*The Annals of Probability*
1995, Vol. 23, No. 4, 1671–1691

# EXPONENTIAL AND UNIFORM ERGODICITY OF MARKOV PROCESSES[1]

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General characterizations of geometric convergence for Markov chains in discrete time on a general state space have been developed recently in considerable detail. Here we develop a similar theory for $\varphi$-irreducible continuous time processes and consider the following types of criteria for geometric convergence:

1. the existence of exponentially bounded hitting times on one and then all suitably "small" sets;
2. the existence of "Foster–Lyapunov" or "drift" conditions for any one and then all skeleton and resolvent chains;
3. the existence of drift conditions on the extended generator $\tilde{\mathscr{A}}$ of the process.

We use the identity $\tilde{\mathscr{A}}R_\beta = \beta(R_\beta - I)$ connecting the extended generator and the resolvent kernels $R_\beta$ to show that, under a suitable aperiodicity assumption, exponential convergence is completely equivalent to any of criteria 1–3. These conditions yield criteria for exponential convergence of unbounded as well as bounded functions of the chain. They enable us to identify the dependence of the convergence on the initial state of the chain and also to illustrate that in general some smoothing is required to ensure convergence of unbounded functions.

**1. Introduction.** In this paper we consider a continuous time Markov process $\mathbf{\Phi} = \Phi_t$, $t \in \mathbb{R}_+$, on a topological space $\mathsf{X}$. Our goal is to characterize exponential convergence for the process: if $P^t(x, A) = \mathsf{P}_x(\Phi_t \in A)$ and $\pi$ is an invariant measure for $P^t$, then we will give several sets of equivalent conditions, each of which imply $\mathbf{\Phi}$ is "exponentially ergodic" in the sense that there exists an invariant measure $\pi$ satisfying

$$(1) \qquad \|P^t(x,\cdot) - \pi\| \le M(x)\rho^t, \qquad t \ge 0,$$

for some finite $M(x)$ and some $\rho < 1$, where $\|\cdot\|$ is the total variation norm.

These characterizations of exponential ergodicity, as given in Sections 5, 6 and 7, are in terms of the following criteria:

1. geometric ergodicity of the embedded skeletons and the resolvent chains; that is, the discrete time chains with transition laws defined, respectively,

Received January 1994; revised December 1994.

[1]Work supported in part by NSF Grant DMS-92-05687, EPRI Contract RP 8030-15 and NSF Grants ECS92-16487 and ECS 94-03742.

*AMS* 1991 *subject classification.* 60J10.

*Key words and phrases.* Convergence rates, irreducible Markov processes, geometric ergodicity, Lyapunov functions, drift conditions, resolvents, generators.

as $P^T(x, A)$ for fixed $T > 0$ and

$$R_\beta(x, A) = \int_0^\infty \beta e^{-\beta t} P^t(x, A)\, dt, \qquad x \in \mathsf{X},\ A \in \mathscr{B}(\mathsf{X}); \tag{2}$$

2. drift conditions on the extended generator of the process, and on the skeletons and the resolvent chains, all of which are shown to be equivalent themselves;
3. behavior of hitting times for the process on suitable small sets and, in particular, exponential bounds on those hitting times.

Some initial results related to statement 1 were developed in [25], where it was shown that exponential ergodicity of the process $\mathbf{\Phi}$ follows from the geometric ergodicity of the embedded skeletons or, under appropriate continuity conditions (in $t$) on the semigroup $P^t$, of a form of simultaneous geometric ergodicity of the resolvent chains. These simultaneity and continuity conditions are shown to be redundant in Section 6.

In [17] it was shown that, as in statement 2, a drift condition on the generator is sufficient to guarantee exponential ergodicity for the process. This generalized results known for countable spaces [27] and for diffusion processes [8] to quite general models, and as shown in (e.g.) [17, 24], the conditions on the generator then provide practical criteria for evaluating the exponential convergence of specific models. Here we show that such a drift for the extended generator is also necessary for exponential ergodicity.

The conclusions in this paper thus strengthen the known results considerably, since they show all of these approaches to be essentially equivalent. Moreover, once the connections between the drift and regularity conditions are established, we may then deduce continuous time ergodicity results stronger than those in (1) from their discrete time counterparts; in particular, we can show that for appropriate (unbounded) functions $f$, we have

$$\|P^t(x, \cdot) - \pi\|_f \le M(x)\rho^t, \qquad t \ge 0, \tag{3}$$

where $\pi$ is an invariant probability measure, and the $f$-norm $\|\cdot\|_f$ is defined for any signed measure $\mu$ by $\|\mu\|_f := \sup_{|g| \le f} |\int \mu(dy) g(y)|$.

**2. Discrete time analogues.** In order to place these results in context, and because we will use the discrete time results directly, we first review briefly the analogous equivalences known in discrete time. Let $\{\Phi_n,\ n \in \mathbb{Z}_+\}$ denote a Markov chain on a space $(\mathsf{X}, \mathscr{B}(\mathsf{X}))$ with the $\sigma$-field countably generated: the theory of such chains is developed in [15].

Suppose the chain has the $\varphi$-irreducibility property that $\varphi(A) > 0$ implies $\mathsf{P}_x(\tau_A < \infty) > 0$ for all $x \in \mathsf{X}$, where $\tau_A := \min\{n \ge 1\colon\ \Phi_n \in A\}$ is the first hitting time on $A$, and $\mathsf{P}_x$ and $\mathsf{E}_x$ denote probability and expectation for the chain with initial state $x$. For such a chain there always exist "small sets" [15, Chapter 5]: that is, sets $C$ such that for some nontrivial probability measure $\nu$ and some $n \ge 1$, $\varepsilon > 0$, the $n$-step transition probability kernel $P^n(x, A) := \mathsf{P}_x(\Phi_n \in A)$ satisfies, for all $x \in C$,

$$P^n(x, A) \ge \varepsilon \nu(A), \qquad A \in \mathscr{B}(\mathsf{X}). \tag{4}$$

We then have [15, Theorem 15.0.1] the following result linking drift toward small sets, hitting times on small sets and rates of convergence of the overall chain.

THEOREM 2.1. *Suppose that the chain* $\mathbf{\Phi}$ *is* $\varphi$*-irreducible and aperiodic. Then the following three conditions are equivalent*:

(a) *There exist some small set* $C$, *some constant* $\pi_C > 0$, *some* $\rho_C < 1$ *and* $M_C < \infty$ *such that, for all* $x \in C$,

$$|P^n(x, C) - \pi_C| \le M_C \rho_C^n. \tag{5}$$

(b) *There exist some small set* $C \in \mathscr{B}(\mathsf{X})$ *and* $\kappa > 1$ *such that the hitting time* $\tau_C$ *on* $C$ *satisfies*

$$\sup_{x \in C} \mathsf{E}_x[\kappa^{\tau_C}] < \infty. \tag{6}$$

(c) *There exist some small set* $C$, *constants* $b < \infty$, $\lambda < 1$ *and a function* $V \ge 1$, *with* $V(x_0) < \infty$ *for some one* $x_0 \in \mathsf{X}$, *satisfying the drift condition*

$$\int P(x, dy) V(y) \le \lambda V(x) + b \mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{7}$$

*Any of these three conditions implies that the set* $S_V = \{x\colon V(x) < \infty\}$ *is absorbing and full* [*i.e., satisfies* $P(x, S_V) = 1$, $x \in S_V$, *and* $\pi(S_V) = 1$], *where* $V$ *is any solution to* (7) *satisfying the conditions of* (c), *and there then exist a unique invariant probability* $\pi$ *and constants* $r > 1$, $D < \infty$, *such that for any* $x \in S_V$,

$$\sum_n r^n \left\| \int P^n(x, \cdot) - \pi \right\|_V \le DV(x). \tag{8}$$

Thus in the discrete case geometric ergodicity, as defined in (8), follows from local geometric convergence as in (5) or geometrically bounded return times as in (6), and, of even more practical importance, is actually equivalent to the existence of a Foster–Lyapunov or drift function $V$ satisfying (7), and that function identifies (a) a set on which convergence takes place, namely, $S_V$; (b) the state-dependent bound $M(x)$ as in (1) or (3) as a constant multiple of $V$; (c) the convergence as holding for all "moments of order less than $V$" as in (8).

In this paper we aim to bring this same level of coherence to the continuous time case.

## 3. Continuous time Markov processes.

We need to develop the appropriate analogues of the drift condition (7) and the hitting time criteria in (6), and to do this we must first give a more formal description of the process structure. We then consider drift conditions in Section 5 and the hitting time conditions in Section 6.

