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BLOCKAGE HYDRODYNAMICS OF ONE-DIMENSIONAL DRIVEN CONSERVATIVE SYSTEMS

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We consider an arbitrary one-dimensional conservative particle system with finite-range interactions and finite site capacity, governed on the hydrodynamic scale by a scalar conservation law with Lipschitz-continuous flux h. A finite-size perturbation restricts the local current to some maximum value ϕ. We show that the perturbed hydrodynamic behavior is entirely determined by ϕ if $\inf(h;\phi)$ is first nondecreasing and then nonincreasing, which we believe is a necessary condition.

1. Introduction. It is known that scalar conservation laws of the form

$$\partial_t \rho + \partial_x h(\rho) = 0 \tag{1}$$

arise as hydrodynamic limits of asymmetric conservative particle systems under Euler time scaling. See, for example, [22] for a celebrated result and [16] for a survey of related literature. Equation (1) and the related particle systems can be viewed as simplified models for a variety of physical phenomena, such as traffic flow [7]. How is (1) modified by a local perturbation of dynamics representing traffic perturbation, for example, a bottleneck? The simplest model was introduced in [12] as a one-dimensional totally asymmetric simple exclusion process (TASEP) where a slow bond has jump rate $\alpha \in [0;1)$ instead of 1. We shall refer to this model as TASEP(α). This and related models have attracted sustained interest in the probability and physics literature over the last years; partly, because they provide examples of phase transition in one-dimensional systems with short-range interactions. See, for example, [18] for a similar ZR(α) model in the case of the zero-range process, [25] for a TASEP(α) with parallel update, [20] for a TASEP with a slow second-class particle.

Stationary response of TASEP(α) to blockage was investigated numerically by Janowski and Lebowitz [12] and analytically by Schütz [25] for parallel update; the latter being exactly solvable as opposed to the former. The common picture is that the perturbation sets a maximum value ϕ for the current through the bottleneck, which entirely determines the behavior of the system. Density values that violate this bound are replaced by a unique jam (i.e., a decreasing shock) with current ϕ. More recently, Jensen [13] determined the optimal perturbation of TASEP that

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stabilizes a given stationary decreasing shock (where optimal is meant in relative entropy cost).

Blockage hydrodynamics were first studied for ZR(α) by Landim [18]. However, the response of this system is different from those considered here, because of *infinite* site capacity and *increasing* flux. The latter property has the simplifying consequence that the perturbation is not felt by the left-hand side of the system; one also benefits from explicit knowledge of invariant measures for the perturbed system. The hydrodynamic profile of TASEP(α) was obtained in [28] by the variational coupling technique due to Seppäläinen (see, e.g., [27] for a detailed introduction thereto), thanks to specific features of the model: the Lax-Hopf formula for (1) when the flux is concave (or convex), echoed by a corresponding microscopic formula for TASEP(α). Due to those particular properties, it is not clear how one-dimensional blockage behavior generically depends on the flux, model and perturbation; and how a unifying formulation (if any) can be given. In general, the flux is not concave (or convex), so there is no variational representation as in [28]; then the problem is entirely new, as one does not even know how to define a candidate hydrodynamic profile. Still for concave flux, the system may not admit a microscopic variational representation, for example, for models with partially asymmetric jumps or jump rates depending on the configuration (see, e.g., [27]); in that case the result of [28] still yields a candidate for hydrodynamics, but a different approach is required for the proof. The same problem occurs even for TASEP under certain more general perturbations than TASEP(α), for example, for a slow bond with state-dependent jump rate (see the third remark in Section 2.5).

In this paper we provide a unified, model-independent approach that does not require convexity and variational coupling, but a much weaker monotonicity assumption for the positions of labelled particles. This assumption contains, in particular, all nearest-neighbor systems for which (1) is known. Variational coupling was shown [24] to be equivalent to a stronger form of monotonicity. Though the problem and approaches are different, the situation is somewhat comparable to [23], where the absence of variational coupling raises a challenging problem. Another difficulty here is that, to the best of our knowledge, there is no existing theory in PDE literature to describe the kind of hydrodynamics we find. Thus, part of our new results is the construction of such a theory, which we hope may be interesting in itself. We consider an arbitrary one-dimensional system ruled by (1), with Lipschitz-continuous flux $h \geq 0$, defined on $[0; K]$, such that $h(0) = h(K) = 0$. K is the maximal density imposed by finite site capacity. Our main result is that the perturbed hydrodynamics is entirely determined by h and the maximal current ϕ, if $\inf(h; \phi)$ is first nondecreasing and then nonincreasing. This is equivalent to $\phi \leq \phi^*$, where ϕ^* is the infimum "genuine" (see Section 2.2) local minimum of h, or $\phi^* = \sup h$ is there is no such local minimum. We believe that for $\phi > \phi^*$, the behavior of the system is no longer determined by the single parameter ϕ. Note that ϕ is not explicit for a given perturbation, because, in general, invariant measures are not known for the perturbed system, even if they are for the unperturbed system. The problem is somewhat comparable to [28] and [23], where the whole flux is unknown, for lack of information about steady states. Nevertheless, the whole range $0 \leq \phi \leq \phi^*$ can be covered by a physically natural family of perturbations.

