The Annals of Probability
1995, Vol. 23, No. 4, 1627–1643

ITERATED LAW OF ITERATED LOGARITHM

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Suppose $\varepsilon \in [0,1)$ and let $\theta_\varepsilon(t) = (1-\varepsilon)\sqrt{2t\ln_2 t}$. Let L_t^ε denote the amount of local time spent by Brownian motion on the curve $\theta_\varepsilon(s)$ before time t. If $\varepsilon > 0$, then $\limsup_{t\to\infty} L_t^\varepsilon/\sqrt{2t\ln_2 t} = 2\varepsilon + o(\varepsilon)$. For $\varepsilon = 0$, a nontrivial lim sup result is obtained when the normalizing function $\sqrt{2t\ln_2 t}$ is replaced by $g(t) = \sqrt{t/\ln_2 t}\ln_3 t$.

Introduction and statement of the results. Let (B_t) be a one-dimensional Brownian motion. If $\theta(t) = \sqrt{2t\ln_2 t}$, the law of the iterated logarithm (LIL) asserts that $\limsup_{t\to\infty}(B_t/\theta(t)) = 1$. A slightly stronger statement may be obtained by applying Kolmogorov's test (see Itô and McKean [4], page 33), namely, for every $\varepsilon \geq 0$ (including $\varepsilon = 0$), (B_t) will hit the curve $\theta_\varepsilon(t) \stackrel{\text{df}}{=} (1-\varepsilon)\theta(t)$ i.o. as t tends to ∞. Our aim is to study the behavior of (B_t) on the curve $\theta_\varepsilon(t)$ for $\varepsilon \geq 0$. How much time will (B_t) spend on θ_ε? More precisely, we will study the local time $(L_t^0(B-\theta_\varepsilon))_{t\geq 0}$ of (B_t) on the curve θ_ε, which is (by definition) the local time of the time-inhomogeneous diffusion $B_t - \theta_\varepsilon(t)$ at the level 0.

By abuse of notation, from now on, $\theta_\varepsilon(t)$ will denote some fixed smooth function equal to $(1-\varepsilon)\sqrt{2t\ln_2 t}$ for $t \geq 100$ and equal to 0 for $t < 50$. Brownian motion accumulates only a finite amount of local time on $\theta_\varepsilon(t)$ before time 100 a.s.

We will normalize the local time so that it is twice as big as that of [5], page 203. As a result, the factor 2 disappears from the statements of Theorem 6.2.23 and formula (6.3.17) of [5]. We shall prove the following result.

THEOREM 1. (i) *For* $\varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{L_t^0(B-\theta_\varepsilon)}{\sqrt{2t\ln_2 t}} = 2\varepsilon + o(\varepsilon) \quad a.s.$$

(ii) *Let* $g(t) = \sqrt{t/\ln_2 t}\ln_3 t$. *Then a.s.*

$$\limsup_{t\to\infty} \frac{L_t^0(B-\theta)}{g(t)} = \frac{3}{2}\sqrt{2}.$$

A well-known theorem says that if we take $\varepsilon = 1$ in Theorem 1, the lim sup is equal to 1 (see [6] or Theorem 2.9.23 and (3.6.28) in [5]).

Received August 1994; revised January 1995.

[1]Research supported in part by NSF Grant DMS-91-00244, Fondef Grant F-11 and AMS Centennial Research Fellowship.

[2]Research supported in part by Fondecyt Grant 1940330.

AMS 1991 *subject classifications*. 60G17, 60J65.

Key words and phrases. Law of iterated logarithm, Brownian motion, local time.

We would like to point out that it is easy to determine the asymptotic behavior of the expectation of $L_t^0(B-\theta_\varepsilon)$. If $p_t(x,y)$ stands for the Brownian transition density, then

$$\mathbb{E}_0 L_t^0(B-\theta_\varepsilon) \approx \int_1^t p_s(0,\theta_\varepsilon(s))\,ds = \int_1^t \frac{1}{\sqrt{2\pi s}}\exp\left(\frac{-\theta_\varepsilon(s)^2}{2s}\right)ds$$
$$(1)\qquad = \int_1^t \frac{1}{\sqrt{2\pi s}}(\ln s)^{-(1-\varepsilon)^2}\,ds \approx K\sqrt{t}(\ln t)^{-(1-\varepsilon)^2}.$$

This asymptotic estimate holds for both positive and negative ε and has no discontinuity at the critical value $\varepsilon = 0$. Note that $L_t^0(B-\theta_\varepsilon)$ grows to infinity as $t \to \infty$ for every fixed $\varepsilon > 0$, while $L_\infty^0(B-\theta_\varepsilon) < \infty$ a.s. for $\varepsilon < 0$.

A calculation similar to (1) shows that $\mathbb{E}_0 L_t^0(B-f_1) \ge \mathbb{E}_0 L_t^0(B-f_2)$ if $0 \le f_1(s) \le f_2(s)$ for all $s \le t$. This does not necessarily imply that the distribution of $L_t^0(B-f_1)$ stochastically dominates that of $L_t^0(B-f_2)$. In fact, there exist functions f_1 and f_2 such that $0 \le f_1(s) \le f_2(s)$ for all $s \le t$ and

$$\mathbb{P}_0(L_t^0(B-f_1) > x) < \mathbb{P}_0(L_t^0(B-f_2) > x)$$

for some $t, x > 0$. The example is not too hard but it would take too much space and so we omit it.

PROBLEM. Determine for which functions f_1 and f_2 satisfying $0 \le f_1(s) \le f_2(s)$ for $s \in (0,\infty)$ we have

$$\mathbb{P}_0(L_t^0(B-f_1) > x) \ge \mathbb{P}_0(L_t^0(B-f_2) > x) \quad \text{for all } t, x > 0.$$

In particular:

(i) Does the inequality hold for $f_1 = \theta_{\varepsilon_1}$ and $f_2 = \theta_{\varepsilon_2}$, with $\varepsilon_1 > \varepsilon_2$?
(ii) Is it enough to assume that both functions f_1 and f_2 are increasing?

Let us mention some results related to ours. In a recent paper, Chan [2] studies the behavior of $t^{-1}\int_0^t \mathbf{1}_{\{B_s > \sqrt{2\gamma s \ln_2 s}\}}\,ds$ (see also an older article by Strassen [13]).

A theorem of Erdős and Révész [3] says that if $\xi(t) = \sup\{s \le t\colon B(s) \ge \theta(s)\}$, then there exists a constant d_0 such that, for any $d > d_0$ and t big enough,

$$\xi(t) \ge t^{1-d\ln_3 t(\ln_2 t)^{-1/2}}.$$

If $d < d_0$, then the opposite inequality is true for infinitely many large t. Shao [11] has determined that $d_0 = 3\sqrt{\pi}$.

Preliminaries. For a given function $h\colon [a,\infty) \to \mathbb{R}$ and $t \ge a$ we denote by h_t the function $h_t(u) = h(t+u)$, $u \ge 0$. Also we will write $\tilde{h}_t(u) = h(t+u) - h(t)$, for $u \ge 0$. Hence, $\tilde{\theta}_{\varepsilon,u}(t) = \theta_\varepsilon(u+t) - \theta_\varepsilon(u)$, for $\varepsilon \ge 0$. We will use K to denote a constant which may take different values from one line to another.

