (1.34) and using Lemma 1.8 we obtain

$$\begin{aligned}P_z^\varepsilon[\tilde{T} < H^\varepsilon_{21r\sqrt{d}}] &\ge P_z^\varepsilon[\{H_V < H^\varepsilon_{20r\sqrt{d}}\} \cap \{\tilde{T}\circ\theta_{H_V} < H^\varepsilon_{r\sqrt{d}}\circ\theta_{H_V}\}]\\ &\ge 2C_1 \cdot \frac{1}{2} = C_1\end{aligned}$$

□

We are now going to show the following lemma.

LEMMA 1.10. *For $z \in \varepsilon Z^d$ we have*

$$E_z^\varepsilon[\exp(M(H_{\mathscr{A}} \wedge \tilde{T}))] \le 1 + \frac{1}{8C(d,M,\rho)}. \tag{1.44}$$

PROOF. With no loss of generality we assume $z \notin \mathscr{A} \cup \mathscr{T}^c \cup \mathscr{H}^\varepsilon$. We denote the successive times of travel of $X^\varepsilon_\bullet$ at distance $\sqrt{r}$ by

$$H^0 := 0, \qquad H^{i+1} := H^i + H^\varepsilon_{\sqrt{r}}\circ\theta_{H^i}, \qquad i \ge 0. \tag{1.45}$$

Thanks to our assumption on z, $H_{\mathscr{A}} \wedge \tilde{T} > 0$ P_z^ε-a.s. and we have

$$
\begin{aligned}
& E_z^\varepsilon[\exp(M(H_{\mathscr{A}} \wedge \tilde{T}))] \\
& \quad = \sum_{k \geq 0} E_z^\varepsilon[\exp(M(H_{\mathscr{A}} \wedge \tilde{T}));\ H^k < H_{\mathscr{A}} \wedge \tilde{T} \leq H^{k+1}] \\
& \quad \leq \sum_{k \geq 0} E_z^\varepsilon[\exp(MH^k)\, I_{\{H_{\mathscr{A}} \wedge \tilde{T} > H^k\}} \cdot E^\varepsilon_{X^\varepsilon_{H^k}}[\exp(MH^\varepsilon_{\sqrt{r}})]] \\
& \quad = \sum_{k \geq 0} E_z^\varepsilon[\exp(MH^k);\ H_{\mathscr{A}} \wedge \tilde{T} > H^k] \cdot E_0^\varepsilon[\exp(MH^\varepsilon_{\sqrt{r}})].
\end{aligned} \tag{1.46}
$$

We now look at the first factor in the last sum. We have, for $k \geq 1$,

$$
\begin{aligned}
& E_z^\varepsilon[\exp(MH^k);\ H_{\mathscr{A}} \wedge \tilde{T} > H^k] \\
& \quad = E_z^\varepsilon[\exp(MH^{k-1})\exp(MH^\varepsilon_{\sqrt{r}} \circ \theta_{H^{k-1}});\ H_{\mathscr{A}} \wedge \tilde{T} > H^k] \\
& \quad \leq E_z^\varepsilon\big[\exp(MH^{k-1})\big(E^\varepsilon_{X_{H^{k-1}}}[H^\varepsilon_{\sqrt{r}} < H_{\mathscr{A}} \wedge \tilde{T}] \\
& \qquad\qquad + E^\varepsilon_{X_{H^{k-1}}}[\exp(MH^\varepsilon_{\sqrt{r}})] - 1\big);\ H_{\mathscr{A}} \wedge \tilde{T} > H^{k-1}\big].
\end{aligned} \tag{1.47}
$$

On the set $\{H_{k-1} < H_{\mathscr{A}} \wedge \tilde{T}\}$ we have $X_{H^{k-1}} \in \mathscr{A}^c$, so using (1.43) and the strong Markov property at the successive times of travel at distance $21r\sqrt{d}$ we find

$$
E^\varepsilon_{X_{H^{k-1}}}\big[H^\varepsilon_{\sqrt{r}} < H_{\mathscr{A}} \wedge \tilde{T}\big] < (1 - C_1)^{[1/(22\sqrt{dr})]} \leq \beta \quad [\text{by } 1.35] \tag{1.48}
$$

since $d(X_0^\varepsilon, X^\varepsilon_{H^\varepsilon_{21r\sqrt{d}}}) \leq 21r\sqrt{d} + \varepsilon$ and $[\sqrt{r}/(21r\sqrt{d} + \varepsilon)] \geq [1/(22\sqrt{dr})]$. Using (1.48) and (1.35) we see that the r.h.s. of the last inequality in (1.47) is smaller than

$$
2\beta E_z^\varepsilon[\exp(MH^{k-1});\ H^{k-1} < H_{\mathscr{A}} \wedge \tilde{T}] \leq (2\beta)^k.
$$

Substituting this in (1.46) and using (1.35) as well as the definition of β in (1.33) we finally have

$$
\begin{aligned}
E_z^\varepsilon[\exp(M(H_{\mathscr{A}} \wedge \tilde{T}))] & \leq (1 + \beta) \sum_{k=0}^{\infty} (2\beta)^k \\
& = 1 + \frac{3\beta}{1 - 2\beta} \\
& \leq 1 + \frac{1}{8C(d, M, \rho)}.
\end{aligned}
$$

□

1.8. *The proof of Theorem* 1.4. We now apply the results of the previous subsection to give the promised bound on $E_z^\varepsilon[\exp\{(\lambda^\varepsilon(\Theta_b) \wedge M - \rho)\tilde{T}\}]$ for $z \in R^d$. As before, we use the notation for the site case. It is enough to study the case where $\lambda := \lambda^\varepsilon(\Theta_b) \wedge M - \rho > 0$. In this case we introduce the stopping time

$$\tau := \begin{cases} H_r^\varepsilon \wedge \tilde{T}, & \text{if } X_0 \in \mathscr{A}^1, \\ H_{\mathscr{A}} \wedge \tilde{T}, & \text{if } X_0 \notin \mathscr{A}^1. \end{cases}$$

We denote by T_b the exit time from Θ_b and set

$$S_0 = 0, \qquad S_1 := \tau \circ \theta_{T_b} + T_b, \qquad S_{k+1} = S_k + S_1 \circ \theta_{S_k}, \qquad k \geq 1,$$

$$J := \inf\{k \geq 0,\ X_{S_k} \in \mathscr{T}^c \cup \mathscr{H}^\varepsilon\}.$$

The S_k are a.s. finite, thanks to Lemmas 1.10 and 1.5. We shall see that we have

$$E_z^\varepsilon[\exp(\lambda S_k);\ \{J > k\}] \leq \left(\tfrac{1}{4}\right)^k, \qquad k \geq 0, \tag{1.49}$$

from which it follows that J is finite almost surely, since $\lambda > 0$. We now see that

$$\begin{aligned} &E_z^\varepsilon[\exp(\lambda \tilde{T})] \\ &\quad \leq E_z^\varepsilon[\exp(\lambda S_J)] = \sum_{k=0}^{\infty} E_z^\varepsilon[\exp(\lambda S_k);\ J = k] \\ &\quad \leq 1 + \sum_{k=1}^{\infty} E_z^\varepsilon[\exp(\lambda S_k;\ \{J > k-1\}] \\ &\quad \leq 1 + \sum_{k=1}^{\infty} E_z^\varepsilon\Bigg[\exp(\lambda S_{k-1}) E_{X^\varepsilon_{S_{k-1}}}^\varepsilon \Big[\exp(\lambda T_b) E_{X^\varepsilon_{T_b}}^\varepsilon [\exp(\lambda\tau)]\Big]; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \{J > k-1\}\Bigg]. \end{aligned}$$

Using Lemma 1.10, Lemma 1.5 and (1.49) we see that the right member of the last inequality is smaller than

$$1 + 2C \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = 1 + \frac{8}{3} C$$

and this yields the claim of Theorem 1.4.

We still have to give the proof of (1.49). Let $\mathscr{E}_k$ be the set $\{J > k\}$. Then for $k \geq 1$,

$$\begin{aligned} &E_z^\varepsilon[\exp(\lambda S_k);\ \mathscr{E}_k] \\ &\quad = E_z^\varepsilon\Big[\exp(\lambda S_{k-1}) E_{X^\varepsilon_{S_{k-1}}}^\varepsilon [\exp(\lambda S_1);\ \mathscr{E}_1];\ \mathscr{E}_{k-1}\Big] \\ &\quad \leq E_z^\varepsilon\Bigg[\exp(\lambda S_{k-1}) E_{X^\varepsilon_{S_{k-1}}}^\varepsilon \Big[\exp(\lambda T_b) E_{X^\varepsilon_{T_b}}^\varepsilon [\exp(\lambda\tau) I_{\{X_\tau^\varepsilon \notin \mathscr{T}^c \cup \mathscr{H}^\varepsilon\}}]\Big]; \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{E}_{k-1}\Bigg] \end{aligned} \tag{1.50}$$

Observe that

$$
\begin{aligned}
(1.51)\qquad & E^{\varepsilon}_{X^{\varepsilon}_{S_{k-1}}}\left[\exp(\lambda T_b) E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[\exp(\lambda\tau) I_{\{X^{\varepsilon}_{\tau}\notin \mathscr{T}^c\cup \mathscr{H}^{\varepsilon}\}}]\right] \\
&\quad \le E^{\varepsilon}_{X^{\varepsilon}_{S_{k-1}}}\left[\exp(\lambda T_b)\left(E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[\exp(\lambda\tau)] - 1 + E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[I_{\{X^{\varepsilon}_{\tau}\notin \mathscr{T}^c\cup \mathscr{H}^{\varepsilon}\}}]\right)\right].
\end{aligned}
$$

To estimate the r.h.s. of (1.51) we first show

$$
(1.52)\qquad E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[e^{\lambda\tau}] - 1 \le \frac{1}{8C}.
$$

If $X^{\varepsilon}_{T_b} \notin \mathscr{A}^1$, our claim follows from (1.44). In the other case we have

$$
\begin{aligned}
E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[\exp(\lambda\tau)] &= E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[\exp(\lambda(H^{\varepsilon}_r \wedge \tilde{T}))] \\
&\le E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[\exp(\lambda H^{\varepsilon}_{\sqrt{r}})] \\
&\le 1 + \beta \le 1 + \frac{1}{8C}.
\end{aligned}
$$

