The Annals of Probability
1995, Vol. 23, No. 4, 1748–1754

ON THE LARGE TIME GROWTH RATE OF THE SUPPORT OF SUPERCRITICAL SUPER-BROWNIAN MOTION

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Consider the supercritical super-Brownian motion $X(t,\cdot)$ on R^d corresponding to the evolution equation

$$u_t = \frac{D}{2}\Delta u + u - u^2.$$

We obtain rather tight bounds on $P_\mu(X(s, B_n^c(0)) = 0$, for all $s \in [0,t])$ and on $P_\mu(X(t, B_n^c(0)) = 0)$, for large n, where P_μ denotes the measure corresponding to the supercritical super-Brownian motion starting from the finite measure, μ, $B_n(0) \subset R^d$ denotes the ball of radius n centered at the origin and $B_n^c(0)$ denotes its complement. In particular, we show, for example, that if μ is a compactly supported, finite measure on R^d, then

$$\lim_{n\to\infty} P_\mu\big(X(t, B_n^c(0)) = 0, \text{for all } t \in [0, \gamma n]\big) = 1 \quad \text{if } \gamma < (2D)^{-1/2}$$

and

$$\lim_{n\to\infty} P_\mu\big(X(\gamma n, B_n^c(0)) = 0 | \text{the process survives}\big) = 0 \quad \text{if } \gamma > (2D)^{-1/2}.$$

In this article, we investigate the large time growth rate of the support of supercritical super-Brownian motion. In particular, we show that the linear growth rate known to hold for classical binary branching Brownian motion continues to hold with the same constant in the present context. We recall the construction of the supercritical super-Brownian motion. For each positive integer n, consider N_n particles, each of mass $1/n$, starting at points $x_i^{(n)} \in R^d$, $i = 1,\dots,N_n$, and performing independent branching d-dimensional Brownian motions corresponding to the generator $(D/2)\Delta$, with branching rate cn, $c > 0$, and with branching distribution $\{p_k^{(n)}\}_{k=0}^\infty$. Assume that $\sum_{k=0}^\infty k p_k^{(n)} = 1 + b/n$, $b > 0$, and that $\sum_{k=0}^\infty (k-1)^2 p_k^{(n)} = a + o(1)$, $a > 0$, as $n \to \infty$. Let $N_n(t)$ denote the total number of particles alive at time t and denote their positions by $x_i^{(n)}(t)$, $i = 1,\dots,N_n(t)$. Recalling that the mass of each particle is $1/n$, define as follows the measure-valued process $X_n(t) = X_n(t,\cdot)$ on $\mathscr{M}(R^d)$, the space of finite measures on R^d:

$$X_n(t,B) = \frac{1}{n}\sum_{i=1}^{N_n(t)} 1_B\big(x_i^{(n)}(t)\big), \qquad B \in \mathscr{B}(R^d).$$

It has been shown that if $X_n(0,\cdot) = (1/n)\sum_{i=1}^{N_n}\delta_{x_i^{(n)}}(\cdot)$ converges weakly as $n \to \infty$ to a limiting measure, $\mu \in \mathscr{M}(R^d)$, then the measure-valued process $X_n(t)$ converges weakly to a limiting measure-valued process, $X(t) = X(t,\cdot)$,

Received March 1994; revised November 1994.
AMS 1991 *subject classifications*. 60J80, 60J25.
Key words and phrases. Super-Brownian motion, supercritical, measure-valued process.

which can be characterized uniquely as the solution to the following martingale problem. [In the sequel, $\langle f, X(t)\rangle$ denotes $\int_{R^d} f(x)X(t,dx)$.]

MG. $X(t) = X(t,\cdot)$ is an $\mathscr{M}(R^d)$-valued process such that $X(0) = \mu$ a.s. and, for each $f \in C_b^\infty(R^d)$,

$$M_f(t) \equiv \langle f, X(t)\rangle - \int_0^t \left\langle \frac{D}{2}\Delta f, X(s)\right\rangle ds - cb\int_0^t \langle f, X(s)\rangle\, ds \text{ is a} \tag{1}$$
$$\text{martingale with increasing process } \langle M_f\rangle_t = 2ca\int_0^t \langle f^2, X(s)\rangle\, ds.$$

(See, e.g., [6] or [1].)