LEMMA 1. (i) *If* $\gamma = \theta'_{\varepsilon,u}(t)$, *then*

$$\mathbb{P}_0(L_t^0(B - \tilde{\theta}_{\varepsilon,u}) \geq x) \leq e^{-x\gamma}.$$

(ii) *Let* $\gamma = \tilde{\theta}'_{\varepsilon,u}(t)$ *and* $\Lambda = \exp(-\frac{1}{2}\int_0^t(\tilde{\theta}'_{\varepsilon,u}(s))^2\,ds)$. *Assume that* $x, t > 0$, $x\gamma \geq 2$ *and* $-Mx + \gamma t \geq 0$, *for some* $M > 1$. *There exists* $K = K(M)$ *such that*

$$\mathbb{P}_0(L_t^0(B - \tilde{\theta}_{\varepsilon,u}) \geq x) \geq K\Lambda \exp(\gamma^2 t/2)\exp(-x\gamma).$$

PROOF. (i) The first part of the proof will use excursion theory. The standard version of excursion theory deals with excursions from a fixed set, that is, from a set which does not depend on time or ω. We want to consider excursions of B from $\tilde{\theta}_{\varepsilon,u}(s)$, that is, excursions from a set which changes with time. In order to be able to apply this version of excursion theory, we will consider space–time Brownian motion X. The state space of X is $\mathbb{R} \times [0,\infty)$. The process X is Markov. Given the starting point (x, s_1), the distribution of X is that of $\{(x + B_s, s_1 + s), s \geq 0\}$, where B is the standard Brownian motion staring from 0. We will consider excursions of X from the set $\Gamma = \{(\tilde{\theta}_{\varepsilon,u}(s), s), s \geq 0\}$ which is nonrandom and which does not depend on time.

Here are some elements of excursion theory for X we will need in our proof. In order to keep the proof reasonably short, our review will be quite sketchy. We are using the results of [7]. For various presentations of excursion theory see [1], [5], [9], [10] or [12]. For $(x,s) \in \Gamma$, an excursion law $H^{(x,s)}$ is a σ-finite measure on the space of paths $\mathscr{C}$ which take values in $\mathbb{R} \times [0,\infty)$, are continuous until a death time ζ and then remain in a coffin state Δ. The measure $H^{(x,s)}$ is supported on the set of paths which start from (x,s), do not intersect Γ until ζ and approach Γ at $\zeta-$. The measure $H^{(x,s)}$ is strong Markov with respect to the transition probabilities of X killed upon hitting Γ.

An "exit system formula" given below involves excursion laws $H^{(x,s)}$ and an additive functional L_s, the local time of X on Γ. Let $\mu(v) = \inf\{s > 0\colon L_s > v\}$, $\eta_v = \inf\{s > v\colon X(s) \in \Gamma\} - v$ and

$$e_v(s) = \begin{cases} X(v+s), & \text{if } s < \eta_v \text{ and } X_v \in \Gamma,\\ \Delta, & \text{otherwise.}\end{cases}$$

Here is a special case of an exit system formula found in [7]:

$$\begin{aligned} E^{(x,s)} \sum_{0<v<\infty} Z_v f(e_v) &= E^{(x,s)} \int_0^\infty Z_v H^{X(v)}(f)\,dL_v \\ &= E^{(x,s)} \int_0^\infty Z_{\mu(v)} H^{X(\mu(v))}(f)\,dv, \end{aligned}$$

for all $(x,s) \in \mathbb{R} \times [0,\infty)$, all positive predictable processes Z and positive measurable functions f defined on $\mathscr{C}$ which vanish on paths equal identically to Δ.

Next we are going to discuss the normalization of L_s and excursion laws $H^{(x,s)}$. Excursions of X from Γ correspond to excursions of B from $\tilde{\theta}_{\varepsilon,u}(s)$ and these in turn correspond to excursions of $B(s) - \tilde{\theta}_{\varepsilon,u}(s)$ from 0. The processes $B(s) - \tilde{\theta}_{\varepsilon,u}(s)$ and $B(s)$ have mutually absolutely continuous distributions on every fixed finite interval. Hence, the local time of $B(s) - \tilde{\theta}_{\varepsilon,u}(s)$ at 0 has the same representation in terms of small excursions as that for the local time of B at 0. We now normalize the local time L_s of X on Γ so that it is equal to the local time of $B(s) - \tilde{\theta}_{\varepsilon,u}(s)$ at 0. Recall that our local time is twice that of [5] and note that Theorem 6.2.23 of [5] deals with excursions of reflected rather than standard Brownian motion. If we take this into account, we see that according to Theorem 6.2.23 of [5], the number of excursions of X from Γ which hit $\Gamma_\delta = \{(\tilde{\theta}_{\varepsilon,u}(s) - \delta, s),\ s \geq 0\}$ before time $\mu(v)$ is equal to $v/(2\delta) + o(1/\delta)$. This and the exit system formula imply that the $H^{(x,s)}$-measure of paths which hit Γ_δ must be $1/(2\delta) + o(1/\delta)$.

Recall t and u from the statement of Lemma 1(i). Fix some $(x, s) \in \Gamma$, $s < t$, and consider the process X under $H^{(x,s)}$. We will find a lower bound for the $H^{(x,s)}$-measure of the paths that do not return to Γ before t. We will apply the strong Markov property at the hitting time of Γ_δ by X, say, v. If $v \geq t$, then of course the excursion does not return to Γ before t.

Suppose that $v < t$. Note that the derivative of $\tilde{\theta}_{\varepsilon,u}$ is a decreasing function. A straight line M passing through the point $(v, \tilde{\theta}_{\varepsilon,u}(v))$ with slope equal to γ lies below the graph of $\tilde{\theta}_{\varepsilon,u}$ on the interval $[v, t]$. The probability that a standard Brownian motion starting from the point $\tilde{\theta}_{\varepsilon,u}(v) - \delta$ at time $v \in [0, t)$ will not hit the graph of $\tilde{\theta}_{\varepsilon,u}$ before time t is not less than the probability that it will never hit M. This and Exercise 4.3.13 of [5], page 265, imply that this probability is bounded below by $1 - e^{-2\delta\gamma}$. The strong Markov property applied at v implies that $(1 - e^{-2\delta\gamma})(1/(2\delta) + o(1/\delta))$ is a lower bound for the $H^{(x,s)}$-measure of the paths that do not return to Γ before t. Since $\delta > 0$ can be taken arbitrarily small, γ is a lower bound for this quantity.

Let U be the starting time of the first (and only) excursion of X from Γ which approaches Γ at its lifetime after time t. A standard application of the exit system formula shows that $\mu(U)$ is an exponential variable and the probability that $\mu(U)$ is greater than or equal to x is less than or equal to $\exp(-x\gamma)$. This is equivalent to saying that the probability that the Brownian excursion from the graph of $\tilde{\theta}_{\varepsilon,u}$ straddling t starts after the time when the local time $L^0_\cdot(B - \tilde{\theta}_{\varepsilon,u})$ accumulates x units, is less than or equal to $\exp(-x\gamma)$. This in turn is equivalent to the statement of Lemma 1(i).

(ii) If $F\colon C[0, t] \to \mathbb{R}$ is a bounded measurable function, then, by Girsanov's theorem,

$$\begin{aligned}\mathbb{E}_0^{\mathbb{P}}(F(B - \tilde{\theta}_{\varepsilon,u})) &= \mathbb{E}_0^{\mathbb{Q}}\left(\exp\left(-\int_0^t \tilde{\theta}'_{\varepsilon,u}(s)\, dW_s - \tfrac{1}{2}\int_0^t (\tilde{\theta}'_{\varepsilon,u}(s))^2\, ds\right) F(W)\right)\\ &= \mathbb{E}_0^{\mathbb{Q}}\left(\Lambda \exp\left(-\int_0^t \tilde{\theta}'_{\varepsilon,u}(s)\, dW_s\right) F(W)\right),\end{aligned}$$

where under $\mathbb{Q}$, W is a Brownian motion starting from 0. This and integration by parts yield

$$\begin{aligned}
&\mathbb{P}_0(L_t^0(B-\tilde{\theta}_{\varepsilon,u}) \geq x)\\
&\quad = \mathbb{E}_0^{\mathbb{Q}}\left(\Lambda \exp\left(-\int_0^t \tilde{\theta}'_{\varepsilon,u}(s)\,dW_s\right);\ L_t^0(W) \geq x\right)\\
&\quad = \mathbb{E}_0^{\mathbb{Q}}\left(\Lambda \exp\left(-W_t\tilde{\theta}'_{\varepsilon,u}(t) + \int_0^t W_s\tilde{\theta}''_{\varepsilon,u}(s)\,ds\right);\ L_t^0(W) \geq x\right) \qquad (2)\\
&\quad \geq \mathbb{E}_0^{\mathbb{Q}}\left(\Lambda \exp\left(-W_t\tilde{\theta}'_{\varepsilon,u}(t) + \int_0^t W_s\tilde{\theta}''_{\varepsilon,u}(s)\,ds\right);\ L_t^0(W) \geq x;\ W_t < 0\right).
\end{aligned}$$

