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EIGENVALUES OF THE NATURAL RANDOM WALK ON THE BURNSIDE GROUP $B(3, n)$

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In this paper we give sharp bounds on the eigenvalues of the natural random walk on the Burnside group $B(3, n)$. Most of the argument uses established geometric techniques for eigenvalue bounds. However, the most interesting bound, the upper bound on the second largest eigenvalue, cannot be done by existing techniques. To give a bound we use a novel method for bounding the eigenvalues of a random walk on a group G (or equivalently its Cayley graph). This method works by choosing eigenvectors which fall into representations of an Abelian normal subgroup of G. One is then left with a large number (one for each representation) of easier problems to analyze.

1. Introduction. In this paper we will give sharp bounds on the eigenvalues of the random walk on the Burnside group $B(3, n)$ with the obvious generating set. Recall that the Burnside group $B(r, n)$ is generated by n elements $x_1, x_2, \dots, x_n$ and satisfies the relations that $g^r = 1$ for all g. For $r = 2$, the Burnside group $B(2, n) \simeq (\mathbf{Z}/2\mathbf{Z})^n$ is Abelian. The random walk on $B(2, n)$ generated by $\{x_1, x_2, \dots, x_n\}$ is therefore the usual random walk on the hypercube. This random walk is classical; in particular, all the eigenvalues of this random walk are known exactly. The second largest eigenvalue is $\beta_1 = 1 - 2/n$. For $r = 3$, the Burnside group $B(3, n)$ is finite and nilpotent of order

$$|B(3, n)| = 3^{n + \binom{n}{2} + \binom{n}{3}} = 3^{(n^3 + 5n)/6}.$$

A more detailed description is given below. The Burnside groups $B(4, n)$ and $B(6, n)$ are known to be finite as well, but the detailed structure of these groups is not known. For larger values of r many of the Burnside groups are known to be infinite.

The random walk on $B(3, n)$ is interesting for a number of reasons. First it fits into the general program of analyzing random walks on nilpotent groups begun by Diaconis and Saloff-Coste [1, 2]. These papers give bounds on the eigenvalues and rates of convergence of random walks on groups satisfying certain growth conditions, including nilpotent groups. In particular, suppose G is a nilpotent group with a symmetric set of E generators (containing the identity), class number l and diameter γ. Let P be the transition probability matrix for the associated random walk on G and U the uniform distribution

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on G. They show ([2], Corollary 5.3) that there are constants $B = B(l, E)$ and $C(l, E)$ such that

$$\|P^n - U\|_{\mathrm{Var}} \le Be^{-c} \quad \text{if } n = (1+c)\gamma^2 E \text{ and } c > 0$$

and

$$\|P^n - U\|_{\mathrm{Var}} \ge \tfrac{1}{2}e^{-c} \quad \text{if } n = c\gamma^2/C.$$

These results show that for many families of nilpotent groups (ones with E and l fixed), order γ^2 steps are necessary and sufficient to achieve randomness. However, for the Burnside groups $B(3, n)$ the parameter E is increasing. The dependence of the constants in [2] on E are such that one cannot get good bounds for $B(3, n)$ from these results. In fact, we will show that for $B(3, n)$ dramatically different results hold; that is, less than γ^2 steps suffice to achieve randomness. Furthermore, the example $B(3, n)$ is also of independent interest because of the universal property of $B(3, n)$. The results of this paper automatically give bounds on the eigenvalues of the random walk on any group all of whose (nonidentity) elements have order 3.

Since $B(3, n)$ has large diameter, the types of geometric bounds on the second largest eigenvalue given in [3] cannot hope to give sharp results (this vague statement is made precise by Proposition 3 and Corollary 3.1 below). To get around this, our argument will make crucial use of the fact that $B(3, n)$ has a large Abelian subgroup (namely, its commutator subgroup $[B(3,n), B(3, n)]$). Therefore, the eigenfunctions of the transition probability matrix for the random walk may be chosen to lie in irreducible representations of this Abelian subgroup. We will show that as a result they are eigenfunctions of a matrix which can be regarded as the transition probability matrix of a twisted random walk on $B(3, n)/[B(3, n), B(3, n)] \simeq (\mathbf{Z}/3\mathbf{Z})^n$. We must now deal with a more complicated "random walk," but on a much simpler graph. These simpler problems are done by applying Cauchy–Schwarz and closely related upper bounds in a geometric way. The method used works in quite a number of situations. For example, one can give bounds on eigenvalues of random walks on nilpotent groups of small class number. The terminology in this paper has been set forth fairly generally, but the other applications and more generalities about the method are given in [7].