The other estimate we need is

$$
(1.53)\qquad E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[I_{\{X^{\varepsilon}_{\tau}\notin \mathscr{T}^c\cup \mathscr{H}^{\varepsilon}\}}] \le \frac{1}{8C}.
$$

To see this we have to distinguish three cases: If $X^{\varepsilon}_{T_b} \in \mathscr{T}^c$, then the left-hand side of (1.53) is equal to 0. If $X^{\varepsilon}_{T_b} \in \mathscr{A}^1 \cup \mathscr{T}$, we have, by the same argument as in the proof of Lemma 1.8,

$$
\begin{aligned}
E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[I_{\{X^{\varepsilon}_{\tau}\notin \mathscr{T}^c\cup \mathscr{H}^{\varepsilon}\}}] &= P^{\varepsilon}_{X^{\varepsilon}_{T_b}}[H^{\varepsilon}_r < \tilde{T}] \\
&\le P^{\varepsilon}_{X^{\varepsilon}_{T_b}}[H^{\varepsilon}_{(10\sqrt{d})^{q+1}b\varepsilon + b\varepsilon} < \tilde{T}] \\
&\le (1-\alpha)^q \le \frac{1}{8C}.
\end{aligned}
$$

If $X^{\varepsilon}_{T_b} \in (\mathscr{A}^1)^c \cap \mathscr{T}$, we see, using (1.43) and the strong Markov property,

$$
\begin{aligned}
E^{\varepsilon}_{X^{\varepsilon}_{T_b}}[I_{\{X^{\varepsilon}_{\tau}\notin \mathscr{T}^c\cup \mathscr{H}^{\varepsilon}\}}] &= P^{\varepsilon}_{X^{\varepsilon}_{T_b}}[\tilde{T} > H_{\mathscr{A}}] \\
&\le (1 - C_1)^{[1/22r\sqrt{d}]} \\
&\le \beta < \frac{1}{8C} \quad \text{[by (1.35)]}.
\end{aligned}
$$

Combining (1.50), (1.51), (1.52) and (1.53) and using Lemma 1.5, we obtain for $k \ge 1$,

$$
\begin{aligned}
E^{\varepsilon}_z[\exp(\lambda S_k);\ \{J > k\}] &\le E^{\varepsilon}_z\left[\exp(\lambda S_{k-1}) E^{\varepsilon}_{X^{\varepsilon}_{S_{k-1}}}\left[\frac{1}{4C}\exp(\lambda T_b)\right]\right] \\
&\le \frac{1}{4} E^{\varepsilon}_z[\exp(\lambda S_{k-1});\ J > k-1]
\end{aligned}
$$

and iterating this yields (1.49). □

1.9. *Lower bounds on eigenvalues.* We shall now apply our result to derive lower bounds on eigenvalues in the same spirit as in [13]. We introduce the semigroups associated with the process killed by the true obstacles in $\mathscr{T}$. In the site case, we let $\tilde{P}_t$ stand for this semigroup, that is, we define

$$\tilde{P}_t\colon f \mapsto \tilde{P}_t f := E^{\varepsilon}_{\bullet}[f(X^{\varepsilon}_t);\ t < \tilde{T}],$$

which is a C_0 semigroup on the subspace of $L^2(\varepsilon Z^d, \#^{\varepsilon})$ consisting of functions which vanish on the set $\mathscr{H}^{\varepsilon}$ and outside of $\mathscr{T}$. The generator L^{ε} is the discrete Laplacian [see (1.15)] and we denote its lowest eigenvalue by $\lambda^{\varepsilon}(\mathscr{H}^{\varepsilon}, \mathscr{T}) \geq 0$.

In the edge case we define

$$\tilde{P}^*_t\colon f \mapsto \tilde{P}^*_t f := E^{\varepsilon}_{\bullet}[f(X^{\varepsilon}_t);\ t < \tilde{T}^*],$$

which is a C_0 semigroup on the subspace of $L^2(\varepsilon Z^d, \#^{\varepsilon})$ consisting of all functions which vanish outside of $\mathscr{T}$. Its generator is L^{ε}_*, where

$$L^{\varepsilon}_* f(z) := \begin{cases} -\dfrac{1}{2d\varepsilon^2} \displaystyle\sum_{\substack{|e|=\varepsilon \\ (z,z+e)\notin \mathscr{H}^{\varepsilon}_*}} f(z+e) + \dfrac{1}{\varepsilon^2} f(z), & \text{if } z \in \mathscr{T}^{\varepsilon} \\ 0, & \text{otherwise.} \end{cases}$$

We denote the lowest eigenvalue of L^{ε}_* by $\lambda^{\varepsilon}_*(\mathscr{H}^{\varepsilon}_*, \mathscr{T}) \geq 0$. We can also introduce the corresponding Dirichlet forms. In fact we have, in the site case,

$$\mathscr{E}(f,f) = (L^{\varepsilon} f, f)_{\varepsilon} = \frac{1}{4\,d\varepsilon^2} \sum_{x\in\varepsilon Z^d} \sum_{|e|=\varepsilon} (f(x+e) - f(x))^2 \tag{1.54}$$

and similarly, in the edge case,

$$\begin{aligned} \mathscr{E}_*(f,f) &= (L^{\varepsilon}_* f, f)_{\varepsilon} \\ &= \frac{1}{4\,d\varepsilon^2} \sum_{x\in\varepsilon Z^d} \sum_{\substack{|e|=\varepsilon \\ (x,x+e)\notin \mathscr{H}^{\varepsilon}_*}} (f(x+e) - f(x))^2 \\ &\quad + \frac{1}{2\,d\varepsilon^2} \sum_{x\in\varepsilon Z^d} f(x)^2 \#\{e\colon |e| = \varepsilon,\ (x, x+e) \in H^{\varepsilon}_*\}. \end{aligned} \tag{1.55}$$

For the principal eigenvalues we have the following variational formulae:

$$\begin{aligned} &\lambda^{\varepsilon}(\mathscr{H}^{\varepsilon}, \mathscr{T}) \\ &\quad = \inf\{\mathscr{E}(f,f)\colon f \in L^2(\varepsilon Z^d, \#^{\varepsilon}),\ \operatorname{supp} f \subset \mathscr{T} \cap (\mathscr{H}^{\varepsilon})^c,\ \|f\|_{2;\varepsilon} = 1\}, \end{aligned} \tag{1.56}$$

$$\begin{aligned} &\lambda^{\varepsilon}_*(\mathscr{H}^{\varepsilon}_*, \mathscr{T}) \\ &\quad = \inf\{\mathscr{E}_*(f,f)\colon f \in L^2(\varepsilon Z^d, \#^{\varepsilon}),\ \operatorname{supp} f \subset \mathscr{T},\ \|f\|_{2;\varepsilon} = 1\}. \end{aligned} \tag{1.57}$$

We can now show that when r is small, for sufficiently small ε the principal eigenvalue $\lambda^{\varepsilon}(\mathscr{H}^{\varepsilon}, \mathscr{T})$ [respectively $\lambda^{\varepsilon}_*(\mathscr{H}^{\varepsilon}_*, \mathscr{T})$ in the edge case] is not really bigger than the principal eigenvalue $\lambda^{\varepsilon}(\Theta_b)$, provided $\lambda^{\varepsilon}(\mathscr{H}^{\varepsilon}, \mathscr{T})$ [resp. $\lambda^{\varepsilon}_*(\mathscr{H}^{\varepsilon}_*, \mathscr{T})$] has a reasonable value.

COROLLARY 1.11. *For $M > 0$ we have*

$$\lim_{r\to 0} \sup_{\substack{b>2\\ 0<\delta<1}} \limsup_{\varepsilon\to 0} \sup_{\mathscr{H}^\varepsilon,\mathscr{T}} (\lambda^\varepsilon(\Theta_b) \wedge M - \lambda^\varepsilon(\mathscr{H}^\varepsilon, \mathscr{T}) \wedge M)_+ = 0, \tag{1.58}$$

$$\lim_{r\to 0} \sup_{\substack{b>2\\ 0<\delta<1}} \limsup_{\varepsilon\to 0} \sup_{\mathscr{H}_*^\varepsilon,\mathscr{T}} (\lambda^\varepsilon(\Theta_b) \wedge M - \lambda_*^\varepsilon(\mathscr{H}_*^\varepsilon, \mathscr{T}) \wedge M)_+ = 0. \tag{1.59}$$

PROOF. We prove (1.58). The proof of (1.59) is completely analogous. Let us assume that for some $M > 0$ the expression above is strictly positive. That is,

$$(\exists\ \rho > 0)(\exists\ M > 0)(\forall\ r_0 > 0)(\exists\ r < r_0)$$
$$(\exists\ b > 2, \delta)(\forall\ \varepsilon_0 > 0)(\exists\ \varepsilon < \varepsilon_0)(\exists\ \mathscr{H}^\varepsilon)(\exists\ \mathscr{T})$$

so that

$$\lambda^\varepsilon(\Theta_b) \wedge M - \rho > \lambda^\varepsilon(\mathscr{H}^\varepsilon, \mathscr{T}).$$

With no loss of generality, we can assume that there is a nonnegative function f in $L^2(\varepsilon Z^d)$ with norm 1 and compact support in $\mathscr{T} \cap (\mathscr{H}^\varepsilon)^c$ such that

$$\lambda^\varepsilon(\mathscr{H}^\varepsilon, \mathscr{T}) \leq \mathscr{E}(f, f) < \lambda^\varepsilon(\Theta_b) \wedge M - \rho.$$

Let $r(s, x, y)$ be the transition density of $\tilde{P}_t$ and $E_\varepsilon(\cdot)$ a resolution of the identity corresponding to this semigroup. We set $E^\varepsilon_{f,f}(\cdot) := (f, E_\varepsilon(\cdot) f)_\varepsilon$. With $\lambda := \lambda^\varepsilon(\Theta_b) \wedge M - \rho$ we see that

$$\begin{aligned}
\infty &= \int_0^\infty ds\, \lambda \exp(\{\lambda - \int_0^\infty \mu\, E^\varepsilon_{f,f}(d\mu)\} s) \\
&\leq \int_0^\infty ds\, \lambda e^{\lambda s} \int_0^\infty e^{-\mu s} E^\varepsilon_{f,f}(d\mu) \quad \text{(by Jensen's inequality)} \\
&= \int_0^\infty ds\, \lambda e^{\lambda s} \sum_{x\in\mathscr{T}^\varepsilon} \sum_{y\in\mathscr{T}^\varepsilon} r(s, x, y) f(x) f(y) \\
&= \int_0^\infty ds\, \lambda e^{\lambda s} \sum_{x\in\mathscr{T}^\varepsilon} f(x) \tilde{P}_s f(x) \\
&\leq \int_0^\infty ds\, \lambda e^{\lambda s} \sum_{x\in\mathscr{T}^\varepsilon} f(x) \|f\|_\infty P_x[\tilde{T} > s] \\
&= \sum_{x\in\mathscr{T}^\varepsilon} E^\varepsilon_x\left[\int_0^{\tilde{T}} \lambda e^{\lambda s}\, ds\right] f(x)\, \|f\|_{\infty;\varepsilon} \\
&\leq \sup_{x\in\mathscr{T}^\varepsilon} E^\varepsilon_x[e^{\lambda \tilde{T}}] \|f\|_{\infty;\varepsilon}\, \|f\|_{1;\varepsilon}.
\end{aligned}$$

By Theorem 1.4 we can assume that our parameters are chosen such that the r.h.s. of the last inequality remains finite, which is a contradiction. □

1.10. *Two more lemmas.* We now prove the two lemmas we used in the definition of constants appearing in Section 1.5. The first one refers to the discrete time random walk on Z^d. Let us recall that for $u > 0$ we denote by H_u^0 the exit time of this process from the ball $B(0,u)$.

