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THE "TRUE" SELF-AVOIDING WALK WITH BOND REPULSION ON $\mathbb{Z}$: LIMIT THEOREMS[1]

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The "true" self-avoiding walk with bond repulsion is a nearest neighbor random walk on $\mathbb{Z}$, for which the probability of jumping along a bond of the lattice is proportional to $\exp(-g\cdot$ number of previous jumps along that bond). First we prove a limit theorem for the distribution of the local time process of this walk. Using this result, later we prove a local limit theorem, as $A\to\infty$, for the distribution of $A^{-2/3}X_{\theta_{s/A}}$, where $\theta_{s/A}$ is a random time distributed geometrically with mean $e^{-s/A}(1-e^{-s/A})^{-1}=A/s+O(1)$. As a by-product we also obtain an apparently new identity related to Brownian excursions and Bessel bridges.

1. Introduction and results. In the present paper we consider a "true" self-avoiding walk with bond repulsion (abbreviated BTSAW) X_i on the one-dimensional integer lattice $\mathbb{Z}$, defined as follows. The walk starts from the origin of the lattice and at time $i+1$ it jumps to one of the two neighboring sites of X_i, so that the probability of jumping along a bond of the lattice is proportional to

$$\exp(-g\cdot \text{number of previous jumps along that bond}).$$

More formally, for a nearest neighbor walk $\underline{x}_0^i=(x_0,x_1,\dots,x_i)$ and a lattice site $y\in\mathbb{Z}$ we define

$$r(y|\underline{x}_0^i)=\#\left\{0\le j<i:\ x_j=y-1,\ x_{j+1}=y\right\}, \tag{1.1}$$

$$l(y|\underline{x}_0^i)=\#\left\{0\le j<i:\ x_j=y,\ x_{j+1}=y-1\right\}, \tag{1.2}$$

$$v(y|\underline{x}_0^i)=r(y|\underline{x}_0^i)+l(y|\underline{x}_0^i). \tag{1.3}$$

Writing $e^{-g}=\lambda\in(0,1)$, the walk is governed by the law

$$\mathbf{P}\left(X_{i+1}=x_i+1\,|\,\underline{X}_0^i=\underline{x}_0^i\right)=\frac{\lambda^{v(x_i+1|\underline{x}_0^i)}}{\lambda^{v(x_i+1|\underline{x}_0^i)}+\lambda^{v(x_i|\underline{x}_0^i)}}, \tag{1.4}$$

$$\mathbf{P}\left(X_{i+1}=x_i-1\,|\,\underline{X}_0^i=\underline{x}_0^i\right)=\frac{\lambda^{v(x_i|\underline{x}_0^i)}}{\lambda^{v(x_i+1|\underline{x}_0^i)}+\lambda^{v(x_i|\underline{x}_0^i)}}. \tag{1.5}$$

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The only difference from the "orthodox" true self-avoiding walk with site repulsion (abbreviated STSAW) [see Amit, Parisi and Peliti (1983)] is that here we count the local time spent on edges, while the jump probabilities of STSAW are determined by the local time spent on sites. We expect that the physical phenomena should be very similar in the two cases. Based on a nonrigorous renormalization group argument it has been conjectured in Amit, Parisi and Peliti (1983) that the upper critical dimension of STSAW is $d_c = 2$. That is, in more than two dimensions the STSAW behaves diffusively, like an ordinary random walk, with logarithmic corrections in two dimensions. Computer simulations of the same authors seem to agree with this conjecture. It was natural to expect overdiffusive behavior below the critical dimension, that is, in $d = 1$. In Peliti and Pietronero (1987) the one-dimensional problem was considered. In that paper, based on nonrigorous scaling arguments, the authors argue that, for late times, the variance of the STSAW should behave like

$$\mathbf{E}(X_t^2) \sim t^{2\nu}, \tag{1.6}$$

with $\nu = 2/3$ in one dimension. They also cite numerical simulations strongly supporting this conjecture. For a review of the problem, see also Lawler (1991) and Madras and Slade (1993). We do not know about the existence of any rigorous results concerning true self-avoiding walks. However, let us mention here two related problems:

1. The self-avoiding walk problem (which is, strictly speaking, not a random walk in the sense of stochastic process evolving in time) has a long history, and there are deep and technically difficult results concerning it. See, for example, Brydges and Spencer (1985), Hara and Slade (1992), Madras and Slade (1993), Bolthausen (1990) and further references cited there. It is particularly interesting to compare Bolthausen's result on "ballistic behavior" of the one-dimensional weakly self-avoiding random walk with our limit theorems proved in the present paper.
2. On the other end of the spectrum are the problems related to self-attracting rather than self-repelling walks, a typical example being the so-called reinforced random walk [see, e.g., Pemantle (1988) or, for a one-dimensional problem closer to our present paper, Davis (1990)].

In the present paper we make assertion (1.6) rigorous: we formulate and prove limit theorems with rather explicit limiting distributions for the local time and position of BTSAW. Our results are in agreement with the physicists' conjectures and numerical results. The results are formulated in Sections 1.1, 1.2 and 1.3.

Let $|W_y|$, $y \in (-\infty, \infty)$, be a (two-sided) reflected Brownian motion with an arbitrary starting point $|W_0| = h \in [0, \infty)$. For $x \in [0, \infty)$ define

$$\omega_x^- = \sup\{y < x \colon |W_y| = 0\}, \tag{1.7}$$

$$\omega_x^+ = \inf\{y > x \colon |W_y| = 0\} \tag{1.8}$$

and

$$(1.9) \qquad T_x = \int_{\omega_0^-}^{\omega_x^+} |W_y|\, dy.$$

That is, ω_x^- (respectively, ω_x^+) is the starting (respectively, ending) point of the excursion straddling $x \in [0,\infty)$, and T_x is the area under the curve $|W_y|$ between ω_0^- and ω_x^+.

1.1. *The local time process and hitting times.* Our first result is a limit theorem for the local time process of the BTSAW X_i, stopped at appropriately defined stopping times. Let $k \in \mathbb{Z} \cap (0,\infty)$ and $m \in \mathbb{N}$. Denote by $T^{>}_{k,m}$ (respectively, $T^{<}_{k,m}$) the time of the $(m+1)$th arrival to the lattice site k from $k-1$ (respectively, from $k+1$):

$$(1.10) \qquad T^{>}_{k,-1} = 0 = T^{<}_{k,-1},$$

$$(1.11) \qquad T^{>}_{k,m+1} = \min\{i > T^{>}_{k,m}\colon\ X_{i-1} = k-1,\ X_i = k\},$$

$$(1.12) \qquad T^{<}_{k,m+1} = \min\{i > T^{<}_{k,m}\colon\ X_{i-1} = k+1,\ X_i = k\}.$$

Most of the forthcoming formulas and results are identically valid for both superscripts $>$ and $<$. In these cases the superscript $*$ stands for either $<$ or $>$.

We consider the following (bond) local time processes of the BTSAW, stopped at $T^*_{k,m}$:

$$(1.13) \qquad S^*_{k,m}(l) = l\big(k - l \,|\, \underline{X}_0^{T^*_{k,m}}\big).$$

Note that $S^*_{k,m}(l)$ is roughly half of the total number of jumps across the bond $\langle k-l-1, k-l\rangle$, performed by the walk stopped at $T^*_{k,m}$:

$$(1.14) \qquad v\Big(k - l \,|\, \underline{X}_0^{T^*_{k,m}}\Big) = \begin{cases} 2S^*_{k,m}(l) + 1, & \text{for } 0 \le l < k, \\ 2S^*_{k,m}(l), & \text{for } l < 0 \text{ or } l \ge k. \end{cases}$$

Denote

$$(1.15) \qquad \omega^{*-}_{k,m} = \omega^-\big(S^*_{k,m}\big) = \max\big\{l < 0\colon\ S^*_{k,m}(l) = 0\big\},$$

$$(1.16) \qquad \omega^{*+}_{k,m} = \omega^+\big(S^*_{k,m}\big) = \min\big\{l \ge k\colon\ S^*_{k,m}(l) = 0\big\}.$$

In plain words, $k - \omega^{*+}_{k,m}$ (respectively, $k - \omega^{*-}_{k,m} - 1$) is the leftmost (respectively, rightmost) site visited by the stopped walk $\underline{X}_0^{T^*_{k,m}}$.

From (1.14) it clearly follows that

$$(1.17) \qquad T^*_{k,m} = 2 \sum_{l=\omega^{*-}_{k,m}}^{\omega^{*+}_{k,m}} S^*_{k,m}(l) + k.$$

Looking at the formal definitions only, in principle, these local times or hitting times might be infinite. That is, it could happen that the site $k \in \mathbb{Z}$ is never hit. The following proposition ensures that, with probability 1, this is *not* the case.

PROPOSITION 1. *With probability* 1, *for any* $k \in \mathbb{Z}$, $m \in \mathbb{N}$, $l \in \mathbb{Z}$ *and* $*$ *standing for* $>$ *or* $<$,

$$S^*_{k,m}(l) < \infty, \tag{1.18}$$

$$\omega^{*+}_{k,m} < \infty, \tag{1.19}$$

$$\omega^{*-}_{k,m} < \infty, \tag{1.20}$$

$$T^*_{k,m} < \infty. \tag{1.21}$$

Relation (1.21) directly follows from (1.17)–(1.20), (1.18) will be proved at the end of Section 2, and (1.19) and (1.20) in Proposition 2 of Section 4.

The following theorem describes the precise asymptotics of the local time *process* $S^*_{k,m}(\cdot)$, properly scaled:

THEOREM 1. *Let* $x \in [0,\infty)$, *let* $h \geq 0$ *and let* $*$ *stand for* $>$ *or* $<$. *Then*

$$\begin{aligned}&\left(\frac{\omega^{*-}_{[Ax],[\sqrt{A}\sigma h]}}{A}, \frac{\omega^{*+}_{[Ax],[\sqrt{A}\sigma h]}}{A}, \frac{S^*_{[Ax],[\sqrt{A}\sigma h]}([Ay])}{\sigma\sqrt{A}}: \right.\\ &\qquad\left. \frac{\omega^{*-}_{[Ax],[\sqrt{A}\sigma h]}}{A} \leq y \leq \frac{\omega^{*+}_{[Ax],[\sqrt{A}\sigma h]}}{A}\right)\\ &\qquad\Rightarrow \left(\omega^-_0, \omega^+_x, |W_y| \colon \omega^-_0 \leq y \leq \omega^+_x \,\middle|\, |W_0| = h\right)\end{aligned} \tag{1.22}$$

in $\mathbb{R}_- \times \mathbb{R}_+ \times D(-\infty,\infty)$ *as* $A \to \infty$, *where*

$$\sigma^2 = \frac{\sum_{z\in\mathbb{Z}} z^2 \lambda^{z^2}}{\sum_{z\in\mathbb{Z}} \lambda^{z^2}}. \tag{1.23}$$

[For the notion of convergence in distribution in the function space involved see Billingsley (1968) and Lindvall (1973).]

REMARK. It is instructive to compare this behavior with that of a simple symmetric random walk on $\mathbb{Z}$. The analogous result for the simple symmetric random walk is formulated in the Ray–Knight theory of local times. According to these classical results, in the ordinary random walk case, the proper scaling is

$$\frac{S^*_{[Ax],[Ah]}([Ay])}{A} \tag{1.24}$$

and the limiting process is a squared Bessel process. For a thorough analysis of the limiting process, see, for example, Revuz and Yor [(1991), Chapter XI].

The scaling and weak convergence in this case follows from diffusion approximation of Galton–Watson processes; see Kawazu and Watanabe (1971), Kurtz (1978) and references cited in these works.

A straightforward consequence of Proposition 1 is the recurrence of the random walk considered:

COROLLARY 1. *The BTSAW visits infinitely often every lattice site, almost surely.*

From (1.17) and (1.22) directly follows the limit theorem for the hitting times:

COROLLARY 2. *For $x \in [0, \infty)$, $h \geq 0$ and $*$ standing for $>$ or $<$,*

$$\frac{T^*_{[Ax],[\sqrt{A}\sigma h]}}{2\sigma A^{3/2}} \Rightarrow (T_x \mid |W_0| = h). \tag{1.25}$$

This last limit law shows that the BTSAW behaves *overdiffusively* indeed, the suggested rate of diffusion being $X_t \sim t^{2/3}$.

1.2. *A by-product: an identity concerning Brownian excursions and Bessel bridges.* Our second result can be formulated in terms of Brownian motion, without any reference to the BTSAW. It is an apparently new identity concerning Brownian excursions and Bessel bridges.