Formally, we assume that $\mathbf{\Phi} = \{\Phi_t\colon t \in \mathbb{R}_+\}$ is a nonexplosive Borel right process with transition semigroup $(P^t)$ on a locally compact, separable metric space $(\mathsf{X}, \mathscr{B}(\mathsf{X}))$, and that $\mathscr{B}(\mathsf{X})$ is the Borel field on $\mathsf{X}$. The reader is referred to [1] and [23] for details of the existence and structure of such processes; criteria for nonexplosivity are given in [17].

The operator $P^t$ acts on bounded measurable functions $f$ and $\sigma$-finite measures $\mu$ on $\mathsf{X}$ via

$$P^t f(x) = \int_{\mathsf{X}} P^t(x, dy) f(y), \qquad \mu P^t(A) = \int_{\mathsf{X}} \mu(dx) P^t(x, A).$$

A $\sigma$-finite measure $\pi$ on $\mathscr{B}(\mathsf{X})$ with the property $\pi = \pi P^t$ for all $t \geq 0$ will be called *invariant*.

For any $\sigma$-finite measure $\varphi$ the process $\mathbf{\Phi}$ is called *$\varphi$-irreducible* if $\varphi(B) > 0 \Rightarrow \mathsf{E}_x[\eta_B] > 0$, $x \in \mathsf{X}$, where $\eta_B$ denotes the occupancy time, defined as $\eta_B := \int_0^\infty \mathbb{1}\{\Phi_t \in B\}\, dt$. As in the discrete time setting, if $\mathbf{\Phi}$ is $\varphi$-irreducible, then there exists a *maximal* irreducibility measure $\psi$ such that $\nu$ is absolutely continuous with respect to $\psi$ for any other irreducibility measure $\nu$ [15, 28]. We shall write $\mathscr{B}^+(\mathsf{X})$ for the collection of all measurable subsets $A \subset \mathsf{X}$ such that $\psi(A) > 0$. When there is an invariant probability measure $\pi$, then $\pi$ and $\psi$ are mutually absolutely continuous and we can thus identify $\mathscr{B}^+(\mathsf{X}) = \{A \in \mathscr{B}(\mathsf{X}) \colon \pi(A) > 0\}$.

We now describe sampled chains and the associated class of subsets of $\mathsf{X}$ called petite sets; these play the role of small sets, as in Theorem 2.1, in describing the "center" of the space. To define these, as in [16], suppose that $a$ is a probability distribution on $\mathbb{R}_+$ and define the Markov transition function $K_a$ for the chain sampled by $a$ as

$$K_a(x, A) := \int P^t(x, A) a(dt), \qquad x \in \mathsf{X},\ A \in \mathscr{B}(\mathsf{X}). \tag{9}$$

Two particular sampled chains with kernels $K_a$ which are fundamental in the continuous time context are *skeleton chains* and *resolvent chains*, with $K_a = P^T$, $R_\beta$, respectively. We will consider both of these in developing stability properties of the process itself, although we note that in [26] it is shown that other embedded chains may equally serve to characterize the behavior of $\mathbf{\Phi}$. In the special case of the resolvent with transition law $R_1$, we denote the chain by $\check{\mathbf{\Phi}} = \{\check{\Phi}_k\}$.

A nonempty set $C \in \mathscr{B}(\mathsf{X})$ is called *$\nu_a$-petite* if $\nu_a$ is a nontrivial measure on $\mathscr{B}(\mathsf{X})$ and $a$ is a sampling distribution on $(0, \infty)$ satisfying

$$K_a(x, \cdot) \geq \nu_a(\cdot), \qquad x \in C. \tag{10}$$

In the common situation where the specific measure $\nu_a$ is not relevant, we simply call the set *petite*. When the sampling distribution $a$ is degenerate, we will adopt standard discrete time usage and call the set $C$ *small*.

Petite sets are not rare: for a $\psi$-irreducible chain, every set in $\mathscr{B}^+(\mathsf{X})$ contains a petite set [17]. We note explicitly that if a set is petite for the process, then it is petite for any resolvent simultaneously, with only the sampling measure changing.

In many applications, all nonempty compact sets can be shown to be petite, and this gives a sound intuition for these sets. Criteria to ensure this identification of compacta as petite may be phrased in terms of a "stochastic controllability" condition in models such as diffusion processes or analogous discrete time nonlinear state space models (cf. [15], Chapter 7, or [9, 20, 19, 12]). More general continuity conditions which imply that compact sets are petite are given in [16].

A $\psi$-irreducible Markov process $\mathbf{\Phi}$ will be called *aperiodic* if for some small set $C \in \mathscr{B}^+(\mathsf{X})$ there exists a $T$ such that $P^t(x, C) > 0$ for all $t \geq T$ and all $x \in C$. Under aperiodicity, any petite set is small for the process and is small for any $T$-skeleton (see the corollary to Theorem 8.1 of [16]). Characterizations of aperiodicity which are (apparently) much weaker than this definition are given in [16], where the structure of periodic chains is also described in detail. Below, it will be appropriate to assume aperiodicity whenever we consider convergence of transition probabilities as in (1) or (3).

The Markov process $\mathbf{\Phi}$ is called *f-exponentially ergodic*, where $f$ is a measurable function from the state space $\mathsf{X}$ to $[1, \infty)$, if the $f$-norm converges as in (3). The Markov process is called simply *exponentially ergodic* if it is $f$-exponentially ergodic for at least one $f \geq 1$, as in (1). Note that any exponentially ergodic chain is automatically a $\pi$-irreducible positive Harris recurrent chain [7, 16].

As strong as $f$-exponential ergodicity may appear, we will find it appropriate to set our results within the seemingly stronger context of *V-uniform ergodicity*, a recent generalization [6, 17, 15] of classical uniform ergodicity (or quasicompactness, as it is often called [22]). When $V \geq 1$ is a measurable function on $\mathsf{X}$, we define $V$-uniform ergodicity by requiring that

$$\|P^t(x, \cdot) - \pi\|_V \leq V(x) D \rho^t, \qquad t \geq 0, \tag{11}$$

for some $D < \infty$, $\rho < 1$; that is, (3) holds with $M(x) = DV(x)$ for some constant $D$. This approach has the advantage that convergence takes place in the operator norm defined for any "kernel" $A$ by

$$|||A|||_V := \sup_{x \in \mathsf{X}} [V(x)]^{-1} \|A(x, \cdot)\|_V.$$

For discrete time chains, as is shown in Theorem 2.1, $V$-uniform ergodicity is in fact equivalent to seemingly weaker formulations of exponential ergodicity. As in the discrete time framework, the exponential convergence (11) is equivalent to an exponential rate of mixing for the process (see [15], Theorem 16.1.5).

**4. Generators and resolvents.** Frequently, the characteristics used in practice to define the process are not couched in terms of the semigroup $P^t$ or of the embedded resolvent or skeleton chains, but rather of some form of *generator* for the process. There are several different versions of generators: for our purpose it is convenient to adopt the following definition, which is a slightly restricted form of that in Davis [2]. We denote by $D(\tilde{\mathscr{A}})$ the set of all functions $f\colon \mathsf{X} \times \mathbb{R}_+ \to \mathbb{R}$ for which there exists a measurable function $g\colon \mathsf{X} \times \mathbb{R}_+ \to \mathbb{R}$ such that, for each $x \in \mathsf{X}$, $t > 0$,

$$\mathsf{E}_x[f(\Phi_t, t)] = f(x,0) + \mathsf{E}_x\left[\int_0^t g(\Phi_s, s)\, ds\right], \tag{12}$$

$$\int_0^t \mathsf{E}_x[|g(\Phi_s, s)|]\, ds < \infty. \tag{13}$$

We write $\tilde{\mathscr{A}} f := g$ and call $\tilde{\mathscr{A}}$ the *extended generator* of $\mathbf{\Phi}$; $D(\tilde{\mathscr{A}})$ is called the domain of $\tilde{\mathscr{A}}$.

This defines an extension of the infinitesimal generator for Hunt processes. When the process is explosive, then the expression (12) is difficult to interpret and may be meaningless: this is one of our main reasons for restricting to nonexplosive processes. Conditions for nonexplosivity based upon the extended generator are given in [17].

The generator and resolvent essentially characterize one another. This observation will be useful when we develop converses to the drift criteria for regularity.