Let $B(3, n)$ be the Burnside group $B(3, n) = \langle x_1, x_2, \dots, x_n : g^3 = 1 \text{ for all } g \in B(3, n)\rangle$, and let $S = \{x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}\}$ be the obvious generating set. A nice discussion of $B(3, n)$ can be found in [4], pages 320–324. For our purposes the salient features of $B(3, n)$ will be the following. Commutators in $B(3, n)$ satisfy the identities

$$[y, x] = [x, y]^{-1} = [x^{-1}, y] = [y, x^{-1}]^{-1}.$$

Double commutators satisfy the identities

$$[[x, y], z] = [[y, z], x] = [[z, x], y] = [[y, x], z]^{-1}.$$

All higher commutators vanish. As a consequence, any element of $B(3, n)$ can be written (in fact by [4] uniquely) in the form

$$(1)\qquad \begin{aligned} g = {} & x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}[x_1, x_2]^{b_{12}}[x_1, x_3]^{b_{13}}\cdots[x_{n-1}, x_n]^{b_{n-1,n}} \\ & \times[[x_1, x_2], x_3]^{c_{123}}\cdots[[x_{n-2}, x_{n-1}], x_n]^{c_{n-2,n-1,n}}, \end{aligned}$$

for $a_1, a_2, \ldots, c_{n-2,n-1,n} \in \mathbf{Z}/3\mathbf{Z}$. Hence

$$|B(3,n)| = 3^{n+\binom{n}{2}+\binom{n}{3}} = 3^{(n^3+5n)/6},$$

which we will denote by N. The commutator subgroup of $B(3, n)$, which we will denote by H, is Abelian and consists of exactly the elements of the form

$$(2)\qquad \begin{aligned} h = {} & [x_1, x_2]^{b_{12}}[x_1, x_3]^{b_{13}}\cdots[x_{n-1}, x_n]^{b_{n-1,n}} \\ & \times[[x_1, x_2], x_3]^{c_{123}}\cdots[[x_{n-2}, x_{n-1}], x_n]^{c_{n-2,n-1,n}}. \end{aligned}$$

Let P be the transition probability matrix for the associated random walk on $B(3, n)$ (i.e., randomly left multiply by an element of S) and let $1 = \beta_0 > \beta_1 \geq \cdots \geq \beta_{N-1} > -1$ be the eigenvalues of P. Let Γ denote the corresponding Cayley graph. The main goal of this paper will be to show that the following bounds on the eigenvalues hold.

THEOREM 1. (i) $1 - 1/(8n) \geq \beta_1 \geq 1 - 3/(2n)$.
(ii) $-1/2 \geq \beta_{N-1} \geq -7/9$.

Let U denote the uniform distribution on $B(3, n)$ and let $\|\cdot\|_{\mathrm{Var}}$ denote the bounded variation distance between probabilities on $B(3, n)$. As an easy corollary, if we apply Proposition 3 of [3], we get the following bound on the convergence rate of the random walk on $B(3, n)$.

COROLLARY 1.1. *If* $k = (2\log 3/3)n^4 + (10\log 3/3)n^2 + 8sn$, *then* $2\|P^k - U\|_{\mathrm{Var}} < e^{-s}$.

As remarked above, these results contrast sharply with the results of [2]. For the Burnside group $B(3, n)$, the diameter γ must be at least $Cn^3/\log n$ since we require words of at least this length in our $2n$ generators to generate all $3^{(n^3+5n)/6}$ elements of $B(3, n)$. [This bound cannot be too far off since the representation of the elements given in (1) above shows the diameter is at most Cn^3.] Therefore, γ^2 steps for $B(3, n)$ means at least $Cn^6(\log n)^{-2}$ steps and at most Cn^6 steps. Thus the convergence is substantially faster than one would guess from naively extrapolating the results of [2].