For any initial condition $|W_0| = h$, T_x defined in (1.9) clearly has an absolutely continuous distribution. Let

$$\varrho(t; x, h) = \frac{\mathbf{P}\big(T_x \in (t, t+dt) \mid |W_0| = h\big)}{dt} \tag{1.26}$$

be the density of the distribution of T_x. From scaling the Brownian motion, we easily get

$$\alpha\varrho(\alpha t; \alpha^{2/3}x, \alpha^{1/3}h) = \varrho(t; x, h) \tag{1.27}$$

for any $\alpha > 0$. Define $\mathbb{R}_+ \times \mathbb{R} \ni (t, x) \mapsto \varphi(t, x) \in \mathbb{R}_+$ as follows:

$$\varphi(t, x) = \int_0^\infty \varrho\left(\frac{t}{2}; |x|, h\right) \mathrm{d}h. \tag{1.28}$$

The finiteness of the integral on the right-hand side will be seen soon. The scaling property (1.27) of ϱ implies

$$\alpha^{2/3}\varphi(\alpha t, \alpha^{2/3}x) = \varphi(t, x). \tag{1.29}$$

We shall denote by $\hat{\varrho}$ and $\hat{\varphi}$ the Laplace transforms of ϱ and φ:

$$\hat{\varrho}(s; x, h) = s\int_0^\infty \exp(-st)\varrho(t; x, h)\, dt = s\,\mathbf{E}\big(\exp\{-sT_x\} \mid |W_0| = h\big), \tag{1.30}$$

$$\hat{\varphi}(s, x) = s\int_0^\infty \exp(-st)\varphi(t, x)\, dt = \int_0^\infty \hat{\varrho}(2s; |x|, h)\, dh. \tag{1.31}$$

These functions scale as follows:

$$\alpha\hat{\varrho}(\alpha^{-1}s; \alpha^{2/3}x, \alpha^{1/3}h) = \hat{\varrho}(s; x, h), \tag{1.32}$$

$$\alpha^{2/3}\hat{\varphi}(\alpha^{-1}s, \alpha^{2/3}x) = \hat{\varphi}(s, x). \tag{1.33}$$

THEOREM 2. *Given* $t \in (0, \infty)$ [*respectively,* $s \in (0, \infty)$] *fixed,* $x \mapsto \varphi(t, x)$ [*respectively,* $x \mapsto \hat{\varphi}(s, x)$] *is a probability density. That is, for any* $t \in (0, \infty)$ [*respectively,* $s \in (0, \infty)$],

$$\int_{-\infty}^{\infty} \varphi(t, x)\, dx = 1, \tag{1.34}$$

$$\int_{-\infty}^{\infty} \hat{\varphi}(s, x)\, dx = 1. \tag{1.35}$$

REMARK. Equations (1.34) and (1.35) are, of course, equivalent statements: $\hat{\varphi}(s, \cdot)$ is the distribution $\varphi(t, \cdot)$ observed at a "random time" of exponential distribution with mean value s^{-1}. Furthermore, given the scaling property (1.33) of $\hat{\varphi}$ it is enough to prove (1.35) for one particular value of s, say $s = 1$. The statement of this theorem is quite surprising, since we could not find any direct intuitive way of proving it. The proof relies partly on some probabilistic results concerning Brownian excursion theory and Bessel bridges and, on the other hand, on rather messy integrations involving Bessel and hypergeometric functions.

1.3. *Local limit theorem for the position process.* We are ready now to formulate our main result. We denote by $P(n, k)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, the distribution of our BTSAW at time n,

$$P(n, k) = \mathbf{P}(X_n = k), \tag{1.36}$$

and by $R(s, k)$, $s \in \mathbb{R}_+$, $k \in \mathbb{Z}$, the distribution of the BTSAW observed at an independent random time of geometric distribution with mean $e^{-s}(1 - e^{-s})^{-1}$,

$$R(s, k) = (1 - e^{-s}) \sum_{n=0}^{\infty} e^{-sn} P(n, k). \tag{1.37}$$

We define the rescaled "densities" of the above distributions

$$\varphi_A(t, x) = A^{2/3} P([At], [A^{2/3}x]), \tag{1.38}$$

$$\hat{\varphi}_A(s, x) = A^{2/3} R(A^{-1}s, [A^{2/3}x]), \tag{1.39}$$

$t, s \in \mathbb{R}_+$, $x \in \mathbb{R}$. It is straightforward that $\hat{\varphi}_A$ is exactly the Laplace transform of φ_A:

$$\hat{\varphi}_A(s, x) = s \int_0^{\infty} e^{-st} \varphi_A(t, x)\, dt. \tag{1.40}$$

Our main result is the following theorem.

THEOREM 3. *For any $s \in \mathbb{R}_+$ and almost all $x \in \mathbb{R}$,*

$$\hat{\varphi}_A(s,x) \to \sigma^{2/3}\hat{\varphi}(s,\sigma^{2/3}x) \tag{1.41}$$

as $A \to \infty$. The positive constant σ is explicitly given in (1.23).

REMARK. This is of course a *local limit theorem* for the BTSAW observed at an independent random time $\theta_{s/A}$ of geometric distribution with mean $e^{-s/A}(1-e^{-s/A})^{-1}$. In particular, the (integral) limit law

$$\mathbf{P}\big(A^{-2/3}X_{\theta_{s/A}} < x\big) \to \int_{-\infty}^{\sigma^{2/3}x} \hat{\varphi}(s,y)\,dy \tag{1.42}$$

follows. This is a little bit short of stating the limit theorem for deterministic time

$$\mathbf{P}\left(A^{-2/3}X_{[At]} < x\right) \to \int_{-\infty}^{\sigma^{2/3}x} \varphi(t,y)\,dy, \tag{1.43}$$

but, of course, we can conclude that if $X_{[At]}$ has any scaling limit, then (1.43) also holds.

The paper contains five more sections. In Section 2 we describe the local time process $S^*_{k,m}(\cdot)$ as a random walk on $\mathbb{Z}_+$. In Section 3 we investigate in more detail an auxiliary Markov chain arising naturally in the previous description. Finally in Sections 4, 5 and 6 we prove, in turn, Theorems 1, 2 and 3 stated above.

2. The local time process as a Markov chain. For sake of definiteness we consider the case of superscript $>$; that is, we stop the BTSAW at the hitting time $T^{>}_{k,m}$. The case of superscript $<$ is done in an identical way, with slight changes in the definition of "system of spanning steps," (2.2) below.

The clue to the proof of Theorem 1 is the observation that, with $k > 0$ and $m \in \mathbb{N}$ fixed, the local time process $S^{>}_{k,m}(\cdot)$ defined in (1.13) is a Markov chain on the state space $\mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$. Apparently this trick has its origin in Knight (1963) and has been rediscovered several times since then [see, e.g., Kesten, Kozlov and Spitzer (1975)]. However, as opposed to the previous applications of this trick, the Markov process arising in our case will be more complicated than a branching process [Knight (1963)] or a branching process with random offspring distribution [Kesten, Kozlov and Spitzer (1975)].

A finite walk $\underline{x}_0^i$ which hits k for the $(m+1)$th time, coming from the left, at time i,

$$\begin{gathered} 0 = x_0, x_1, \ldots, x_{i-1}, x_i = k, \\ \#\{0 < j \le i\colon\ x_{j-1} = k-1, x_j = k\} = m+1 \end{gathered} \tag{2.1}$$

determines uniquely the finite sequence

$$\Sigma(0) = m,$$

$$\underline{\sigma}(l) = (\sigma_0(l), \dots, \sigma_{\Sigma(l-1)}(l)), \qquad \Sigma(l) = \sum_{r=0}^{\Sigma(l-1)} \sigma_r(l), \qquad \text{for } l = 1, 2, \dots, k,$$

$$\underline{\sigma}(l) = (\sigma_1(l), \dots, \sigma_{\Sigma(l-1)}(l)), \qquad \Sigma(l) = \sum_{r=1}^{\Sigma(l-1)} \sigma_r(l), \qquad \text{for } l = k+1, k+2, \dots, \tag{2.2}$$

$$\underline{\sigma}(l) = (\sigma_1(l), \dots, \sigma_{\Sigma(l+1)}(l)), \qquad \Sigma(l) = \sum_{r=1}^{\Sigma(l+1)} \sigma_r(l), \qquad \text{for } l = -1, -2, \dots,$$

where, for $l > 0$, $\sigma_r(l)$ is the number of steps $(k-l) \to (k-l-1)$ between the rth and $(r+1)$th steps $(k-l+1) \to (k-l)$ and, for $l < 0$, $\sigma_r(l)$ is the number of steps $(k-l-1) \to (k-l)$ between the rth and $(r+1)$th steps $(k-l-2) \to (k-l-1)$, performed by the walk. We shall refer to the sequence (2.2) as the system of spanning steps of the walk (2.1). Finiteness of the system of spanning steps means that there exist $l_{\max} > k$ and $l_{\min} < 0$ such that $\Sigma(l_{\max}) = 0 = \Sigma(l_{\min})$. This correspondence between finite walks hitting k and finite systems of spanning steps is *one-to-one*: given the sequence (2.2), one can reconstruct the complete walk (2.1) uniquely. Clearly, the system of spanning steps is defined so that, for $\underline{X}_0^{T^{>}_{k,m}} = \underline{x}_0^i$,

$$S^{>}_{k,m}(l) = \Sigma(l), \qquad l \in \mathbb{Z}. \tag{2.3}$$

Let the finite walk (2.1) be given, and let (2.2) be the corresponding system of spanning steps. Since $T^{>}_{k,m}$ is a stopping time, the probability that the BTSAW coincides with (2.1) until $T^{>}_{k,m}$ is

$$\mathbf{P}\Big(\underline{X}_0^{T^{>}_{k,m}} = \underline{x}_0^i\Big) = \prod_{j=1}^{i} w_j\big(\underline{x}_0^j\big), \tag{2.4}$$

where, according to (1.4) and (1.5), the weight of the jth step is

$$w_j\big(\underline{x}_0^j\big) = \frac{\lambda^{v((x_{j-1}+x_j+1)/2 | \underline{x}_0^{j-1})}}{\lambda^{v(x_{j-1} | \underline{x}_0^{j-1})} + \lambda^{v(x_{j-1}+1 | \underline{x}_0^{j-1})}}. \tag{2.5}$$

Now, rearranging the product in (2.4) we get

$$\mathbf{P}\Big(\underline{X}_0^{T^{>}_{k,m}} = \underline{x}_0^i\Big) = \prod_{l \in \mathbb{Z}} \bigg\{ \prod_{1 \le j \le i} \big(\mathbb{1}\{x_{j-1} = k - l\} w_j\big(\underline{x}_0^j\big) + \mathbb{1}\{x_{j-1} \ne k - l\}\big) \bigg\}. \tag{2.6}$$

This last rearranged product can be written in terms of the system of spanning steps $\underline{\sigma}(l)$ as

$$\tag{2.7} \begin{aligned} \mathbf{P}\left(\underline{X}_0^{T^>_{k,m}} = \underline{x}_0^i\right) &= \left[\prod_{l<0} \mathscr{R}(\Sigma(l+1); \underline{\sigma}(l))\right]\left[\prod_{l=1}^{k-1} \mathscr{P}(\Sigma(l-1); \underline{\sigma}(l))\right] \\ &\quad \times \mathscr{Q}(\Sigma(k-1); \underline{\sigma}(k))\left[\prod_{l>k} \mathscr{R}(\Sigma(l-1); \underline{\sigma}(l))\right] \end{aligned}$$

with transition probabilities

$$\tag{2.8} \mathscr{P}(\Sigma; \underline{\sigma}') = \prod_{r=0}^{\Sigma}\left[\prod_{i=1}^{\sigma'_r} \frac{\lambda^{2(\sum_{s=0}^{r-1}\sigma'_s + i - 1)+1}}{\lambda^{2r} + \lambda^{2(\sum_{s=0}^{r-1}\sigma'_s + i-1)+1}}\right] \frac{\lambda^{2r}}{\lambda^{2r} + \lambda^{2\sum_{s=0}^{r}\sigma'_s + 1}},$$

$$\tag{2.9} \mathscr{Q}(\Sigma; \underline{\sigma}') = \prod_{r=0}^{\Sigma}\left[\prod_{i=1}^{\sigma'_r} \frac{\lambda^{2(\sum_{s=0}^{r-1}\sigma'_s + i - 1)}}{\lambda^{2r} + \lambda^{2(\sum_{s=0}^{r-1}\sigma'_s + i-1)}}\right] \frac{\lambda^{2r}}{\lambda^{2r} + \lambda^{2\sum_{s=0}^{r}\sigma'_s}},$$

$$\tag{2.10} \mathscr{R}(\Sigma; \underline{\sigma}') = \prod_{r=1}^{\Sigma}\left[\prod_{i=1}^{\sigma'_r} \frac{\lambda^{2(\sum_{s=1}^{r-1}\sigma'_s + i - 1)}}{\lambda^{2r-1} + \lambda^{2(\sum_{s=1}^{r-1}\sigma'_s + i-1)}}\right] \frac{\lambda^{2r-1}}{\lambda^{2r-1} + \lambda^{2\sum_{s=1}^{r}\sigma'_s}}.$$

From (2.3) and (2.7) we see that $S^>_{k,m}(l)$ and $S^>_{k,m}(-l)$, $l = 0, 1, \dots$, are indeed two independent Markov chains on the state space $\mathbb{Z}_+$, with initial condition $S^>_{k,m}(0) = m$. The chains are homogeneous in the intervals $l \le 0$, $0 \le l < k$, $l = k$ and $l > k$.