Let U be the last zero of W before t. By the reflection principle, the distribution of $\{W_s,\ 0 \leq s \leq U\}$ is symmetric given the value of U and the amount of local time at zero at time U. Note that $\tilde{\theta}''_{\varepsilon,u}(s) < 0$. Hence, the distribution of $\int_0^U W_s\tilde{\theta}''_{\varepsilon,u}(s)\,ds$ is symmetric and $\int_U^t W_s\tilde{\theta}''_{\varepsilon,u}(s)\,ds$ is nonnegative assuming $W_t < 0$. Thus the probability that $\int_0^t W_s\tilde{\theta}''_{\varepsilon,u}(s)\,ds$ is positive is at least 1/2 given the event that $W_t < 0$. It follows that (2) is not less than

$$(1/2)\mathbb{E}_0^{\mathbb{Q}}(\Lambda \exp(-W_t\tilde{\theta}'_{\varepsilon,u}(t));\ L_t^0(W) \geq x;\ W_t < 0).$$

Recall that $\gamma = \tilde{\theta}'_{\varepsilon,u}(t)$. Karatzas and Shreve [5, page 420] give an explicit formula for the joint density of the Brownian motion and local time. We use this formula and the substitution $v = (a - b + \gamma t)/\sqrt{t}$ to write

$$\begin{aligned}
&\mathbb{P}_0(L_t^0(B-\tilde{\theta}_{\varepsilon,u}) \geq x)\\
&\quad \geq \left(\frac{1}{2}\right)\Lambda \int_x^\infty \int_{-\infty}^0 \exp(-\gamma a)\frac{b-a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(b-a)^2}{2t}\right)da\,db\\
&\quad = \left(\frac{1}{2}\right)\Lambda \int_x^\infty \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right)\int_{-\infty}^0 \frac{b-a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-b+\gamma t)^2}{2t}\right)da\,db\\
&\quad = \left(\frac{1}{2}\right)\Lambda \int_x^\infty \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right)\int_{-\infty}^{(-b+\gamma t)/\sqrt{t}} \frac{\gamma t - v\sqrt{t}}{\sqrt{2\pi t^3}} \exp\left(\frac{-v^2}{2}\right)\sqrt{t}\,dv\,db\\
&\quad = \left(\frac{1}{2}\right)\Lambda \int_x^\infty \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right)\int_{-\infty}^{(-b+\gamma t)/\sqrt{t}} \frac{\gamma}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right)dv\,db\\
&\qquad - \left(\frac{1}{2}\right)\Lambda \int_x^\infty \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right)\int_{-\infty}^{(-b+\gamma t)/\sqrt{t}} \frac{v}{\sqrt{2\pi t}} \exp\left(\frac{-v^2}{2}\right)dv\,db\\
&\quad \geq \left(\frac{1}{2}\right)\Lambda \int_x^\infty \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right)\int_{-\infty}^{(-b+\gamma t)/\sqrt{t}} \frac{\gamma}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right)dv\,db\\
&\qquad - \left(\frac{1}{2}\right)\Lambda \int_x^\infty \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right)\int_{-\infty}^{\infty} \frac{v}{\sqrt{2\pi t}} \exp\left(\frac{-v^2}{2}\right)dv\,db\\
&\quad = \left(\frac{1}{2}\right)\Lambda \int_x^\infty \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right)\int_{-\infty}^{(-b+\gamma t)/\sqrt{t}} \frac{\gamma}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right)dv\,db.
\end{aligned}$$

Now assume that $M > 1$, $-Mx + \gamma t \ge 0$ and $x\gamma \ge 2$. Then $-b + \gamma t \ge 0$ for all $b \in [x, Mx]$ and so

$$\begin{aligned}
&\mathbb{P}_0(L_t^0(B - \tilde{\theta}_{\varepsilon,u}) \ge x) \\
&\quad \ge K\Lambda \int_x^{\infty} \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right) \int_{-\infty}^{(-b+\gamma t)/\sqrt{t}} \frac{\gamma}{\sqrt{2\pi}} \exp\left(\frac{-v^2}{2}\right) dv\, db \\
&\quad \ge K\Lambda \int_x^{Mx} \exp\left(-b\gamma + \frac{\gamma^2 t}{2}\right) \gamma\, db \\
&\quad = K\Lambda \exp\left(\frac{\gamma^2 t}{2}\right) \exp(-x\gamma)\big(1 - \exp(-(M-1)x\gamma)\big) \\
&\quad \ge K\Lambda \exp\left(\frac{\gamma^2 t}{2}\right) \exp(-x\gamma). \qquad \square
\end{aligned}$$

PROOF OF THEOREM 1. We shall divide the proof into four sections which are more or less independent. Throughout the proof of Theorem 1 we shall assume $0 < \varepsilon < 1/2$.

The lower bound for the curve θ_ε. We start by introducing a number of parameters whose values will be chosen later in the proof. We will consider $\chi > 1$, $q = \chi/\varepsilon$, $\lambda \in (0, 9\chi/(1-\varepsilon)) \subset (0, 18\chi)$ and $\alpha = \lambda + q$. In this proof, u and v will be related by $v = uq/\alpha$ and we will typically assume that $u \in (\alpha^n, \alpha^{n+1})$. Let $x = \beta\varepsilon\sqrt{2u \ln_2 u}$, with $\beta = \lambda(1-\varepsilon)/(4\chi)$. Let $\mathscr{F}_t$ be the σ-field generated by $\{B_s, 0 \le s \le t\}$. Since the local time is a nondecreasing process, the Markov property implies that

$$\begin{aligned}
J_n &\stackrel{\mathrm{df}}{=} \mathbb{P}_0\big(\exists\, u \in (\alpha^n, \alpha^{n+1})\colon L_u^0(B - \theta_\varepsilon) \ge x \text{ or } L_u^0(-B + \theta_\varepsilon) \ge x \mid \mathscr{F}_{\alpha^n}\big) \\
&\ge \mathbb{P}_{|B_{\alpha^n}|}\big(\exists\, u \in (\alpha^n, \alpha^{n+1})\colon L_{u-\alpha^n}^0(B - \theta_{\varepsilon,\alpha^n}) \ge x \\
&\qquad\qquad \text{or } L_{u-\alpha^n}^0(-B + \theta_{\varepsilon,\alpha^n}) \ge x \mid \mathscr{F}_{\alpha^n}\big) 1_{|B_{\alpha^n}| \le a_n},
\end{aligned}$$

where $a_n = (1+\varepsilon)\sqrt{2\alpha^n \ln n}$.