2. The easy bounds. The lower bound on β_1 and the upper bound on β_{N-1} are easy. Let χ: $(\mathbf{Z}/3\mathbf{Z})^n \to \mathbf{C}$ be a character of $(\mathbf{Z}/3\mathbf{Z})^n$ and consider the composition

$$\psi\colon B(3,n) \to B(3,n)/H = (\mathbf{Z}/3\mathbf{Z})^n \to_\chi \mathbf{C}.$$

This function ψ is an eigenfunction of P, and its eigenvalue is the same as the eigenvalue of χ for the usual random walk on $(\mathbf{Z}/3\mathbf{Z})^n$. Therefore, the eigenvalues $1 - 3k/(2n)$, $k = 0, 1, \dots, n$, of the usual random walk on $(\mathbf{Z}/3\mathbf{Z})^n$ are among the eigenvalues for P and, in particular, $1 - 3/(2n)$ and $-1/2$ are among them.

The lower bound on β_{N-1} is a direct application of Proposition 2 in [3]. Instead of doing the calculation directly, we will prove an amusing reformulation of Proposition 2. A number of interesting examples are easily handled by this reformulation; for example, Lemma 1 of [6] is a special case of this proposition.

PROPOSITION 2. *Let S be a generating set for the group G with $1 \notin S$, S closed under taking inverses, $|S| = d$ and $|G| = N$. Let P be the transition probability matrix for the random walk on G given by S and let $1 = \beta_0 > \beta_1 \geq \cdots \geq \beta_{N-1} \geq -1$ be the eigenvalues of P. Let k_m be the number of elements of S with order m. Then*

$$\beta_{N-1} \geq -1 + \frac{2}{d} \sum_{m \text{ odd}} \frac{k_m}{m^2}.$$

PROOF. To apply Proposition 2 of [3] we must choose for every vertex x of G a collection of paths from x to itself of odd length with total weight 1. For each element $g \in S$ of odd order, take the path $x, gx, g^2x, \dots, g^{\operatorname{ord}(g)-1}x$, x with weight $C \operatorname{ord}(g)^{-2}$. The total weight will be 1 if and only if $C^{-1} = \sum_{m \text{ odd}} (k_m/m^2)$. Then Proposition 2 of [3] says that $\beta_{N-1} \geq -1 + 2/\iota$, where in our case their formula for ι simplifies to $\iota = d \max_e \sum_\sigma |\sigma| \operatorname{wt}(\sigma)$, where the maximum is taken over directed edges and the sum is over all the paths containing that edge. If the edge e is $[x, gx]$ and $\operatorname{ord}(g)$ is odd, then e is on exactly $\operatorname{ord}(g)$ paths, all of length $\operatorname{ord}(g)$ and all of weight $C \operatorname{ord}(g)^{-2}$. If $\operatorname{ord}(g)$ is even, then e is not on any path. Hence $\iota = dC$ and $\beta_{N-1} \geq -1 + (2/d)\sum_{m \text{ odd}} (k_m/m^2)$. □

COROLLARY 2.1. *For the random walk on $B(3, n)$, $\beta_{N-1} \geq -7/9$.*

PROOF. The generating set S for $B(3, n)$ has $2n$ elements all of order 3, so $d = 2n$ and $\sum_{m \text{ odd}} (k_m/m^2) = 2n/9$. Therefore, we get the desired bound. □

Before turning to the proof of the remaining bound, the upper bound on β_1, it is worth noting that the geometric bounds of [3] cannot give sharp bounds on the eigenvalues [though they do at least give some bound, even bounds on $B(4, n)$ and $B(6, n)$]. To see this we have the following proposition. (A similar result holds for the bounds coming from path bounds on the Cheeger constant [5].)

PROPOSITION 3. *Let P be the transition probability matrix for the random walk on a connected, regular graph $\Gamma = (V, E)$. Let D be the expected squared* *distance between points of V. Suppose B is the upper bound on β_1 coming from* [3], *Proposition* 1 *or* 1′, *for some choice of paths. Then* $B \geq 1 - D^{-1}$.

PROOF. This fact is contained in the proof of Proposition 4 in [3]. For any choice of paths we get a lower bound on κ by replacing the maximum over edges by the average over edges. The calculation in [3], Proposition 4, then gives $\kappa \geq D$, as desired. □

COROLLARY 3.1. *If* Γ *above is vertex-transitive, then* $B \geq 1 - 4\operatorname{diam}(\Gamma)^{-2}$.