REMARK. In the case of STSAW this rearrangement of the product, that is, the transcription of (2.4) to (2.7), cannot be performed. This is the step where the proof of a similar result for the STSAW fails.

The transition probabilities $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{R}$ look quite threatening at first sight, but we shall see soon that they have a transparent interpretation and can be tamed. In order to see this, we define two auxiliary Markov chains on $\mathbb{Z}$ which will help us to understand the random walk $S^>_{k,m}(\cdot)$ better.

For $z \in \mathbb{Z}$ let

$$\tag{2.11} p(z) = \frac{\lambda^{2z+1}}{1+\lambda^{2z+1}}, \qquad q(z) = \frac{1}{1+\lambda^{2z+1}},$$

$$\tag{2.12} \tilde{p}(z) = \frac{\lambda^{2z}}{1+\lambda^{2z}}, \qquad \tilde{q}(z) = \frac{1}{1+\lambda^{2z}}$$

and, for $x, y \in \mathbb{Z}$,

$$\tag{2.13} P(x,y) = \begin{cases} \prod_{z=x}^{y} p(z) q(y+1), & \text{if } x-1 \le y, \\ 0, & \text{if } x-1 > y, \end{cases}$$

$$\tilde{P}(x,y) = \begin{cases} \prod_{z=x}^{y} \tilde{p}(z)\tilde{q}(y+1), & \text{if } x-1 \le y, \\ 0, & \text{if } x-1 > y. \end{cases} \tag{2.14}$$

(When $x = y + 1$, the empty product is by definition equal to 1.) From $\prod_{z=x}^{\infty} p(z) = 0 = \prod_{z=x}^{\infty} \tilde{p}(z)$ it follows that P and $\tilde{P}$ are the transition matrices of two Markov chains η_r and $\tilde{\eta}_r$, $r = 0, 1, 2, \dots$, on the state space $\mathbb{Z}$. On the right-hand side of (2.8) [respectively, (2.10)] we have exactly the probability distribution of the first $\Sigma + 1$ steps (respectively, the first Σ steps) of the Markov chain $\eta_{\cdot}$ starting from 0 (respectively, starting from -1). On the right-hand side of (2.9) we have the probability distribution of the first $\Sigma + 1$ steps of the Markov chain $\tilde{\eta}_{\cdot}$ starting from 0. More precisely,

$$\mathscr{P}(\Sigma; \underline{\sigma}') = \mathbf{P}(\eta_{r+1} - \eta_r = \sigma'_r - 1,\ r = 0, \dots, \Sigma \mid \eta_0 = 0), \tag{2.15}$$

$$\mathscr{Q}(\Sigma; \underline{\sigma}') = \mathbf{P}(\tilde{\eta}_{r+1} - \tilde{\eta}_r = \sigma'_r - 1,\ r = 0, \dots, \Sigma \mid \tilde{\eta}_0 = 0), \tag{2.16}$$

$$\mathscr{R}(\Sigma; \underline{\sigma}') = \mathbf{P}(\eta_r - \eta_{r-1} = \sigma'_r - 1,\ r = 1, \dots, \Sigma \mid \eta_0 = -1). \tag{2.17}$$

Thus, denoting by $\xi^{>}_{k,m}(l)$ the lth step of the random walk $S^{>}_{k,m}(\cdot)$, that is,

$$\xi^{>}_{k,m}(l) = S^{>}_{k,m}(l) - S^{>}_{k,m}(l-1), \qquad l > 0, \tag{2.18}$$

$$\xi^{>}_{k,m}(l) = S^{>}_{k,m}(l) - S^{>}_{k,m}(l+1), \qquad l < 0, \tag{2.19}$$

we get

$$\mathbf{P}\left(\xi^{>}_{k,m}(l) = x \,\|\, S^{>}_{k,m}(l-1) = n\right) = P^{n+1}(0, x-1) \qquad \text{for } l = 1, 2, \dots, k-1, \tag{2.20}$$

$$\mathbf{P}\left(\xi^{>}_{k,m}(l) = x \,\|\, S^{>}_{k,m}(l-1) = n\right) = \tilde{P}^{n+1}(0, x-1) \quad \text{for } l = k, \tag{2.21}$$

$$\mathbf{P}\left(\xi^{>}_{k,m}(l) = x \,\|\, S^{>}_{k,m}(l-1) = n\right) = P^{n}(-1, x-1) \qquad \text{for } l = k+1, k+2, \dots, \tag{2.22}$$

$$\mathbf{P}\left(\xi^{>}_{k,m}(l) = x \,\|\, S^{>}_{k,m}(l+1) = n\right) = P^{n}(-1, x-1) \qquad \text{for } l = -1, -2, \dots. \tag{2.23}$$

PROOF OF (1.18) IN PROPOSITION 1. As $S^{>}_{k,m}(0) = m < \infty$ and (2.20)–(2.23) are bona fide transition probabilities, that is,

$$\mathbf{P}\left(\xi^{>}_{k,m}(l) \in [-n, \infty) \| S^{>}_{k,m}(l-1) = n < \infty\right) = 1 \quad \text{for } l > 0, \tag{2.24}$$

$$\mathbf{P}\left(\xi^{>}_{k,m}(l) \in [-n, \infty) \| S^{>}_{k,m}(l+1) = n < \infty\right) = 1 \quad \text{for } l < 0, \tag{2.25}$$

the local times $S^{>}_{k,m}(l) \in \mathbb{Z}_+$, $l \in \mathbb{Z}$, are almost surely finite. Relations (1.19) and (1.20) will be proved in Proposition 2 in Section 4. □

3. The auxiliary Markov chains: technical lemmas. In this section we summarize the properties of the Markov chains η_i and $\tilde{\eta}_i$ needed for the proof of Theorem 1.

LEMMA 1. *The unique stationary distribution of the Markov chain* η, *respectively*, $\tilde{\eta}$, *is*

$$\rho(x) = \frac{\lambda^{(x+1)^2}}{\sum_{z\in\mathbb{Z}} \lambda^{(z+1)^2}}, \qquad x \in \mathbb{Z}, \tag{3.1}$$

respectively,

$$\tilde{\rho}(x) = \frac{\lambda^{(x+1/2)^2}}{\sum_{z\in\mathbb{Z}} \lambda^{(z+1/2)^2}}, \qquad x \in \mathbb{Z}. \tag{3.2}$$

There exist constants $C_1 < \infty$ *and* $C_2 > 0$ *such that the following exponential bounds hold*:

$$\sum_{y\in\mathbb{Z}} |P^n(0,y) - \rho(y)| < C_1 \exp(-C_2 n), \tag{3.3}$$

$$\sum_{y\in\mathbb{Z}} |P^n(-1,y) - \rho(y)| < C_1 \exp(-C_2 n), \tag{3.4}$$

$$\sum_{y\in\mathbb{Z}} |\tilde{P}^n(0,y) - \tilde{\rho}(y)| < C_1 \exp(-C_2 n). \tag{3.5}$$

REMARKS. Denote by π the following probability distribution on $\mathbb{Z}$:

$$\pi(x) = \rho(x-1) = \frac{\lambda^{x^2}}{\sum_{z\in\mathbb{Z}} \lambda^{z^2}}, \qquad x \in \mathbb{Z}. \tag{3.6}$$

According to (2.20), (2.22), (2.23), (3.3) and (3.4), the distribution of the steps $\xi^{>}_{k,m}(l)$, $l \neq k$, of the process $S^{>}_{k,m}(l)$ will be *typically very close* (in variation distance) to the distribution π. This observation is the clue to the coupling argument used in the next section. On the other hand, (3.5) provides a uniform stochastic bound on the size of the single exceptional step $\xi^{>}_{k,m}(k)$, ensuring that this single step will have no effect whatsoever on the limiting process.

PROOF OF LEMMA 1. We prove (3.1), (3.3) and (3.4). An identical proof works for (3.2) and (3.5), too. The identity

$$P(x,y) = \begin{cases} \dfrac{1}{\rho(x)}\left[p(x) \displaystyle\prod_{z=x+1}^{y+1} q(z)\right]\rho(y), & \text{for } x-1 \le y, \\ 0, & \text{for } x-1 > y, \end{cases} \tag{3.7}$$

is straightforward. Hence,

$$\sum_{x\in\mathbb{Z}} \rho(x)P(x,y) = \left[\sum_{x\le y+1} p(x) \prod_{z=x+1}^{y+1} q(z)\right]\rho(y) = \rho(y), \tag{3.8}$$

which proves that ρ is indeed a stationary distribution of the chain η_r, $r = 0, 1, 2 \dots$.

The proof of uniqueness and exponential convergence (3.3) and (3.4) is a bit lengthy, but consists of standard procedures. First of all notice that, due to stationarity of the distribution ρ, the inequality

$$P^n(x, y) \le \frac{\rho(y)}{\rho(x)} \tag{3.9}$$

holds, which proves a superexponential bound, uniform in n, on the rate of decay of the tails of the distributions $P^n(x, \cdot)$. In consequence, all the expectations below make sense.

Denote by $\theta_\pm$ and σ the following stopping times:

$$\theta_+ = \min\{n \ge 0\colon\ \eta_n \ge 0\}, \tag{3.10}$$

$$\theta_- = \min\{n \ge 0\colon\ \eta_n \le 0\}, \tag{3.11}$$

$$\sigma \ = \min\{n \ge 1\colon\ \eta_n = 0\}. \tag{3.12}$$

According to Theorem 6.14 and Example 5.5(a) of Nummelin (1984), the uniqueness of the stationary distribution and the exponential convergence (3.3) and (3.4) follow from

$$\mathbf{E}\big(\exp(C_2\sigma) \| \eta_0 = 0\big) < \infty, \tag{3.13}$$

with $C_2 > 0$. So our goal is to prove (3.13). In the following expression we use the fact that the Markov chain $\eta_\cdot$ does not jump more than one lattice step to the left:

$$\begin{aligned}
&\mathbf{E}\big(\exp(C_2\sigma) | \eta_0 = 0\big)\\
&\quad = \exp(C_2) \sum_{y \ge 0} P(0, y)\, \mathbf{E}\big(\exp(C_2\theta_-) | \eta_0 = y\big)\\
&\qquad + \exp(C_2) P(0, -1) \sum_{y \ge 0} \mathbf{E}\big(\exp(C_2\theta_+)\, \mathbb{1}(\eta_{\theta_+} = y) | \eta_0 = -1\big)\\
&\qquad\qquad \times \mathbf{E}\big(\exp(C_2\theta_-) | \eta_0 = y\big).
\end{aligned} \tag{3.14}$$

Next we use a special consequence of the structure (2.13) of the transition kernel P. Namely, it is an easy computation to check that, given $\eta_0 = x < 0$, the random variables θ_+ and η_{θ_+} are independent and, for $y \ge 0$,

$$\begin{aligned}
&\mathbf{E}\big(\exp(C_2\theta_+)\, \mathbb{1}(\eta_{\theta_+} = y) | \eta_0 = -1\big)\\
&\quad = \frac{P(0, y)}{1 - P(0, -1)}\, \mathbf{E}\big(\exp(C_2\theta_+) | \eta_0 = -1\big).
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15),

$$\begin{aligned}(3.16)\qquad &\mathbf{E}\big(\exp(C_2\sigma)\mid \eta_0=0\big)\\ &\quad=\exp(C_2)\sum_{y\ge 0}P(0,y)\,\mathbf{E}\big(\exp(C_2\theta_-)|\eta_0=y\big)\\ &\qquad\times\left[1+\frac{P(0,-1)}{1-P(0,-1)}\mathbf{E}\big(\exp(C_2\theta_+)|\eta_0=-1\big)\right].\end{aligned}$$

From a dominated convergence argument, we can see that

$$(3.17)\qquad P(-\infty,y)=\lim_{x\to-\infty}P(x,y)=\prod_{z=-\infty}^{y+1}p(z)q(y+1)$$

is a probability distribution on $\mathbb{Z}$. Let $x_1\le x_2$. Then, for $y\ge x_2-1$,

$$(3.18)\qquad P(x_1,y)=\left[\sum_{z\ge x_2-1}P(x_1,z)\right]P(x_2,y),$$

which implies that the distributions $P(x,\cdot)$ are stochastically ordered: for $x_1 < x_2$ and arbitrary y_0,

$$(3.19)\qquad \sum_{y\ge y_0}P(-\infty,y)<\sum_{y\ge y_0}P(x_1,y)<\sum_{y\ge y_0}P(x_2,y).$$

From (3.19) it follows that

$$(3.20)\qquad \mathbf{P}(\theta_+>n\mid \eta_0=-1)\le r^n,\quad \text{where } r=\sum_{y<0}P(-\infty,y)<1.$$

Hence

$$(3.21)\qquad \mathbf{E}\big(\exp(C_3\theta_+)\mid\eta_0=-1\big)<\infty$$

for $C_3 < -\log r$.