For a fixed constant $\beta > 0$ we also require the kernel $U_\beta$, defined by $U_\beta = \beta^{-1} R_\beta$. These kernels have the interpretations

$$R_\beta(x, f) = \mathsf{E}_x[f(\Phi_\xi)], \tag{14}$$

$$U_\beta(x, f) = \mathsf{E}_x\left[\int_0^\xi f(\Phi_s)\, ds\right], \tag{15}$$

where $\xi$ is an exponentially distributed random variable, independent of $\mathbf{\Phi}$, with mean $1/\beta$. We now generalize the definition of $R_\beta, U_\beta$ so that $\beta$, the rate of the "sampling time," may depend upon the value of the state $\Phi_t$. Let $h$ be a bounded measurable function on $\mathsf{X}$ and set

$$R_h(x, f) = \mathsf{E}_x\left[\int_0^\infty \exp\left\{-\int_0^t h(\Phi_s)\, ds\right\} h(\Phi_t) f(\Phi_t)\, dt\right], \tag{16}$$

$$U_h(x, f) = \mathsf{E}_x\left[\int_0^\infty \exp\left\{-\int_0^t h(\Phi_s)\, ds\right\} f(\Phi_t)\, dt\right]. \tag{17}$$

A function $f\colon \mathsf{X} \to \mathbb{R}$ is in the *domain of* $R_h$ if $R_h(x, |f|)$ is finite for all $x \in \mathsf{X}$. These kernels were introduced by Neveu [21], where $h$ is taken to be strictly positive. Setting $U_h(x, E) \equiv E_h \mathbb{1}_E(x)$ defines a kernel on $(\mathsf{X}, \mathscr{B}(\mathsf{X}))$, which takes on finite values if $h$ satisfies $\inf_{x \in \mathsf{X}} h(x) > 0$, although such a bound on $h$ is not necessary in general.

These kernels have an interpretation which is entirely analogous to (14) and (15) which will lead to several new results below. Let $\xi$ denote an exponential random variable with unit mean which is independent of the process, and define for any $h, r > 0$,

$$\tilde{\tau}_{h,r} = \inf\left\{t > 0\colon \int_0^t h(\Phi_s)\, ds \geq r^{-1}\xi\right\}. \tag{18}$$

When $h = \mathbb{1}_C$ for some set $C$, we use $\tilde{\tau}_{C,r}$. We then have, for any positive $h$ and any $f$ in the domain of $R_h, U_h$,

$$R_h(x,f) = \mathsf{E}_x\big[f(\Phi_{\tilde{\tau}_{h,1}})\big], \tag{19}$$

$$U_h(x,f) = \mathsf{E}_x\left[\int_0^{\tilde{\tau}_{h,1}} f(\Phi_s)\,ds\right]. \tag{20}$$

These formulae follow directly from the identity $\mathsf{P}(\tilde{\tau}_{h,r} > t \mid \Phi_0^\infty) = \exp(-r\int_0^t h(\Phi_s)\,ds)$.

LEMMA 4.1. *Let $h, g$ be bounded, measurable functions on $\mathsf{X}$ with $h \geq g$, and let $f\colon \mathsf{X} \to \mathbb{R}$ be in the domain of $U_g$. Then $U_h$ and $U_g$ satisfy the resolvent equation*

$$U_g f - U_h f = U_h I_{h-g} U_g f = U_g I_{h-g} U_h f, \tag{21}$$

*where the "multiplication kernel" $I_{h-g}$ is defined as $I_{h-g}(x, A) := (h(x) - g(x))\mathbb{1}_A(x)$.*

PROOF. Note that the theorem statement differs from Lemma 1.3.1 of Kunita [10] only in that the functions $h$ and $g$ may take on negative values. However, a close examination shows that the proof there is still valid with this modification. $\square$

Lemma 4.1 immediately gives the usual resolvent identity, that $U_\beta = U_\alpha + (\alpha - \beta)U_\alpha U_\beta$ for any $\alpha > \beta > 0$. From this it follows that

$$R_\beta = \frac{\beta}{\alpha}\sum_{n=0}^{\infty}\left(\frac{\alpha-\beta}{\alpha}\right)^n R_\alpha^{n+1}. \tag{22}$$

These relations will allow us to transfer properties from one resolvent to another in several results below. Then to connect these results back to geometric ergodicity for the process, we will use the kernel $U_C^r$, which is defined for $r > 1$ by

$$U_C^r = U_h \quad \text{for } h = \mathbb{1}_C - [r-1]\mathbb{1}_{C^c}.$$

This has the sample path interpretation, just as in (20),

$$U_C^r(x,f) = \mathsf{E}_x\left[\int_0^{\tilde{\tau}_{C,r}} e^{(r-1)t} f(\Phi_t)\,dt\right]. \tag{23}$$

As an immediate consequence of the resolvent equation, we obtain an analogous interpretation of $U_C^r$ in terms of the resolvent chain. Define the hitting time to a set $C$ by the resolvent chain (with $\beta = 1$) by

$$\check{\tau}_C = \min\{k \geq 1 \colon \check{\Phi}_k \in C\}. \tag{24}$$

LEMMA 4.2. *Suppose $C$ is any set in $\mathscr{B}(\mathsf{X})$ and $r \geq 1$. The kernel $U_C^r$ satisfies the following identity for any positive function $f$ on $\mathsf{X}$*:

$$\mathsf{E}_x\left[\int_0^{\tilde{\tau}_{C,r}} f(\Phi_t) e^{(r-1)t}\,dt\right] = U_C^r(x, f) = \mathsf{E}_x\left[\sum_{k=1}^{\check{\tau}_C} r^{k-1} f(\check{\Phi}_k)\right]. \tag{25}$$

PROOF. The first identity is just (23).

We will prove the second equation for $|r| < 1$; the general case follows by analytic continuation. Substituting $h = 1$ and $g = \mathbb{1}_C - [r-1]\mathbb{1}_{C^c}$ in (21), we have

$$U_C^r = R_1 + U_C^r(1 - (\mathbb{1}_C - [r-1]\mathbb{1}_{C^c}))R_1 = R_1 + rU_C^r \mathbb{1}_{C^c} R_1. \tag{26}$$

Iterating (26) gives us

$$U_C^r = r^n U_C^r (I_{C^c} R_1)^n + \sum_{k=1}^{n} r^{k-1} [R_1 I_{C^c}]^{k-1} R_1,$$

and letting $n \to \infty$ yields, for $|r| < 1$,

$$U_C^r = \sum_{k=1}^{\infty} r^{k-1} [R_1 I_{C^c}]^{k-1} R_1. \tag{27}$$

Now using the interpretation that

$$[R_1 I_{C^c}]^{k-1} R_1(x, B) = \mathsf{P}_x\left(\check{\Phi}_k \in B; \check{\tau}_C \geq k\right),$$

we see that (27) gives the form (25) as required. □

The following key lemma characterizes the extended differential generator for the process $\mathbf{\Phi}$ in terms of the resolvent chains of the process and allows us to compare continuous and discrete time drift operations, as will be seen in Theorem 5.1 and other results below. This result may be applied to many situations outside the scope of this paper.

LEMMA 4.3. (a) *For any $\beta$ and any measurable $f$ in the domain of $U_\beta$, the extended generator satisfies the identity*

$$\tilde{\mathscr{A}} U_\beta f = -f + \beta U_\beta f. \tag{28}$$

(b) *For any bounded measurable $g$ and any measurable $f$ in the domain of $U_g$, the extended generator satisfies*

$$\tilde{\mathscr{A}} U_g f = -f + I_g U_g f. \tag{29}$$

PROOF. For the strong or weak generator, Lemma 4.3(a) is known [see (2.4) in [5] or page 342 of [11]). We have not seen a proof for the extended generator, although the proof of the result is similar to Lemma 3.2 of [5], page 174, and will not be included here.

Result (b) follows from (a) and the resolvent equation $U_g = U_\beta + U_\beta I_{\beta - g} U_g$, where $\beta$ is chosen so that $\beta \geq g(x)$ for all $x$. □

**5. Stability in terms of drifts.** In this section we consider forms of stability, analogous to (7), which involve a *mean drift* for some function $V(\Phi_t)$ toward the "center" of the space, as defined by petite sets.