COROLLARY 3.2. *For the random walk on* $B(3, n)$ *defined above, let B be any upper bound on* β_1 *coming from* [3], *Proposition* 1 *or* 1′. *Then* $B \geq 1 - C(\log n)^2 n^{-6}$.

3. The upper bound on β_1. For the upper bound on β_1, we need the following observation. Let $H \subset B(3, n)$ be the commutator subgroup of $B(3, n)$. Recall from our discussion above that H is Abelian [in fact, isomorphic to $(\mathbf{Z}/3\mathbf{Z})^{(n^3-n)/6}$]. We chose the random walk to be given by left multiplication by generators; therefore, H acts on Γ by right multiplication. Hence the eigenfunctions of P may be chosen to lie in irreducible representations of H (which are of course one-dimensional since H is Abelian). Pick one such representation $\rho\colon H \to \mathbf{C}$ and let ψ be an eigenfunction of P which lies in that representation; that is, $\psi(gh) = \psi(g)\rho(h)$ for all $g \in B(3, n)$ and $h \in H$. If we fix coset representatives for $B(3, n)/H$, then ψ is determined by its values on the coset representatives. That is, we can view ψ as a function on $B(3, n)/H = (\mathbf{Z}/3\mathbf{Z})^n$.

Explicitly let $[g]$ denote the coset containing g and let $t_{[g]}$ be the coset representative. Define $\bar{\psi}\colon B(3, n)/H \simeq (\mathbf{Z}/3\mathbf{Z})^n \to \mathbf{C}$ by $\bar{\psi}([g]) = \psi(t_{[g]})$. We can recover ψ from $\bar{\psi}$ by the identity $\psi(g) = \bar{\psi}([g])\rho(t_{[g]}^{-1}g)$. In terms of $\bar{\psi}$, the equation $P\psi = \lambda\psi$ becomes

$$\begin{aligned} \lambda\bar{\psi}([g]) = \lambda\psi(t_{[g]}) &= \frac{1}{2n}\sum_{i=1}^{n}\left\{\psi(x_i t_{[g]}) + \psi(x_i^{-1}t_{[g]})\right\} \\ &= \frac{1}{2n}\sum_{i=1}^{n}\Big\{\psi(t_{[x_i g]})\rho\big(t_{[x_i g]}^{-1}x_i t_{[g]}\big) \\ &\qquad + \psi(t_{[x_i^{-1}g]})\rho\big(t_{[x_i^{-1}g]}^{-1}x_i^{-1}t_{[g]}\big)\Big\} \\ &= \frac{1}{2n}\sum_{i=1}^{n}\Big\{\bar{\psi}([x_i g])\rho\big(t_{[x_i g]}^{-1}x_i t_{[g]}\big) \\ &\qquad + \bar{\psi}([x_i^{-1}g])\rho\big(t_{[x_i^{-1}g]}^{-1}x_i^{-1}t_{[g]}\big)\Big\}. \end{aligned} \tag{3}$$

That is, $\bar{\psi}$ is an eigenfunction of a "random walk on $(\mathbf{Z}/3\mathbf{Z})^n$ with phases on the edges." Let $\bar{P}$ denote the corresponding matrix. The phases are values of ρ and hence are all 1, $\omega = (-1 + i\sqrt{3})/2$ or $\omega^2 = (-1 - i\sqrt{3})/2$. If ρ is the trivial representation of H, then all the phases are 1. Thus $\overline{\psi}$ is an eigenfunction of the usual random walk on $(\mathbf{Z}/3\mathbf{Z})^n$ as encountered above. The eigenvalues are therefore $1 - 3k/(2n)$ for $k = 0, 1, \dots, n$. Except for the $k = 0$ eigenvalue, which is β_0, these all satisfy the bound claimed in Theorem 1. Thus we are reduced to bounding the eigenvalues of $\overline{P}$ away from 1 for all nontrivial representations ρ of H. We will do so by showing that for any vector v we have $v^T\overline{P}v \le c|v|^2$ for some $c \le 1 - 1/(8n)$.