On the other hand, since $p(z) > p(z+1)$ [see (2.11)], the distributions $P(x,x+\cdot)$ are stochastically ordered in the opposite sense: for $x_1 < x_2$ and arbitrary $y_0\ge 0$,

$$(3.22)\quad \sum_{y\ge y_0}P(x_1,x_1+y)=\prod_{z=0}^{y_0}p(x_1+z)>\prod_{z=0}^{y_0}p(x_2+z)=\sum_{y\ge y_0}P(x_2,x_2+y).$$

Hence, for $y\ge 0$,

$$(3.23)\qquad \mathbf{P}\Big(\min_{0\le j\le n}\eta_j\le 0\mid \eta_0=y\Big)\ge\mathbf{P}\Big(\min_{0\le j\le n}\sum_{i=1}^{j}\zeta_i\le -y\Big),$$

where the ζ_i's are i.i.d. random variables with distribution

$$(3.24)\qquad \mathbf{P}(\zeta_i=x)=P(0,x).$$

Now, from (3.9) it easily follows that

$$(3.25)\qquad \forall\,\alpha\in\mathbb{R},\qquad \sum_{x\in\mathbb{Z}}P(0,x)e^{\alpha x}<\infty.$$

The map $\lambda \mapsto P_\lambda(0,\cdot)$ is stochastically monotonic, too, in the sense that $0 \le \lambda_1 < \lambda_2 \le 1$ implies

$$\forall\, y \ge 0, \qquad \sum_{x\ge y} P_{\lambda_1}(0,x) = \prod_{z=0}^{y} \frac{\lambda_1^{2z+1}}{1+\lambda_1^{2z+1}} < \prod_{z=0}^{y} \frac{\lambda_2^{2z+1}}{1+\lambda_2^{2z+1}} = \sum_{x\ge y} P_{\lambda_2}(0,x). \tag{3.26}$$

Hence,

$$\forall\, \lambda \in (0,1), \qquad \sum_{x=-1}^{\infty} P(0,x)x < \sum_{x=-1}^{\infty} \frac{1}{2^{x+2}} x = 0. \tag{3.27}$$

From (3.23)–(3.25) and (3.27), via a supermartingale argument, it follows that if $y \ge 0$,

$$\mathbf{E}\big(\exp(C_4\theta_-)|\eta_0 = y\big) \le \exp(\beta y) \tag{3.28}$$

with some $C_4 > 0$ and $\beta < \infty$.

Now (3.13), with $C_2 = \min\{C_3, C_4\}$, follows from (3.16), (3.21), (3.25) and (3.28). □

The next lemma establishes a superexponential bound on the rate of decay of the right tails of the distributions $P^n(0,\cdot)$ and $P^n(-1,\cdot)$, uniform in n. This bound is much stronger than what we actually need in the proof of Theorem 1.

LEMMA 2. *There exists a constant* $C_5 < \infty$ *such that, for any* $n \ge 0$ *and* $x \ge 0$,

$$P^n(0,x+1) \le C_5 \lambda^x P^n(0,x), \tag{3.29}$$

$$P^n(-1,x+1) \le C_5 \lambda^x P^n(-1,x). \tag{3.30}$$

REMARK. Similar bounds can be established for the left tails of the distributions, too, but we do not need them in the proof of Theorem 1.

PROOF OF LEMMA 2. We prove the bound (3.29). The second one is proved in an identical way. We apply induction on n and x. Inequality (3.29) clearly holds for $n = 0$ and any $x \ge 0$. By (3.1) and (3.3) we have $\lim_{n\to\infty} P^n(0,1)/P^n(0,0) = \lambda^3 < \infty$ and hence

$$C_5 = \sup_{n\ge 0} \frac{P^n(0,1)}{P^n(0,0)} < \infty. \tag{3.31}$$

Thus, (3.29) holds for $\forall\, n \ge 0$ and $x = 0$. We proceed now by induction. Given an arbitrary probability distribution r on $\mathbb{Z}$, the following identity holds:

$$[rP](x+1) = \frac{p(x+1)}{q(x+1)} q(x+2)[rP](x) + q(x+2)r(x+2). \tag{3.32}$$

Assume that (3.29) holds for $(n, x+2)$ and for $(n+1, x)$. Using (3.32) we get

$$
\begin{aligned}
P^{n+1}(0,x+2) &= \frac{p(x+2)}{q(x+2)} q(x+3) P^{n+1}(0,x+1) \\
&\quad + q(x+3) P^{n}(0,x+3) \\
&\le \frac{p(x+2)}{q(x+2)} q(x+3) C_5 \lambda^{x} P^{n+1}(0,x) \\
&\quad + q(x+3) C_5 \lambda^{x+2} P^{n}(0,x+2) \\
&= C_5 \lambda^{x+1} \frac{\lambda q(x+3)}{q(x+2)} \left[\frac{p(x+1)}{q(x+1)} q(x+2) P^{n+1}(0,x) \right. \\
&\qquad\qquad \left. + q(x+2) P^{n}(0,x+2) \right] \\
&= C_5 \lambda^{x+1} \frac{\lambda q(x+3)}{q(x+2)} P^{n+1}(0,x+1).
\end{aligned} \tag{3.33}
$$

Since

$$
\frac{\lambda q(x+3)}{q(x+2)} = \frac{\lambda(1+\lambda^{2x+5})}{1+\lambda^{2x+7}} < 1, \tag{3.34}
$$

(3.33) yields (3.29) for $(n+1, x+1)$. □

4. Proof of Theorem 1. In the proofs we use only some of the qualitative features (formulated in the lemmas of the previous section) and not the explicit form of the transition probabilities of the random walk $S^{>}_{k,m}(\cdot)$. We are going to formulate and prove Theorem 4, below, in these more general terms. Theorem 1 will follow directly from Theorem 4, as a concrete application.

As in Section 1, $A > 0$ will denote the scaling parameter. For $A > 0$ let

$$
S_A(l) = S_A(0) + \sum_{j=1}^{l} \xi_A(j), \qquad l \in \mathbb{N}, \tag{4.1}
$$

be a space–time inhomogeneous random walk on $\mathbb{Z}_+$ with the following law:

$$
\mathbf{P}\big(\xi_A(l) = x \mid S_A(l-1) = n\big) = \pi_A(x|n,l), \tag{4.2}
$$

$$
\sum_{x \ge -n} \pi_A(x|n,l) = 1. \tag{4.3}
$$

REMARK. Compare (4.2) and (4.3) with (2.20)–(2.23). The time inhomogeneity and dependence on A of the law of the random walk is just a nuisance we have to live with. As all the estimates below will be uniform in l and A, this will cause no real trouble.

For $r \in \mathbb{R}_+$ we define the following stopping time of the random walk $S_A(\cdot)$:

$$\omega_{[Ar]} = \inf\{l \geq [Ar]: \ S_A(l) = 0\}. \tag{4.4}$$

Now we formulate some conditions on the behavior of the step distributions $\pi_A(\cdot|n,l)$. All the constants arising in various inequalities below are absolute constants not depending on A or l.

CONDITION 1 (Existence of an asymptotic step distribution). The step distributions $\pi_A(\cdot|n,l)$ converge in $\ell_1(\mathbb{Z})$, exponentially fast as $n \to \infty$, to an asymptotic distribution π. That is, there are two constants $C_1 < \infty$ and $C_2 > 0$ such that

$$\sum_{x \in \mathbb{Z}} |\pi_A(x|n,l) - \pi(x)| < C_1 \exp(-C_2 n). \tag{4.5}$$

The asymptotic distribution is symmetric,

$$\pi(-x) = \pi(x), \tag{4.6}$$

and its moments of any order are assumed finite. We denote by σ^2 its variance:

$$\sum_{x \in \mathbb{Z}} x^2 \pi(x) = \sigma^2 \in (0, \infty). \tag{4.7}$$

REMARK. Relation (4.5) should be compared with (3.3) and (3.4); (4.6) and the moment conditions with the explicit form (3.6). This condition will hold (uniformly in l and A) for all but one *exceptional* step. See Remark (i) following Theorem 4. The symmetry condition (4.6) is not really needed; $\sum_{x \in \mathbb{Z}} x\pi(x) = 0$ would be sufficient. We assume (4.6) only for shortening the argument [see the observation after (4.28) and the rightmost inequality in (A4.2)].

CONDITION 2 (Uniform decay of the tails of the step distributions). The tails of the distributions $\pi_A(\cdot|n,l)$ are uniformly exponentially bounded. That is, there are two constants $C_6 < \infty$ and $C_7 > 0$ such that

$$\pi_A(x|n,l) \leq C_6 \exp(-C_7|x|), \qquad l \in \mathbb{N}. \tag{4.8}$$

For technical purposes we formulate an even stronger condition on the decay of the right tail of the distributions $\pi_A(\cdot|n,l)$: there is an $x_0 \in \mathbb{Z}$ such that, for $x \geq x_0$,

$$\pi_A(x+1|n,l) < \exp(-C_7)\pi_A(x|n,l). \tag{4.9}$$

REMARK. Compare (4.8) with (3.9), and (4.9) with (3.29) and (3.30).

CONDITION 3 (Uniform nontrapping condition). The random walk is not trapped in a bounded domain or in a domain away from the origin. That is, there is a constant $C_8 > 0$ such that

$$\forall\, n \geq 0, \qquad \sum_{x>0} \pi_A(x|n,l) > C_8 > 0, \tag{4.10}$$

$$\forall\, n > 0, \qquad \sum_{x<0} \pi_A(x|n,l) > C_8 > 0. \tag{4.11}$$

REMARK. Strictly speaking, in our concrete case, (4.10) does not hold for $l > [Ar]$ and $n = 0$. However, as the walk is stopped at $\omega_{[Ar]}$, this fact does not make any difference.

Denote by θ_I the exit time from the interval $I \subset \mathbb{Z}_+$:

$$\theta_I = \min\{l \geq 0: \ S_A(l) \notin I\}. \tag{4.12}$$

LEMMA 3. *Let $S_A(\cdot)$ be a random walk on $\mathbb{Z}_+$ for which all conditions* (4.5)–(4.11) *hold.*

(i) *There exists a constant $C_9 < \infty$ such that, for any $0 < n < b$ in $\mathbb{Z}_+$,*

$$\mathbf{P}\big(S_A(\theta_{(0,b)}) = 0 \,|\, S_A(0) = n\big) > \frac{b-n}{b+C_9}. \tag{4.13}$$

(ii) *There exists a constant $C_{10} < \infty$ such that, for any $0 \leq n < b$ in $\mathbb{Z}_+$,*

$$\mathbf{E}\big(\theta_{[0,b)} \,|\, S_A(0) = n\big) < C_{10} b^3. \tag{4.14}$$

Since the proof of this lemma is a rather standard application of submartingale techniques and the optional sampling theorem [see Breiman (1968)] and has no relevance to the rest of the proofs, we postpone it to the appendix at the end of this section. We should remark here that in (4.14) a better bound $\sim b^2$ can be proved with some more work. However, this bound is sufficient for our purposes.

PROPOSITION 2. *Under the same conditions,*

$$\mathbf{P}(\omega_{[Ar]} < \infty) = 1. \tag{4.15}$$

PROOF. Given (4.13), this is evident. This statement also yields (1.19) and (1.20) and thus completes the proof of Proposition 1. □

The following theorem is the natural general formulation of Theorem 1.

THEOREM 4. *Let $S_A(\cdot)$, $A > 0$, be random walks on $\mathbb{Z}_+$. Assume that the following hold:*

(a) *Conditions* (4.5)–(4.11) *hold for all but one exceptional step, say, l_A.*
(b) *The tail estimate* (4.8) *holds for the exceptional step, too.*

If

$$\frac{S_A(0)}{\sigma\sqrt{A}} \to h, \tag{4.16}$$

then

$$\left(\frac{\omega_{[Ar]}}{A}, \frac{S_A([Ay])}{\sigma\sqrt{A}}: \ 0 \leq y \leq \frac{\omega_{[Ar]}}{A}\right) \Rightarrow \big(\omega_r^+, |W_y|: \ 0 \leq y \leq \omega_r^+ \,|\, |W_0| = h\big) \tag{4.17}$$

in $\mathbb{R}_+ \times D[0,\infty)$, *as* $A \to \infty$; σ^2 *is the variance of the asymptotic step distribution given in* (4.7).