Letting $\{P^t\colon t \in \mathbb{R}_+\}$, $\{R_\beta\colon \beta \in \mathbb{R}_+\}$ and $\tilde{\mathscr{A}}$ denote, respectively, the skeleton kernels, the resolvent kernels and the extended generator for the process, three intuitively reasonable sets of drift conditions toward a petite set $C$ may be written as follows:

($\mathscr{D}_T$) DRIFT FOR SKELETON CHAINS. For some $T > 0$ there exist constants $\lambda(s)$, bounded for $s \in (0, T]$ and with $\lambda(T) < 1$, some $b < \infty$, a petite set $C$ in $\mathscr{B}(\mathsf{X})$ and a function $V_T \geq 1$, such that

$$P^s V_T \leq \lambda(s) V_T + b\,\mathbb{1}_C, \qquad s \leq T.$$

($\check{\mathscr{D}}_\beta$) DRIFT FOR RESOLVENT CHAINS. For some $\lambda < 1$, $b < \infty$, $\beta > 0$, a petite set $C$ in $\mathscr{B}(\mathsf{X})$ and a function $V_\beta \geq 1$,

$$R_\beta V_\beta \leq \lambda V_\beta + b\,\mathbb{1}_C.$$

($\tilde{\mathscr{D}}$) DRIFT FOR THE EXTENDED GENERATOR. For constants $b, c > 0$, a petite set $C$ in $\mathscr{B}(\mathsf{X})$ and a function $\tilde{V} \geq 1$,

$$\tilde{\mathscr{A}}\tilde{V} \leq -c\tilde{V} + b\,\mathbb{1}_C.$$

We note that condition ($\tilde{\mathscr{D}}$) is identical to the condition that $\tilde{\mathscr{A}}\tilde{V} \leq -c_0\tilde{V} + b_0$ for finite positive constants $c_0, b_0$, when the function $\tilde{V}$ is *unbounded off petite sets* (i.e., when the sublevel set $\{x\colon V(x) \leq n\}$ is either empty or petite for each $n$). The other two conditions have similar counterparts.

In the next section we will identify a set of solutions to these three drift conditions, which involves the hitting times on petite sets. Our goal here is to show that the drift conditions are all essentially equivalent and then to show that any one of them suffices for an appropriate form of geometric ergodicity to hold.

THEOREM 5.1. *The following relations hold for any Markov process* $\mathbf{\Phi}$:

(a) *Suppose that* ($\check{\mathscr{D}}_\beta$) *holds for some one* $\beta$ *and for some function* $V_\beta \geq 1$. *Then* ($\tilde{\mathscr{D}}$) *holds with* $\tilde{V} = R_\beta V_\beta$.

(b) *Suppose* ($\check{\mathscr{D}}_\beta$) *holds for some one* $\beta$ *and for some function* $V_\beta \geq 1$. *Then* ($\mathscr{D}_T$) *holds for every* $T$ *with* $V_T = R_\beta V_\beta$ *and* $\lambda(s)$ *can be chosen continuous in* $s$ *with* $\lambda(T) < 1$.

(c) *Suppose* ($\tilde{\mathscr{D}}$) *holds. Then* ($\check{\mathscr{D}}_\beta$) *holds for any* $\beta$ *and with* $V_\beta = \tilde{V}$.

(d) *Suppose* ($\tilde{\mathscr{D}}$) *holds. Then* ($\mathscr{D}_T$) *holds for any* $T$ *and with* $V_T = \tilde{V}$.

(e) *Suppose* ($\mathscr{D}_T$) *holds for some one* $T > 0$. *Then* ($\check{\mathscr{D}}_\beta$) *holds for some* $\beta_0 > 0$, *with* $V_{\beta_0} = V_T$, *and hence for all* $\beta$ *with* $V_\beta = R_{\beta_0} V_T$.

(f) *Suppose* ($\mathscr{D}_T$) *holds for some one* $T > 0$. *Then* ($\tilde{\mathscr{D}}$) *holds with* $\tilde{V} = R_{\beta_0} V_T$ *for some one* $\beta_0$.

*Consequently, the three drift criteria* $(\check{\mathscr{D}}_\beta)$–$(\tilde{\mathscr{D}})$ *are all equivalent, although the function* $V$ *and the petite set* $C$ *may differ at each appearance.*

PROOF. To prove (a), suppose that $(\check{\mathscr{D}}_\beta)$ holds and define $\tilde{V} = R_\beta V_\beta$. By Lemma 4.3,

$$\begin{aligned}\tilde{\mathscr{A}}\tilde{V} &= \beta\tilde{\mathscr{A}}U_\beta V_\beta = \beta(R_\beta - I)V_\beta \\ &\le -\beta(\lambda^{-1} - 1)\tilde{V} + \lambda^{-1}\beta b \mathbb{1}_C,\end{aligned} \tag{30}$$

which gives $(\tilde{\mathscr{D}})$. Conversely, to prove (c) note that if $(\tilde{\mathscr{D}})$ holds for some $\tilde{V}$ in the domain of the extended generator, then using Dynkin's formula exactly as in the proof of Theorem 6.1 of [17], we have for all $t$,

$$e^{ct}P^t\tilde{V} \le \tilde{V} + b\int_0^t P^s(x,C)e^{cs}\,ds. \tag{31}$$

Integrating the bound (31) gives

$$\begin{aligned}R_\beta\tilde{V}(x) &\le [\beta/(\beta + c)]\tilde{V}(x) + b\beta\int_0^\infty e^{-\beta t}\int_0^t P^s(x,C)e^{c(s-t)}\,ds\,dt \\ &= [\beta/(\beta + c)]\tilde{V}(x) + [b/(\beta + c)]R_\beta(x,C).\end{aligned} \tag{32}$$

Now denote, for fixed $\varepsilon$, the set $C_\varepsilon = \{x\colon R_\beta(x,C) > \varepsilon(\beta + c)/b\}$. From Proposition 5.5.4 of [15] we know that $C_\varepsilon$ is petite, and with this choice of petite set, with $\varepsilon$ sufficiently small, $(\check{\mathscr{D}}_\beta)$ holds as required.

Using (31) we obtain (d) in a similar manner: for all $t$ we can write (31) as

$$P^t\tilde{V} \le e^{-ct}\tilde{V} + b\int_0^t P^s(x,C)e^{c(s-t)}\,ds. \tag{33}$$

If for fixed $t > 0$ we use the sampling kernel with density $a(s) = e^{c(s-t)}/[1 - e^{-t}]$, then we have $(\mathscr{D}_T)$ provided we use the petite set where $K_a(x,C) \ge \varepsilon c/b[1 - e^{-t}]$.

To prove (b), we have that if $(\check{\mathscr{D}}_\beta)$ holds, then from (a), $(\tilde{\mathscr{D}})$ holds with $\tilde{V} = R_\beta V_\beta$, and then from (d) we obtain the result. Note that we can in fact take, for any $\varepsilon$ and appropriate choice of $C$,

$$\lambda(s) = \exp(-\beta(\lambda^{-1} - 1)s + \varepsilon),$$

which gives the required structure on the function $\lambda(s)$.

To see (e), observe that if $(\mathscr{D}_T)$ holds, then by iteration one can show that

$$P^tV_T \le M\lambda^n V_T + \frac{Mb}{1-\lambda} + b,$$

where $\lambda = \lambda(T)$, $M = \max_{0\le s\le T}\lambda(s)$ and $t$ is taken of the form $t = nT + s$ for some $0 \le s \le T$. It then follows that, for $\beta_0$ sufficiently small,

$$R_{\beta_0}V_T \le \lambda_{\beta_0}V_T + \frac{Mb}{1-\lambda} + b,$$

for some $\lambda_{\beta_0} < 1$.

Any function satisfying $(\mathscr{D}_T)$ is unbounded off petite sets as in Lemma 15.2.2 of [15]. It then follows from the above bound on $R_{\beta_0}$ that $(\check{\mathscr{D}}_{\beta_0})$ holds by applying Lemma 15.2.8 of [15].

Finally, we have (f) by combining (e) with (a). □

Drift in any of these senses toward a petite set implies exponential ergodicity for an aperiodic process; conversely, exponential ergodicity entails that all of these drift conditions are satisfied, as we now show. Thus the continuous time analogues of the discrete time criterion (c) in Theorem 2.1 all hold.

THEOREM 5.2. *For a $\psi$-irreducible, aperiodic Markov process*:

(a) *If $(\check{\mathscr{D}}_\beta)$ holds, then $\mathbf{\Phi}$ is $R_\beta V_\beta$-uniformly ergodic.*
(b) *If $(\mathscr{D}_T)$ holds, then $\mathbf{\Phi}$ is $V_T$-uniformly ergodic.*
(c) *If $(\tilde{\mathscr{D}})$ holds, then $\mathbf{\Phi}$ is $\tilde{V}$-uniformly ergodic.*

PROOF. First suppose that (b) holds. Then by aperiodicity, the set $C$ is small for the skeleton chain $\{\Phi_{kT}: k \in \mathbb{Z}_+\}$ (see the corollary to Theorem 8.1 of [16]). Hence from $(\mathscr{D}_T)$ and Theorem 2.1,

$$|||P^{nT} - \pi|||_{V_T} \le D\rho^n, \qquad n \in \mathbb{Z}_+,$$

where $D < \infty$ and $\rho < 1$. By $(\mathscr{D}_T)$ we also have, for some constant $M$,

$$P^s V \le MV, \qquad 0 \le s \le T,$$

and it follows from the submultiplicative property of the operator norm that for any $t \in \mathbb{R}_+$, with $t$ taken of the form $t = nT$ for some $0 \le s \le T$,

$$|||P^t - \pi|||_{V_T} \le |||P^{nT} - \pi|||_{V_T}\,|||P^s|||_{V_T} \le MB\rho^n, \qquad n \in \mathbb{Z}_+,$$

which implies that the process is $V_T$-uniformly ergodic.