The solution to the above martingale problem is known as supercritical super-Brownian motion. (If $b = 0$, then the process is called critical super-Brownian motion.) Probabilities and expectations with respect to the above supercritical super-Brownian motion satisfying MG will be denoted by P_μ and E_μ. The independence built into the approximating particle system manifests itself in the limiting supercritical super-Brownian motion via the following log-Laplace equation. For nonnegative $g, \psi \in C_c(R^d)$,

$$E_\mu \exp\left(-\langle g, X(t)\rangle - \int_0^t \langle \psi, X(s)\rangle\, ds\right) = \exp\left(-\int_{R^d} u(x,t)\mu(dx)\right), \tag{2}$$

where u is the unique mild solution to the following evolution equation:

$$u_t = \frac{D}{2}\Delta u + cbu - cau^2 + \psi, \qquad (x,t) \in R^d \times [0,\infty),$$
$$u(\cdot,0) = g, \qquad u \ge 0, \qquad u(\cdot,t) \in C_0(R^d) \quad \text{for } t > 0.$$

where $C_0(R^d) = \{w \in C(R^d): \lim_{|x|\to\infty} w(x) = 0\}$. This was proved in [3] for the case of critical super-Brownian motion; the same type of proof works in the present case; see also [1]. If u satisfies the above equation, then $v(x,t) \equiv (a/b)u(x, t/(cb))$ satisfies the same equation with $ca = cb = 1$, and with D, g and ψ replaced by $D/(cb)$, $(a/b)g$ and $(a/(cb^2))\psi$. Thus, in the sequel, we will make the normalization $ca = cb = 1$. We record the corresponding equation for future reference:

$$u_t = \frac{D}{2}\Delta u + u - u^2 + \psi, \qquad (x,t) \in R^d \times [0,\infty). \tag{3}$$
$$u(\cdot,0) = g(\cdot), \qquad u \ge 0, \qquad u(\cdot,t) \in C_0(R^d).$$

We note that, for $\psi, g \in C_c^\infty(R^d)$, it follows from standard regularity results for parabolic equations that $u(x,t)$ is a classical solution to (3).

Let $Z(t) = \langle 1, X(t)\rangle$ denote the total mass process. Substituting $f \equiv 1$ in (1), it follows that under P_μ, $Z(t)$ is a one-dimensional diffusion on $[0,\infty)$ corresponding to the operator $x(d^2/dx^2) + x(d/dx)$ and satisfying $Z(0) = \mu(R^d)$. As is well known, the probability that a diffusion corresponding to the operator $x(d^2/dx^2) + x(d/dx)$ and starting from $y \in (0,n)$ reaches 0 before reaching n is given by $u_n(y)$, where u_n solves $xu_n'' + xu_n' = 0$, $u_n(0) = 1$, $u_n(n) = 0$. Solving this gives $u_n(y) = (e^{-y} - e^{-n})/(1 - e^{-n})$. Letting $n \to \infty$, we conclude that, starting from y, the probability of the diffusion ever hitting 0 is e^{-y}. Furthermore, by a theorem due to Yamada and Watanabe [7], it follows that, upon hitting zero, the diffusion remains there forever. Applying this to $Z(t)$ gives

$$(4)\qquad \begin{aligned} P_\mu(Z(t) > 0, \forall t \geq 0) &= 1 - \exp(-\mu(R^d)),\\ P_\mu(Z(t) = 0, \text{for all large } t) &= \exp(-\mu(R^d)). \end{aligned}$$

If $Z(t) > 0$, for all $t \geq 0$, we will say that the supercritical super-Brownian motion survives, while if $Z(t) = 0$, for all large t, we will say that it dies out.