For $n = 2$ we can easily give such a bound (but a little too easily to give the full picture). The two nontrivial representations of $H = \mathbf{Z}/3\mathbf{Z}$ differ by complex conjugation; hence the associated matrices $\overline{P}$ will have the same eigenvalues. Calculating using (3) above, one sees that $\overline{\psi}$ on $(\mathbf{Z}/3\mathbf{Z})^2$ will be an eigenfunction of the 9×9 Hermitian matrix

$$\overline{P} = \frac{1}{4}\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \omega & \omega & 1 & 0 & 0 \\ 0 & 1 & 0 & \overline{\omega} & 0 & \omega & 0 & 1 & 0 \\ 0 & 0 & 1 & \overline{\omega} & \overline{\omega} & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \overline{\omega} & \overline{\omega} \\ 0 & 1 & 0 & 0 & 1 & 0 & \omega & 0 & \overline{\omega} \\ 0 & 0 & 1 & 0 & 0 & 1 & \omega & \omega & 0 \end{pmatrix}.$$

As remarked above, this matrix $\overline{P}$ has an interesting property. It is the transition probability matrix for the usual random walk on $(\mathbf{Z}/3\mathbf{Z})^2$ with some phases added to the edges as shown in Figure 1. As such it is the sum of nine nearly identical contributions from the nine squares that make up $(\mathbf{Z}/3\mathbf{Z})^2$ [here we are thinking of $(\mathbf{Z}/3\mathbf{Z})^2$ as lying on the torus]. Explicitly, $\overline{P} = \frac{1}{2}(Q_1 + Q_2 + \cdots + Q_9)$, where Q_1 is the 9×9 matrix that has zeroes in rows $3, 6, 7, 8, 9$ and columns $3, 6, 7, 8, 9$ but otherwise agrees with $\overline{P}$, and similarly for $Q_2, \dots, Q_9$. The matrices Q_i are all conjugate and all involve only four of the coordinates. They are all, up to conjugation and adding rows

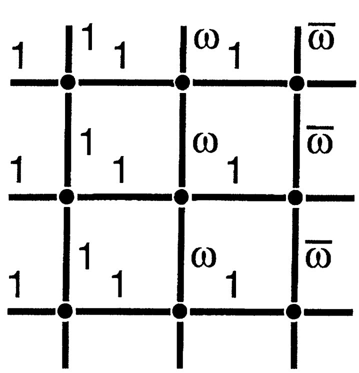


FIG. 1. *The graph underlying $\overline{P}$ drawn as though on a torus.*

and columns of zeroes, the same as the 4×4 matrix

$$\hat{Q} = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & \omega \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \overline{\omega} & 0 & 1 & 0 \end{pmatrix}.$$

The largest eigenvalue of this matrix is $\sqrt{3}/4$; therefore

$$v^T Q_1 v \leq \sqrt{3}\left(v_1^2 + v_2^2 + v_4^2 + v_5^2\right)/4,$$

and similarly for the other eight Q_i's. Summing over i we get $v^T \overline{P} v \leq \sqrt{3}|v|^2/2$. Therefore, the largest eigenvalue of $\overline{P}$ is bounded above by $\sqrt{3}/2 < 1 - 1/16$.

One difficulty in extending the bound above for $B(3,2)$ to $B(3,n)$ is that describing the intermediate matrices is cumbersome with this method. Instead note that we can encode a Hermitian matrix by a graph with some weights. This encoding is only a slight generalization of the encoding one gets by saying any regular graph (without weights) encodes the transition probability matrix for the usual random walk. This is another reason why it is convenient to think of these intermediate steps as being new "random walks" with some nonstandard features.

Consider the following data: a graph $\Gamma = (V, E)$, a positive real number d, a function w from the directed edges to $\mathbf{C}$ with $w([y, x]) = \overline{w([x, y])}$ and a function s from the vertices to $[0, d]$. Call such a collection of data $\Gamma = (\Gamma, d, w, s)$ a twisted graph. We will interpret this data as follows. The graph Γ is the underlying graph for our "random walk with phases on the edges." In our application Γ will always be a simple graph. We will regard d as roughly the degree of the graph. The function w we will regard as giving the weight on the edge. Our weights will generally be in $\{z \in \mathbf{C}: |z| = 1\}$ and in this case we will also refer to them as the phase on the edge. We regard $s(x)$ as giving the weight for remaining stationary at x and $s(x)/d$ as the probability of remaining stationary. In our examples we will always have $s(x)$ in the range $[0, d - \Sigma_y |w([x, y])|]$. We may regard $a(x) = d - \Sigma_y |w([x, y])| - s(x)$ as the weight for a particle at x to disappear, or be absorbed, and $a(x)/d$ is the probability a particle at x will be absorbed.