Let $T_n = \inf\{t \ge 0\colon |B_t| = \theta_{\varepsilon,\alpha^n}(t)\}$. If $T_n \le \alpha^n(q-1)$, then $T_n\alpha/q \le \alpha^{n+1}$. The strong Markov property applied at T_n gives

$$\begin{aligned}
J_n &\ge 1_{|B_{\alpha^n}| \le a_n} \mathbb{E}_{|B_{\alpha^n}|}\big(1_{T_n \le \alpha^n(q-1)} 1_{B(T_n) \ge 0} \mathbb{P}_0(L_{T_n\alpha/q - T_n}^0(B - \tilde{\theta}_{\varepsilon,T_n}) \ge x)\big) \\
&\quad + 1_{|B_{\alpha^n}| \le a_n} \mathbb{E}_{|B_{\alpha^n}|}\big(1_{T_n \le \alpha^n(q-1)} 1_{B(T_n) \le 0} \mathbb{P}_0(L_{T_n\alpha/q - T_n}^0(-B + \tilde{\theta}_{\varepsilon,T_n}) \ge x)\big).
\end{aligned} \tag{3}$$

In our estimates below, we will assume that $v \in (0, \alpha^n(q-1))$ and $u = v\alpha/q$. We can think about v as a generic value of T_n, and hence we can combine our estimates with (3).

First we are going to deal with the local time term. Let $\gamma = \theta_\varepsilon'(u)$. We would like to have

$$-Mx + \gamma(u - v) \ge 0 \tag{4}$$

for some $M > 1$ in order to apply Lemma 1(ii). For every fixed $b > 1$ and sufficiently large s we have

$$(1-\varepsilon)\sqrt{\frac{\ln_2 s}{2s}} \le \theta'_\varepsilon(s) \le b(1-\varepsilon)\sqrt{\frac{\ln_2 s}{2s}}. \tag{5}$$

Inequality (4) will hold if

$$-M\beta\varepsilon\sqrt{2u\ln_2 u} + (1-\varepsilon)\sqrt{\frac{\ln_2 u}{2u}}(u-v) \ge 0.$$

This is equivalent to each of the following inequalities:

$$\begin{gathered} -M\beta\varepsilon\sqrt{2} + (1-\varepsilon)\sqrt{1/2}(1-q/\alpha) \ge 0; \\ -M\beta\varepsilon\sqrt{2} + (1-\varepsilon)\sqrt{1/2}\varepsilon\lambda/(\chi+\varepsilon\lambda) \ge 0; \\ \beta \le (1/2M)(1-\varepsilon)\lambda/(\chi+\varepsilon\lambda); \\ [(1-\varepsilon)/(4\chi)]\lambda \le (1/2M)(1-\varepsilon)\lambda/(\chi+\varepsilon\lambda); \\ 1/(4\chi) \le (1/2M)/(\chi+\varepsilon\lambda); \\ M \le 2\chi/(\chi+\varepsilon\lambda). \end{gathered} \tag{6}$$

The last inequality is satisfied for every fixed $\chi > 1$ and $M = 3/2$ when $\varepsilon > 0$ is sufficiently small.

Let $\Lambda = \exp(-\frac{1}{2}\int_0^{u-v}(\tilde{\theta}'_{\varepsilon,v}(s))^2\,ds)$. If (6) and (4) are satisfied, then we obtain from Lemma 1(ii),

$$\mathbb{P}_0(L^0_{u-v}(B-\tilde{\theta}_{\varepsilon,v}) \ge x) \ge K\Lambda\exp\left(\frac{\gamma^2(u-v)}{2}\right)\exp(-x\gamma). \tag{7}$$

For all $x > 0$ we have $\ln x \le x-1$ so $\ln(\alpha/q) \le (\alpha-q)/q$. Choose a constant $b > 1$ in (5). For any $b_1 > b^2$ and large n,

$$\begin{aligned} \Lambda &= \exp\left(-\tfrac{1}{2}\int_0^{u-v}(\tilde{\theta}'_{\varepsilon,v}(s))^2\,ds\right) \\ &= \exp\left(-\tfrac{1}{2}\int_v^{u}(\tilde{\theta}'_{\varepsilon}(s))^2\,ds\right) \\ &\ge \exp\left(-\int_v^{u} b^2(1-\varepsilon)^2(\ln_2 s/4s)\,ds\right) \\ &\ge \exp\left(-b^2(1-\varepsilon)^2\ln_2 u\int_v^{u}(ds/4s)\,ds\right) \\ &= \exp\left(-\tfrac{1}{4}b^2(1-\varepsilon)^2\ln_2 u\ln(u/v)\right) \\ &= \exp\left(-\tfrac{1}{4}b^2(1-\varepsilon)^2\ln_2 u\ln(\alpha/q)\right) \\ &\ge \exp\left(-\tfrac{1}{4}b^2(1-\varepsilon)^2((\alpha-q)/q)\ln_2 u\right) \\ &\ge \exp(-(1/(4\chi))b_1(1-\varepsilon)^2\varepsilon\lambda\ln n) \\ &= n^{-b_1(1-\varepsilon)^2\varepsilon\lambda/(4\chi)}. \end{aligned} \tag{8}$$

Next we bound the second factor in (7):

$$\exp(\gamma^2(u-v)/2) \geq \exp\bigl((\ln_2 u/2u)u(1-q/\alpha)/2\bigr) \geq n^{(\alpha-q)/(4\alpha)} = n^{\varepsilon\lambda/[4(\chi+\varepsilon\lambda)]}. \tag{9}$$

The last factor in (7) may be estimated as follows using (5):

$$\exp(-x\gamma) \geq \exp\left(-\beta\varepsilon\sqrt{2u\ln_2 u}\,b(1-\varepsilon)\sqrt{\frac{\ln_2 u}{2u}}\right) = n^{-\beta\varepsilon b(1-\varepsilon)}. \tag{10}$$

Combining (7)–(10) gives

$$\mathbb{P}_0(L^0_{u-v}(B-\tilde{\theta}_{\varepsilon,v}) \geq x) \geq Kn^{-\mathscr{R}},$$

where

$$\mathscr{R} = b_1(1-\varepsilon)^2\varepsilon\lambda/(4\chi) - \varepsilon\lambda/4(\chi+\varepsilon\lambda) + \beta\varepsilon b(1-\varepsilon).$$

Observe that, on the set $\{|B_{\alpha^n}| \leq a_n\}$,

$$\begin{aligned}
&\mathbb{P}_{|B_{\alpha^n}|}(T_n \leq \alpha^n(q-1);\ B_{T_n} \geq 0)\\
&\quad = \left(\frac{1}{2}\right)\mathbb{P}_{|B_{\alpha^n}|}(T_n \leq \alpha^n(q-1))\\
&\quad \geq \left(\frac{1}{2}\right)\mathbb{P}_0\bigl(\operatorname{sgn}(B_{\alpha^n})(B_{\alpha^n q} - B_{\alpha^n}) \geq \theta_\varepsilon(\alpha^n q)\bigr)\\
&\quad = \left(\frac{1}{2}\right)\mathbb{P}_0\left(B_1 \geq \left(1+O\left(\frac{1}{\ln n}\right)\right)(1-\varepsilon)\sqrt{\frac{2q}{q-1}\ln n}\right)\\
&\quad \geq \frac{Kn^{-(1-\varepsilon)^2 q/(q-1)}}{\sqrt{\ln n}}.
\end{aligned} \tag{11}$$

Now we choose the parameters. Fix arbitrary $\beta < \beta_1 < \beta_2 < 2$. Find χ so large that

$$(1-\varepsilon)^2\frac{q}{q-1} = (1-\varepsilon)^2\left(1+\frac{\varepsilon}{\chi-\varepsilon}\right) < 1-\beta_2\varepsilon \tag{12}$$

for sufficiently small ε. Next we choose $b, b_1 > 1$ so that $\mathscr{R} < \beta_1\varepsilon$ for small $\varepsilon > 0$ and so we have

$$\mathbb{P}_0\bigl(L^0_{u-v}(B-\tilde{\theta}_{\varepsilon,v}) \geq x\bigr) \geq Kn^{-\beta_1\varepsilon}.$$

This, (3), (11) and (12) imply that, for small ε and large n,

$$J_n \geq \frac{Kn^{-(1-(\beta_2-\beta_1)\varepsilon)}}{\sqrt{\ln n}} 1_{|B_{\alpha^n}| \leq a_n}.$$