Without convexity and variational representation, an interesting problem is to *characterize* the hydrodynamic profile. It is known that (1) has no strong solutions in general; besides, choosing from the infinity of weak solutions with given Cauchy datum requires an additional *entropy condition* to select the physical solution, called the *entropy solution*. This solution describes the hydrodynamics of the *unperturbed* system. We shall see that the microscopic perturbation does not modify the hydrodynamic equation (1), as the system still selects a weak solution thereto; but it does modify the choice of the relevant solution. We prove that the hydrodynamic limit is characterized by a modified entropy condition that specifies a new set of admissible shocks at the bottleneck. It may alternatively be viewed as a special kind of boundary condition. For this condition we prove original existence and uniqueness results. We only require Lipschitz continuity of the flux, which is the standard condition for well-posedness of (1) in Kružkov's theory [17]. In the present paper where $\phi \leq \phi^*$, the condition depends only on ϕ. It states that densities or shocks with current greater than ϕ should be replaced by a decreasing shock with current ϕ. As opposed to TASEP(α), there may be several such shocks if h has flat segments or local minima with value ϕ. We believe that, for $\phi \leq \phi^*$, this modified entropy condition is the only one that induces a maximal current ϕ; while for $\phi > \phi^*$, there should be an infinity of such entropy conditions. This suggests (as mentioned above) that the behavior of the system cannot be determined only by ϕ in the latter case.

The paper is organized as follows. In Section 2 we define the framework, state the main results and discuss three classes of examples: nearest-neighbor attractive systems, nearest-neighbor ASEP with speed change and nearest-available neighbor ASEP. In Section 3 we introduce the modified entropy condition and state properties of the corresponding solution; as an illustration, we explicitely construct Riemann solutions. In Section 4 we state comparison lemmas that will be the basis for proving hydrodynamics. Section 5 is devoted to the proof of the hydrodynamic behavior. The main idea is to compare the perturbed system with suitable free systems on either side of the perturbation; the key ingredient is a coupling lemma (Lemma 4.6), which states that for given dynamics on a half-line, the macroscopic profile is entirely determined by the incoming flux. Existence and uniqueness for modified entropy conditions are proved in Section 6, which may at first be skipped by the reader mainly interested in particle systems. There we rely on material from conservation laws theory (see [10, 17, 29]) with some original ideas. Lemma 6.4, though technically elementary, is conceptually crucial, as it both explains well-posedness of the modified entropy condition and the restriction $\phi \leq \phi^*$.

2. The framework and results.

Set notation. $\mathbb{N}^\star, \mathbb{R}^*, \mathbb{R}^+, \mathbb{R}^{+*}, \mathbb{R}^-, \mathbb{R}^{-*}$, respectively, denote the sets of positive integers and nonzero (resp. nonnegative, positive, nonpositive, negative) real numbers.

The configuration space. We consider particle systems on the one-dimensional lattice $\mathbb{Z}$, with at most K particles per site. Particles are labelled increasingly from left to right, and we denote by $\sigma(n)$ the position of the particle with label n. If for some $n \in \mathbb{Z}$, there is no more particle to the right (left) of $\sigma(n)$, then we set $\sigma(m) = +\infty$ $(-\infty)$ for every $m > n$ $(m < n)$. We denote by $\mathcal{E}_d$ the set of such configurations $\sigma = (\sigma(n), n \in \mathbb{Z})$, called *distinguishable*. They will always be denoted by σ or τ. We write $\sigma \le \tau$ iff $\sigma(n) \le \tau(n)$ for every $n \in \mathbb{Z}$. For a configuration $\sigma(\tau)$, we denote by $\eta(x)$ $(\zeta(x))$ the number of particles at site x. We denote by $\mathcal{E}_u = \{0, \dots, K\}^{\mathbb{Z}}$ the set of configurations $\eta = (\eta(x), x \in \mathbb{Z})$, called *undistinguishable*. We write $\eta \le \zeta$ iff $\eta(x) \le \zeta(x)$ for every $x \in \mathbb{Z}$. For every $y \in \mathbb{Z}$, the space shift τ_y is defined respectively on a distinguishable configuration σ and an undistinguishable configuration η by $\tau_y\sigma(n) = \sigma(n) - y$ and $\tau_y\eta(x) = \eta(x + y)$.