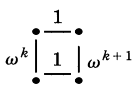
We will say a Hermitian matrix A' dominates another A if $v^T A v \leq v^T A' v$ for all v. Similarly we will say one twisted graph dominates another if the vertex sets agree and the corresponding Hermitian matrix dominates the other. Note that we can easily build twisted graphs which dominate others. If $\mathbf{X}$ is a sub(twisted graph) of Γ, $\mathbf{X}'$ dominates $\mathbf{X}$ and Γ' is the twisted graph we get by replacing $\mathbf{X}$ in Γ by $\mathbf{X}'$, then Γ' dominates Γ. The easiest example of this is Cauchy–Schwarz applied to an edge. The twisted graph

$$\bullet\!\overset{1}{\text{———}}\!\bullet \qquad (s = 0)$$

is dominated by the twisted graph

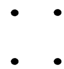
$$\underset{\bullet}{s = 1 - \varepsilon} \qquad \underset{\bullet}{s = 1/(1 - \varepsilon)}$$

for any $0 \le \varepsilon < 1$. Therefore we may do this replacement to any edge (with any starting phase in fact) to get a dominating twisted graph. This fact, if not this terminology, was used extensively in [6]. Note that removing an edge symmetrically with Cauchy–Schwarz leaves the absorption weight unchanged. The other basic example we need for this paper is the one used above: the twisted graph



$(s = 0)$

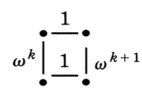
is dominated by the twisted graph



with $s = \sqrt{3}$ at every vertex.

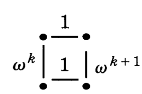
As a first easy example of how this terminology can be used to give eigenvalue bounds, consider the following example (which will be used below). Suppose our twisted graph has as its underlying graph a 3-cube, $d = 3$, $s = 0$ and $w(e) \in \{z \in \mathbf{C}: |z|^2 = 1\}$ for all directed edges e.

LEMMA 4. *If the twisted graph* Γ *is as above and any face of the cube is of the form*



then Γ *is dominated by the twisted graph whose underlying graph is eight points with no edges*, $d = 3$ *and* $s = 23/8$ *at every vertex.*

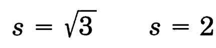
PROOF. By the above, the face of the form



is dominated by

(with $s = \sqrt{3}$ at every vertex) and the opposite face is necessarily dominated by

(with $s = 2$ at every vertex) since every edge weight is of unit norm. Therefore, Γ is dominated by the disjoint union of four twisted graphs of the form



for some $|\eta| = 1$. Applying Cauchy–Schwarz asymmetrically to the edge, this twisted graph is dominated by the twisted graph consisting of two vertices with weights $\sqrt{3} + 1/(1 - \varepsilon)$ and $3 - \varepsilon$, respectively. Taking $\varepsilon = 1/8$ gives the near optimum value of $s = 23/8$. □

We will use these definitions to give eigenvalue bounds as follows. Call a twisted graph diagonal if the underlying graph has no edges (or each edge has weight 0). Then we have the following obvious lemma.

LEMMA 5. *If the twisted graph* Γ *is dominated by the diagonal twisted graph* $\Gamma' = (\Gamma', d', w', s')$ *and* λ *is any eigenvalue of* Γ, *then* $\lambda \le \max_{v \in \Gamma'} s'(v)/d'$.

With this terminology the remaining steps in the bound on β_1 are straightforward. Fix a nontrivial representation ρ: $H \to \mathbf{C}$ and let ψ be an eigenfunction of P in that representation; that is, $\psi(gh) = \psi(g)\rho(h)$ for all $g \in B(3, n)$ and $h \in H$. Recall from (2) above that any element of H can be written uniquely as

$$h = [x_1, x_2]^{b_{12}}[x_1, x_3]^{b_{13}} \cdots [x_{n-1}, x_n]^{b_{n-1,n}}$$
$$\times [[x_1, x_2], x_3]^{c_{123}} \cdots [[x_{n-2}, x_{n-1}], x_n]^{c_{n-2,n-1,n}}$$

and with respect to this decomposition,

$$\rho(h) = \omega^{\alpha_{12}b_{12} + \alpha_{13}b_{13} + \cdots + \alpha_{n-1,n}b_{n-1,n} + \gamma_{123}c_{123} + \cdots + \gamma_{n-2,n-1,n}c_{n-2,n-1,n}}$$