REMARKS. (i) The existence of the exceptional step at time l_A is just another minor nuisance. We have to include it since in our concrete application the single step $\xi_{k,m}(k)$ has slightly different behavior than the rest. Condition (4.8) [which follows from (2.21), (3.2) and (3.9)] ensures that this single step will have no effect on the limiting procedure. Actually we could easily include much more, $o(A^{1/2-\varepsilon})$, exceptional steps.

(ii) The outline of the proof is quite simple. We consider the walk $S_A(l)$ in the (very long!) time interval $l \in [0, A^{1+\varepsilon})$. Due to the exponential convergence (4.5) of the step distributions, as long as $S_A(\cdot)$ stays above the threshold

$$b_A = [A^\varepsilon], \tag{4.18}$$

the trajectory will be that of a homogeneous random walk with step distribution π, with very high probability. On the other hand, with the help of (4.14), we prove that in the time interval considered (of length $A^{1+\varepsilon}$), the total amount of time spent by $S_A(\cdot)$ below the threshold b_A will be rather small, of $o(A^{1/2+5\varepsilon})$, with overwhelming probability. Joining these two arguments, we couple to our original random walk $S_A(\cdot)$ a reflected homogeneous random walk, $Y_A(\cdot)$, with constant step distribution π, so that the supremum distance of the two processes is $o(A^{1/4+3\varepsilon})$ and the difference of their $\omega_{[Ar]}$ stopping times is $o(A^{1/2+5\varepsilon})$, with probability converging to 1.

PROOF OF THEOREM 4. From (4.5) it follows that we can couple to our random walk $S_A(\cdot)$ a sequence of i.i.d. random variables $\zeta_A(l)$, $l = 1, 2, \dots$, with distribution π, so that

$$\mathbf{P}\big(\zeta_A(l+1) \neq \xi_A(l+1) \mid S_A(l) = n\big) < C_1 \exp(-C_2 n). \tag{4.19}$$

We denote by $\sigma_A(l)$ the amount of time spent above the threshold level b_A, before time l, less the exceptional moment:

$$\sigma_A(l) = \#\{0 \le j < l\colon\ [j \neq l_A] \wedge [S_A(j) > b_A]\}. \tag{4.20}$$

We define the following sequence of sampling times:

$$\begin{aligned} &\tau_A(-1) = -1,\\ &\tau_A(l) = \min\{j > \tau_A(l-1)\colon\ [j \neq l_A] \wedge [S_A(j) > b_A]\},\\ &\qquad\qquad 0 \le l < \infty. \end{aligned} \tag{4.21}$$

Note that τ_A is the inverse function of σ_A in the sense that

$$\sigma_A(\tau_A(l)) = l \quad \text{for } 0 \le l < \infty. \tag{4.22}$$

Let

$$\tilde{\xi}_A(l) = \xi_A(\tau_A(l-1)+1) \tag{4.23}$$

and

$$\tilde{\zeta}_A(l) = \zeta_A(\tau_A(l-1)+1). \tag{4.24}$$

As the $\tilde{\zeta}_A$'s are selected according to an increasing sequence of sampling times, they are still i.i.d. random variables with the common distribution π. On the other hand, (4.19) and (4.18) imply

$$\forall\, l \in \mathbb{N}, \qquad \mathbf{P}\big(\tilde{\xi}_A(l) \neq \tilde{\zeta}_A(l)\big) < C_1 \exp(-C_2 A^{\varepsilon}). \tag{4.25}$$

We define two "truncated" processes $\tilde{S}_A(\cdot)$ and $\tilde{Y}_A(\cdot)$:

$$\tilde{S}_A(0) = (S_A(0) - b_A) \vee 0, \qquad \tilde{S}_A(l) = \big(\tilde{S}_A(l-1) + \tilde{\xi}_A(l)\big) \vee 0, \tag{4.26}$$

$$\tilde{Y}_A(0) = (S_A(0) - b_A) \vee 0, \qquad \tilde{Y}_A(l) = \big(\tilde{Y}_A(l-1) + \tilde{\zeta}_A(l)\big) \vee 0. \tag{4.27}$$

Finally, let the random walk Y_A be defined as follows:

$$Y_A(0) = (S_A(0) - b_A) \vee 0, \qquad Y_A(l) = |Y_A(l-1) + \tilde{\zeta}_A(l)|. \tag{4.28}$$

Due to the symmetry of the distribution π, $Y_A(\cdot)$ is a homogeneous random walk reflected at the origin and $\tilde{Y}_A(\cdot)$ differs from $Y_A(\cdot)$ only by the cutoffs of the overshootings of the origin: using the constructions (4.27) and (4.28) one can easily check

$$0 \le Y_A(l) - \tilde{Y}_A(l) \le \max_{1 \le j \le l} |\tilde{\zeta}_A(j)|. \tag{4.29}$$

We also need the $\omega_{[Ar]}$ stopping times of the processes $\tilde{Y}_A(\cdot)$ and $Y_A(\cdot)$:

$$\omega'_{[Ar]} = \inf\{l \ge [Ar]\colon\ \tilde{Y}_A(l) = 0\}, \tag{4.30}$$

$$\omega''_{[Ar]} = \inf\{l \ge [Ar]\colon\ Y_A(l) = 0\}. \tag{4.31}$$

The standard invariance principle holds for $Y_A(\cdot)$ [see Billingsley (1968) and Lindvall (1973)], and, consequently,

$$\begin{aligned} &\left(\frac{\omega''_{[Ar]}}{A}, \frac{Y_A([Ay])}{\sigma\sqrt{A}}\colon\ 0 \le y < \infty\right) \\ &\qquad \Rightarrow (\omega_r^+, |W_y|\colon\ 0 \le y < \infty \mid |W_0| = h) \quad \text{in } \mathbb{R} \times D[0,\infty). \end{aligned} \tag{4.32}$$

Due to the closeness of the $\tilde{Y}_A(\cdot)$ and $Y_A(\cdot)$ paths, (4.29), the same convergence in distribution is easily proved for the process $\tilde{Y}_A(\cdot)$:

$$\begin{aligned} &\left(\frac{\omega'_{[Ar]}}{A}, \frac{\tilde{Y}_A([Ay])}{\sigma\sqrt{A}}\colon\ 0 \le y < \infty\right) \\ &\qquad \Rightarrow (\omega_r^+, |W_y|\colon\ 0 \le y < \infty \mid |W_0| = h) \quad \text{in } \mathbb{R} \times D[0,\infty). \end{aligned} \tag{4.33}$$

We omit the details of this straightforward step.

We shall prove the theorem by proving

$$A^{-1/2} \max_{0\le l\le A^{1+\varepsilon}} |S_A(l) - \tilde{Y}_A(l)| \to_{\mathbf{P}} 0 \tag{4.34}$$

and

$$A^{-1}\left|\omega_{[Ar]} - \omega'_{[Ar]}\right| \to_{\mathbf{P}} 0. \tag{4.35}$$

Consider first (4.34):

$$\begin{aligned}
&\max_{0\le l\le A^{1+\varepsilon}} |S_A(l) - \tilde{Y}_A(l)| \\
&\qquad \le \max_{0\le l\le A^{1+\varepsilon}} |S_A(l) - \tilde{S}_A(\sigma_A(l))| \\
&\qquad\quad + \max_{0\le l\le A^{1+\varepsilon}} |\tilde{S}_A(\sigma_A(l)) - \tilde{Y}_A(\sigma_A(l))| \\
&\qquad\quad + \max_{0\le l\le A^{1+\varepsilon}} |\tilde{Y}_A(\sigma_A(l)) - \tilde{Y}_A(l)|.
\end{aligned} \tag{4.36}$$

In order to estimate the three terms on the right-hand side of (4.36), we define the following events:

$$\mathscr{A}_A = \left\{ \max_{1\le l\le A^{1+\varepsilon}} |\xi_A(l)| < A^{\varepsilon} \right\}, \tag{4.37}$$

$$\mathscr{B}_A = \{\tilde{\xi}_A(l) = \tilde{\zeta}_A(l),\ 1 \le l \le A^{1+\varepsilon}\}, \tag{4.38}$$

$$\mathscr{C}_A = \{A^{1+\varepsilon} - \sigma_A(A^{1+\varepsilon}) < A^{1/2+5\varepsilon}\}, \tag{4.39}$$

$$\begin{aligned}
\mathscr{D}_A = \big\{\max\{|\tilde{Y}_A(j) - \tilde{Y}_A(l)|\colon\ 0 \le j,\ l \le A^{1+\varepsilon},\ |j-l| < A^{1/2+5\varepsilon}\} \\
< A^{1/4+3\varepsilon}\big\},
\end{aligned} \tag{4.40}$$

$$\mathscr{E}_A = \left\{\#\{l\colon\ 0 \le l \le A^{1+\varepsilon},\ \tilde{Y}_A(l) = 0\} < A^{1/2+\varepsilon}\right\}. \tag{4.41}$$

The proof will consist of showing that the probabilities of these events converge to 1 as $A \to \infty$. [Four more auxiliary events will be introduced later, when proving (4.35).] By construction of the process $\tilde{S}_A(\cdot)$ for $0 \le l \le A^{1+\varepsilon}$ we have

$$\begin{aligned}
|S_A(l) - \tilde{S}_A(\sigma_A(l))| &\le b_A + \max_{1<j\le l}(\xi_A(j) \vee 0) + |\xi_A(l_A)| \\
&\le b_A + 2 \max_{1<j\le A^{1+\varepsilon}} |\xi_A(j)|.
\end{aligned} \tag{4.42}$$

So the first term on the right-hand side of (4.36) is easily evaluated:

$$\mathbf{P}\Big(\max_{0\le l\le A^{1+\varepsilon}} |S(l) - \tilde{S}_A(\sigma_A(l))| < 3A^{\varepsilon} \Big) \ge \mathbf{P}(\mathscr{A}_A) \to 1 \tag{4.43}$$

as $A \to \infty$. The convergence on the right-hand side of (4.43) follows from the uniform bound (4.8) on the decay of the tails of step distributions.

The second term on the right-hand side of (4.36) is even simpler:

$$\mathbf{P}\Big(\max_{0\le l\le A^{1+\varepsilon}} |\tilde{S}_A(\sigma_A(l)) - \tilde{Y}_A(\sigma_A(l))| = 0\Big) \ge \mathbf{P}(\mathscr{B}_A) \to 1 \tag{4.44}$$

as $A \to \infty$, by the exponential closeness (4.25).

To estimate the last term on the right-hand side of (4.36), notice that

$$\mathscr{C}_A \cap \mathscr{D}_A \subset \Big\{\max_{0\le l\le A^{1+\varepsilon}} \big|\tilde{Y}_A(\sigma_A(l)) - \tilde{Y}_A(l)\big| < A^{1/4+3\varepsilon}\Big\}, \tag{4.45}$$

so the estimates

$$\mathbf{P}(\mathscr{C}_A) \to 1, \tag{4.46}$$

$$\mathbf{P}(\mathscr{D}_A) \to 1, \tag{4.47}$$

as $A \to \infty$, are still wanted. The convergence (4.47) and

$$\mathbf{P}(\mathscr{E}_A) \to 1, \tag{4.48}$$

as $A \to \infty$, are well established facts about homogeneous random walks: The proof of (4.47) is based on an adaptation of the method of proof of Theorem 8.3 of Billingsley (1968) (the finiteness of high moments of the distribution π is needed here), and (4.48) follows from an estimate on the number of crossings of the origin by a recurrent homogeneous random walk [for more details on this, see, e.g., Spitzer (1964), Section 17]. As the proofs of (4.47) and (4.48) are quite standard, we do not give the details here.

Finally

$$\mathbf{P}(\mathscr{C}_A \mid \mathscr{E}_A \cap \mathscr{B}_A) \to 1 \quad \text{as } A \to \infty \tag{4.49}$$

is a consequence of Lemma 3(ii), (4.14): notice that in $\mathscr{E}_A \cap \mathscr{B}_A$ there are altogether less than $A^{1/2+\varepsilon}$ visits of $S_A(l)$, $0 \le l \le A^{1+\varepsilon}$, to the interval $[0, b_A)$ and (4.14) provides an upper bound on the expectation of the duration of each of these visits. In consequence, the conditional expectation of the total time spent below the threshold level b_A is bounded as follows:

$$\begin{aligned} &\mathbf{E}\big(A^{1+\varepsilon} - \sigma_A(A^{1+\varepsilon}) \mid \mathscr{E}_A \cap \mathscr{B}_A\big) \\ &\quad \le A^{1/2+\varepsilon} \max_{0\le n<b_A} \mathbf{E}\big(\theta_{[0,b_A)} \mid S_A(0) = n\big) \\ &\quad \le C_{10} A^{1/2+4\varepsilon}. \end{aligned} \tag{4.50}$$

Hence (4.49) follows from Markov's inequality.