If (a) or (c) holds, then (b) holds with $V_T = R_\beta V_\beta$ or $V_T = \tilde{V}$, respectively, from Theorem 5.1, and hence $V_T$-uniform ergodicity holds in these cases also. □

Result (b) was used to establish geometric ergodicity for a class of generalized Jackson networks in [13]. Several queuing models and diffusion models are analyzed in [16] using a formulation of (c).

As a consequence of Theorems 5.1 and 5.2, we have a number of further criteria for exponential ergodicity extending in particular those in [26], where it was required that all resolvents be "uniformly" exponentially ergodic to deduce properties of the process.

THEOREM 5.3. *For a $\psi$-irreducible, aperiodic Markov process the following are equivalent*:

(a) *The $T$-skeleton is geometrically ergodic for some one and then any $T > 0$.*

(b) *The $R_\beta$-resolvent is geometrically ergodic for some one and then any* $\beta > 0$.

(c) $\mathbf{\Phi}$ *is exponentially ergodic.*

PROOF. To infer (c) from (a) one may apply the contractivity of the total variation norm, as in the proof of the previous theorem. That (c) implies (a) is immediate.

Now from criterion (c) of Theorem 2.1, for any $\beta$ we have (b) for the resolvent $R_\beta$ if and only if we have $(\check{\mathscr{D}}_\beta)$; this implies (c) from Theorem 5.2(a). Finally, one can calculate directly that (c) gives (b), or instead use the fact that (a) implies $(\mathscr{D}_T)$, which implies $(\check{\mathscr{D}}_\beta)$ from Theorem 5.1. $\square$

We note that if there exists some small set $C$, some constant $\pi_C > 0$, some $\rho_C < 1$ and $M_C < \infty$ such that, for all $x \in C$,

$$|P^t(x,C) - \pi_C| \le M_C \rho_C^t, \tag{34}$$

then by applying criterion (a) of Theorem 2.1 to any $T$-skeleton we have that (a) holds on some full absorbing set $S_V$. It immediately follows that (b) and (c) also hold with the process restricted to $S_V$.

**6. Drift conditions and exponential regularity.** In this section we consider relations between the drift conditions above and forms of stability in terms of "exponential regularity" or boundedness of the mean return time to the center of the state space, again defined in terms of petite sets. In the final section we will then establish a surprisingly strong solidarity between regularity for the resolvent chain and the process, and this then gives new criteria for $V$-uniform ergodicity.

Our first result shows that the drift conditions in the previous section provide explicit bounds on the exponential behavior of the hitting times on the sets $C$ involved. To do this we need appropriate definitions of hitting times on petite sets in continuous time. The hitting times to a set $A$ are defined as

$$\tau_A = \inf\{t \ge 0 : \Phi_t \in A\}, \qquad \tau_A(\delta) = \inf\{t \ge \delta : \Phi_t \in A\}. \tag{35}$$

We now have the following theorem:

THEOREM 6.1. *Suppose that $(\tilde{\mathscr{D}})$ is satisfied for some $V \ge 1$ and some set $C$. Then for any $\eta \le c$,*

$$V(x) \ge \mathsf{E}_x[e^{\eta\tau_C}] + (c - \eta)\mathsf{E}_x\left[\int_0^{\tau_C} e^{\eta t} V(\Phi_t)\, dt\right]. \tag{36}$$

PROOF. As in the proof of Theorem 6.1 of [17], the product rule applied to the extended space–time generator for the function $g(x,t) = e^{\eta t}V(x)$ gives

$$\begin{aligned} \tilde{\mathscr{A}} g(x,t) &= e^{\eta t}\tilde{\mathscr{A}} V(x) + \eta e^{\eta t} V(x) \\ &\le (\eta - c)e^{\eta t}V(x) + b\mathbb{1}_C e^{\eta t} \end{aligned} \tag{37}$$

from $(\tilde{\mathscr{D}})$. Now use Dynkin's formula (see [17], Section 1.3) with the stopping time $\tau^n = \inf(\tau_C, n, \sup(t\colon V(\Phi_t) \le n))$ for fixed $n$ to get

$$\begin{aligned} \mathsf{E}_x\big[e^{\eta\tau^n} V(\Phi_{\tau^n})\big] &= V(x) + \int_0^{\tau^n} \tilde{\mathscr{A}} g(x,t)\,dt \\ &\le V(x) + (\eta - c)\mathsf{E}_x \int_0^{\tau^n} e^{\eta t} V(\Phi_t)\,dt. \end{aligned} \tag{38}$$

Since $V \ge 1$ we thus have

$$V(x) \ge \mathsf{E}_x[e^{\eta\tau^n}] + (c-\eta)\mathsf{E}_x \int_0^{\tau^n} e^{\eta t} V(\Phi_t)\,dt.$$

The process $\mathbf{\Phi}$ is nonexplosive so that $\sup(t\colon V(\Phi_t) \le n) \to \infty$ as $n \to \infty$, and hence also $\tau^n \to \tau_C$. Hence our result follows by Fatou's lemma as $n \to \infty$. □

Our next result shows that the solutions to the drift inequalities not only provide bounds to hitting times, but that functions of the hitting times themselves provide solutions to the drift inequalities. This is identical to the situation found in discrete time [15, Chapter 15].

THEOREM 6.2. *Suppose there exists a function $f \ge 1$ and a closed set $C \in \mathscr{B}(\mathsf{X})$ for which*

$$V_0(x) := 1 + \mathsf{E}_x\left[\int_0^{\tau_C(\delta)} e^{\eta t} f(\Phi_t)\,dt\right] < \infty, \qquad x \in C^c, \tag{39}$$

*and*

$$V_0(x) < M < \infty, \qquad x \in C, \tag{40}$$

*for some constants $\delta, \eta > 0$, $M < \infty$. Then for the function $V_0(x)$:*

(a) *if $C$ is petite, then $V_0$ satisfies $(\mathscr{D}_T)$ for any $T > 0$;*
(b) *if $C$ is petite, then $V_0$ satisfies $(\check{\mathscr{D}}_\beta)$ for any $\beta > 0$;*
(c) *for some $\eta_0$, the lower bound $V_0(x) \ge \eta_0 R_1 f(x)$ holds for all $x$.*

PROOF. By assumption we have that $V_0$ is bounded on $C$, and arguing as in Lemma 4.1 of [14], we have, for some $r_1 < \infty$, $d_1 < \infty$,

$$P^T V_0(x) \le d_1 r_1^T, \qquad x \in C, T \ge 0. \tag{41}$$

This bound may be refined substantially. First write

$$\begin{aligned} P^T V_0(x) &= e^{-\eta T}\mathsf{E}_x\left[\int_T^{\tau_C(T)} f(\Phi_t) e^{\eta t}\,dt\right] + 1 \\ &= e^{-\eta T}\mathsf{E}_x\left[\int_T^{\tau_C} f(\Phi_t) e^{\eta t}\,dt\, \mathbb{1}(\tau_C \ge T)\right] \\ &\quad + e^{-\eta T}\mathsf{E}_x\left[\int_T^{\tau_C(T)} f(\Phi_t) e^{\eta t}\,dt\, \mathbb{1}(\tau_C < T)\right] + 1. \end{aligned} \tag{42}$$

The last term is bounded as follows. First note that on the event $\{\tau_C < T\}$,

$$\int_T^{\tau_C(T)} f(\Phi_t) e^{\eta t}\,dt \le \theta^{\tau_C} \int_0^{\tau_C(T)} f(\Phi_t) e^{\eta t}\,dt,$$

where $\theta$ is the shift operator on the sample space [15]. From this and the Markov property, we have

$$\mathsf{E}_x\left[\int_T^{\tau_C(T)} f(\Phi_t)e^{\eta t}\,dt\ \mathbb{1}(\tau_C < T)\right] \le \mathsf{E}_x\left[\mathsf{E}_{\Phi_{\tau_C}}\left[\int_0^{\tau_C(T)} f(\Phi_t)e^{\eta t}\,dt\right]\right] \le d_1 r_1^T, \tag{43}$$

where we are using (41) and the assumption that $C$ is closed, so that $\Phi_{\tau_C} \in C$.