LEMMA 1.12. *For $s > 0$ we have*

$$\inf_{l\geq 1} \inf_{\substack{z\in B(0,l+1)\\ A\in \mathscr{A}_l^s}} P_z[H_A < H^0_{10\sqrt{d}l}] = \gamma > 0 \quad (l \text{ is a real parameter}), \tag{1.60}$$

where $\mathscr{A}_l^s := \{A \subset \overline{B}(0,l)\colon \#A \geq s\,\#\overline{B}(0,l)\}$.

PROOF. For $u \in R^+$ we denote by $g(u,x,y)$ the Green's function of S_n killed outside of $B(0,u)$. That is, $g(u,x,y) = \sum_{i=0}^{\infty} P_x[S_i = y,\ i < H_u^0]$. At this point we remark that by translation invariance we have

$$g(u,x,y) = g_{-y}(u, x-y, 0), \tag{1.61}$$

where $g_{-y}(u,\cdot,\cdot)$ denotes the Green's function of the process killed outside of the ball $B(-y,u)$. We now have, for $l \geq 1$, $z \in B(0,l+1)$ and $A \in \mathscr{A}_l^s$,

$$\begin{aligned} P_z[H_A < H^0_{10\sqrt{d}l}] &= \int_A g(10\sqrt{d}l, z, y)\, e(dy) \\ &\geq \inf_{z,y\in B(0,2l)} g(10\sqrt{d}l, z, y)\ \mathrm{Cap}_{10\sqrt{d}l}(A). \end{aligned} \tag{1.62}$$

Here e stands for the relative equilibrum measure of A to the ball $B(0, 10\sqrt{d}l)$ and $\mathrm{Cap}_{10\sqrt{d}l}(A)$ for the relative capacity of A in this ball. By Dirichlet's principle we know that

$$\left(\mathrm{Cap}_{10\sqrt{d}l}(A)\right)^{-1} = \inf_{\mu} \int_A g(10\sqrt{d}l, z, y)\, \mu(dz)\, \mu(dy), \tag{1.63}$$

where μ runs over all probability measures on A.

Let us first treat the case of *recurrent random walk* ($d \leq 2$). Then the potential kernel

$$a(x) := \sum_{i=0}^{\infty} P_0[S_i = 0] - P_0[S_i = x] \tag{1.64}$$

exists for all $x \in Z^d$, and we have (see [8], Proposition 1.6.3)

$$g(u,z,y) = E_z[a(S_{H_u^0} - y)] - a(z-y). \tag{1.65}$$

First we look at the case $d = 1$. Then we know that $a(x) = |x|$, $x \in Z$. Pick $l \geq 1$. For $z, y \in B(0,2l)$ we have $|S_{H_{10l}} - y| \geq 8l$ and $|z-y| \leq 4l$ and, therefore,

$$\inf_{z,y\in B(0,2l)} g(10l, z, y) \geq 4l.$$

Pick $A \in \mathscr{A}_l^s$ and $x \in A$. Then the relative energy of the Dirac measure in x is

$$I_l(\delta_x) = g(10l, x, x) \le 11l + 1$$

since $g(10l, x, x) = |S_{H_{10l}} - x| \le 11l + 1$ if $x \in \overline{B}(0, l)$. Thus by (1.63) we have $\mathrm{Cap}_{10l}(A) \ge 1/(11l + 1)$ and using (1.62) we obtain

$$P_z[H_A < H_{10l}] \ge \frac{4l}{11l + 1} \ge \frac{1}{3} > 0. \tag{1.66}$$

Consider now the case $d = 2$: For $x, y \in B(0, 2l)$ we have, by (1.61),

$$\begin{aligned} g(10\sqrt{2}l, x, y) &= g_{-y}(10\sqrt{2}l, x - y, 0) \\ &\ge g(10l, x - y, 0) \end{aligned} \tag{1.67}$$

since $B(0, 10l) \subset B(-y, 10\sqrt{2}l)$. Thus by (1.62) we have to show

$$\inf_{l \ge 1} \inf_{z \in B(0,4l)} g(10l, z, 0)\, \mathrm{Cap}_{10\sqrt{2}l}(A) > 0. \tag{1.68}$$

It is a classical result (see [10], Chapter III.12, Proposition 3) that

$$a(z) = \frac{2}{\pi} \log |z| + k + o(1) \qquad (|z| \to \infty), \tag{1.69}$$

where k is a constant, which can be explicitly calculated, but we will not need the exact value of k. Let $\eta > 0$. Then we can find a number $l_0 \in N$, such that for $|z| > l_0$,

$$\frac{2}{\pi} \log |z| + k - \eta \le a(z) \le \frac{2}{\pi} \log |z| + k + \eta. \tag{1.70}$$

We now have

$$\inf_{l \le l_0} \inf_{z \in B(0,4l)} \inf_{A \in \mathscr{A}_l^s} g(10l, z, 0)\, \mathrm{Cap}_{10\sqrt{2}l}(A) \ge \mathrm{const} > 0 \tag{1.71}$$

since we have only finitely many cases.

In the case $l > l_0$ we give separate estimates on the Green's function and the capacity. We will show that both are strictly positive. For the Green's function we obtain

$$\inf_{l > l_0} \inf_{z \in B(0,l_0)} g(10l, z, 0) \ge \inf_{z \in B(0,l_0)} g(10l_0, z, 0) > 0. \tag{1.72}$$

Using (1.65) and (1.70) we see that we also have

$$\begin{aligned} \inf_{l > l_0} \inf_{\substack{z \in B(0,4l) \\ |z| > l_0}} g(10l, z, 0) &\ge \inf_{l > l_0} \inf_{\substack{z \in B(0,4l) \\ |z| > l_0}} \frac{2}{\pi}(\log(10l) - \log |z|) - 2\eta \\ &\ge \frac{2}{\pi} \log\left(\frac{10}{4\sqrt{2}}\right) - 2\eta > 0 \end{aligned} \tag{1.73}$$

since η was arbitrary.

To estimate the capacity we use again (1.63). Pick $A \in \mathscr{A}_l^s$ and let μ be the discrete uniform distribution on A. Then we have for the relative energy

$$\tag{1.74} \begin{aligned} I_l(\mu) &= (\#A)^{-2} \sum_{x,y\in A} g(10\sqrt{2}l, x, y) \\ &\le \sup_{x\in A} (\#A)^{-1} \sum_{y\in A} g(10\sqrt{2}l, x, y). \end{aligned}$$

We can assume $l_0 > 5$ and so using (1.61) and (1.10), the last expression becomes less than

$$\tag{1.75} \begin{aligned} &\frac{1}{s}\frac{1}{\#\overline{B}(0,l)} \sum_{z\in\overline{B}(0,2l)} g(11\sqrt{2}l, 0, z) \\ &\le \frac{16}{s}\frac{1}{\#\overline{B}(0,2l)} \sum_{z\in\overline{B}(0,2l)} g(11\sqrt{2}l, 0, z) \\ &\le \frac{16}{s}\frac{1}{\#\overline{B}(0,2l)} \sum_{z\in\overline{B}(0,l_0)} g(11\sqrt{2}l, 0, z) \\ &\quad + \frac{16}{s}\frac{1}{\#\overline{B}(0,2l)} \sum_{z:\ l_0<d(0,z)\le 2l} g(11\sqrt{2}l, 0, z). \end{aligned}$$

The first term in the last expression is a finite sum and therefore less than

$$\mathrm{const}(\#\overline{B}(0,2l))^{-1} g(11\sqrt{2}l, 0, 0),$$

which remains finite for $l \to \infty$ thanks to (1.65) and (1.70).

To estimate the second term we remark that $|S_{H^0_{11\sqrt{2}l}}| \le \sqrt{2}(11\sqrt{2}l+1) \le 24l$, so using (1.65), (1.70) and $l > l_0$ we see that if $|z| > l_0$,

$$\tag{1.76} g(11\sqrt{2}l, z, 0) = E_z[a(S_{H^0_{11\sqrt{2}l}})] - a(z)$$

$$\tag{1.77} \begin{aligned} &\le \frac{2}{\pi}(\log(24l) - \log|z|) + 2\eta \\ &< \frac{2}{\pi}\log\left(\frac{25l}{|z|}\right) \end{aligned}$$

since we can assume that $e^{\pi\eta} < 25/24$. Therefore the second term in the last line of (1.75) is less than

$$\frac{16}{s}\frac{1}{\#\overline{B}(0,2l)} \sum_{z:\ l_0<d(0,z)\le 2l} \log\left(\frac{25l}{|z|}\right) \le \mathrm{const} \sum_{k=1}^{2l} \frac{k}{(2l)^2} \log\left(\frac{25l}{k}\right)$$

since the cardinality of $\{z\colon k = d(0,z)\}$ grows linearly with k. The last expression is a Riemann sum convergent to the finite integral

$$\mathrm{const} \int_0^1 x \log\left(\frac{25}{2x}\right) dx < \infty$$

and therefore uniformly bounded by some constant. We obtain $\sup_{l>l_0} I_l(\mu) < \infty$ and our claim follows by (1.63).