The standard LIL implies that $|B_{\alpha^n}| \leq a_n$ eventually. Since $1-(\beta_2-\beta_1)\varepsilon < 1$, we deduce, using a generalized Borel–Cantelli lemma (see Neveu [8], page 152, Corollaire VII-2-6), that, for infinitely many n,

$$L_u^0(B-\theta_\varepsilon) \geq x = \beta\varepsilon\sqrt{2u \ln_2 u}$$

or

$$L_u^0(-B+\theta_\varepsilon) \geq x = \beta\varepsilon\sqrt{2u \ln_2 u},$$

from which we have

$$\limsup_{t\to\infty} \frac{L_t^0(B-\theta_\varepsilon)}{\sqrt{2t\ln_2 t}} \geq \beta\varepsilon$$

or

$$\limsup_{t\to\infty} \frac{L_t^0(-B+\theta_\varepsilon)}{\sqrt{2t\ln_2 t}} \geq \beta\varepsilon.$$

An easy argument based on the symmetry of the Brownian motion allows us to deduce

$$\limsup_{t\to\infty} \frac{L_t^0(B-\theta_\varepsilon)}{\sqrt{2t\ln_2 t}} \geq \beta\varepsilon \quad \text{a.s.}$$

for every $\beta < 2$ and $\varepsilon < \varepsilon_0(\beta)$.

The upper bound for the curve θ_ε. First we outline the idea of the proof of the upper bound. We start with an estimate of the probability that Brownian motion will hit θ_ε between times α^n and α^{n+1}. This estimate is used to find an upper bound for the probability that the local time increments over several consecutive intervals $[\alpha^{n+k-1}, \alpha^{n+k}]$ are large (the precise meaning of "large" will be made clear below). An application of the Borel–Cantelli lemma shows that, starting at some random N, the increments are not too large. It turns out that the sum of the increments is sufficiently small to yield the upper bound in Theorem 1(i).

Take some $\alpha > 1$ and define $T_n = \inf\{t \geq \alpha^n \colon B_t = \theta_\varepsilon(t)\}$. Then

$$\begin{aligned}
&\mathbb{P}_0\big(L^0_{\alpha^{n+1}}(B-\theta_\varepsilon) - L^0_{\alpha^n}(B-\theta_\varepsilon) \geq x\big)\\
&\quad = \mathbb{P}_0\big(T_n \leq \alpha^{n+1};\ L^0_{\alpha^{n+1}}(B-\theta_\varepsilon) - L^0_{T_n}(B-\theta_\varepsilon) \geq x\big)\\
&\quad = \mathbb{P}_0\big(T_n \leq \alpha^{n+1};\ \mathbb{P}_0(L^0_{\alpha^{n+1}-T_n}(B-\tilde\theta_{\varepsilon,T_n}) \geq x \mid \mathscr{F}_{T_n})\big).
\end{aligned}$$

First we will estimate $\mathbb{P}_0(T_n \leq \alpha^{n+1})$. To this end, take an integer $M > 2\alpha$ and consider $q_i = 1 + (i-1)\alpha/M$ for $i = 1, \dots, M+1$. Let $I_i = [q_i\alpha^n, q_{i+1}\alpha^n]$. Recall that

$$\int_z^\infty \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-y^2}{2t}\right) dy \leq \frac{\sqrt{t}}{z\sqrt{2\pi}} \exp\left(\frac{-z^2}{2t}\right)$$

for $z > 0$. We have

$$\begin{aligned}\mathbb{P}_0(T_n \le \alpha^{n+1}) &= \sum_{i=1}^{M} \mathbb{P}_0(T_n \in I_i) \le \sum_{i=1}^{M} \mathbb{P}_0\Big(\max_{0\le t\le q_{i+1}\alpha^n} B_t \ge \theta_\varepsilon(q_i\alpha^n)\Big)\\ &\le \sum_{i=1}^{M} 2\mathbb{P}_0\big(B_{q_{i+1}\alpha^n} \ge \theta_\varepsilon(q_i\alpha^n)\big)\\ &\le \sum_{i=1}^{M} 2\frac{\sqrt{q_{i+1}\alpha^n}}{\theta_\varepsilon(q_i\alpha^n)\sqrt{2\pi}} \exp\bigg(-\frac{(\theta_\varepsilon(q_i\alpha^n))^2}{2q_{i+1}\alpha^n}\bigg)\\ &\le \sum_{i=1}^{M} K\sqrt{\frac{q_{i+1}}{q_i \ln n}}\, n^{-(1-\varepsilon)^2 q_i/q_{i+1}}.\end{aligned}$$

Take an arbitrarily large $b < 1$ and fix a large integer M so that $q_i/q_{i+1} > b$ for all $i \le M$. Then

$$\mathbb{P}_0(T_n \le \alpha^{n+1}) \le K n^{-(1-\varepsilon)^2 b},$$

where K depends only on b.

Let $\gamma_n = \theta'_\varepsilon(\alpha^{n+1})$ and $x = c\sqrt{2\alpha^n \ln n}$. Lemma 1(i) implies that, for every $s \in [\alpha^n, \alpha^{n+1}]$, $b < 1$ and large n,

$$\begin{aligned}\mathbb{P}_0\big(L^0_{\alpha^{n+1}-s}(B - \tilde\theta_{\varepsilon,s}) \ge x\big) &\le \exp(-x\gamma_n)\\ &\le \exp\bigg(-cb\sqrt{2\alpha^n \ln n}(1-\varepsilon)\sqrt{\frac{\ln n}{2\alpha^{n+1}}}\bigg)\\ &\le n^{-(1-\varepsilon)cb/\sqrt{\alpha}}.\end{aligned}$$

Fix some integer $j \ge 1$ and suppose that $\beta_1, \beta_2, \dots, \beta_j > 0$. Let $\tilde\beta = \sum_{k=1}^{j} \beta_k$ and $x_k = x_k(n) = \beta_k \varepsilon\sqrt{2\alpha^{n+k-1}\ln(n+k-1)}$. By applying the strong Markov property at $T_n, T_{n+1}, \dots, T_{n+j-1}$ we obtain

$$\begin{aligned}&\mathbb{P}_0\bigg(\bigcap_{k=1}^{j}\{L^0_{\alpha^{n+k}}(B-\theta_\varepsilon) - L^0_{\alpha^{n+k-1}}(B-\theta_\varepsilon) \ge x_k\}\bigg)\\ &\quad\le \mathbb{P}_0(T_n \le \alpha^{n+1}) \prod_{k=1}^{j} \max_{s\in[\alpha^{n+k-1},\alpha^{n+k}]} \mathbb{P}_0\big(L^0_{\alpha^{n+k}-s}(B-\tilde\theta_{\varepsilon,s}) \ge x_k\big)\\ &\quad\le K n^{-(1-\varepsilon)^2 b} n^{-\mathscr{R}},\end{aligned}$$

where

$$\mathscr{R} = \sum_{k=1}^{j}(1-\varepsilon)\beta_k \varepsilon b/\sqrt{\alpha} = (1-\varepsilon)\tilde\beta\varepsilon b/\sqrt{\alpha}.$$

Fix some small $\delta > 0$ and $\beta > 2$. If

$$L^0_{\alpha^{n+j}}(B-\theta_\varepsilon) - L^0_{\alpha^n}(B-\theta_\varepsilon) \ge \beta\varepsilon\sqrt{2\alpha^{n+j-1}\ln(n+j-1)},$$

then there must exist nonnegative integers $i_k \le \beta/\delta$ such that

$$L^0_{\alpha^{n+k}}(B-\theta_\varepsilon) - L^0_{\alpha^{n+k-1}}(B-\theta_\varepsilon) \ge \beta_k \varepsilon \sqrt{2\alpha^{n+j-1}\ln(n+j-1)} \ge x_k,$$