Combining (43) with (42) gives

$$P^T V_0 \le e^{-\eta T}\left(V_0 - \int_0^T P^s f\,ds\right) + d_1 r_1^T e^{-\eta T} + 1, \tag{44}$$

which proves (a).

Multiplying both sides of this bound by $\alpha e^{-\alpha T}$ and integrating over $T \ge 0$, we have, for any $\alpha > 1$ sufficiently large, constants $\lambda < 1$, $b < \infty$, $d < \infty$, such that

$$R_\alpha V_0 \le \lambda V_0 + b - dR_\beta f, \tag{45}$$

where $\beta = \eta + \alpha > \alpha$. This shows that $(\check{\mathscr{D}}_\alpha)$ holds.

To obtain $(\check{\mathscr{D}}_{\alpha_0})$ for $\alpha_0 < \alpha$, observe that by (22), with $\gamma(n) = (\alpha_0/\alpha)[(\alpha - \alpha_0)/\alpha]^n$,

$$R_{\alpha_0} = \sum_{k=0}^{\infty} \gamma(n) R_\alpha^{n+1}.$$

By replacing (45) with the cruder bound $R_\alpha V_0 \le \lambda V_0 + b$ and, hence, also

$$R_\alpha^n V_0 \le \lambda^n V_0 + \frac{b}{1-\lambda}, \qquad n \ge 1,$$

we have

$$R_{\alpha_0} V_0 \le \lambda_1 V_0 + b, \tag{46}$$

where $\lambda_1 = \sum_{n=1}^{\infty} \gamma(n)\lambda^n < 1$, which establishes (b).

We now prove (c). From (45) we have that $R_\alpha V_0 \le V_0 + b - dR_\beta f$ and, hence, also

$$0 \le R_\alpha^{n+1} V_0 \le V_0 + (n+1)b - dR_\alpha^n R_\beta f, \qquad n \ge 0. \tag{47}$$

Multiplying both sides of this equation by $\alpha^{-1}(1-\alpha^{-1})^n$, summing over $n$ and applying (22) gives, for some $d_2, b_2$,

$$0 \le V_0 + b_2 - d_2 R_1 R_\beta f = V_0 + b_2 - \frac{d_2}{\beta - 1}(\beta R_1 f - R_\beta f),$$

where the equality follows from the resolvent equation. By (47) with $n = 0$ we have that $R_\beta f \le c_2 V_0$ for some constant $c_2$, and this gives the desired relationship between $R_1 f$ and $V_0$:

$$R_1 f \le \frac{\beta - 1}{d_2\,\beta}(V_0 + b_2) + \frac{1}{\beta} c_2 V_0 \le c_3 V_0, \tag{48}$$

where $c_3$ is a finite constant. This establishes the theorem. □

As a corollary to Theorem 6.2, we obtain solutions to ($\tilde{\mathscr{D}}$):

THEOREM 6.3. *Under the conditions of Theorem 6.2, either of the functions*

$$V_T^* = \int_0^T P^s V_0 \, ds \quad \text{or} \quad V_\beta^* = R_\beta V_0$$

*satisfies* ($\tilde{\mathscr{D}}$).

PROOF. From the definition of the generator and Lemma 4.3 we have the pair of identities

$$\tilde{\mathscr{A}} \int_0^T P^s = P^T - I, \qquad \tilde{\mathscr{A}} R_\beta = \beta(R_\beta - I).$$

The result follows directly from these and Theorem 6.2(a) and (b). □

The following result gives a more exact solution to the drift inequality ($\tilde{\mathscr{D}}$).

THEOREM 6.4. *Suppose that, for some $r > 1$, a measurable function $f \geq 1$ and a set $C \in \mathscr{B}(\mathsf{X})$, we have that the function*

$$V_1(x) := U_C^r(x, f) = \mathsf{E}_x\left[\int_0^{\tilde{\tau}_{C,r}} e^{(r-1)t} f(\Phi_t)\, dt\right]$$

*is finite-valued and bounded on $C$. Then*:

(a) *if $C$ is petite, then $V_1$ satisfies* ($\tilde{\mathscr{D}}$);
(b) *for some $\eta_1$, the lower bound $V_1(x) \geq \eta_1 R_1 f(x)$ holds for all $x$.*

PROOF. From (26) we have that $U_C^r f \geq R_1 f$, which proves (b).
To prove (a) we apply Lemma 4.3 with $g = \mathbb{1}_C - [r-1]\mathbb{1}_{C^c}$:

$$\tilde{\mathscr{A}} U_C^r f = -f + g U_C^r = -f - (r-1) I_{C^c} U_C^r f + I_C U_C^r f.$$

If $U_C^r f(x)$ is bounded on $C$, then this gives ($\tilde{\mathscr{D}}$). □

We now connect the mean hitting times of $\mathbf{\Phi}$ with mean hitting times for the resolvent chain. This result shows that the conditions of Theorems 6.2 and 6.4 are essentially equivalent.

THEOREM 6.5. (a) *For any $r \geq 1$, $\delta > 0$ and any set $C$,*

$$\mathsf{E}_x\left[\int_0^{\tau_C(\delta)} f(\Phi_t) e^{[r-1]t}\, dt\right] \leq e^{\delta r} \mathsf{E}_x\left[\sum_{k=1}^{\check{\tau}_C} r^{k-1} f(\check{\Phi}_k)\right], \qquad x \in \mathsf{X}. \tag{49}$$

(b) *Suppose that* $\Phi$ *is* $\psi$*-irreducible,* $C \in \mathscr{B}^+(\mathsf{X})$ *is closed and petite and that, for some* $\delta_0 > 0$, $r_0 \geq 1$,

$$\sup_{x \in C} \mathsf{E}_x\left[\int_0^{\tau_C(\delta_0)} \exp((r_0 - 1)t) f(\Phi_t)\, dt\right] < \infty. \tag{50}$$

*If* $r_0 > 1$ *in* (50), *then for all sufficiently small* $r > 1$ *and all* $\delta > 0$, *there exists* $b(\delta) < \infty$ *such that, for all* $x$,

$$\mathsf{E}_x\left[\sum_{k=1}^{\check{\tau}_C} r^{k-1} f(\check{\Phi}_k)\right] \leq b(\delta) \mathsf{E}_x\left[\int_0^{\tau_C(\delta)} f(\Phi_t) e^{[r-1]t}\, dt\right]. \tag{51}$$

*If* $r_0 = 1$ *in* (50), *then* (51) *holds with* $r = 1$.

Proof. To see (a), note that with $\eta = r - 1$,

$$\begin{aligned} U_C^r(x, f) &:= \mathsf{E}_x\left[\int_0^{\infty} \exp\left\{\int_0^t (-\mathbb{1}_C(\Phi_s) + \eta \mathbb{1}_{C^c}(\Phi_s))\, ds\right\} f(\Phi_t)\, dt\right] \\ &\geq \mathsf{E}_x\left[\int_0^{\tau_C(\delta)} \exp\left\{\int_0^t (-\mathbb{1}_C(\Phi_s) + \eta \mathbb{1}_{C^c}(\Phi_s))\, ds\right\} f(\Phi_t)\, dt\right] \\ &\geq \mathsf{E}_x\left[\int_0^{\delta} \exp(-\delta) f(\Phi_t)\, dt\right] + \mathsf{E}_x\left[\int_{\delta}^{\tau_C(\delta)} \exp(\eta t) f(\Phi_t)\, dt\right] \\ &\geq \exp(-\delta(1 + \eta)) \mathsf{E}_x\left[\int_0^{\tau_C(\delta)} \exp(\eta t) f(\Phi_t)\, dt\right], \end{aligned}$$

which proves (49) from (25).

Result (b) is given as Theorem A.3.3 of [3] and we omit the rather lengthy proof. □

If one is willing to accept a slightly weaker conclusion, it is possible to remove the assumption in Theorem 6.5(b) that $\psi(C) > 0$. This may seem a somewhat technical improvement, but in practice it is of some considerable value [4].