We now treat the case of *transient random walk* ($d \geq 3$): Using (1.61) we obtain

$$\inf_{l\geq 1} \inf_{x,y\in B(0,2l)} g(10\sqrt{d}l, x, y) \geq \inf_{l\geq 1} \inf_{z\in B(0,4l)} g(8\sqrt{d}l, z, 0) \tag{1.78}$$

since for $y \in B(0, 2l)$ we have $B(0, 8\sqrt{d}l) \subset B(-y, 10\sqrt{d}l)$. For transient random walk the free Green's function $g(x, y)$ exists and we have (see [8], Proposition 1.5.8)

$$g(u, x, y) = g(x, y) - E_x[g(S_{H_u^0}, y)]. \tag{1.79}$$

The asymptotic behavior of $g(\cdot, \cdot)$ is well known; we have, in fact (see [8], Theorem 1.5.4),

$$g(0, z) \sim a_d |z|^{2-d} \quad \text{as } |z| \to \infty, \tag{1.80}$$

a_d being a constant depending only on dimension. So there is a number l_0 such that for $|z| > l_0$,

$$\frac{9}{10} a_d |z|^{2-d} \leq g(0, z) \leq \frac{11}{10} a_d |z|^{2-d}. \tag{1.81}$$

By the same argument as in (1.71) it is enough to look at the case $l > l_0$. As in (1.72) we have

$$\inf_{l>l_0} \inf_{z\in B(0,l_0)} g(8\sqrt{d}l, z, 0) > 0. \tag{1.82}$$

Using (1.79) and (1.81) we have, for $l > l_0$,

$$\inf_{\substack{z\in B(0,4l)\\ |z|>l_0}} g(8\sqrt{d}l, z, 0) \geq \frac{1}{10} a_d [9(4l\sqrt{d})^{2-d} - 11(8l\sqrt{d})^{2-d}] \geq k_d\, l^{2-d}, \tag{1.83}$$

where $k_d = (1/10) a_d [9(4\sqrt{d})^{2-d} - 11(8\sqrt{d})^{2-d}] > 0$. Our claim will follow if we show that there is a constant $\tilde{k}_d > 0$, such that for $l > l_0$,

$$\sup_{A\in\mathscr{A}_l^s} \mathrm{Cap}_{10\sqrt{d}l}(A) \geq \tilde{k}_d l^{d-2}. \tag{1.84}$$

We use the same strategy as in the recurrent case. We pick $A \in \mathscr{A}_l^s$ and let μ be the discrete uniform distribution on A. Then we have for the energy

$$\begin{aligned} I_l(\mu) &= (\#A)^{-2} \sum_{x,y\in A} g(10\sqrt{d}l, x, y) \\ &\leq (\#A)^{-2} \sum_{x,y\in A} g(x, y). \end{aligned} \tag{1.85}$$

Assuming $l_0 > 5$ we see by (1.10) and and (1.81) that the last expression is less than

$$s^{-1}4^{-d}\frac{1}{\#\bar{B}(0,2l)}\sum_{z\in\bar{B}(0,2l)} g(0,z)$$
$$\leq \text{const}\, l^{-d} \sum_{z\in\bar{B}(0,l_0)} g(0,z) + \text{const}\, l^{-d} \sum_{\substack{z\in\tilde{B}(0,2l)\\ |z|>l_0}} |z|^{2-d}.$$

Since the cardinality of the set $\{z \in Z^d,\ k = d(0,z)\}$ grows with k as a polynomial of degree $d-1$, the right-hand side of the last inequality is less than

$$\text{const}\, l^{-d}\sum_{k=1}^{2l} k^{2-d}k^{d-1} \leq \text{const}\, l^{2-d} \tag{1.86}$$

and by (1.63) our claim (1.84) is established. □

The next lemma justifies the choice of r in (1.35).

LEMMA 1.13. *Let M be an arbitrary positive number. Then for every $\beta > 0$ there is $r > 0$ such that*

$$\sup_{\varepsilon>0} E_0^{\varepsilon}[\exp(MH^{\varepsilon}_{\sqrt{r}})] \leq 1+\beta. \tag{1.87}$$

PROOF. Since $H^{\varepsilon}_{\sqrt{r}} \sim_d rH_1^{\varepsilon/\sqrt{r}}$, (1.87) is equivalent to

$$\sup_{\varepsilon>0} E_0^{\varepsilon}[\exp(MrH_1^{\varepsilon})] \leq 1+\beta \tag{1.88}$$

for r sufficiently small. Observe that for r small enough we have

$$\sup_{1\leq\varepsilon<\infty} E_0^{\varepsilon}[\exp(MrH_1^{\varepsilon})] \leq 1+\beta \tag{1.89}$$

since one exits after one jump. Therefore it is enough to show that for sufficiently small r we have

$$\sup_{0<\varepsilon<1} E_0^{\varepsilon}[\exp(MrH_1^{\varepsilon})] \leq 1+\beta. \tag{1.90}$$

It is enough to treat the one-dimensional case. Then the generator of X_t^{ε} is $-\triangle_{\varepsilon}$, and for smooth functions $f\colon R \to R$ we have [see (1.30)]

$$\triangle_{\varepsilon} f(x) = \tfrac{1}{2}\int_0^1\int_0^1 f''(x-\varepsilon+\varepsilon u+\varepsilon v)\,du\,dv.$$

We choose $f(x) := \cos(\pi x/9)$. Let $g\colon R \to R$ be a smooth function such that:

(i) $g(x) = f(x)$ on the interval $[-3,3]$.
(ii) $g(x) \geq \frac{1}{4}$ on R.
(iii) g is constant outside $[-5,5]$.

We know (see [7]) that $M_t := g(X_t^\varepsilon)\exp\{-\int_0^t \triangle_\varepsilon g/g(X_s^\varepsilon)\,ds\}$ is a martingale. For the stopped martingale we have

$$M_{H_1^\varepsilon \wedge t} \geq \frac{1}{2}\exp\left\{\frac{1}{2}\int_0^{H_1^\varepsilon \wedge t} \frac{\pi^2}{162}\,ds\right\}$$

since for $0 < \varepsilon < 1$ and $|x| < 1 + \varepsilon$ we have $g(x) > \frac{1}{2}$ and

$$\inf_{u,v\in[0,1]} \frac{-g''(x-\varepsilon+\varepsilon u+\varepsilon v)}{g(x)} \geq \frac{1}{2}\cdot\frac{\pi^2}{81}. \tag{1.91}$$

By the optional sampling theorem we have

$$\begin{aligned} 1 &= E^\varepsilon[M_{t\wedge H_1^\varepsilon}] \\ &\geq E^\varepsilon\left[\frac{1}{2}\exp\left\{\frac{\pi^2}{162}(H_1^\varepsilon \wedge t)\right\}\right] \\ &\geq \frac{1}{2}e^{ct}\cdot P^\varepsilon[H_1^\varepsilon > t] \end{aligned}$$

with $c := \pi^2/162$. Consequently

$$\begin{aligned} E_0^\varepsilon[\exp(MrH_1^\varepsilon)] &= 1 + \int_0^\infty Mr\exp(Mrs)P^\varepsilon[H_1^\varepsilon > s]\,ds \\ &\leq 1 + \int_0^\infty Mr\exp(Mrs)2\exp(-cs)\,ds. \end{aligned}$$

The last expression converges to 1 as $r \to 0$, and this yields our claim. □

2. Application to a trapping problem.

2.1. *Definitions and statement of results.* We shall now apply the results of Section 1 to the trapping problem described in the Introduction. Our simple random walk $(S_t)_{t\geq 0}$ will now move among random killing obstacles which are sites (resp., bonds) of the d-dimensional lattice. We assume that each site (resp., bond) has the same probability p to be an obstacle, independently of the other sites (resp., bonds). More formally, the law of the obstacle configuration in the site case is described by the product measure $\mathbb{P} := \mu^{\otimes Z^d}$ on $!\Omega := \{0,1\}^{Z^d}$ (with the product σ-algebra), where μ is the Bernoulli probability measure on $\{0,1\}$ with $\mu(1) = p$ and $\mu(0) = 1-p$. In the edge case we denote by $\mathbb{P}_*$ the analogous measure on $\{0,1\}^{E^d}$.

We denote the entrance time of $S_\bullet$ into the obstacle set by T and T_*, respectively. We also introduce R_t (resp., R_t^*), the number of distinct sites (resp., bonds) visited by the random walk up to time t. Finally we define $\nu > 0$ via

$$p = 1 - e^{-\nu}. \tag{2.1}$$

Our main result is the following theorem.

THEOREM 2.1. *We have*

$$\lim_{t\to\infty} t^{-d/(d+2)} \log \mathbb{P}_* \otimes P_0[T_* > t] = \lim_{t\to\infty} t^{-d/(d+2)} \log E_0[\exp(-\nu R_t^*)] = -c_*(d,\nu), \tag{2.2}$$

$$\lim_{t\to\infty} t^{-d/(d+2)} \log \mathbb{P} \otimes P_0[T > t] = \lim_{t\to\infty} t^{-d/(d+2)} \log E_0[\exp(-\nu R_t)] = -c(d,\nu), \tag{2.3}$$

where

$$c(d,\nu) = \inf_U \{\nu|U| + \lambda(U)\}, \tag{2.4}$$

$$c_*(d,\nu) = \inf_U \{d\nu|U| + \lambda(U)\}. \tag{2.5}$$

In both cases U runs over all open bounded subsets of R^d with $|\partial U| = 0$ and $\lambda(U)$ stands for the principal Dirichlet eigenvalue of $-1/2d\triangle$ in U.