$\beta_k = i_k\delta$, for $k = 1, \dots, j$, and $\tilde{\beta} \ge \beta - j\delta$. The probability of

$$\bigcap_{k=1}^{j} \{L^0_{\alpha^{n+k}}(B-\theta_\varepsilon) - L^0_{\alpha^{n+k-1}}(B-\theta_\varepsilon) \ge x_k\}$$

for every such j-tuple $(\beta_1, \dots, \beta_j)$ is bounded by $Kn^{-(1-\varepsilon)^2 b} n^{-\mathscr{R}}$ with $\mathscr{R} = (1-\varepsilon)(\beta - j\delta)\varepsilon b/\sqrt{\alpha}$. The restriction $i_k \le \beta/\delta$ implies that there are only a finite number of j-tuples $(\beta_1, \dots, \beta_j)$ and so

$$\begin{aligned}\mathbb{P}_0\Big(L^0_{\alpha^{n+j}}(B-\theta_\varepsilon) - L^0_{\alpha^n}(B-\theta_\varepsilon) &\ge \beta\varepsilon\sqrt{2\alpha^{n+j-1}\ln(n+j-1)}\Big)\\ &\le Kn^{-(1-\varepsilon)^2 b} n^{-\mathscr{R}},\end{aligned}$$

with $\mathscr{R} = (1-\varepsilon)(\beta - j\delta)\varepsilon b/\sqrt{\alpha}$, for large n. Now take any $a > 0$. Recall that $\beta > 2$. One can find $\alpha > 1$, $b < 1$ and small $\delta > 0$ depending on j so that, for small $\varepsilon > 0$,

$$n^{-(1-\varepsilon)^2 b} n^{-\mathscr{R}} \le Kn^{-1-a}.$$

Let $y_n = \beta\varepsilon\sqrt{2\alpha^n \ln n}$. The Borel–Cantelli lemma now implies that

$$\Delta^j_n \overset{\text{df}}{=} L^0_{\alpha^{n+j}}(B-\theta_\varepsilon) - L^0_{\alpha^n}(B-\theta_\varepsilon) < y_{n+j-1}$$

eventually. In particular, for $j = 1$ we obtain

$$\Delta_n \overset{\text{df}}{=} L^0_{\alpha^{n+1}}(B-\theta_\varepsilon) - L^0_{\alpha^n}(B-\theta_\varepsilon) < y_n$$

eventually. We let $N = \inf\{n\colon \Delta_k \le y_k \text{ and } \Delta^j_k \le y_k \ \forall\ k \ge n\}$. Then, for some $b_1 > 1$, all $\alpha^{n+j-1} < t \le \alpha^{n+j}$ and sufficiently large $n > N$, we have

$$\begin{aligned}L^0_t(B-\theta_\varepsilon) &\le L^0_{\alpha^N}(B-\theta_\varepsilon) + \left[\sum_{k=N}^{n-1} L^0_{\alpha^{k+1}}(B-\theta_\varepsilon) - L^0_{\alpha^k}(B-\theta_\varepsilon)\right]\\ &\quad + L^0_{\alpha^{n+j}}(B-\theta_\varepsilon) - L^0_{\alpha^n}(B-\theta_\varepsilon)\\ &\le L^0_{\alpha^N}(B-\theta_\varepsilon) + y_{n+j-1} + \sum_{k=N}^{n} y_k\\ &\le L^0_{\alpha^N}(B-\theta_\varepsilon) + y_{n+j-1} + \sum_{k=0}^{n} y_k\\ &\le L^0_{\alpha^N}(B-\theta_\varepsilon) + \beta\varepsilon\sqrt{2\alpha^{n+j-1}\ln(n+j-1)} + \beta\varepsilon\sqrt{2\ln n}\sum_{k=0}^{n}(\sqrt{\alpha})^k\end{aligned}$$

$$\le L^0_{\alpha^N}(B-\theta_\varepsilon) + \beta\varepsilon\sqrt{2\alpha^{n+j-1}\ln(n+j-1)} + \frac{\beta\varepsilon\sqrt{\alpha}}{\sqrt{\alpha}-1}\sqrt{2\alpha^n \ln n}$$
$$\le L^0_{\alpha^N}(B-\theta_\varepsilon) + \left(1+\frac{b_1\sqrt{\alpha}}{\sqrt{\alpha}-1}\alpha^{-j/2}\right)\beta\varepsilon\sqrt{2t\ln_2 t}.$$

Since j may be taken arbitrarily large, we deduce that

$$\limsup_{t\to\infty}\frac{L^0_t(B-\theta_\varepsilon)}{\sqrt{2t\ln_2 t}} \le \beta\varepsilon,$$

where β can be an arbitrary number greater than 2 and $\varepsilon < \varepsilon_0(\beta)$.

The lower bound for the critical curve θ. We are going to use a result of Erdös and Révész [3]. For that matter consider $\xi(t) = \sup\{s \le t,\ B_s \ge \theta(s)\}$. Then, for large t: $\xi(t) \ge t^{1-d\ln_3 t(\ln_2 t)^{-1/2}}$ a.s., where d is a large positive constant. Let $\alpha \ge e^e, \beta \ge e^e$ and $\varepsilon > 0$ be fixed numbers and consider

$$t_n = \alpha^{\beta^{n^{(2/3+\varepsilon)}}}.$$

In this way

$$\ln t_n = \beta^{n^{(2/3+\varepsilon)}}\ln\alpha,$$
$$\ln_2 t_n = n^{(2/3+\varepsilon)}\ln\beta + \ln_2\alpha,$$
$$\ln_3 t_n = \left(\frac{2}{3}+\varepsilon\right)\ln n + \ln_2\beta + \ln\left(1+\frac{\ln_2\alpha}{n^{2/3+\varepsilon}\ln\beta}\right).$$

It is not hard to check that $\xi(t_{n+1}) \ge t_n$ for large n. Therefore, for t large enough there is an s in the interval $I = [t^{1-d\ln_3 t(\ln_2 t)^{-1/2}}, t]$ for which $B_s \ge \theta(s)$. In a similar way we will have that there is an s' in the same interval for which $B_{s'} \le -\theta(s')$. Thus there exists an instant $u \in I$, where $B_u = \theta(u)$. Hence, letting $T_n = \inf\{t \ge t_n,\ B_t = \theta(t)\}$, we have, for large enough n,

$$T_n \le t_{n+1}.$$

Fix some $M > 80$ and let $h(u) = Mu\ln_3 u/\ln_2 u$. We have

$$\mathbb{P}_0\big(T_n \le t_{n+1},\ L^0_{T_n+h(T_n)}(B-\theta) - L^0_{T_n}(B-\theta) \ge cg(T_n) \mid \mathscr{F}_{T_n}\big)$$
$$= 1_{T_n\le t_{n+1}} H(T_n),$$

where $H(u) = \mathbb{P}_0(L^0_{h(u)}(B-\tilde\theta_u) \ge cg(u))$. Now, for $t_n \le u \le t_{n+1}$ and large n we have

$$-2cg(u) + \theta'(u+h(u))h(u) \ge -2cg(u) + \frac{1}{2}\theta'(u)\frac{Mu\ln_3 u}{\ln_2 u}$$
$$\ge -2c\sqrt{\frac{u}{\ln_2 u}}\ln_3 u + \sqrt{\frac{\ln_2 u}{2u}}\frac{Mu\ln_3 u}{2\ln_2 u}.$$

This quantity is nonnegative if n is large enough, for any fixed $c < 10 < M/8$.