Construct $\{\psi_{n,m}\}_{n,m=1}^{\infty} \subset C^\infty([0,\infty))$ such that $\psi_{n,m} \leq \psi_{n,m+1}$ and such that

$$\begin{aligned} \psi_{n,m}(r) &= 0 \quad \text{if } 0 \leq r \leq n,\\ \psi_{n,m}(r) &= m \quad \text{if } n + \frac{1}{m} \leq r \leq n + m,\\ \psi_{n,m}(r) &= 0 \quad \text{if } r \geq n + m + 1, \end{aligned}$$

and

$$0 \leq \psi_{n,m} \leq m.$$

Let $u_{n,m}$ denote the solution to (3) with $\psi(x) = \psi_{n,m}(|x|)$ and $g = 0$. From the symmetry of g and ψ, the symmetry of (3) and the uniqueness of the solution to (3), it follows that $u_{n,m}$ is radially symmetric. Thus, in the sequel we will write $u_{n,m} = u_{n,m}(r,t)$, $r \geq 0$, $t \geq 0$. From (2), it also follows that $u_{n,m}$ is monotone nondecreasing in m. Define

$$(5)\qquad u_n(r,t) \equiv \lim_{m\to\infty} u_{n,m}(r,t) \quad \text{for } (r,t) \in [0,\infty) \times [0,\infty).$$

Let $B_n(0)$ denote the ball of radius r centered at the origin and let $B_n^c(0)$ denote its complement. Using (2), (3) and (5) and letting $\hat{\psi}_{n,m}(x) = \psi_{n,m}(|x|)$, $x \in R^d$, we obtain

$$(6)\qquad \begin{aligned} &P_\mu(X(s, B_n^c(0)) = 0, \text{for all } s \in [0,t])\\ &\quad = P_\mu\left(\int_0^t X(s, B_n^c(0))\, ds = 0\right) = \lim_{m\to\infty} E_\mu \exp\left(-\int_0^t \langle \hat{\psi}_{n,m}, X(s)\rangle\, ds\right)\\ &\quad = \lim_{m\to\infty} \exp\left(-\int_{R^d} u_{n,m}(|x|,t)\,\mu(dx)\right) = \exp\left(-\int_{R^d} u_n(|x|,t)\,\mu(dx)\right). \end{aligned}$$

We now reverse the order of ψ and g. Let $\{g_{n,m}\}_{n,m=1}^{\infty} \subset C^\infty([0,\infty))$ satisfy $g_{n,m} \leq g_{n,m+1}$ and

$$\begin{aligned} g_{n,m}(r) &= 0 \quad \text{if } 0 \leq r \leq n,\\ g_{n,m}(r) &= m \quad \text{if } n + \frac{1}{m} \leq r \leq n + m,\\ g_{n,m}(r) &= 0 \quad \text{if } r \geq n + m + 1, \end{aligned}$$

and

$$0 \leq g_{n,m} \leq m,$$

and let $U_{n,m}$ denote the solution to (3) with $\psi(x) = 0$ and $g(x) = g_{n,m}(|x|)$. The same considerations used above for $u_{n,m}$ show that $U_{n,m}$ is radially symmetric and monotone nondecreasing in m. Thus, in the sequel, we will write $U_{n,m} = U_{n,m}(r,t)$, $r \geq 0$, $t \geq 0$. Define

$$(7) \qquad U_n(r,t) \equiv \lim_{m\to\infty} U_{n,m}(r,t) \quad \text{for } (r,t) \in [0,\infty) \times [0,\infty).$$

By a calculation similar to (6), it follows that

$$(8) \qquad P_\mu(X(t, B_n^c(0)) = 0) = \exp\left(-\int_{R^d} U_n(|x|,t)\,\mu(dx)\right).$$

From (6) and (8), it follows that the growth rate of the support of the supercritical super-Brownian motion is governed by the asymptotic behavior as $n, t \to \infty$ of the functions u_n and U_n in (5) and (7). The behavior of u_n and U_n was investigated by the author in [5]; the relevant results are summarized in Proposition A and Theorem A below.