Before giving the proof, we reformulate our results in a scaled form which is more comfortable to handle. We adopt $t^{1/(d+2)}$ and $t^{2/(d+2)}$ as new space and time units. So we have to study a random walk on $t^{-1/(d+2)}Z^d$ until time $s = t^{d/(d+2)}$. Let us introduce $\varepsilon := t^{-1/(d+2)}$ and denote by P^ε the law of the rescaled process (i.e., we use the notation of Section 1) and by $\mathbb{P}^\varepsilon$ the law of the corresponding obstacle configuration on εZ^d. So we have to show

$$\lim_{s\to\infty} s^{-1} \log \mathbb{P}^\varepsilon \otimes P_0^\varepsilon[T > s] = -c(d,\nu), \tag{2.6}$$

$$\lim_{s\to\infty} s^{-1} \log \mathbb{P}_*^\varepsilon \otimes P_0^\varepsilon[T_* > s] = -c_*(d,\nu). \tag{2.7}$$

2.2. *The lower bound.* We start with the lower bound part of the proof. Let $a > 0$ and let B_a be the Euclidean ball $\{x \in R^d\colon |x| < a\}$. Then we have

$$\mathbb{P}^\varepsilon \otimes P_0^\varepsilon[T > s] = (\#^\varepsilon B_a)^{-1} \sum_{z\in B_a^\varepsilon} \mathbb{P}^\varepsilon \otimes P_z^\varepsilon[T > s] \geq (\#^\varepsilon B_a)^{-1} \sum_{z\in B_a^\varepsilon} P_z^\varepsilon[T_{B_a} > s]\mathbb{P}^\varepsilon[N(B_a) = 0] \tag{2.8}$$

and for edge obstacles we obtain the same lower bound with $\mathbb{P}^\varepsilon[N(B_a) = 0]$ replaced by $\mathbb{P}_*^\varepsilon[N_*(B_a) = 0]$, where $N(B_a)$ and $N_*(B_a)$, respectively, denote the number of obstacles in B_a. We now consider $L^2(B_a^\varepsilon, \varepsilon^d \#^\varepsilon)$ and the inner product

$$(f,g)_\varepsilon := \varepsilon^d \sum_{z\in B_a^\varepsilon} f(z)g(z). \tag{2.9}$$

Observe that this inner product is different from that used in Section 1. Using the semigroup $P_t := P_t^{B_a}$ associated with the random walk killed outside of

B_a we obtain

$$\begin{aligned}\mathbb{P}^\varepsilon \otimes P_0^\varepsilon[T > s] &\geq \frac{\varepsilon^{-d}}{\#^\varepsilon B_a} \exp(-\nu \#^\varepsilon B_a(P_s 1, 1)_\varepsilon \\ &= \frac{\varepsilon^{-d}}{\#^\varepsilon B_a} \exp(-\nu \#^\varepsilon B_a) \sum_i (\Phi_i^\varepsilon, 1)_\varepsilon^2 \exp(-s\lambda_i^\varepsilon) \\ &\geq \frac{\varepsilon^{-d}}{\#^\varepsilon B_a} \exp(-\nu \#^\varepsilon B_a) \exp(-s\lambda_1^\varepsilon)(\Phi_1^\varepsilon, 1)_\varepsilon^2 \\ &\geq \frac{\varepsilon^{-d}}{\#^\varepsilon B_a} \exp(-\nu \#^\varepsilon B_a) \exp(-s\lambda_1^\varepsilon).\end{aligned}$$

Here Φ_i^ε stands for the ith normalized eigenfunction of the generator of P_t and λ_i^ε stands for the corresponding eigenvalue. In the last inequality we have used that $\Phi_1^\varepsilon \geq 0$ and therefore

$$(\Phi_1^\varepsilon, 1)_\varepsilon = \|\Phi_1^\varepsilon\|_{1;\varepsilon} \geq \|\Phi_1^\varepsilon\|_{2;\varepsilon} = 1. \tag{2.10}$$

In the edge case we obtain, similarly,

$$\mathbb{P}_*^\varepsilon \otimes P_0^\varepsilon[T_* > s] \geq \frac{\varepsilon^{-d}}{\#^\varepsilon B_a} \exp(-\nu \#_*^\varepsilon B_a) \exp(-s\lambda_1^\varepsilon). \tag{2.11}$$

We now need a comparison of the counting measure with the Euclidean volume. For $z \in B_a^\varepsilon$ we set $C_z := \prod_{i=1}^d [z_i, z_i + \varepsilon]$ and $C := \bigcup_{C_z \cap B_a^\varepsilon \neq \varnothing} C_z$. Then $B_a \subset C \subset B_{a+\varepsilon\sqrt{d}}$ and we have

$$\#^\varepsilon B_a \leq \varepsilon^{-d}|C| \leq \varepsilon^{-d}|B_{a+\varepsilon\sqrt{d}}| \quad \text{and} \quad \#_*^\varepsilon B_a \leq d\varepsilon^{-d}|C| \leq d\varepsilon^{-d}|B_{a+\varepsilon\sqrt{d}}|.$$

Therefore we obtain

$$\mathbb{P}^\varepsilon \otimes P_0^\varepsilon[T > s] \geq \frac{\varepsilon^{-d}}{\#^\varepsilon B_a} \exp\{-(\nu s|B_{a+\varepsilon\sqrt{d}}| + s\lambda^\varepsilon)\}, \tag{2.12}$$

$$\mathbb{P}_*^\varepsilon \otimes P_0^\varepsilon[T_* > s] \geq \frac{\varepsilon^{-d}}{\#^\varepsilon B_a} \exp\{-(\nu sd|B_{a+\varepsilon\sqrt{d}}| + s\lambda^\varepsilon)\}. \tag{2.13}$$

We shall prove [see (2.39) in Section 2.4] that

$$\limsup_{\varepsilon \to 0} \lambda^\varepsilon(B_a) \leq \lambda(B_a). \tag{2.14}$$

Since the infimum in (2.4) is attained if U is a ball, (2.12) and (2.13) together with (2.14) imply the lower bound.

2.3. *The upper bound.* We now prove the upper bound in several steps. We shall state our claims in the site case notation without mentioning each time that in the bond case one has to replace c by c_* and $\mathbb{P}$ by $\mathbb{P}_*$. The proofs are of course given for both cases.

With the help of Proposition 1.7 we set $\mathscr{T} := (-N[s], N[s])^d$, where N is a number large enough so that

$$\limsup_{s\to\infty} s^{-1} \log P_0^\varepsilon[T_{\mathscr{T}} \le s] \le -c(d,\nu) - 1. \tag{2.15}$$

We denote by $\tilde{\mathscr{T}}$ the set $(-N[s]-1, N[s]+1)^d$. As already mentioned in the Introduction, we derive the upper bound via the following inequality, which holds for any positive number n_0:

$$\begin{aligned} &\mathbb{P}^\varepsilon \otimes P_0^\varepsilon[T > s] \\ &\quad \le P_0^\varepsilon[T_{\mathscr{T}} \le s] + \mathbb{P}^\varepsilon \otimes P_0^\varepsilon[T \wedge T_{\mathscr{T}} > s;\ |\mathscr{A} \cap \tilde{\mathscr{T}}| \le n_0] \\ &\qquad + \mathbb{P}^\varepsilon[|\mathscr{A} \cap \tilde{\mathscr{T}}| > n_0]. \end{aligned} \tag{2.16}$$

Thanks to (2.15), the first term is negligible for our purpose. The second term will be estimated by the enlargement technique developed in Section 1 and we shall use our covering Lemma 1.1 to estimate the third term. We begin with the following lemma.

LEMMA 2.2.

$$\forall\, r > 0, \qquad \limsup_{n_0\to\infty}\ \limsup_{\substack{b\to\infty \\ \delta\to 0}}\ \limsup_{s\to\infty} s^{-1} \log \mathbb{P}^\varepsilon[|\mathscr{A} \cap \tilde{\mathscr{T}}| \ge n_0] = -\infty. \tag{2.17}$$

PROOF. We use the notation of Section 1 and denote by U_m ($\tilde{U}_m$) the complement of all closed subboxes in C_m where an obstacle (a good obstacle) falls. Then we have by Lemma 1.1,

$$\#^\varepsilon \tilde{U}_m \le \#^\varepsilon U_m + \delta \#^\varepsilon C_m. \tag{2.18}$$

Therefore we see that

$$\begin{aligned} \{C_m \text{ is a clearing}\} &= \{\#^\varepsilon \tilde{U}_m \ge 4^{-d} \#^\varepsilon \overline{B}(0,r)\} \\ &\subset \{\#^\varepsilon U_m \ge 4^{-d} \#^\varepsilon \overline{B}(0,r) - \delta \#^\varepsilon C_m\}. \end{aligned} \tag{2.19}$$

It now follows that $\mathbb{P}^\varepsilon[|\mathscr{A} \cap \tilde{\mathscr{T}}| \ge n_0]$ is smaller than

$$\begin{aligned} &\#(\text{choices of } n_0 \text{ distinct boxes in } \tilde{\mathscr{T}}) \\ &\quad \times \mathbb{P}^\varepsilon\Bigg[\bigcap_{m\in\mathscr{M}} \{\#^\varepsilon U_m \ge 4^{-d} \#^\varepsilon \overline{B}(0,r) - \delta \#^\varepsilon C_m\}\Bigg], \end{aligned} \tag{2.20}$$

where $\mathscr{M}$ is a subset of $[-N[s]-1, N[s]]^d$ with n_0 elements. For each m the number of possibilities for U_m is at most $2^{(s^{1/d}/(b-1)+1)^d}$. Therefore the expression above is less than

$$(2Ns+2)^{dn_0}\Big[2^{(s^{1/d}/(b-1)+1)^d} \sup_{m\in\mathscr{M}} \sup_{U_m} \mathbb{P}^\varepsilon[N(U_m) = 0]\Big]^{n_0}, \tag{2.21}$$

where, for fixed $m \in \mathscr{M}$, U_m runs over the complement of union of subboxes in the interior of C_m with volume bigger than $4^{-d} \#^\varepsilon \overline{B}(0,r) - \delta \#^\varepsilon C_m$. In the edge case we have the same upper bound with N replaced by N_*. We now have in the site case

$$\limsup_{s\to\infty} s^{-1}\log \mathbb{P}^\varepsilon[|\mathscr{A}\cap\tilde{\mathscr{T}}|\geq n_0]$$
$$\leq (\log 2)n_0(b-1)^{-d} - \liminf_{s\to\infty} \nu s^{-1} n_0\big(4^{-d}\#^\varepsilon \overline{B}(0,r) - \delta \sup_m \#^\varepsilon C_m\big).$$