Equations (4.36) and (4.43)–(4.49) imply (4.34).

Now we prove (4.35). First let us define the event

$$\mathscr{F}_A = \Big\{\omega'_{[Ar]} < \frac{A^{1+\varepsilon}}{2}\Big\}. \tag{4.51}$$

From the weak convergence of $\omega'_{[Ar]}/A$ in (4.33) it clearly follows that

$$\mathbf{P}(\mathscr{F}_A) \to 1, \tag{4.52}$$

as $A \to \infty$. Using (4.22) we easily get

$$\mathscr{F}_A \cap \mathscr{C}_A \subset \{\tau_A(\omega'_{[Ar]}) - \omega'_{[Ar]} \le A^{1/2+5\varepsilon}\} \stackrel{\text{def}}{=} \mathscr{G}_A, \tag{4.53}$$

and hence

$$\mathbf{P}(\mathscr{G}_A) \to 1, \tag{4.54}$$

as $A \to \infty$. Let

$$\mathscr{H}_A = \{|\omega_{[Ar]} - \omega'_{[Ar]}| \le 2A^{1/2+5\varepsilon}\}. \tag{4.55}$$

We partition $\mathscr{H}_A^c$ in the following way:

$$\begin{aligned} \mathscr{H}_A^c = {} & (\mathscr{H}_A^c \cap \{\sigma_A(\omega_{[Ar]}) \le \omega_{[Ar]} \le \omega'_{[Ar]}\}) \\ & \cup (\mathscr{H}_A^c \cap \{\omega'_{[Ar]} \le \tau_A(\omega'_{[Ar]}) < \omega_{[Ar]}\}) \\ & \cup (\mathscr{H}_A^c \cap \{\omega'_{[Ar]} < \omega_{[Ar]} \le \tau_A(\omega'_{[Ar]})\}). \end{aligned} \tag{4.56}$$

The last two events in (4.56) are denoted as follows:

$$\mathscr{K}_A = \{\omega'_{[Ar]} - \sigma_A(\omega_{[Ar]}) \le 2A^{1/2+5\varepsilon}\}, \tag{4.57}$$

$$\mathscr{L}_A = \{\omega_{[Ar]} - \tau_A(\omega'_{[Ar]}) \le A^{1/2+5\varepsilon}\}. \tag{4.58}$$

The following inclusions clearly hold:

$$\mathscr{H}_A^c \cap \{\sigma_A(\omega_{[Ar]}) \le \omega_{[Ar]} \le \omega'_{[Ar]}\} \subset \mathscr{K}_A^c, \tag{4.59}$$

$$\mathscr{H}_A^c \cap \{\omega'_{[Ar]} \le \tau_A(\omega'_{[Ar]}) < \omega_{[Ar]}\} \subset \mathscr{G}_A^c \cup \mathscr{L}_A^c, \tag{4.60}$$

$$\mathscr{H}_A^c \cap \{\omega'_{[Ar]} < \omega_{[Ar]} \le \tau_A(\omega'_{[Ar]})\} \subset \mathscr{G}_A^c. \tag{4.61}$$

Given (4.54), in order to prove

$$\mathbf{P}(\mathscr{H}_A) \to 1, \tag{4.62}$$

we have to show that

$$\mathbf{P}(\mathscr{K}_A^c) \to 0, \tag{4.63}$$

$$\mathbf{P}(\mathscr{L}_A^c) \to 0, \tag{4.64}$$

as $A \to \infty$. Note that on $\mathscr{A}_A \cap \mathscr{B}_A \cap \mathscr{C}_A \cap \mathscr{F}_A \cap \mathscr{K}_A^c$ we have $\omega_{[Ar]} \le A^{1+\varepsilon}$, and so using the arguments of (4.43) and (4.44) we get

$$\begin{aligned} & \mathscr{A}_A \cap \mathscr{B}_A \cap \mathscr{C}_A \cap \mathscr{F}_A \cap \mathscr{K}_A^c \\ & \quad \subset \{\tilde{Y}_A(\sigma_A(\omega_{[Ar]})) \le 3A^\varepsilon\} \cap \{\omega'_{[Ar]} > \sigma_A(\omega_{[Ar]}) + 2A^{1/2+5\varepsilon}\}. \end{aligned} \tag{4.65}$$

For very similar reasons,

$$\begin{aligned} & \mathscr{A}_A \cap \mathscr{B}_A \cap \mathscr{C}_A \cap \mathscr{F}_A \cap \mathscr{L}_A^c \\ & \quad \subset \{S_A(\tau_A(\omega'_{[Ar]})) \le 3A^\varepsilon\} \cap \{\omega_{[Ar]} > \tau_A(\omega'_{[Ar]}) + A^{1/2+5\varepsilon}\}. \end{aligned} \tag{4.66}$$

We prove that the probabilities of the events appearing on the right-hand side of (4.65) and (4.66) go to zero. From the construction (4.24) and (4.27) it follows that $\{\tilde{\zeta}_A(\sigma_A(\omega_{[Ar]}) + l),\ l = 1,2,\dots\}$ are independent of $\{\tilde{\zeta}_A(l),\ l = 1,2,\dots,\sigma_A(\omega_{[Ar]})\}$, and thus from (4.65) we get

$$\begin{aligned}(4.67)\quad &\mathbf{P}(\mathscr{A}_A \cap \mathscr{B}_A \cap \mathscr{C}_A \cap \mathscr{F}_A \cap \mathscr{K}_A^c)\\ &\quad \le \mathbf{P}\big(\min\{l\colon\ \tilde{Y}_A(l) = 0\} > 2A^{1/2+5\varepsilon} \mid \tilde{Y}_A(0) \le 3A^{\varepsilon}\big) \to 0\end{aligned}$$

as $A \to \infty$. Hence (4.63). Also from the construction of the coupling we can see that $\tau_A(\omega'_{[Ar]})$ is a stopping time and from (4.66) we get now

$$\begin{aligned}(4.68)\quad &\mathbf{P}(\mathscr{A}_A \cap \mathscr{B}_A \cap \mathscr{C}_A \cap \mathscr{F}_A \cap \mathscr{L}_A^c)\\ &\quad \le \mathbf{P}\big(\min\{l\colon\ S_A(l) = 0\} > A^{1/2+5\varepsilon} \mid S_A(0) \le 3A^{\varepsilon}\big).\end{aligned}$$

From Lemma 3, (4.13) and (4.14), it easily follows that

$$(4.69)\qquad \mathbf{P}\big(\theta_{(0,\infty)} < A^{7\varepsilon} \mid S_A(0) \le 3A^{\varepsilon}\big) \to 1$$

as $A \to \infty$. Note that $\theta_{(0,\infty)}$ is exactly the hitting time of $0 \in \mathbb{Z}_+$. Thus (4.64) follows from (4.68) and (4.69).

So (4.62) [and consequently (4.35)] is proved now. □

PROOF OF THEOREM 1. Given the explicit form (2.20)–(2.23) of the transition probabilities and Lemmas 1 and 2 of Section 3, Theorem 1 becomes simply a concrete case of Theorem 4. We apply Theorem 4 twice. First, take

$$(4.70)\qquad S_A(l) = S_{[Ax],[\sqrt{A}\sigma h]}(l), \qquad l = 0,1,2,\dots,$$

$r = x$ and the exceptional step at time $l_A = [Ax]$. Second, take

$$(4.71)\qquad S_A(l) = S_{[Ax],[\sqrt{A}\sigma h]}(-l), \qquad l = 0,1,2,\dots,$$

$r = 0$ and no exceptional step. Strictly speaking, the nontrapping condition (4.10) does not hold for $n = 0$ and $l > [Ax]$ (respectively, $l > 0$). However, as the walk is stopped at $\omega_{[Ax]}$ (respectively, at $\omega_{[A0]}$), this does not make any difference. □

Appendix to Section 4.

PROOF OF LEMMA 3. The proof of (4.13) and (4.14) is a standard application of submartingale techniques and the optional sampling theorem [see Breiman (1968)].

General ingredients of the proof. Conditions (4.5), (4.6) and (4.8) imply

$$(\text{A4.1})\qquad \left|\sum_{x\in\mathbb{Z}} \pi_A(x|n,l)x\right| < C_{11}\exp(-C_{12}n),$$

with $C_{11} < \infty$ and $C_{12} = C_2/2$, and

$$(\text{A4.2})\qquad \liminf_{n\to\infty} \sum_{x\in\mathbb{Z}} \pi_A(x|n,l)\exp(-C_{12}x) \ge \sum_{x\in\mathbb{Z}} \pi(x)\exp(-C_{12}x) > 1.$$

Here once again we used the symmetry (4.6) of the asymptotic distribution π. Given (A4.2) we can choose $a_1 \in \mathbb{Z}_+$ and a finite constant $C_{13} < \infty$ such that, for $n \geq a_1$,

$$C_{13}\left(\sum_{x\in\mathbb{Z}} \pi_A(x|n,l)\exp(-C_{12}x) - 1\right) \geq C_{11}. \tag{A4.3}$$

It is straightforward to check that from (A4.3) and (A4.1) it follows that both

$$\pm S(l) + C_{13}\exp(-C_{12}S(l)), \qquad 0 \leq l \leq \theta_{(a_1,\infty)}, \tag{A4.4}$$

are submartingales.

Using the strong exponential bound (4.9) on the decay rate of the right tails, we can get the following bounds on the "overshoots" above any level b: for any $n \in (a,b) \subset \mathbb{Z}_+$,

$$\mathbf{E}\big(S(\theta_{(a,b)}) \,|\, [S(0) = n] \wedge [S(\theta_{(a,b)}) \geq b]\big) \leq (b + C_{14}), \tag{A4.5}$$

$$\mathbf{E}\big(S^2(\theta_{(a,b)}) \,|\, [S(0) = n] \wedge [S(\theta_{(a,b)}) > b]\big) \leq (b + C_{14})^2, \tag{A4.6}$$

with some $C_{14} < \infty$.

PROOF OF (4.13). Let $a_1 < m < b$. Considering the submartingale

$$-S(l) + C_{13}\exp(-C_{12}S(l)) + b - C_{13}\exp(-C_{12}b), \qquad 0 \leq l \leq \theta_{(a_1,b)}, \tag{A4.7}$$

we get

$$\begin{aligned} b - m &< b - m + C_{13}(\exp(-C_{12}m) - \exp(-C_{12}b)) \\ &\leq \mathbf{E}\big(-S(\theta_{(a_1,b)}) + C_{13}\exp(-C_{12}S(\theta_{(a_1,b)})) \\ &\qquad + b - C_{13}\exp(-C_{12}b) \,|\, S(0) = m\big) \\ &\leq \big(C_{13} + b - C_{13}\exp(-C_{12}b)\big)\,\mathbf{P}\big(S(\theta_{(a_1,b)}) \leq a_1 \,|\, S(0) = m\big). \end{aligned} \tag{A4.8}$$

[In the last inequality we used the monotonicity of (A4.7).] From (A4.8), for $a_1 < m < b$, we have

$$\mathbf{P}\big(S(\theta_{(a_1,b)}) \leq a_1 \,|\, S(0) = m\big) > \frac{b - m}{b + C_{13}}. \tag{A4.9}$$

Using this inequality, for $0 < n \leq a_1$ we get

$$\begin{aligned} &\mathbf{P}\big(S(\theta_{(0,b)}) = 0 \,|\, S(0) = n\big) \\ &\quad \geq \mathbf{P}\big(S(\theta_{(0,a_1]}) = 0 \,|\, S(0) = n\big) + \mathbf{P}\big(S(\theta_{(0,a_1]}) > a_1 \,|\, S(0) = n\big) \\ &\qquad \times \left(\sum_{m>a_1} \mathbf{P}\big(S(\theta_{(0,a_1]}) = m \,|\, [S(0) = n] \wedge [S(\theta_{(0,a_1]}) > a_1]\big)\frac{b - m}{b + C_{13}}\right) \\ &\qquad \times \min_{0<n'\leq a_1} \mathbf{P}\big(S(\theta_{(0,b)}) = 0 \,|\, S(0) = n'\big). \end{aligned} \tag{A4.10}$$