PROPOSITION A. (i) *There exists a unique positive radial solution* $\phi_n(|x|) \in C^2(B_n(0))$ *to the equation*

$$\frac{D}{2}\Delta\phi = \phi^2 - \phi \quad in\ B_n(0),$$
$$\lim_{|x|\to n} \phi(x) = \infty,$$
$$\phi \geq 1.$$

Furthermore,

$$\phi_n(r) \leq 1 + \frac{12Dn^2}{(n^2 - r^2)^2}.$$

(ii) *For each* $\rho \in (0, \sqrt{2})$, *there exists a unique positive increasing solution* $f_\rho \in C^2([0,\infty))$ *to the equation*

$$\tfrac{1}{2}f'' - \rho f' + f - f^2 = 0, \qquad x \geq 0,$$
$$f(0) = 0, \qquad \lim_{x\to\infty} f(x) = 1.$$

THEOREM A. *Let* u_n *and* U_n *be as in* (5) *and* (7) *and let* ϕ_n *and* f_ρ *be as in Proposition* A. *Then the following hold*:

(a)(i) *For each* $\varepsilon > 0$, *there exists a constant* $c_\varepsilon > 0$ *and an* n_ε *such that, for* $n \geq n_\varepsilon$,

$$u_n(r,t) \leq \phi_n(r)\exp\left(-\left(\frac{(n-r)^2}{2D(1+\varepsilon)t} - t - c_\varepsilon\right)^+\right),$$
$$(r,t) \in [0,n) \times [0,\infty).$$

(ii) *For* $\rho \in (0, \sqrt{2})$ *and* $n > 0$,

$$u_n(r,t) \ge f_\rho\left(\left(\rho t + \frac{r-n}{\sqrt{D}}\right)^+\right)\phi_n(r), \qquad (r,t) \in [0,n) \times [0,\infty).$$

(b)(i) *For each* $\varepsilon > 0$, *there exists a constant* $c_\varepsilon > 0$ *such that, for all* $n > 0$,

$$U_n(r,t) \le \begin{cases} (1-\exp(-t))^{-1}\exp\left(-\left(\dfrac{(n-r)^2}{2D(1+\varepsilon)t} - t - c_\varepsilon\right)^+\right), \\ \qquad (r,t) \in [0,n) \times [0,\infty), \\ (1-\exp(-t))^{-1}, \\ \qquad (r,t) \in [n,\infty) \times [0,\infty). \end{cases}$$

(ii) *For* $\rho \in (0, \sqrt{2})$ *and* $n > 0$,

$$U_n(r,t) \ge f_\rho\left(\left(\rho t + \frac{r-n}{\sqrt{D}}\right)^+\right)(1-e^{-t})^{-1}, \qquad (r,t) \in [0,\infty) \times [0,\infty).$$

Using Proposition A and Theorem A, we can prove the following theorem concerning the asymptotic growth rate of the support of the supercritical super-Brownian motion.

THEOREM 1. (a) *Let* $\mu \in \mathscr{M}(R^d)$ *have compact support. Then the following hold*:

(i)

$$\lim_{n\to\infty} P_\mu\big(X(t, B_n^c(0)) = 0, \text{ for all } t \in [0, \gamma n]\big) = 1 \quad \text{if } \gamma < (2D)^{-1/2}.$$

(ii)

$$\lim_{n\to\infty} P_\mu\big(X(\gamma n, B_n^c(0)) = 0 \mid \text{the process survives}\big) = 0 \quad \text{if } \gamma > (2D)^{-1/2};$$

(b) *Let* $\{x_n\}_{n=1}^\infty \subset R^d$ *satisfy* $|x_n| = n - l_n$, *where* $\lim_{n\to\infty} l_n = \infty$. *Then*:

(i)

$$\lim_{n\to\infty} P_{\delta_{x_n}}\big(X(t, B_n^c(0)) = 0, \text{ for all } t \in [0, \gamma l_n]\big) = 1 \quad \text{if } \gamma < (2D)^{-1/2};$$

(ii)

$$\lim_{n\to\infty} P_{\delta_{x_n}}\big(X(\gamma l_n, B_n^c(0)) = 0 \mid \text{the process survives}\big) = 0 \quad \text{if } \gamma > (2D)^{-1/2}.$$

(c) *Let* $\{m_n\}_{n=1}^\infty, \{t_n\}_{n=1}^\infty \subset (0,\infty)$ *and let* $\{x_n\}_{n=1}^\infty \subset R^d$ *satisfy* $|x_n| = n - l_n$, *where* $\lim_{n\to\infty} l_n = \infty$. *If*

$$\lim_{n\to\infty} m_n \exp\left(-\left(\frac{l_n^2}{2D(1+\varepsilon)t_n} - t_n\right)^+\right) = 0 \quad \text{for some } \varepsilon > 0,$$

then

$$\lim_{n\to\infty} P_{m_n\delta_{x_n}}\big(X(t, B_n^c(0)) = 0 \quad \text{for all } t \in [0, t_n]\big) = 1.$$