Observe that there is a positive constant $k_1(d)$ such that $\#^\varepsilon \overline{B}(0,r)\geq k_1 r^d\varepsilon^{-d}$ and we have $\sup_m \#^\varepsilon C_m \leq \frac{3}{2}\varepsilon^{-d}$ for ε small enough. Using $s^{-1}=\varepsilon^d$ we obtain

$$\limsup_{n_0\to\infty}\ \limsup_{\substack{b\to\infty\\ \delta\to 0}}\ \limsup_{s\to\infty} s^{-1}\log \mathbb{P}^\varepsilon[|\mathscr{A}\cap\tilde{\mathscr{T}}|\geq n_0]$$
$$\leq \limsup_{n_0\to\infty}\ \limsup_{\substack{b\to\infty\\ \delta\to 0}} (\log 2)n_0(b-1)^{-d} - \nu n_0\left(4^{-d}r^d k_1 - \frac{3\delta}{2}\right) = -\infty.$$

To treat the edge case we have to compare $\#_*^\varepsilon U_m$ with $\#^\varepsilon U_m$. In a sufficiently large box we have approximately d times more bonds than sites. As we shall see later [see (2.35)], we have, in fact,

$$\#_*^\varepsilon U_m \geq \left(\frac{b-3}{b}\right)^{2d} d(\#^\varepsilon U_m - db\varepsilon^{1-d}), \tag{2.22}$$

where the last term in the bracket is a correction coming from the "last layer of subboxes." We mention at this point that for the following calculation a much weaker estimate would be enough, but since we need this strong version later, we also use it now. We now see by (2.22) that, for b big enough,

$$\mathbb{P}_*^\varepsilon[N_*(U_m)=0] = \exp\{-\nu\#_*^\varepsilon U_m\} \leq \exp\big\{-\tfrac{1}{2}\nu d(\#^\varepsilon U_m - db\varepsilon^{1-d})\big\}. \tag{2.23}$$

Similarly as in the site case we obtain

$$\limsup_{s\to\infty} s^{-1}\log \mathbb{P}_*^\varepsilon[|\mathscr{A}\cap\tilde{\mathscr{T}}|\geq n_0]$$
$$\leq (\log 2)n_0(b-1)^{-d}$$
$$- \liminf_{s\to\infty} \tfrac{1}{2}\nu\, d s^{-1} n_0\big(4^{-d}\#^\varepsilon \overline{B}(0,r) - \delta\sup_m \#^\varepsilon C_m - db\,\varepsilon^{1-d}\big)$$

and by the same argument as above, our claim in the edge case also follows. □

The next lemma enables us to control the second term at the r.h.s. of (2.16). Using the notation introduced in Section 1, we write $\tilde{T}$ for $T\wedge T_{\mathscr{T}}$.

LEMMA 2.3.

$$\limsup_{r\to 0}\ \limsup_{n_0\to\infty}\ \limsup_{\substack{b\to\infty\\ \delta\to 0}}\ \limsup_{s\to\infty}\{s^{-1} \times \log \mathbb{P}^\varepsilon\otimes P_0^\varepsilon[\tilde{T}>s,\ |\mathscr{A}\cap\tilde{\mathscr{T}}|\leq n_0]\} \leq -c(d,\nu). \tag{2.24}$$

PROOF. We denote the limit appearing in (2.24) by $\widetilde{\lim}$. We pick $M > 0$ and $\rho > 0$. Then we have

$$\begin{aligned}
&\widetilde{\lim}\, s^{-1} \log \mathbb{P}^\varepsilon \otimes P_0^\varepsilon[\tilde{T} > s,\ |\mathscr{A} \cap \tilde{\mathscr{T}}| \le n_0] \\
&\quad = \widetilde{\lim}\, s^{-1} \log \mathbb{E}^\varepsilon[P_0^\varepsilon[\tilde{T} > s];\ |\mathscr{A} \cap \tilde{\mathscr{T}}| \le n_0] \\
&\quad \le \widetilde{\lim}\, s^{-1} \log \mathbb{E}^\varepsilon[\exp(-(\lambda^\varepsilon(\Theta_b) \wedge M - \rho)_+ s) \\
&\qquad\qquad \times \sup_\omega E_0^\varepsilon[\exp((\lambda^\varepsilon(\Theta_b) \wedge M - \rho)_+ \tilde{T})]; \\
&\qquad\qquad\qquad |\mathscr{A} \cap \tilde{\mathscr{T}}| \le n_0] \\
&\quad \le \widetilde{\lim}\, s^{-1} \log(\sup_\omega E_0^\varepsilon[\exp((\lambda^\varepsilon(\Theta_b) \wedge M - \rho)_+ \tilde{T})]) \\
&\qquad + \widetilde{\lim}\, s^{-1} \log \mathbb{E}^\varepsilon[\exp(-(\lambda^\varepsilon(\Theta_b) \wedge M - \rho)_+ s);\ |\mathscr{A} \cap \tilde{\mathscr{T}}| \le n_0].
\end{aligned}$$

Using Theorem 1.4 we see that the first expression is zero, so it remains to show

$$\limsup_{\substack{\rho \to 0 \\ M \to \infty}} \widetilde{\lim}\, s^{-1} \log \mathbb{E}^\varepsilon[\exp(-(\lambda^\varepsilon(\Theta_b) \wedge M - \rho)_+ s);\ |\mathscr{A} \cap \tilde{\mathscr{T}}| \le n_0] \le \begin{cases} -c(d,\nu) & \text{(site case)}, \\ -c_*(d,\nu) & \text{(edge case)}. \end{cases} \tag{2.25}$$

For simplicity we set $\widetilde{\widetilde{\lim}}$ for the limit appearing in (2.25). We now introduce the set $D := \mathscr{T} \cap (\bigcup_{\overline{C}_m \cap \mathscr{A} \neq \varnothing} C_m)^\circ$. That is, D is obtained by deleting in $\mathscr{T}$ all closed boxes $\overline{C}_m$ which are not neighbors of $\mathscr{A}$. We also define U (resp., $\tilde{U}$) as the complement in D of the union over $m \in Z^d$ of closed subboxes intersecting the interior of C_m and containing a point (resp., a good point) of C_m. We then have, for a suitable constant $c(d) \ge 0$,

$$\#_\varepsilon \tilde{U} \le \sum_{\overline{C}_m \cap \mathscr{A} \neq \varnothing} \#_\varepsilon \tilde{U}_m + n_0\, c(d)\, \varepsilon^{1-d}, \tag{2.26}$$

where the second term arises because of the fact that the sites on the boundary of C_m never belong to $\tilde{U}_m$; however, they could eventually belong to U. We now have $\Theta_b \subset \tilde{U}$, $|\mathscr{A} \cap \tilde{\mathscr{T}}| \le n_0$ and, thanks to Lemma 1.1,

$$U \subset \tilde{U} \subset D, \qquad \#^\varepsilon \tilde{U} \le \#^\varepsilon U + \delta n_0 3^d \sup_m \#^\varepsilon C_m + n_0\, c(d)\, \varepsilon^{1-d}. \tag{2.27}$$

We see that the number of possibilities for D grows at most polynomially in s for fixed n_0, and for fixed D the number of possibilities for U and $\tilde{U}$ is smaller than $2^{2n_0 3^d (s^{1/d}/(b-1)+1)^d}$. Observe that $\lambda^\varepsilon(\Theta_b) \ge \lambda^\varepsilon(\tilde{U})$. Thus (2.25) will follow if we show

$$\widetilde{\widetilde{\lim}} \sup_{U,\tilde{U},D} \{-(\lambda^\varepsilon(\tilde{U}) \wedge M - \rho)_+ + s^{-1} \log \mathbb{P}^\varepsilon(N(U) = 0)\} \\ \leq -c(d,\nu), \tag{2.28}$$

$$\widetilde{\widetilde{\lim}} \sup_{U,\tilde{U},D} \{-(\lambda^\varepsilon(\tilde{U}) \wedge M - \rho)_+ + s^{-1} \log \mathbb{P}^\varepsilon(N_*(U) = 0)\} \\ \leq -c_*(d,\nu), \tag{2.29}$$

where U, $\tilde{U}$ and D satisfy (2.27). Using the fact that

$$\mathbb{P}^\varepsilon(N(U) = 0) = \exp(-\nu \#^\varepsilon U) \\ \leq \exp\{-\nu(\#^\varepsilon \tilde{U} - \delta 3^d n_0 \sup_m \#^\varepsilon C_m - n_0 c(d) \varepsilon^{1-d})\}, \tag{2.30}$$

the left-hand side of (2.28) is less than

$$\begin{aligned} &\widetilde{\widetilde{\lim}} \sup_{U,\tilde{U},D} \bigg\{-(\lambda^\varepsilon(\tilde{U}) \wedge M - \rho)_+ \\ &\qquad - \nu s^{-1}\bigg(\#^\varepsilon \tilde{U} - \frac{3}{2}\delta n_0 3^d s - n_0 c(d) s^{1-1/d}\bigg)\bigg\} \\ &\leq \widetilde{\widetilde{\lim}} \sup_{\tilde{U} \subset D} \{-\lambda^\varepsilon(\tilde{U}) \wedge M - \nu s^{-1} \#^\varepsilon \tilde{U}\} \\ &\leq \widetilde{\widetilde{\lim}} \sup_{\tilde{U} \subset D} \{-((\lambda^\varepsilon(\tilde{U}) + \nu s^{-1} \#^\varepsilon \tilde{U}) \wedge M)\}, \end{aligned} \tag{2.31}$$

where the supremum is taken over all open subsets $\tilde{U} \subset D$ obtained by taking the complement in D of certain closed subboxes in $\overline{D}$.