for some $\alpha_{12}, \ldots, \alpha_{n-1,n}, \gamma_{123}, \ldots, \gamma_{n-2,n-1,n} \in \mathbf{Z}/3\mathbf{Z}$, not all 0. We are free to act on H by any isomorphism of $B(3, n)$ without changing the eigenvalues. (It is remotely possible one could actually use symmetry to reduce the problem to a small enough set of representations to give a complete list of eigenvalues.) Since we can permute or invert the elements $x_1, \ldots, x_n$ by isomorphisms of $B(3, n)$, we may assume one of the following holds:

1. $\alpha_{12} = 1$ and $\gamma_{123} = 1$;
2. $\alpha_{12} = 1$ and $\gamma_{12k} = 0$ for all $k = 3, \ldots, n$;
3. $\alpha_{ij} = 0$ for all $1 \le i < j \le n$ and $\gamma_{123} = 1$.

Choose coset representatives for $B(3, n)/H$ to be $x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ for $a_1, a_2, \ldots, a_n \in \mathbf{Z}/3\mathbf{Z}$. If we view ψ as $\bar{\psi}$ a function on $B(3, n)/H = (\mathbf{Z}/3\mathbf{Z})^n$ as above, then $\bar{\psi}$ is an eigenfunction of the Hermitian matrix associated with the twisted graph Γ whose underlying graph is $(\mathbf{Z}/3\mathbf{Z})^n$, $d = 2n$, $s = 0$ and the weights always phases. In fact, from (3) above the weight on the edge from $[g]$ to $[x_i g]$ is $\rho(t^{-1}_{[x_i g]} x_i t_{[g]})$. Thus the following lemma gives the weights.

LEMMA 6. *In* $B(3, n)$, $x_i x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = x_1^{a_1} \cdots x_i^{a_i+1} \cdots x_n^{a_n} h$, *where*

$$h = [x_1, x_i]^{-a_1}[x_2, x_i]^{-a_2} \cdots [x_{i-1}, x_i]^{-a_{i-1}}$$
$$\times [[x_1, x_i], x_2]^{-a_1 a_2} \cdots [[x_1, x_i], x_n]^{-a_1 a_n}$$
$$\times [[x_2, x_i], x_3]^{-a_2 a_3} \cdots [[x_{i-1}, x_i], x_n]^{-a_{i-1} a_n}.$$

That is, the exponent of $[x_k, x_i]$ *in* h *is* $-a_k$ *if* $k < i$ *and* 0 *if* $k > i$, *and the exponent of* $[[x_k, x_i], x_j] = [[x_k, x_j], x_i]^{-1}$ *is* $-a_k a_j$ *if* $k < i, k < j$ *and* $j \ne i$, *and* 0 *otherwise. (Note that to put this into our standard form, we must choose the first form if* $i < j$ *and the second if* $i > j$.)

PROOF. Calculate using the identities for commutators in $B(3, n)$ from [4] mentioned above. □

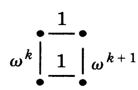
Now we are ready to find a diagonal twisted graph which dominates Γ. The underlying graph of Γ can be edge-decomposed into the following pieces: for each $i \geq 4$, a collection of 3^{n-1} 3-cycles corresponding to multiplication by x_i and 3^n cubes (each with weight $1/4$). A typical cube has its eight vertices of the form

$$\{x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_n^{a_n},\ x_1^{a_1+1}x_2^{a_2}x_3^{a_3}\cdots x_n^{a_n},\ x_1^{a_1}x_2^{a_2+1}x_3^{a_3}\cdots x_n^{a_n},\\ x_1^{a_1+1}x_2^{a_2+1}x_3^{a_3}\cdots x_n^{a_n},\ x_1^{a_1}x_2^{a_2}x_3^{a_3+1}\cdots x_n^{a_n},\ x_1^{a_1+1}x_2^{a_2}x_3^{a_3+1}\cdots x_n^{a_n},\\ x_1^{a_1}x_2^{a_2+1}x_3^{a_3+1}\cdots x_n^{a_n},\ x_1^{a_1+1}x_2^{a_2+1}x_3^{a_3+1}\cdots x_n^{a_n}\}.$$

Each vertex is in exactly eight such cubes and each edge generated by one of $\{x_1, x_2, x_3\}$ is in exactly four such cubes. [If one views each copy of $(\mathbf{Z}/3\mathbf{Z})^3$ generated by x_1, x_2 and x_3 as lying on a 3-torus $(S^1)^3$, then these are the cubes into which the graph divides the 3-torus.]