Let $\Lambda = \exp(-\frac{1}{2}\int_u^{u+h(u)}(\theta'(s))^2\,ds)$ and $\gamma = \theta'(u+h(u))$. We obtain from Lemma 1(ii),

$$H(u) \geq K\Lambda\exp(\gamma^2 h(u)/2)\exp(-cg(u)\gamma). \tag{13}$$

We have

$$\begin{aligned}\Lambda\exp(\gamma^2 h(u)/2) &= \exp\biggl(-\tfrac{1}{2}\int_u^{u+h(u)}(\theta'(s)^2-\gamma^2)\,ds\biggr)\\ &\geq \exp\Bigl(-(1/2)h(u)\max_{u\leq s\leq u+h(u)}(\theta'(s)^2-\gamma^2)\Bigr)\\ &\geq \exp\bigl(-(1/2)h(u)(\theta'(u)^2-\gamma^2)\bigr).\end{aligned}$$

Note that

$$\begin{aligned}\theta'(u)^2-\gamma^2 &= \frac{\ln_2 u}{2u}\biggl(1+\frac{1}{\ln u\ln_2 u}\biggr)^2 - \frac{\ln_2(u+h(u))}{2u(1+M\ln_3 u/\ln_2 u)}\\ &\qquad\times\biggl(1+\frac{1}{\ln(u+h(u))\ln_2(u+h(u))}\biggr)^2\\ &= \frac{\ln_2 u}{2u}\biggl[\biggl(1+\frac{1}{\ln u\ln_2 u}\biggr)^2 - \frac{\ln_2(u+h(u))}{\ln_2 u(1+M\ln_3 u/\ln_2 u)}\\ &\qquad\times\biggl(1+\frac{1}{\ln(u+h(u))\ln_2(u+h(u))}\biggr)^2\biggr],\end{aligned}$$

where the expression in the square brackets approaches 0 as u goes to infinity. Hence, for arbitrary $b>0$ and large u,

$$\Lambda\exp\biggl(\frac{\gamma^2 h(u)}{2}\biggr) \geq \exp\biggl(-\frac{1}{2}\frac{Mu\ln_3 u}{\ln_2 u}b\frac{\ln_2 u}{2u}\biggr) = \exp\biggl(-\biggl(\frac{Mb}{4}\biggr)\ln_3 u\biggr). \tag{14}$$

As for the last factor in (13), we have, for arbitrary $b_2 > b_1 > 1$ and sufficiently large u,

$$\begin{aligned}\exp(-cg(u)\gamma) &\geq \exp\Biggl(-cb_1\sqrt{\frac{u}{\ln_2 u}}\ln_3 u\sqrt{\frac{\ln_2(u+h(u))}{2(u+h(u))}}\Biggr)\\ &\geq \exp\biggl(-\biggl(\frac{b_2 c}{\sqrt{2}}\biggr)\ln_3 u\biggr).\end{aligned}$$

This combined with (13) and (14) yields

$$\begin{aligned}H(u) &\geq K\exp\bigl(-(Mb/4+b_2c/\sqrt{2})\ln_3 u\bigr)\\ &\geq K\exp\bigl(-(Mb/4+b_2c/\sqrt{2})\ln_3 t_{n+1}\bigr) \geq Kn^{-(Mb/4+b_2c/\sqrt{2})(2/3+\varepsilon)}.\end{aligned}$$

For an arbitrary $c < \sqrt{2}(\frac{2}{3}+\varepsilon)^{-1}$ we can find $b>0$ and $b_2>1$ so that

$$(Mb/4+b_2c/\sqrt{2})\bigl(\tfrac{2}{3}+\varepsilon\bigr) < 1.$$

Then

$$\sum_{n \text{ even}} \mathbb{P}_0\big(T_n \le t_{n+1},\ L^0_{T_n+h(T_n)}(B-\theta) - L^0_{T_n}(B-\theta) \ge cg(T_n) \mid \mathscr{F}_{T_n}\big)$$
$$= \infty \quad \text{a.s.}$$

Since $T_n + h(T_n) \le T_{n+2}$ and $T_n + h(T_n)$ is a stopping time, we get from the generalized Borel–Cantelli lemma [8, page 152] that $\{T_n \le t_{n+1};\ L^0_{T_n+h(T_n)}(B-\theta) \ge cg(T_n)\}$ occurs i.o.

Given that $(T_n + h(T_n))/T_n \to 1$ as $n \to \infty$, we deduce that

$$\limsup_{n\to\infty} \frac{L^0_{T_n+h(T_n)}(B-\theta)}{g(T_n+h(T_n))} \ge c$$

and, therefore,

$$\limsup_{t\to\infty} \frac{L^0_t(B-\theta)}{g(t)} \ge c.$$

Since the inequality holds for all $c < \sqrt{2}(\frac{2}{3}+\varepsilon)^{-1}$ and $\varepsilon > 0$ is arbitrarily small,

$$\limsup_{t\to\infty} \frac{L^0_t(B-\theta)}{g(t)} \ge \frac{3}{2}\sqrt{2}.$$

The upper bound for the critical curve θ. We proceed as in the case θ_ε. Let $\alpha > 1$ and $T_n = \inf\{t \ge \alpha^n\colon B_t = \theta(t)\}$. We have

$$\begin{aligned}&\mathbb{P}_0\big(L^0_{\alpha^{n+1}}(B-\theta) - L^0_{\alpha^n}(B-\theta) \ge x\big)\\ &\quad = \mathbb{P}_0\big(T_n \le \alpha^{n+1};\ \mathbb{P}_0(L^0_{\alpha^{n+1}-T_n}(B-\tilde{\theta}_{T_n}) \ge x \mid \mathscr{F}_{T_n})\big).\end{aligned}$$

Let $v_n = \alpha^n/\ln n$ and consider $q_i = ((i-1)v_n)/\alpha^n + 1$ for $i = 1,\dots,s_n \overset{\text{df}}{=} [(\alpha^n(\alpha-1))/v_n] + 2$. If $I_i = [\alpha^n q_i, \alpha^n q_{i+1}]$,

$$\begin{aligned}\mathbb{P}_0(T_n \le \alpha^{n+1}) &= \sum_{i=1}^{s_n} \mathbb{P}_0(T_n \in I_i) \le \sum_{i=1}^{s_n} \mathbb{P}_0\Big(\max_{0\le t\le q_{i+1}\alpha^n} B_t \ge \theta(q_i\alpha^n)\Big)\\
&\le \sum_{i=1}^{s_n} 2\mathbb{P}_0(B_{q_{i+1}\alpha^n} \ge \theta(q_i\alpha^n))\\
&\le \sum_{i=1}^{s_n} 2\frac{\sqrt{q_{i+1}\alpha^n}}{\theta(q_i\alpha^n)\sqrt{2\pi}} \exp\Big(-\frac{(\theta(q_i\alpha^n))^2}{2q_{i+1}\alpha^n}\Big)\\
&\le \sum_{i=1}^{s_n} K\sqrt{\frac{q_{i+1}}{q_i \ln n}}\, n^{-q_i/q_{i+1}}\\
&\le K s_n \frac{1}{\sqrt{\ln n}} n^{-1/q_2}\\
&\le K\sqrt{\ln n}\cdot n^{-1}.\end{aligned}$$

Let $\gamma_n = \theta'(\alpha^{n+1})$ and $x = c\sqrt{\alpha^n/\ln n}\ln_2 n$. Lemma 1(i) implies that, for $u \in [\alpha^n, \alpha^{n+1}]$ and an arbitrary $b < 1$,

$$\begin{aligned}\mathbb{P}_0\big(L^0_{\alpha^{n+1}-u}(B-\tilde{\theta}_u) \geq x\big) &\leq \exp(-x\gamma_n)\\ &\leq K\exp\Big(-bc\sqrt{\alpha^n/\ln n}\ln_2 n\sqrt{(\ln n)/(2\alpha^{n+1})}\Big)\\ &\leq K\exp(-bc\ln_2 n/\sqrt{2\alpha}) = K(\ln n)^{-bc/\sqrt{2\alpha}}.\end{aligned}$$