Theorem 6.6. *Suppose* $\Phi$ *is* $\psi$*-irreducible and let* $C \in \mathscr{B}(\mathsf{X})$ *be a closed petite set, not necessarily in* $\mathscr{B}^+(\mathsf{X})$. *Let* $f \geq 1$ *be a function bounded on* $C$, *such that, for some* $r > 1$, $\delta > 0$,

$$\sup_{x \in C} \mathsf{E}_x\left[\int_0^{\tau_C(\delta)} f(\Phi_t) e^{[r-1]t}\, dt\right] < \infty. \tag{52}$$

*Then for some* $r_0 > 1$, $b < \infty$ *and a petite set* $C_0 \in \mathscr{B}^+(\mathsf{X})$,

$$\mathsf{E}_x\left[\sum_{k=1}^{\check{\tau}_{C_0}} r_0^{k-1} f(\check{\Phi}_k)\right] \leq b\left(1 + \mathsf{E}_x\left[\int_0^{\tau_C} f(\Phi_t) e^{[r-1]t}\, dt\right]\right). \tag{53}$$

PROOF. Let $V_0(x) = 1 + \mathsf{E}_x[\int_0^{\tau_C} e^{[r-1]t} f(\Phi_t)\,dt]$. By Theorem 6.2 we have that $R_1 f \le c_0 V_0$ for some constant $c_0$, and we have that $(\check{\mathscr{D}}_\beta)$ holds with this $V_0$ for $\beta = 1$. By Theorem 15.2.5 of [15] we then have a petite set $C_0 \in \mathscr{B}^+(\mathsf{X})$, for which $V_0$ is bounded on $C_0$, and for some $r_0 > 1$,

$$\mathsf{E}_x\left[\sum_{k=0}^{\check{\tau}_{C_0}-1} V_0(\check{\Phi}_k) r_0^k\right] \le dV_0(x), \qquad x \in \mathsf{X}.$$

Since $R_1 f \le c_0 V_0$, we also have

$$\mathsf{E}_x\left[\sum_{k=0}^{\check{\tau}_{C_0}-1} R_1 f(\check{\Phi}_k) r_0^k\right] \le c_0\, dV_0(x), \qquad x \in \mathsf{X}.$$

By the Markov property, the LHS is exactly $\mathsf{E}_x[\sum_{k=1}^{\check{\tau}_{C_0}} f(\check{\Phi}_k) r_0^{k-1}]$, which gives the desired bound. □

**7. Exponential regularity and exponential ergodicity.** The results of the previous section lead us to consider the following conditions, which describe the existence of "strongly" regular sets for the resolvent and for the process. The final goal of this paper is to show in rather more detail that, as suggested by the results above, these are essentially equivalent to each other and to the drift conditions of the previous section. This gives a full description of exponential ergodicity in terms of return times.

$(\check{\mathscr{R}}_r)$ GEOMETRIC REGULARITY OF THE RESOLVENT CHAIN. For some $r \ge 1$ and some petite set $C \in \mathscr{B}(\mathsf{X})$, there exist $b > 0$ and a function $\check{V} \ge 1$ bounded on $C$ such that

$$\mathsf{E}_x\left[\sum_{k=1}^{\check{\tau}_C} r^{k-1} \check{V}(\check{\Phi}_k)\right] \le b\check{V}(x), \qquad x \in \mathsf{X}.$$

$(\mathscr{R}_\eta)$ EXPONENTIAL REGULARITY OF THE PROCESS. For some $\eta \ge 0$ and some closed petite set $C \in \mathscr{B}^+(\mathsf{X})$, there exist constants $\delta, b > 0$ and a function $V_\eta \ge 1$ with $V_\eta$ bounded on $C$, such that

$$\mathsf{E}_x\left[\int_0^{\tau_C(\delta)} V_\eta(\Phi_s) e^{\eta s}\, ds\right] \le bV_\eta(x).$$

We will again show that these regularity conditions are equivalent. Recall that from Lemma 4.2 we have the identity $U_C^r(x, \check{V}) = \mathsf{E}_x[\sum_{k=1}^{\check{\tau}_C} r^{k-1} \check{V}(\check{\Phi}_k)]$, so that when $(\check{\mathscr{R}}_r)$ holds we have a solution to $(\check{\mathscr{D}})$, by Theorem 6.4. We will show that in fact the regularity conditions are equivalent to the drift properties $(\mathscr{D}_T)$–$(\tilde{\mathscr{D}})$ for petite sets. Perhaps the least intuitive part of this result is that $(\mathscr{R}_0)$ or $(\check{\mathscr{R}}_1)$ actually suffices for the conditions with exponential or geometric weighting to hold.

THEOREM 7.1. *Suppose that* $\mathbf{\Phi}$ *is* $\psi$*-irreducible and let* $C$ *be a closed petite set in* $\mathscr{B}^+(\mathsf{X})$. *The following are equivalent, for any function* $V \geq 1$.

(a) *The regularity condition* $(\check{\mathscr{R}}_1)$ *holds for* $\check{V} = V$.
(b) *The regularity condition* $(\check{\mathscr{R}}_r)$ *holds for* $\check{V} = V$ *and some* $r > 1$.
(c) *The regularity condition* $(\mathscr{R}_0)$ *holds for* $V_0 = V$.
(d) *The regularity condition* $(\mathscr{R}_\eta)$ *holds for* $V_\eta = V$ *and some* $\eta > 0$.
(e) *The drift condition* $(\check{\mathscr{D}}_1)$ *holds for a function* $V_1$ *equivalent to* $V$ *in the sense that, for constants* $0 < c_1, c_2 < \infty$,

$$c_1 V \leq V_1 \leq c_2 V. \tag{54}$$

PROOF. It is obvious that (b) $\Rightarrow$ (a). In discrete time the fact that the "nongeometric" form (a) implies the geometric form (b) is shown in [18], using the contraction properties of the $V$-norm as first observed on countable spaces in [6].

For the discrete time resolvent we also have the equivalence (b) $\Leftrightarrow$ (e) from Theorems 15.2.4 and 15.2.6 of [15].

The equivalence of (a) and (c) follows from the bounds obtained in Proposition 4.3 of [14], provided $C$ is closed and petite.

Trivially (d) $\Rightarrow$ (c), and if we can show (b) $\Rightarrow$ (d), then we are done. Suppose then that (b) holds for some $r > 1$. By Theorem 6.5(a), for any $\delta$,

$$b\check{V}(x) \geq \mathsf{E}_x\left[\sum_{k=1}^{\check{\tau}_C} r^{k-1}\check{V}\big(\check{\Phi}_k\big)\right] \geq e^{-\delta r}\mathsf{E}_x\left[\int_0^{\tau_C(\delta)}\check{V}(\Phi_s)e^{[r-1]s}\,ds\right], \qquad x \in \mathsf{X},$$

and so (d) holds as required. □

We can now show that the existence of a single petite set satisfying the regularity or drift criteria leads to regularity of the hitting times on all sets $B \in \mathscr{B}^+(\mathsf{X})$.

THEOREM 7.2. *Suppose that any of the drift criteria* $(\check{\mathscr{D}}_\beta)$–$(\tilde{\mathscr{D}})$ *or the regularity criteria* $(\check{\mathscr{R}}_r)$–$(\mathscr{R}_\eta)$ *hold, and let* $V$ *denote the function used in the assumed drift or regularity criterion. Then* $\mathbf{\Phi}$ *is* $V$*-exponentially regular in the sense that, for any* $B \in \mathscr{B}^+(\mathsf{X})$ *and any* $\delta > 0$, *there exist* $c = c(B, \delta) < \infty$ *and* $\eta = \eta(B, \delta) > 0$ *such that*

$$\mathsf{E}_x\left[\int_0^{\tau_B(\delta)} V(\Phi_s)e^{\eta s}\,ds\right] \leq cV(x).$$

PROOF. If any of $(\check{\mathscr{D}}_\beta)$–$(\tilde{\mathscr{D}})$ or $(\check{\mathscr{R}}_r)$–$(\mathscr{R}_\eta)$ hold, then $(\check{\mathscr{D}}_1)$ holds with the function $R_1 V_0$, where $V_0$ is *equivalent* to the function used in the drift or regularity criterion. That is, for constants $0 < c_1, c_2 < \infty$,

$$c_1 V_0 \leq V \leq c_2 V_0.$$

This is immediate from Theorems 5.1 and 7.1. Hence with $V_* := \gamma V_0 + R_1 V_0$,

$\gamma > 0$, we have for some $\lambda < 1$ and a petite set $C$,

$$\begin{aligned} R_1 V_* &= \gamma R_1 V_0 + R_1 R_1 V_0 \\ &\le \gamma R_1 V_0 + \lambda R_1 V_0 + b \mathbb{1}_C \\ &\le [\gamma + \lambda] V_* + b \mathbb{1}_C. \end{aligned}$$

We thus have that $(\check{\mathscr{D}}_1)$ holds with the function $V_*$ whenever $0 < \gamma < 1 - \lambda$.