LEMMA 7. *Each of these cubes satisfies the hypotheses of Lemma* 4; *hence each is dominated by a diagonal twisted graph with* $s(v) \leq 23/8$.

PROOF. We need only check that at least one of the faces of the cube has the form



(the other hypotheses being clear). The edges generated by x_1 always have weight 1 since by Lemma 6 or inspection we have $x_1 t_{[g]} = t_{[x_1 g]}$. The edge from $x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_n^{a_n}$ to $x_1^{a_1}x_2^{a_2+1}x_3^{a_3}\cdots x_n^{a_n}$ has, by Lemma 6 and the definition of ρ, weight

$$\rho\left([x_1, x_2]^{-a_1}[[x_1, x_2], x_3]^{-a_1a_3}\cdots[[x_1, x_2], x_n]^{-a_1a_n}\right)\\ = \omega^{-\alpha_{12}a_1-\gamma_{123}a_1a_3-\cdots-\gamma_{12n}a_1a_n}.$$

Note that this can be rewritten as η^{a_1}, where $\eta = \omega^{-\alpha_{12}-\gamma_{123}a_3-\cdots-\gamma_{12n}a_n}$. Similarly, the weight on the edge from

$$x_1^{a_1+1}x_2^{a_2}x_3^{a_3}\cdots x_n^{a_n} \quad \text{to} \quad x_1^{a_1+1}x_2^{a_2+1}x_3^{a_3}\cdots x_n^{a_n}$$

is η^{a_1+1}. Therefore, unless $\eta = 1$ the face

$$\{x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots x_n^{a_n},\ x_1^{a_1+1}x_2^{a_2}x_3^{a_3}\cdots x_n^{a_n},\ x_1^{a_1}x_2^{a_2+1}x_3^{a_3}\cdots x_n^{a_n},\\ x_1^{a_1+1}x_2^{a_2+1}x_3^{a_3}\cdots x_n^{a_n}\}$$

has the desired form. Similarly, the face $\{x_1^{a_1}x_2^{a_2}x_3^{a_3+1}\cdots x_n^{a_n}, x_1^{a_1+1}x_2^{a_2}x_3^{a_3+1}\cdots x_n^{a_n}, x_1^{a_1}x_2^{a_2+1}x_3^{a_3+1}\cdots x_n^{a_n}, x_1^{a_1+1}x_2^{a_2+1}x_3^{a_3+1}\cdots x_n^{a_n}\}$ will have the desired form unless $\eta' = \omega^{-\alpha_{12}-\gamma_{123}(a_3+1)-\cdots-\gamma_{12n}a_n} = 1$. Recall that we assumed by symmetry that one of the following holds:

1. $\alpha_{12} = 1$ and $\gamma_{123} = 1$;
2. $\alpha_{12} = 1$ and $\gamma_{12k} = 0$ for all $k = 3, \dots, n$;
3. $\alpha_{ij} = 0$ for all $1 \le i < j \le n$ and $\gamma_{123} = 1$.

If 2 holds, then $\eta = \eta' = \omega^{-1} \neq 1$, both these faces have the desired form and hence we are done. If either 1 or 3 holds, then $\eta(\eta')^{-1} = \omega \neq 1$ and hence at least one of these two faces has the desired form. In any event, Lemma 6 applies. □

Therefore, the twisted graph Γ is dominated by the twisted graph where all the edges generated by x_1, x_2 and x_3 have been replaced with $s = 8(1/4)(23/8) = 23/4$ on every vertex. Since the remaining 3-cycles all have weights of norm 1, each is dominated by the twisted graph with $s = 2$ on every vertex the 3-cycle goes through. Replacing every 3-cycle in this fashion, we see that Γ is dominated by the diagonal twisted graph with $d = 2n$ and $s = 2(n-3) + 23/4 = 2n - 1/4$ on every vertex. Thus by Lemma 5 every eigenvalue of Γ is bounded above by $1 - 1/(8n)$. This completes the bounds. □

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