Inequalities (A4.5) and (A4.10) yield

$$
\begin{aligned}
&\mathbf{P}\big(S(\theta_{(0,b)}) = 0 \mid S(0) = n\big) \\
&\qquad \geq \mathbf{P}\big(S(\theta_{(0,a_1]}) = 0 \mid S(0) = n\big) \\
&\qquad\quad + \mathbf{P}\big(S(\theta_{(0,a_1]}) > a_1 \mid S(0) = n\big) \frac{b - a_1 - 1 - C_{14}}{b + C_{13}} \\
&\qquad\quad \times \min_{0<n'\leq a_1} \mathbf{P}\big(S(\theta_{(0,b)}) = 0 \mid S(0) = n'\big).
\end{aligned} \tag{A4.11}
$$

The nontrapping condition (4.11) implies

$$
\min_{0<n'\leq a_1} \mathbf{P}\big(S(\theta_{(0,a_1]}) = 0 | S(0) = n'\big) = C_{15} > 0. \tag{A4.12}
$$

From (A4.11) and (A4.12) we get

$$
\begin{aligned}
&\min_{0<n'\leq a_1} \mathbf{P}\big(S(\theta_{(0,b)}) = 0 | S(0) = n'\big) \\
&\qquad \geq \frac{C_{15}(b + C_{13})}{C_{15} b + C_{13} + (1 - C_{15})(a_1 + 1 + C_{14})}.
\end{aligned} \tag{A4.13}
$$

Finally, (A4.9) and (A4.13) yield (4.13), with a suitably chosen C_9. □

PROOF OF (4.14). First we prove a sort of counterpart of (4.13). Namely, there is a constant $C_{16} > 0$ such that, for any $0 < a < n < b$ in $\mathbb{Z}_+$,

$$
\mathbf{P}\big(S_A(\theta_{(a,b)}) \geq b \mid S_A(0) = n\big) > C_{16} \frac{n - a}{b - a}. \tag{A4.14}
$$

Denote $a_2 = \max\{a_1, C_{13}\}$. Let $a_2 \leq a < b$ and now consider the submartingale

$$
S(l) + C_{13} \exp(-C_{12} S(l)) - a - C_{13} \exp(-C_{12} a), \qquad 0 \leq l \leq \theta_{(a,b)}. \tag{A4.15}
$$

Then, using (A4.5) again,

$$
\begin{aligned}
&n + C_{13} \exp(-C_{12} n) - a - C_{13} \exp(-C_{12} a) \\
&\qquad \leq \mathbf{E}\big(S(\theta_{(a,b)}) + C_{13} \exp(-C_{12} S(\theta_{(a,b)})) - a \\
&\qquad\qquad\qquad - C_{13} \exp(-C_{12} a) \mid S(0) = n\big) \\
&\qquad \leq (b + C_{14} + C_{13} - a) \mathbf{P}\big(S(\theta_{(a,b)}) \geq b \mid S(0) = n\big),
\end{aligned} \tag{A4.16}
$$

which implies (A4.14) with some constant $C_{17} > 0$ and $a_2 \leq a < n < b$. In the second inequality of (A4.16) we used the fact that (A4.15) is negative for

$S(l) < a$. Changing the constant C_{17} to the smaller one,

$$C_{16} = \min_{0\le a'<a_2} \min_{a'<n'\le a_2} \mathbf{P}\big(S(\theta_{(a',a_2]}) > a_2 \| S(0) = n'\big) C_{17} > 0, \tag{A4.17}$$

the inequality extends to any $0 < a < n < b$ and (A4.14) is proved.

Conditions (4.5) and (4.10) imply that there exists a positive constant $\underline{\sigma}$ such that

$$0 < \underline{\sigma}^2 \le \sum_{x\in\mathbb{Z}} \pi_A(x|n,l)|x|^2. \tag{A4.18}$$

Now choose $a_3 \in \mathbb{Z}_+$ so that, for $n \ge a_3$,

$$2nC_{11} \exp(-C_{12}n) < \frac{\underline{\sigma}^2}{2}. \tag{A4.19}$$

Then, (A4.1) ensures that, in the domain $n > a_3$,

$$\mathbf{E}\Big(S^2(l+1) - \frac{\underline{\sigma}^2}{2}(l+1) \,|\, S(l) = n\Big) \ge n^2 - \frac{\underline{\sigma}^2}{2} l \tag{A4.20}$$

and consequently

$$S^2(l) - \frac{\underline{\sigma}^2 l}{2}, \qquad l = 0, 1, \dots, \theta_{(a_3,\infty)}, \tag{A4.21}$$

is a submartingale. Hence, with $T > 0$ fixed,

$$\mathbf{E}\left(S^2(\theta_{(a_3,b)} \wedge T) - \frac{\underline{\sigma}^2}{2}(\theta_{(a_3,b)} \wedge T) \,|\, S(0) = n\right) \ge n^2 \ge 0. \tag{A4.22}$$

Now, using the "overshoot bound" (A4.6) we get

$$\mathbf{E}\big(\theta_{(a_3,b)} \,|\, S(0) = n\big) = \lim_{T\to\infty} \mathbf{E}\big(\theta_{(a_3,b)} \wedge T \,|\, S(0) = n\big) \le \frac{2}{\underline{\sigma}^2}(b + C_{14})^2. \tag{A4.23}$$

Finally (A4.14) and (A4.23) lead directly to

$$\begin{aligned} &\mathbf{E}\big(\theta_{[0,b)} \,|\, S(0) = n\big) \\ &\quad \le C_{16}^{-1} b \left(\max_{0\le n'\le a_3} \mathbf{E}\big(\theta_{[0,a_3]} \,|\, S(0) = n'\big) + \frac{4}{\underline{\sigma}^2}(b + C_{14})^2 \right). \end{aligned} \tag{A4.24}$$

From the nontrapping condition (4.10) it follows that $\max_{0\le n'\le a_3} \mathbf{E}(\theta_{[0,a_3]}|S(0) = n')$ is finite. Thus (A4.24) proves (4.14), with a suitably chosen constant C_{10}. □

5. Proof of Theorem 2. Throughout this section we shall use the notation of Revuz and Yor [(1991), Chapters XI and XII]. In particular, we denote by $(U, \mathscr{U})$ the space of excursions of Brownian motion with its natural σ-algebra; by e_s, $s > 0$, we denote the excursion process of Brownian motion; by $n(du)$, we denote Itô's intensity measure on $(U, \mathscr{U})$; and by $\rho_\nu(t)$, $t \in [0, 1]$, we denote the standard Bessel bridge of index ν [i.e., of dimension $\delta = 2(\nu+1)$]

over the time interval $[0,1]$. We shall need the following two functionals of excursions:

$$R(u) = \inf\{s > 0 \mid u(s) = 0\}, \tag{5.1}$$

$$A(u) = \int_0^{R(u)} |u(s)|\, ds \tag{5.2}$$

and the area under the three-dimensional Bessel bridge:

$$\tau = \int_0^1 \rho_{1/2}(s)\, ds. \tag{5.3}$$

Changing the order of integration in (1.31) and (1.35), we easily get

$$\begin{aligned}
&\int_0^\infty \left(\int_0^\infty \hat{\varrho}(1; x, h)\, dh \right) dx \\
&\quad = \int_0^\infty \mathbf{E}\left(\int_0^\infty \exp(-T_x)\, dx \,\Big|\, |W_0| = h \right) dh \\
&\quad = \int_0^\infty \mathbf{E}\big(\exp(-T_0)\omega_0^+ \,\big|\, |W_0| = h\big)\, dh \\
&\qquad + \int_0^\infty \mathbf{E}\big(\exp(-T_0) \,\big|\, |W_0| = h\big)\, dh \\
&\qquad \times \mathbf{E}\left(\int_0^\infty \exp(-T_x)\, dx \,\Big|\, |W_0| = 0 \right).
\end{aligned} \tag{5.4}$$

In the second step we decomposed the integral with respect to x in two parts (from 0 to ω_0^+ and from ω_0^+ to ∞) and used the strong Markov property of Brownian motion.

Using Bismut's characterization and Itô's representation of the intensity measure of Brownian excursions [Revuz and Yor (1991), Theorems XII.4.7 and XII.4.2], we get

$$\begin{aligned}
\int_0^\infty \mathbf{E}\big(\exp(-T_0)\omega_0^+ \mid |W_0| = h\big)\, dh &= \frac{1}{2} \int_U R^2(u) \exp\{-A(u)\} n(du) \\
&= \frac{1}{4\sqrt{2\pi}} \int_0^\infty \mathbf{E}(\exp(-\tau b^{3/2})) b^{1/2}\, db \\
&= \frac{1}{6\sqrt{2\pi}} \mathbf{E}(\tau^{-1})
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
\int_0^\infty \mathbf{E}\big(\exp(-T_0) \mid |W_0| = h\big)\, dh &= \int_U R(u) \exp\{-A(u)\} n(du) \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\infty \mathbf{E}(\exp(-\tau b^{3/2})) b^{-1/2}\, db \\
&= \frac{\Gamma(1/3)}{3\sqrt{2\pi}} \mathbf{E}(\tau^{-1/3}).
\end{aligned} \tag{5.6}$$

Next we express the second factor in the rightmost term of (5.4):

$$
\begin{aligned}
&\mathbf{E}\biggl(\int_0^\infty \exp(-T_x)\,dx \,\big|\, |W_0| = 0\biggr)\\
&\quad = \mathbf{E}\biggl(\sum_{s>0} \exp\biggl\{-\sum_{t<s} A(e_t)\biggr\} R(e_s)\exp\{-A(e_s)\}\biggr)\\
&\quad = \int_0^\infty \mathbf{E}\biggl(\exp\biggl\{-\sum_{t<s} A(e_t)\biggr\}\biggr)\,ds \cdot \int_U R(u)\exp\{-A(u)\}n(du)\\
&\quad = \int_0^\infty \biggl(\exp\biggl\{-s\int_U \bigl(1-\exp(-A(u))\bigr)n(du)\biggr\}\biggr)\,ds \qquad (5.7)\\
&\qquad \times \int_U R(u)\exp\{-A(u)\}n(du)\\
&\quad = \frac{\int_U R(u)\exp\{-A(u)\}n(du)}{\int_U \bigl(1-\exp(-A(u))\bigr)n(du)}\\
&\quad = \frac{\int_0^\infty \mathbf{E}(\exp(-\tau b^{3/2}))b^{-1/2}\,db}{\int_0^\infty \mathbf{E}(1-\exp(-\tau b^{3/2}))b^{-3/2}\,db}\\
&\quad = \frac{\Gamma(1/3)}{3\Gamma(2/3)}\frac{\mathbf{E}(\tau^{-1/3})}{\mathbf{E}(\tau^{1/3})}.
\end{aligned}
$$

The first equality is a simple transcription of the expression in terms of the excursion process. In the second step we use the so-called master formula for Poisson point processes [see Revuz and Yor (1991), Proposition XII.1.10]. In the third step we use the "exponential formula" for Poisson point processes [see Revuz and Yor (1991), Proposition XII.1.12]. In the fourth step we perform the integration over the variable s. In the fifth equality, Itô's representation is used again. Finally, in the last step the integrations over the variable b are performed.

Finally (5.4)–(5.7) yield

$$
\int_{-\infty}^\infty \hat{\varphi}(s,x)\,dx = \frac{1}{\sqrt{2\pi}}\biggl[\frac{1}{3}\mathbf{E}(\tau^{-1}) + \frac{2}{9}\frac{\Gamma^2(1/3)}{\Gamma(2/3)}\frac{\mathbf{E}^2(\tau^{-1/3})}{\mathbf{E}(\tau^{1/3})}\biggr]. \qquad (5.8)
$$

Let the random variables X and Y be defined as follows:

$$
X = \int_0^1 (\rho_{1/3}(s))^{-2/3}\,ds, \qquad (5.9)
$$

$$
Y = \bigl(\tfrac{3}{2}\bigr)^{2/3}\biggl(\int_0^1 \rho_{1/2}(s)\,ds\biggr)^{-2/3} = \bigl(\tfrac{3}{2}\bigr)^{2/3}\tau^{-2/3}. \qquad (5.10)
$$

According to Biane and Yor (1987) [see also Revuz and Yor (1991), Theorem XI.3.5], for any measurable function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$,

$$
\mathbf{E}(f(Y)) = \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)}\mathbf{E}(X^{1/2}f(X)). \qquad (5.11)
$$

Choosing $f(y) = y^{3/2}$, $y^{1/2}$ and $y^{-1/2}$, we get in turn

$$\mathbf{E}(\tau^{-1}) = \left(\frac{3}{2}\right)^{-1} \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)} \mathbf{E}(X^2), \tag{5.12}$$

$$\mathbf{E}(\tau^{-1/3}) = \left(\frac{3}{2}\right)^{-1/3} \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)} \mathbf{E}(X), \tag{5.13}$$

$$\mathbf{E}(\tau^{1/3}) = \left(\frac{3}{2}\right)^{1/3} \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)}. \tag{5.14}$$