Under $(\check{\mathscr{D}}_1)$ with this function $V_*$ it follows from Theorem 15.3.3 of [15] that, for any $B \in \mathscr{B}^+(\mathsf{X})$, there exists $c_0 = c_0(B)$ and $r = r(B)$ such that

$$U_B^r(x, V_*) \le c_0 V_*(x).$$

By Theorem 6.5(a) it then follows that

$$\mathsf{E}_x\left[\int_0^{\tau_B(\delta)} V_0(\Phi_t) e^{[r-1]t}\, dt\right] \le \gamma^{-1} e^{\delta r} c_0 V_*(x).$$

Since by Theorem 6.2(c) we have that $V_*(x) \le c_1 V_0(x)$, this implies the result. □

The equivalences given in Theorem 7.1 now yield new criteria for $V$-uniform ergodicity and $V$-exponential regularity:

THEOREM 7.3. *For a $\psi$-irreducible, aperiodic Markov process, if any of the equivalent conditions $(\mathscr{R}_r)$–$(\mathscr{R}_\eta)$ hold for some $V$, then the process $\mathbf{\Phi}$ is $R_1 V$-uniformly ergodic.*

PROOF. This follows immediately on applying Theorem 7.1 to obtain $(\check{\mathscr{D}}_1)$, and combining this with Theorem 5.2(a). □

Theorem 7.3 has recently been applied in [4] to obtain exponential ergodicity for a class of feedforward queueing networks.

Finally we show that any function $f$ which is sufficiently regular can be smoothed to give a function $f^*$ for which the process is $f^*$-uniformly ergodic.

THEOREM 7.4. *Suppose that $\mathbf{\Phi}$ is a $\psi$-irreducible, aperiodic Markov process and that $f \ge 1$ satisfies* (39) *and* (40) *for a closed petite set $C \in \mathscr{B}(\mathsf{X})$ and constants $\delta, \eta > 0$, $M < \infty$. Then the process $\mathbf{\Phi}$ is $f^*$-uniformly ergodic for some function $f^*$ equivalent to $R_1 f$.*

PROOF. We will take $f^*$ to be the function $V_0$ defined in (39), for some suitable $\eta < 1$. From Theorem 6.2(c) we have that $V_0$ is bounded below by a multiple of $R_1 f$, and by definition $V_0$ is less than $1 + \mathsf{E}_x \int_0^\infty e^{\eta t} f(\Phi_t)\, dt \le 1 + R_1 f$. Hence $f^*$ is equivalent to $R_1 f$, as required.

By Theorem 6.2(a) we see that $V_0$ satisfies $(\mathscr{D}_T)$ for any $T > 0$ and so from Theorem 5.2(b) we have the required ergodicity. □

To conclude, we note that when considering $f$-exponentially regular processes, we have not been able to establish $f$-geometric ergodicity without imposing additional assumptions on $f$. The following counterexample shows that one must indeed "smooth" the function $f$ in some way, for example, by considering $f^*_\beta = R_\beta f$ or $f^*_T = \int_0^T P^s f\, ds$ as in Theorem 6.3, if one wants to infer $f$-exponential ergodicity from $f$-exponential regularity.

Consider the Markov process $\mathbf{\Phi}$ on the unit circle $\mathsf{X} = S^1$ in the complex plane, with deterministic counterclockwise motion $\Phi_{t+s} = e^{2\pi i t}\Phi_s$ up until the first time that $\Phi = 1$, at which time a jump occurs with probability $1/2$ so that

$$\mathsf{P}(\Phi_{t+} \in A \mid \Phi_t = 1) = 1/2[\delta_1(A) + \mu(A)]$$

(where $\mu$ denotes Lebesgue measure on $\mathsf{X}$ and $\delta_1$ is the Dirac measure concentrated on 1).

This process is $\mu$-irreducible and aperiodic, and the measure $\mu$ is an invariant probability. In fact, the state space $S^1$ is petite for this process, so that $\mathbf{\Phi}$ is uniformly ergodic. Let $\{r_n, n \ge 1\}$ denote an ordering of the rational points on $S^1$, and define $V\colon S^1 \to [0, \infty)$ as

$$V(x) = \begin{cases} 1, & \text{if } x \text{ is irrational,} \\ n, & \text{if } x = r_n. \end{cases}$$

We have that $\mu(V) = 1$ and, moreover, using the petite set $C = S^1$,

$$\mathsf{E}_x\left[\int_0^{\tau_C(\delta)} r^t V(\Phi_t)\, dt\right] = \int_0^\delta r^t\, dt < \infty.$$

This shows that the process satisfies $(\mathscr{R}_\eta)$ with this $V$ for *any* $r > 1$. However, $\mathbf{\Phi}$ is clearly not $V$-geometrically ergodic, since for any $x, n$,

$$\sup_{n \le t \le n+1} \mathsf{E}_x[V(\Phi_t)] \ge 2^{-n-1} \sup_{0 \le \zeta \le 1} V(e^{2\pi i \zeta}) = \infty.$$

## REFERENCES


[1] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic Press, New York.

[2] DAVIS, M. H. A. (1993). *Markov Models and Optimization.* Chapman and Hall, London.

[3] DOWN, D. (1994). Stability of queuing networks. Ph.D. dissertation, Univ. Illinois, Urbana.

[4] DOWN, D. and MEYN, S. P. (1994). Stability of acyclic multiclass queuing networks. *IEEE Trans. Automat. Control.* To appear.

[5] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence.* Wiley, New York.

[6] HORDIJK, A. and SPIEKSMA, F. M. (1992). On ergodicity and recurrence properties of a Markov chain with an application. *Adv. in Appl. Probab.* **24** 343–376.

[7] KASPI, H. and MANDELBAUM, A. (1992). Regenerative closed queuing networks. *Stochastics* **39** 239–258.

[8] KHAS'MINSKII, R. Z. (1980). *Stochastic Stability of Differential Equations.* Sijthoff & Noordhoff, Groningen, Netherlands.

[9] KLIEMANN, W. (1987). Recurrence and invariant measures for degenerate diffusions. *Ann. Probab.* **15** 690–707.

[10] KUNITA, H. (1990). *Stochastic Flows and Stochastic Differential Equations.* Cambridge Univ. Press.

[11] LOÈVE, M. (1978). *Probability Theory II*, 4th ed. Springer, New York.

[12] MEYN, S. P. and CAINES, P. E. (1991). Asymptotic behavior of stochastic systems processing Markovian realizations. *SIAM J. Control Optim.* **29** 535–561.

[13] MEYN, S. P. and DOWN, D. (1994). Stability of generalized Jackson networks. *Ann. Appl. Probab.* **4** 124–148.

[14] MEYN, S. P. and TWEEDIE, R. L. (1993). Generalized resolvents and Harris recurrence of Markov processes. *Contemp. Math.* **149** 227–250.

[15] MEYN, S. P. and TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer, London.

[16] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes II: continuous time processes and sampled chains. *Adv. in Appl. Probab.* **25** 487–517.

[17] MEYN, S. P. and TWEEDIE, R. L. (1993). Stability of Markovian processes III: Foster–Lyapunov criteria for continuous time processes. *Adv. in Appl. Probab.* **25** 518–548.

[18] MEYN, S. P. and TWEEDIE, R. L. (1994). State-dependent criteria for convergence of Markov chains. *Ann. Appl. Probab.* **4** 149–168.

[19] MOKKADEM, A. (1987). Sur un modèle autorégressif non linéaire: ergodicité et ergodicité géométrique. *J. Time Ser. Anal.* **8** 195–204.

[20] MOKKADEM, A. (1988). Ergodicité des systèmes stochastiques polynomiaux en temps discret. In *New Trends on Nonlinear Control Theory. Lecture Notes in Control and Inform. Sci.* **122** 404–413. Springer, Berlin.

[21] NEVEU, J. (1972). Potentiel Markovien récurrent des chaînes de Harris. *Ann. Inst. Fourier (Grenoble)* **22** 7–130.

[22] REVUZ, D. (1984). *Markov Chains*, 2nd ed. North-Holland, Amsterdam.

[23] SHARPE, M. (1988). *General Theory of Markov Processes*. Academic Press, New York.

[24] STRAMER, O., TWEEDIE, R. L. and BROCKWELL, P. J. (1993). Existence and stability of continuous time threshold ARMA models. Unpublished manuscript.

[25] TUOMINEN, P. and TWEEDIE, R. L. (1979). Exponential decay and ergodicity of general Markov processes and their discrete skeletons. *Adv. in Appl. Probab.* **11** 784–803.

[26] TUOMINEN, P. and TWEEDIE, R. L. (1979). The recurrence structure of general Markov processes. *Proc. London Math. Soc.* (*3*) **39** 554–576.

[27] TWEEDIE, R. L. (1981). Criteria for ergodicity, exponential ergodicity and strong ergodicity of Markov processes. *J. Appl. Probab.* **18** 122–130.

[28] TWEEDIE, R. L. (1994). Topological conditions enabling use of Harris methods in discrete and continuous time. *Acta Appl. Math.* **34** 175–188.

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