Fix some integer $j \geq 1$ and suppose that $\beta_1, \beta_2, \ldots, \beta_j > 0$. Let $\tilde{\beta} = \sum_{k=1}^j \beta_k$ and

$$x_k = x_k(n) = \beta_k\sqrt{\frac{\alpha^{n+k-1}}{\ln(n+k-1)}}\ln_2(n+k-1).$$

By applying the strong Markov property at $T_n, T_{n+1}, \ldots, T_{n+j-1}$ we obtain

$$\begin{aligned}&\mathbb{P}_0\bigg(\bigcap_{k=1}^j\big\{L^0_{\alpha^{n+k}}(B-\theta) - L^0_{\alpha^{n+k-1}}(B-\theta) \geq x_k\big\}\bigg)\\ &\quad\leq \mathbb{P}_0(T_n \leq \alpha^{n+1})\prod_{k=1}^j \max_{s\in[\alpha^{n+k-1},\alpha^{n+k}]}\mathbb{P}_0\big(L^0_{\alpha^{n+k}-s}(B-\tilde{\theta}_s) \geq x_k\big)\\ &\quad\leq K\sqrt{\ln n}\cdot n^{-1}(\ln n)^{-\mathscr{R}},\end{aligned}$$

where

$$\mathscr{R} = \sum_{k=1}^j b\beta_k/\sqrt{2\alpha} = b\tilde{\beta}/\sqrt{2\alpha}.$$

Fix some small $\delta > 0$ and $\beta > 3\sqrt{2}/2$. If

$$L^0_{\alpha^{n+j}}(B-\theta) - L^0_{\alpha^n}(B-\theta) \geq \beta\sqrt{\frac{\alpha^{n+j-1}}{\ln(n+j-1)}}\ln_2(n+j-1),$$

then there must exist nonnegative integers $i_k \leq \beta/\delta$ such that

$$L^0_{\alpha^{n+k}}(B-\theta) - L^0_{\alpha^{n+k-1}}(B-\theta) \geq \beta_k\sqrt{\frac{\alpha^{n+j-1}}{\ln(n+j-1)}}\ln_2(n+j-1) \geq x_k,$$

$\beta_k = i_k\delta$, for $k = 1, \ldots, j$, and $\tilde{\beta} \geq \beta - j\delta$. The probability of

$$\bigcap_{k=1}^j\big\{L^0_{\alpha^{n+k}}(B-\theta) - L^0_{\alpha^{n+k-1}}(B-\theta) \geq x_k\big\}$$

for every such j-tuple $(\beta_1, \ldots, \beta_j)$ is bounded by $K\sqrt{\ln n}\cdot n^{-1}(\ln n)^{-\mathscr{R}}$ with $\mathscr{R} = b(\beta - j\delta)/\sqrt{2\alpha}$. The restriction $i_k \leq \beta/\delta$ implies that there are only a finite number of j-tuples $(\beta_1, \dots, \beta_j)$ and so

$$\mathbb{P}_0\left(L^0_{\alpha^{n+j}}(B-\theta) - L^0_{\alpha^n}(B-\theta) \geq \beta\sqrt{\frac{\alpha^{n+j-1}}{\ln(n+j-1)}}\ln_2(n+j-1)\right)$$
$$\leq K\sqrt{\ln n}\cdot n^{-1}(\ln n)^{-\mathscr{R}}$$

with $\mathscr{R} = b(\beta - j\delta)/\sqrt{2\alpha}$, for large n. Now take any $a > 0$. Recall that $\beta > 3\sqrt{2}/2$. One can find $\alpha > 1$, $b < 1$ and small $\delta > 0$ depending on j so that, for small $\varepsilon > 0$,

$$\sqrt{\ln n}\cdot n^{-1}(\ln n)^{-\mathscr{R}} \leq Kn^{-1}(\ln n)^{-1-a}.$$

Let $y_n = \beta\sqrt{\alpha^n/\ln n}\ln_2 n$. The Borel–Cantelli lemma now implies that

$$\Delta^j_n \stackrel{\text{df}}{=} L^0_{\alpha^{n+j}}(B-\theta) - L^0_{\alpha^n}(B-\theta) < y_{n+j-1}$$

eventually. In particular, for $j = 1$ we obtain

$$\Delta_n \stackrel{\text{df}}{=} L^0_{\alpha^{n+1}}(B-\theta) - L^0_{\alpha^n}(B-\theta) < y_n$$

eventually.

Find k_0 such that $\ln_2 k_0/\sqrt{\ln k_0} \leq 1$ and $\ln_2 n/\sqrt{\ln n}$ is a decreasing function for $n \geq k_0$. Let $N = \inf\{n \geq k_0\colon \Delta_k \leq y_k \text{ and } \Delta^j_k \leq y_k\ \forall\ k \geq n\}$. Then, for large m,

$$\begin{aligned}\sum_{k=N}^{m} y_k &\leq \sum_{k=k_0}^{m/2-1} y_k + \sum_{k=m/2}^{m} y_k \leq \sum_{k=k_0}^{m/2}\sqrt{\alpha^k} + \frac{\ln_2(m/2)}{\sqrt{\ln(m/2)}}\sum_{k=m/2}^{m}\sqrt{\alpha^k}\\
&\leq \frac{(\sqrt{\alpha})^{m/2+1}}{\sqrt{\alpha}-1} + (\sqrt{\alpha})^{m/2}\frac{(\sqrt{\alpha})^{m/2+1}}{\sqrt{\alpha}-1}\frac{\ln_2 m}{\sqrt{\ln m}}\left(1 + O\left(\frac{1}{\ln m}\right)\right)\\
&\leq \frac{K\sqrt{\alpha}}{\sqrt{\alpha}-1}\sqrt{\frac{\alpha^m}{\ln m}}\ln_2 m\left(1 + O\left(\frac{1}{\ln m}\right)\right).\end{aligned}$$

Suppose that $\alpha^{m+j-1} < t \leq \alpha^{m+j}$. Let $A = L^0_{\alpha^N}(B-\theta)$. For large m we have

$$\begin{aligned}L^0_t(B-\theta) &\leq A + \left[\sum_{k=N}^{m-1} L^0_{\alpha^{k+1}}(B-\theta) - L^0_{\alpha^k}(B-\theta)\right]\\
&\quad + L^0_{\alpha^{m+j}}(B-\theta) - L^0_{\alpha^m}(B-\theta)\\
&\leq A + y_{m+j-1} + \sum_{k=N}^{m} y_k\\
&\leq A + \beta\sqrt{\frac{\alpha^{m+j-1}}{\ln(m+j-1)}}\ln_2(m+j-1) + \frac{K\sqrt{\alpha}}{\sqrt{\alpha}-1}\sqrt{\frac{\alpha^m}{\ln m}}\ln_2 m\end{aligned}$$

$$\leq A + \left(\beta + K\frac{\sqrt{\alpha}}{\sqrt{\alpha}-1}\alpha^{-j/2}\right)\sqrt{\frac{\alpha^{m+j-1}}{\ln(m+j-1)}}\ln_2(m+j-1)$$

$$\leq A + \left(\beta + K\frac{\sqrt{\alpha}}{\sqrt{\alpha}-1}\alpha^{-j/2}\right)\sqrt{\frac{t}{\ln_2 t}}\ln_3 t.$$

Since j may be an arbitrarily large integer, we obtain

$$\limsup_{t\to\infty}\frac{L_t^0(B-\theta)}{g(t)} \leq \beta$$

for every $\beta > 3\sqrt{2}/2$. □

Acknowledgments. We thank Davar Khoshnevisan and Marc Yor for their most useful advice. We are grateful to the referee for a number of important suggestions.

Note added in proof. Burgess Davis (private communication) has shown by an example that the answer to Problem (ii) is negative.

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