Inserting these expressions into (5.8) we get

$$\int_{-\infty}^{\infty} \hat{\varphi}(s,x)\,dx = \frac{2^{2/3}}{9\Gamma(1/3)}\left[\mathbf{E}(X^2) + \frac{2}{3}\frac{\Gamma^2(1/3)}{\Gamma(2/3)}\mathbf{E}^2(X)\right]. \tag{5.15}$$

We sketch the lengthy calculations of $\mathbf{E}(X)$ and $\mathbf{E}(X^2)$. We shall use the notation of Erdélyi, Magnus, Oberhettinger and Tricomi (1953): we denote by J_ν the Bessel function of order ν, by I_ν the modified Bessel function of order ν and by $\Phi(a, c; \cdot)$ the confluent hypergeometric function with indices a and c. Let $\pi_t(x)$, $t \in (0,1)$, $x \geq 0$ and $\pi_{s,t}(x,y)$, $s, t \in (0,1)$, $s + t < 1$, $x, y \geq 0$ be the densities of the distribution of $\rho_{1/3}(t)$, respectively, of the joint distribution of $(\rho_{1/3}(s), \rho_{1/3}(1-t))$ [see Itô and McKean (1965) or Revuz and Yor (1991)]:

$$\pi_t(x) = \frac{3}{2^{1/3}\Gamma(1/3)} \frac{x^{5/3}}{[t(1-t)]^{4/3}} \exp\left(-\frac{x^2}{2t(1-t)}\right), \tag{5.16}$$

$$\begin{aligned} \pi_{t,s}(x,y) = {} & \frac{3}{2^{1/3}\Gamma(1/3)}\left(\frac{x}{s}\right)^{4/3}\left(\frac{y}{t}\right)^{4/3} \exp\left(-\frac{x^2}{2s}\right)\exp\left(-\frac{y^2}{2t}\right) \\ & \times \frac{1}{1-t-s}\exp\left(-\frac{x^2+y^2}{2(1-t-s)}\right) I_{1/3}\left(\frac{xy}{1-t-s}\right). \end{aligned} \tag{5.17}$$

The first two moments and correlations of $\rho_{1/3}^{-2/3}$ will be denoted as follows:

$$m_1(t) = \mathbf{E}\big((\rho_{1/3}(t))^{-2/3}\big) = \int_0^\infty x^{-2/3}\pi_t(x)\,dx, \tag{5.18}$$

$$m_2(t) = \mathbf{E}\big((\rho_{1/3}(t))^{-4/3}\big) = \int_0^\infty x^{-4/3}\pi_t(x)\,dx, \tag{5.19}$$

$$\begin{aligned} c(s,t) &= \mathbf{E}\big((\rho_{1/3}(s))^{-2/3}(\rho_{1/3}(1-t))^{-2/3}\big) \\ &= \int_0^\infty \int_0^\infty (xy)^{-2/3}\pi_{s,t}(x,y)\,dx\,dy. \end{aligned} \tag{5.20}$$

Clearly

$$\mathbf{E}(X) = \int_0^1 m_1(t)\,dt, \tag{5.21}$$

$$\mathbf{E}(X^2) = 2\int_0^1 \int_0^{1-t} c(s,t)\,ds\,dt. \tag{5.22}$$

Moments m_1 and m_2 are easily computed:

$$m_1(t) = \frac{3}{2^{1/3}\Gamma(1/3)}[t(1-t)]^{-1/3}, \tag{5.23}$$

$$m_2(t) = \frac{3\Gamma(2/3)}{2^{2/3}\Gamma(1/3)}[t(1-t)]^{-2/3}. \tag{5.24}$$

From (5.21) and (5.23),

$$\mathbf{E}(X) = \frac{9\Gamma^2(2/3)}{2^{1/3}\Gamma^2(1/3)}. \tag{5.25}$$

The computation $c(s,t)$ is much more involved. Using identity (7.7.38) of Erdélyi, Magnus, Oberhettinger and Tricomi [(1953), Volume 2], we first rewrite $\pi_{s,t}$:

$$\begin{aligned}\pi_{t,s}(x,y) = {} & \frac{3}{2^{1/3}\Gamma(1/3)}(st)^{-4/3} \\ & \times \int_0^\infty \left[\exp\left(\frac{-(1-s-t)v^2}{2}\right)\left(x^{4/3}\exp\left(-\frac{x^2}{2s}\right)J_{1/3}(xv)\right)\right. \\ & \qquad \times \left.\left(y^{4/3}\exp\left(-\frac{y^2}{2t}\right)J_{1/3}(yv)\right)v\right]dv.\end{aligned} \tag{5.26}$$

Next we use the Hankel transform (8.6.14) from Erdélyi, Magnus, Oberhettinger and Tricomi [(1954), Volume 2] and the identity (6.3.7) from Erdélyi, Magnus, Oberhettinger and Tricomi [(1953), Volume 1] to get (5.27) below [note that the Erdélyi, Magnus, Oberhettinger and Tricomi volumes use both notations $\Phi(a,c,x)$ and $F_1(a,c,x)$ for the same confluent hypergeometric functions]:

$$c(s,t) = \frac{3^3}{2^{2/3}\Gamma^3(1/3)}(st)^{-1/3}\int_0^\infty u^{1/3}e^{-u}\Phi\left(\frac{1}{3},\frac{4}{3};su\right)\Phi\left(\frac{1}{3},\frac{4}{3};tu\right)du. \tag{5.27}$$

In the following step we use the basic integral representation of the confluent hypergeometric function (6.5.1) from Erdélyi, Magnus, Oberhettinger and Tricomi [(1953), Volume 1]:

$$c(s,t) = \frac{1}{2^{2/3}\Gamma^2(1/3)}(st)^{-2/3}\int_0^s\int_0^t (xy)^{-2/3}(1-x-y)^{-4/3}\,dy\,dx. \tag{5.28}$$

In the last expression the integration with respect to the y variable can be performed:

$$c(s,t) = \frac{9}{2^{2/3}\Gamma^2(1/3)} t^{-1/3} \frac{s^{-2/3}}{3} F(s,t), \tag{5.29}$$

where the function F is defined on the domain $\{(s,t) \in (0,1)\times(0,1)\colon s+t < 1\}$ as follows:

$$F(s,t) = \int_0^s x^{-2/3}(1-x)^{-1}(1-t-x)^{-1/3}\,dx. \tag{5.30}$$

Integrating by parts with respect to the s variable, we are led to

$$\begin{aligned}
&\int_0^1 \int_0^{1-t} c(s,t)\,ds\,dt \\
&\quad = 3\int_0^1 (1-t)c(1-t,t)\,dt \\
&\qquad - \frac{9}{2^{2/3}\Gamma^2(1/3)} \int_0^1 \int_0^{1-t} t^{-1/3}s^{-1/3}(1-s)^{-1}(1-t-s)^{-1/3}\,ds\,dt.
\end{aligned} \tag{5.31}$$

Inserting $c(1-t,t) = m_2(t)$ from (5.24) and performing some straightforward transformations in the last integral on the right-hand side of (5.31), we finally get

$$\begin{aligned}
\mathbf{E}(X^2) &= \frac{9\cdot 2^{1/3}\Gamma(2/3)}{\Gamma(1/3)} \int_0^1 t^{-2/3}(1-t)^{1/3}\,dt \\
&\quad - \frac{9\cdot 2^{1/3}}{\Gamma^2(1/3)} \int_0^1 s^{-1/3}(1-s)^{-2/3}\,ds \int_0^1 t^{-1/3}(1-t)^{-1/3}\,dt \\
&= \frac{9\Gamma(1/3)}{2^{2/3}} - \frac{27\cdot 2^{1/3}\Gamma^3(2/3)}{\Gamma^2(1/3)}.
\end{aligned} \tag{5.32}$$

Inserting $\mathbf{E}(X)$ from (5.25) and $\mathbf{E}(X^2)$ from (5.32) into (5.15), we get (1.35) indeed. □

REMARK. According to Darling (1983) and Vervaat (1979) we can express the moment generating function of τ as

$$\mathbf{E}(\exp(-\lambda\tau)) = \lambda\sqrt{2\pi}\sum_{n=1}^{\infty} \exp\left\{-\frac{\lambda^{2/3}\sigma_n}{2^{1/3}}\right\}, \tag{5.33}$$

where σ_n is the nth zero of the Airy function,

$$\mathrm{Ai}(-z) = \tfrac{1}{3}\sqrt{z}\left\{J_{1/3}\left(\tfrac{2}{3}z^{3/2}\right) + J_{-1/3}\left(\tfrac{2}{3}z^{3/2}\right)\right\}. \tag{5.34}$$

[For properties of the Airy functions, see, e.g., Abramowitz and Stegun (1964).] Using the formula

$$\mathbf{E}(\tau^{-p}) = \frac{1}{\Gamma(p)} \int_0^\infty \lambda^{p-1}\,\mathbf{E}(e^{-\lambda\tau})\,d\lambda, \qquad p > 0, \tag{5.35}$$

we get expressions for the negative moments of τ:

$$\mathbf{E}(\tau^{-p}) = 3 \cdot 2^{p/2} \frac{\Gamma(1/2)\Gamma(3(p+1)/2)}{\Gamma(p)} \sum_{n=1}^{\infty} \sigma_n^{-3(p+1)/2}. \tag{5.36}$$

As a by-product of the proof of Theorem 2, from (5.13), (5.25) and (5.36) [respectively, from (5.12), (5.32) and (5.36)], we get the following two identities valid for the zeros of the Airy function $\mathrm{Ai}(-z)$:

$$\sum_{n=1}^{\infty} \sigma_n^{-2} = 3^{2/3} \left(\frac{\Gamma(2/3)}{\Gamma(1/3)} \right)^2 = 0.531457\ldots, \tag{5.37}$$

$$\sum_{n=1}^{\infty} \sigma_n^{-3} = \frac{1}{2} - 3 \left(\frac{\Gamma(2/3)}{\Gamma(1/3)} \right)^3 = 0.112561\ldots. \tag{5.38}$$

6. Proof of Theorem 3. We shall prove (1.41) for $x \geq 0$. For $x \leq 0$ the same proof holds with slightly changed notation.

To prove Theorem 3 we note first that

$$P(n,k) = \mathbf{P}(X_n = k) = \sum_{m=0}^{\infty} [\mathbf{P}(T_{k,m}^{>} = n) + \mathbf{P}(T_{k,m}^{<} = n)]. \tag{6.1}$$

On the other hand, from the definition of $\hat{\varphi}_A$,

$$\hat{\varphi}_A(s,x) = \frac{1 - e^{-s/A}}{s/A} s A^{-1/3} \sum_{n=0}^{\infty} e^{-ns/A} P(n, [A^{2/3}x]). \tag{6.2}$$

Combining (6.1) and (6.2), we are led to

$$\begin{aligned} \hat{\varphi}_A(s,x) = {} & \frac{1 - \exp(-s/A)}{s/A} \\ & \times s A^{-1/3} \sum_{m=0}^{\infty} \big[\mathbf{E}\big(\exp(-sT_{[A^{2/3}x],m}^{>}/A)\big) \\ & \qquad + \mathbf{E}\big(\exp(-sT_{[A^{2/3}x],m}^{<}/A)\big)\big]. \end{aligned} \tag{6.3}$$

Defining

$$\hat{\varrho}_A^{*}(s;x,h) = s\,\mathbf{E}\big(\exp\big(-sT_{[A^{2/3}x],[A^{1/3}\sigma h]}^{*}/(2\sigma A)\big)\big), \tag{6.4}$$

(6.3) reads

$$\hat{\varphi}_A(s,x) = \frac{1 - e^{-s/A}}{s/A} \frac{1}{2} \int_0^{\infty} \big(\hat{\varrho}_A^{>}(2\sigma s;x,h) + \hat{\varrho}_A^{<}(2\sigma s;x,h)\big)\, dh. \tag{6.5}$$

From Corollary 2 of Theorem 1 it follows that, for any $s > 0$, $x \in [0,\infty)$ and $h > 0$,

$$\hat{\varrho}_A^{*}(s;x,h) \to \hat{\varrho}(s;x,h) \tag{6.6}$$

as $A \to \infty$. Relations (6.5) and (6.6) imply, for any $x \in \mathbb{R}$,

$$\liminf_{A\to\infty} \hat{\varphi}_A(s,x) \geq \int_0^\infty \hat{\varrho}(2\sigma s; |x|, h)\, dh = \sigma^{2/3}\hat{\varphi}(s, \sigma^{2/3}x). \tag{6.7}$$

On the other hand, by Theorem 2,

$$\int_{-\infty}^{\infty} \hat{\varphi}_A(s,x)\, dx = 1 = \int_{-\infty}^{\infty} \hat{\varphi}(s,x)\, dx. \tag{6.8}$$

The statement of Theorem 3 follows from (6.7) and (6.8). □

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