To treat the edge case we need a comparison of $\#^\varepsilon_*(U)$ with $\#^\varepsilon(U)$. For each box C_m we consider the set C^1_m, obtained by taking the union of the interior of those closed subboxes of C_m which do not belong to the "last layer" of subboxes. We then set

$$U^1 := U \cap \bigcup_{m \in Z^d} C^1_m. \tag{2.32}$$

So, U^1 is the finite union of disjoint open subboxes W_i of size $\varepsilon(b-1)$. For each subbox W_i we have

$$\#^\varepsilon_*(W_i) = \#^\varepsilon\bigg(W_i \cap \bigcup_{j=1}^d \bigg(\frac{1}{2}\varepsilon e_j + \varepsilon Z^d\bigg)\bigg). \tag{2.33}$$

Using (1.8) we see that

$$([b]-2)^d \leq \#^\varepsilon W_i \leq [b]^d, \qquad ([b]-2)^d \leq \#^\varepsilon \overline{W}_i \leq [b]^d. \tag{2.34}$$

Using these estimates and (2.33), we obtain

$$
\begin{aligned}
\#_*^\varepsilon U &\geq \#_*^\varepsilon U_1 \\
&= \sum_i \#_*^\varepsilon W \\
&\geq \sum_i \left(\frac{[b]-2}{[b]}\right)^d \#_*^\varepsilon \overline{W}_i \\
&\geq \sum_i \left(\frac{[b]-2}{[b]}\right)^{2d} d\#^\varepsilon \overline{W}_i \\
&\geq \left(\frac{b-3}{b}\right)^{2d} d(\#^\varepsilon U - \#^\varepsilon(\text{last layer})) \\
&\geq \left(\frac{b-3}{b}\right)^{2d} d(\#^\varepsilon U - dn_0\, 3^d b\varepsilon^{1-d}).
\end{aligned}
\tag{2.35}
$$

Using (2.35) we obtain for U and $\tilde{U}$ as in (2.27),

$$
\begin{aligned}
&\mathbb{P}_*(N_*(U) = 0) \\
&\quad \leq \exp\left\{-\nu\left(\frac{b-3}{b}\right)^{2d} d(\#^\varepsilon U - dn_0\, 3^d b\varepsilon^{1-d})\right\} \\
&\quad \leq \exp\Bigg\{-\nu\left(\frac{b-3}{b}\right)^{2d} d\Big(\#^\varepsilon \tilde{U} - \delta n_0\, 3^d \sup_m \#^\varepsilon C_m \\
&\qquad\qquad - dn_0\, 3^d b\varepsilon^{1-d} - n_0 c(d)\varepsilon^{1-d}\Big)\Bigg\} \\
&\quad \leq \exp\Bigg\{-\nu d\#^\varepsilon \tilde{U} + \nu d \sup_m \#^\varepsilon C_m \left(1 - \left(\frac{b-3}{b}\right)^{2d}(1 - \delta n_0 3^d)\right) \\
&\qquad\qquad + \nu\, dn_0\, \varepsilon^{1-d}(db\, 3^d + c(d))\Bigg\}.
\end{aligned}
$$

By similar manipulations as in the site case we see that the left-hand side of (2.29) is smaller than

$$
\widetilde{\widetilde{\lim}} \sup_{\tilde{U} \subset D} \{-\lambda^\varepsilon(\tilde{U}) - \nu\, ds^{-1}\, \#^\varepsilon \tilde{U}\}. \tag{2.36}
$$

To establish (2.28) and (2.29) it remains to prove that expression (2.31) is smaller than $-c(d,\nu)$. For this observe that for small ε we have

$$
\lambda^\varepsilon(\tilde{U}) = \inf_{D_j \cap \tilde{U} \neq \varnothing} \lambda^\varepsilon(\tilde{U} \cap D_j), \tag{2.37}
$$

where the D_j are the connected components of D. By construction each D_j is contained in a cubic box Γ of size $2 \cdot 3^d n_0$ and by Lemma 2.5 in the next subsection we have

$$\liminf_{\varepsilon\to 0} \inf_{\tilde U\subset\Gamma}[\lambda^\varepsilon(\tilde U) + \varepsilon^d \nu \#^\varepsilon \tilde U] \geq \inf_{\substack{U\subset R^d\\ |\partial U|=0}}[\lambda(U)+\nu|U|], \tag{2.38}$$

and this yields our claim. □

Combining (2.16) and the estimates (2.15), (2.17) and (2.24) we immediately obtain the desired upper bound.

2.4. *Two approximation lemmas.* We now prove the two lemmas which we have used to compare expressions involving eigenvalues of the discrete Laplacian to the corresponding continuous quantities.

LEMMA 2.4.

$$\limsup_{\varepsilon\to 0} \lambda^\varepsilon(B_a) \leq \lambda(B_a). \tag{2.39}$$

PROOF. Let L^ε be the generator of $P_t^{B_a}$. Then we have

$$\lambda^\varepsilon(B_a) = \inf\{(L^\varepsilon f, f)_\varepsilon;\ \operatorname{supp} f \subset B_a,\ \|f\|_{2;\varepsilon} = 1\}.$$

Let $f \geq 0$ be the normalized [w.r.t. the $L^2(R^d, dx)$ norm] first Dirichlet eigenfunction of $-(1/(2d))\triangle$ in B_a extended by 0 outside B_a. We have

$$(L^\varepsilon f, f)_\varepsilon = \frac{1}{4d}\varepsilon^{d-2} \sum_{x\in B_a^\varepsilon} \sum_{\substack{e\in Z^d\\ |e|=1}} (f(x+\varepsilon e) - f(x))^2. \tag{2.40}$$

It is classical that f has first derivatives on B_a which have a continuous extension to $\overline{B}_a$. Thus by the mean value theorem we have numbers η_e, $0 \leq \eta_e \leq 1$, such that

$$\begin{aligned}(L^\varepsilon f, f)_\varepsilon = {}& \frac{1}{4d}\varepsilon^{d} \sum_{x\in B_{a-\varepsilon}^\varepsilon} \sum_{\substack{e\in Z^d\\ |e|=1}} (\nabla f(x+\eta_e\varepsilon e)\cdot e)^2 \\ &+ \frac{1}{4d}\varepsilon^{d-2} \sum_{x\in B_a^\varepsilon\setminus B_{a-\varepsilon}^\varepsilon} \sum_{\substack{e\in Z^d\\ |e|=1}} (f(x+\varepsilon e) - f(x))^2.\end{aligned}$$

Using the Lipschitz continuity of f we see that the second term goes to zero for $\varepsilon \to 0$. The first term is a convergent Riemann sum, so we see that

$$\limsup_{\varepsilon\to 0}(L^\varepsilon f, f)_\varepsilon = \frac{1}{2d}\int_{R^d} |\nabla f|^2 = \lambda(B_a) \tag{2.41}$$

and this yields our claim since $\lambda^\varepsilon(B_a) \leq (L^\varepsilon f, f)_\varepsilon / \|f\|_{2;\varepsilon}^2$ and $\|f\|_{2;\varepsilon} \to \|f\| = 1$ as $\varepsilon \to 0$. □

A little bit more care is needed in the proof of the next lemma.

LEMMA 2.5. *Let* $\Gamma := \overline{B}(0, 3^d n_0 + 1)$. *Then we have*

$$\limsup_{\varepsilon\to 0} \inf_{\tilde U \subset \Gamma} [\lambda^\varepsilon(\tilde U) + \varepsilon^d \nu \#^\varepsilon \tilde U] \geq \inf_{\substack{U\subset R^d\\ |\partial U|=0}} [\lambda(U) + \nu|U|] \quad \text{with the notation of (2.3).} \tag{2.42}$$

PROOF. Let $C_\varepsilon := \inf_{\tilde U\in\Gamma}[\lambda^\varepsilon(\tilde U) + \varepsilon^d\nu\#^\varepsilon\tilde U]$. There is nothing to prove if the left member of (2.42) is infinite. Otherwise there is a sequence ε_n in R^+ with $\varepsilon_n \to 0$ and a sequence of nonnegative functions f_n defined on $\varepsilon_n Z^d$ with

$$\forall\, n \in N, \qquad \|f_n\|^2_{2;\varepsilon} = \varepsilon_n^d \sum_z f_n^2(z) = 1, \qquad \operatorname{supp} f_n \subset \Gamma, \tag{2.43}$$

$$\sup_n (L^\varepsilon f_n, f_n)_{\varepsilon_n} = \sup_n \varepsilon_n^{d-2} \frac{1}{4d} \sum_{\substack{|e|=\varepsilon_n\\ x\in\Gamma}} [f_n(x+e) - f_n(x)]^2 = \gamma < \infty, \tag{2.44}$$

$$\liminf_{n\to\infty} \{\nu\varepsilon_n^d \#^\varepsilon (f_n > 0) + (L^{\varepsilon_n} f_n, f_n)_{\varepsilon_n}\} = \liminf_{\varepsilon\to 0} C_\varepsilon. \tag{2.45}$$

We now construct a new sequence $\tilde f_n$ of functions defined on R^d as follows. We let $\tilde f_n = f_n$ on $\varepsilon_n Z^d$ and $\tilde f_n$ is defined to be constant on each cube $\prod_{i=1}^d [x_i - \varepsilon_n/2, x_i + \varepsilon_n/2)$, $x \in \varepsilon_n Z^d$. With no loss of generality the $\tilde f_n$ are all supported in $\tilde\Gamma := (2\Gamma)^\circ$.

We first show that $(\tilde f_n)_{n\in N}$ is a relatively compact subset of $L^2(\tilde\Gamma)$. For this it is enough to show (see [1], Theorem 2.21) that $\forall\, \eta > 0\ \exists\, \sigma > 0$ such that

$$\forall\, h \in R^d,\ |h| < \sigma, \qquad \sup_n \int_{R^d} |\tilde f_n(x+h) - \tilde f_n(x)|^2\, dx < \eta. \tag{2.46}$$

It is enough to pick $h = \mu e_i$. For notational convenience we treat e_1. We can also safely assume $\mu \geq 0$. We set $\mu = l\varepsilon_n + r$, $l \geq 0$, $0 \leq r < \varepsilon_n$. Then we have

$$\begin{aligned}\|\tilde f_n(x+h) - \tilde f_n(x)\|_2 &\leq \sum_{k=1}^{l} \|\tilde f_n(x + k\varepsilon_n e_1) - f_n(x + (k-1)\varepsilon_n e_1)\|_2 \\ &\quad + \|\tilde f_n(x+h) - \tilde f_n(x+h-re_1)\|_2\end{aligned}$$

and by translation invariance this is equal to

$$l\left(\int_{\tilde\Gamma} |\tilde f_n(x + e_1\varepsilon_n) - \tilde f_n(x)|^2\, dx\right)^{1/2} + \left(\int_{\tilde\Gamma} |\tilde f_n(x) - \tilde f_n(x - re_1)|^2\, dx\right)^{1/2}. \tag{2.47}$$

The first term is smaller than

$$l\left(\varepsilon_n^d \sum_{z\in\varepsilon Z^d} |f_n(z+e_1\varepsilon_n) - f_n(z)|^2\right)^{1/2} \le l\varepsilon_n\sqrt{4d\gamma} \tag{2.48}$$
$$\le |h|\sqrt{4d\gamma} \quad \text{since } l \le |h|/\varepsilon_n.$$

The second term is smaller than $\sqrt{4\,d\gamma\,r}$, so we obtain

$$\sup_n \int_{R^d} |\tilde{f}_n(x+h) - \tilde{f}_n(x)|^2\,dx \le (|h| + \sqrt{|h|})\sqrt{4d\gamma} \tag{2.49}$$

and this is arbitrarily small if $|h|$ is small. This proves (2.46).