REMARK 1. Part (a) is reminiscent of a well-known result for binary branching Brownian motion. A Brownian motion corresponding to the generator $(D/2)\Delta$ starts at $x_0 \in R^d$ and, at a mean 1 exponential time, dies and bears two offspring in its stead. The two offspring perform independent Brownian motions from their birth location and die and bear two offspring at independent mean 1 exponential times and so forth. Let P_{x_0} denote the probability measure corresponding to this branching process. Then

$$\lim_{n \to \infty} P_{x_0}(\text{all particles alive at time } \gamma n \text{ lie in } (-\infty, n])$$
$$= \begin{cases} 1, & \text{if } \gamma < (2D)^{-1/2}, \\ 0, & \text{if } \gamma > (2D)^{-1/2} \end{cases}$$

(see [4]). A recent paper by Evans and O'Connell [2] was brought to my attention by the referee. In that paper, the authors obtain a representation for supercritical super-Brownian motion in terms of a subcritical super-Brownian motion with immigration, where the immigration is governed by the state of an underlying binary branching Brownian motion. It may be possible to give an alternative proof of Theorem 1, especially parts (a) and (b), via [2] and [4].

REMARK 2. Using Theorem A, Proposition A, (6) and (8), it is easy, of course, to formulate and prove a more general result involving arbitrary sequences of times $\{t_n\}_{n=1}^{\infty}$ and of initial measures $\{\mu_n\}_{n=1}^{\infty} \subset \mathscr{M}(R^d)$.

PROOF OF THEOREM 1. The proofs of (a)(i), (b)(i) and (c) follow immediately from Theorem A(a)(i), Proposition A and (6), by picking $\varepsilon > 0$ sufficiently small depending on $\gamma < (2D)^{-1/2}$. We now prove (a)(ii) and (b)(ii). In order to handle the two cases simultaneously and to maintain simple notation, we will assume that $\mu(R^d) = 1$ in (a)(ii), and we will define $\mu_n \equiv \mu$ in (a)(ii) and $\mu_n = \delta_{x_n}$ in (b)(ii). By (4),

$$P_{\mu_n}(\text{the process dies out}) = e^{-1} \quad \text{for all } n. \tag{9}$$

As noted earlier, under P_{μ_n}, the total mass process is a one-dimensional diffusion on $[0, \infty)$ corresponding to the operator $x(d^2/dx^2) + x(d/dx)$ and starting at $\mu_n(R^d) = 1$. Thus, clearly,

$$\lim_{t \to \infty} P_{\mu_n}(\text{the process dies out at some time in } [t, \infty)) = 0, \quad \text{uniformly in } n. \tag{10}$$

It follows from (a)(i) and (b)(i) that

$$\lim_{n \to \infty} P_{\mu_n}(X(s, B_n^c(0)) = 0, \text{for all } s \in [0, t]) = 1 \quad \text{for each } t > 0. \tag{11}$$

From (9)–(11), we conclude that

$$\lim_{n \to \infty} P_{\mu_n}(X(t_n, B_n^c(0)) = 0, \text{and the process dies out}) = e^{-1}, \quad \text{if } \lim_{n \to \infty} t_n = \infty. \tag{12}$$

From (8), Theorem A(b)(ii) and the fact that $\lim_{x\to\infty} f_\rho(x) = 1$, it follows, by choosing ρ sufficiently close to $\sqrt{2}$ depending on $\gamma > (2D)^{-1/2}$, that

$$\begin{aligned} &\lim_{n\to\infty} P_{\mu_n}\big(X(\gamma n, B_n^c(0)) = 0\big) = e^{-1} \quad \text{in case (a)(ii)}, \\ &\lim_{n\to\infty} P_{\mu_n}\big(X(\gamma l_n, B_n^c(0)) = 0\big) = e^{-1} \quad \text{in case (b)(ii)}. \end{aligned} \tag{13}$$

Now (a)(ii) and (b)(ii) follow from (12) (with $t_n = \gamma n$ or γl_n) and (13).

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