It now follows that there is a subsequence, which we also denote by $\tilde{f}_n$, converging in $L^2(\tilde{\Gamma})$ to a function $\tilde{f}$. Moreover, $\tilde{f} \in H^1(R^d)$ and

$$\mathscr{E}(\tilde{f},\tilde{f}) = \frac{1}{2d}\int |\nabla\tilde{f}|^2\,dx \le \gamma. \tag{2.50}$$

Indeed, by the same argument as before we have

$$\frac{1}{2d}\sum_{i=1}^d \int_{R^d} (\tilde{f}_n(x+\varepsilon_n e_i) - \tilde{f}_n(x))^2\,dx \le \varepsilon_n^2\gamma \tag{2.51}$$

and therefore the Fourier transforms $\widehat{\tilde{f}}_n$ of $\tilde{f}_n$ (resp., $\widehat{\tilde{f}}$, the Fourier transform of $\tilde{f}$) satisfy

$$\frac{1}{2d}\sum_{i=1}^d \int_{R^d} \varepsilon_n^{-2}|\exp(i\varepsilon_n x_i) - 1|^2 |\widehat{\tilde{f}}_n(x)|^2\,dx \le \gamma \tag{2.52}$$

and by Fatou's lemma applied to a subsequence $\widehat{\tilde{f}}_{n_k}$ almost surely converging to $\widehat{\tilde{f}}$,

$$\frac{1}{2d}\sum_{i=1}^d \int_{R^d} x_i^2 |\widehat{\tilde{f}}(x)|^2\,dx \le \gamma, \tag{2.53}$$

which yields (2.50).

Thus, by (2.50) and the lower semicontinuity on $L^2(\tilde{\Gamma})$ of $g \to |1_{\{g>0\}}|$ it follows that

$$\nu|\{\tilde{f} > 0\}| + \mathscr{E}(\tilde{f},\tilde{f}) \le \liminf_{\varepsilon\to 0} C_\varepsilon. \tag{2.54}$$

Now the claim of the lemma can be obtained similarly as in [12, Lemma 3.5]. We give the argument for the reader's convenience. Pick $\alpha > 0$ with $|\{\tilde{f} = \alpha\}| = 0$ and let g_k be a sequence of functions in C^∞ with compact support in $\tilde{\Gamma}$ and $g_k \to \tilde{f}$ in H^1. Then we have, by a "portmanteau" type technique,

$$\lim_{k\to\infty} \nu|\{g_k > \alpha\}| + \frac{1}{2d}\int_{\{g_k>\alpha\}} |\nabla g_k|^2\,dx \tag{2.55}$$
$$= \nu|\{\tilde{f} > \alpha\}| + \frac{1}{2d}\int_{\{\tilde{f}>\alpha\}} |\nabla\tilde{f}|^2\,dx.$$

Thus we see, using Sard's theorem, that for any $\eta > 0$ there exist a function $g \in C^\infty(\tilde{\Gamma})$ and $\alpha > 0$, such that $g = \alpha$ is a regular value of g and

$$\nu|\{g > \alpha\}| + \frac{1}{2d}\int_{\{g>\alpha\}} |\nabla g|^2\, dx \le \liminf_{\varepsilon\to 0} C_\varepsilon + \eta, \tag{2.56}$$

$$\int_{\{g>\alpha\}} (g-\alpha)^2\, dx > \frac{1}{1+\eta}. \tag{2.57}$$

Let V be the smooth open set $\{g > \alpha\}$ and $h := \mu(g-\alpha)I_V$, where μ is choosen such that $\|h\|_{L^2} = 1$. Then we have $|\nabla h|^2 = \mu^2|\nabla g|^2$, $h \in H_0^1(V)$ and from (2.56),

$$\nu|V| + \frac{1}{2d}\int_V \frac{1}{\mu^2}|\nabla h|^2\, dx \le \liminf_{\varepsilon\to 0} C_\varepsilon + \eta. \tag{2.58}$$

Since $1 < 1/(1+\eta) \le 1/\mu^2$ it follows that

$$[\nu|V| + \mathscr{E}(h,h)] \le \big(\liminf_{\varepsilon\to 0} C_\varepsilon + \eta\big)(1+\eta). \tag{2.59}$$

Using the fact that $\mathscr{E}(h,h) \ge \lambda(V)$, we obtain

$$\inf_{\tilde{U}}(\nu|\tilde{U}| + \lambda(\tilde{U})) \le [\nu|V| + \lambda(V)] \tag{2.60}$$

$$\le \big(\liminf_{\varepsilon\to 0} C_\varepsilon + \eta\big)(1+\eta). \tag{2.61}$$

Since η was arbitrary, our claim (2.42) follows. □

We remark at this point for further use (not in this paper) that by the same technique we also obtain the following result:

LEMMA 2.6. *For all positive constants c_1, c_2 we have*

$$\liminf_{\varepsilon\to 0} \inf_{\substack{\varepsilon^d \#^\varepsilon U \le c_1 \\ \operatorname{diam}(U)\le c_2}} \lambda^\varepsilon(U) \ge \inf_{|V|\le c_1} \lambda(V) = \frac{\lambda_d}{R^2}, \tag{2.62}$$

where $\omega_d R^d = c_1$. Here ω_d and λ_d stand, respectively, for the volume of the unit ball in R^d and the principal Dirichlet eigenvalue of $-(1/2d)\triangle$ in this ball.

PROOF. In the same way as in the proof of the preceding lemma, we can construct a function $\tilde{f}$ (the condition on the diameter of U is needed for the relative compactness of the approximating sequence) with unit $L^2(dx)$ norm and with $|\{\tilde{f} > 0\}| \le c_1$, such that the left member of (2.62) is bigger than $\mathscr{E}(\tilde{f},\tilde{f})$. Using the same approximation technique as at the end of the proof of Lemma 2.5, we obtain (2.62). □

2.5. *Some further comments.* Let us give some comments on the result proved in Theorem 2.1. The variational problem (2.4) has a well known solution, in fact, Donsker and Varadhan [5] showed, using the isoperimetric inequality, that the infimum is attained when U is a ball of radius $R_0 = ((2/d)\lambda_d/\nu\omega_d)^{1/(d+2)}$, and its value is

$$c(d,\nu) = (\nu\omega_d)^{2/(d+2)}\left(\frac{d+2}{2}\right)\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)} \tag{2.63}$$

Consequently we have

$$c_*(d,\nu) = c(d,\nu d) = d^{2/(d+2)}c(d,\nu). \tag{2.64}$$

Theorem 2.1 can be interpreted as an estimate on the long time survival probability of the random walk among random obstacles or as an estimate of the Laplace transform of the number of distinct sites (resp., bonds) visited by the random walk. Let us give some more comments on this point. For simplicity we assume $d \geq 3$. The asymptotic behavior of R_t for large t is well known (see [6]). We have, in fact,

$$\lim_{t\to\infty} t^{-1}E[R_t] = \frac{1}{g(0,0)} \tag{2.65}$$

and so by the subadditive ergodic theorem (see [9], page 277)

$$\frac{R_t}{t} \to \frac{1}{g(0,0)} \quad \text{a.s. and in } L^1 \text{ as } t \to \infty, \tag{2.66}$$

and for bonds,

$$\frac{R_t^*}{t} \to \frac{1}{(1/2d)(g(0,0)+g(0,e_1))+1} \quad \text{a.s. and in } L^1 \text{ as } t \to \infty. \tag{2.67}$$

This result is certainly also classical; however, we did not find a precise reference and we sketch its proof for the reader's convenience further below. Using (2.66) and (2.67) we see that the ratio R_n^*/R_n converges almost surely to the constant

$$\zeta := \frac{g(0,0)}{(1/(2d))(g(0,0)+g(0,1))+1},$$

which tends to 1 as $d \to \infty$. Naively one could expect that from (2.1) we can obtain the analogous result for the bonds by replacing ν by $\zeta\nu$, but as (2.64) shows, this is not the case.

Let us finally describe how (2.67) can be obtained. The replacement of continuous time by discrete time random walk is routine. Let R_n^*, $n \in N$, denote the number of distinct bonds visited by the discrete time random walk between time 0 and n and let T_j^* denote the hitting time of the bond $(0, 0+e_j)$,

e_j being the jth basis vector of the canonical basis of R^d. Then we can easily see that

$$\begin{aligned}\sum_{n=1}^{\infty} u^n E[R_n^*] &= \sum_{x\in Z^d}\sum_{j=1}^{d} E_x\left[\sum_{n=1}^{\infty} u^n I_{\{T_j^*\le n\}}\right] \\ &= \sum_{x\in Z^d}\sum_{j=1}^{d} E_x\left[u^{T_j^*}\frac{1}{1-u}\right].\end{aligned} \tag{2.68}$$

We set

$$g_{u,j}^*(x,0) := E_x\left[\sum_{n=1}^{\infty} u^n I_{\{S_n\sim(0,e_j)\}}\right]. \tag{2.69}$$

Using the strong Markov property at the time T_j^* we obtain

$$g_{u,j}^*(x,0) = E_x[u^{T_j^*}](g_{u,j}^*(0,0)+1). \tag{2.70}$$

From (2.68) and (2.70) we deduce that

$$\sum_{n=1}^{\infty} u^n E[R_n^*] = \frac{u}{d(1-u)^2}\sum_{j=1}^{d}\frac{1}{g_{u,j}^*(0,0)+1}$$

and by Karamata's Tauberian theorem for power series (see [2], Corollary 1.7.3), we have

$$E[R_n^*] \sim n\,\frac{1}{d}\sum_{n=1}^{d}\frac{1}{g_{1,j}^*(0,0)+1} = n\,\frac{1}{1/2d(g(0,0)+g(0,1))+1}.$$

So (2.67) follows by by the subadditive ergodic theorem.

Acknowledgment. The author would like to thank Professor Alain-Sol Sznitman for many helpful suggestions.

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