REMARK. The set of admissible configurations could be defined more generally by the *v-exclusion* condition introduced in [24]. In this case the maximal current configuration η^* defined in Section 2.2 should be replaced by the one defined in [24], but the rest of our arguments would be unchanged.

Throughout the paper the word "system" will have various meanings: (i) the random dynamics specified by a graphical construction or a generator as explained below; (ii) a particular process $\eta_. = (\eta_t, t \ge 0)$ or $\sigma_. = (\sigma_t, t \ge 0)$ governed by such dynamics; (iii) when considering hydrodynamic scaling limits, a collection $(\eta_.^N, N \in \mathbb{N}^*)$ of such processes, where N is the scaling parameter and $\eta_.^N = (\eta_t^N, t \ge 0)$. In the latter case, to reduce notation, we denote the system by $[\eta]$ (not to be confused with η without brackets, which denotes a particular configuration). The relevant interpretation will always be clear from the context and notation. We now introduce a graphical construction to define general conservative dynamics on $\mathbb{Z}$ with bounded interaction range and transition rates.

2.1. *The graphical construction.*

The probability space. We shall construct random particle dynamics on a product probability space $\Omega \times \Omega'$. Space Ω is a "dynamical" space of random events that entirely determine the evolution of the system from a given initial configuration; Ω' is an auxiliary space used to randomize initial configurations.

The space Ω' and its underlying probability measure need not be specified but should be "large enough" so that any probability measure on $\mathcal{E}_d$ or $\mathcal{E}_u$ can be realized as a random variable on Ω'. We now proceed to the definition of Ω.

Let $\mathcal{U}$ denote an arbitrary measurable space endowed with some finite nonnegative measure μ. We define a *system of clocks* as a locally finite countable sum $\omega(dt, dx, du)$ of Dirac measures on $\mathbb{R}^{+*} \times \mathbb{Z} \times \mathcal{U}$ such that $\omega(\{(t, x, u)\}) \in \{0; 1\}$ for every (t, x, u). $\omega(\{(t, x, u)\}) = 1$ means that a clock rings at site x and time t producing some "threshold" value u. (t, x, u) will be called an *event* relative to ω. The "dynamical" probability space Ω will be the set of ω's equipped with the filtration $(\mathcal{F}_t,\ t > 0)$, where $\mathcal{F}_t$ is the σ-field generated by the mappings $\omega \mapsto \omega((0; t] \times S \times U)$ for finite subsets S of $\mathbb{Z}$ and measurable subsets U of $\mathcal{U}$. The underlying probability measure, denoted by $\mathbb{P}$, is such that ω is a Poisson measure with intensity $\lambda_{\mathbb{R}^{+*}}(dt)\lambda_{\mathbb{Z}}(dx)\mu(du)$, where λ denotes the counting measure or the Lebesgue measure. For $t > 0$ and $x \in \mathbb{Z}$, we define a time shift $\theta_t\omega$ and a space shift $\tau_x\omega$ as follows: θ_t forgets events prior to time t and resets the time origin to t, while $\tau_x\omega$ sets the space origin at x.

Construction of dynamics. We consider a collection T of *conservative*, *uniformly finite-range* transformations $\eta \mapsto T^{x,u}\eta$ on $\mathcal{E}_u$ with the following interpretation: whenever an event (t, x, u) occurs, the current particle configuration η is replaced with $T^{x,u}\eta$. For simplicity T itself will be called a transformation. By uniformly finite-range, we mean that there is some $r \in \mathbb{N}$ (the interaction range) independent of x, u such that $T^{x,u}$ only acts on sites $x - r$ to $x + r$ without looking at other sites. By conservative, we mean that the total number of particles in these sites is unchanged under $T^{x,u}$. It follows that, if the configuration η is congested on $[x - r; x + r]$, that is, $\eta(y) = K$ for every $y \in [x - r; x + r]$, then $T^{x,u}\eta = \eta$. The transformation T may be defined on distinguishable configurations in a unique way that does not affect the positions of labelled particles outside $[x - r; x + r]$ and maintains increasing labels to the right among particles within $[x - r; x + r]$. For simplicity, we shall still write $T^{x,u}\sigma$ for this transformation. We define the random evolution of a system on $\Omega \times \Omega'$ by the update rule following an ω-event (t, x, u):

$$\eta_t(\omega, \omega') = T^{x,u}\eta_{t^-}(\omega, \omega'), \qquad \sigma_t(\omega, \omega') = T^{x,u}\sigma_{t^-}(\omega, \omega'), \tag{2}$$

where $\eta_0(\omega, \omega') = \eta_0(\omega')$ and $\sigma_0(\omega, \omega') = \sigma_0(\omega')$ are random initial states defined on Ω'. Note that, since all processes are constructed on $\Omega \times \Omega'$, simultaneously evolving systems are coupled in a natural way by using the same system of clocks.

REMARK. It takes some care to give sense to (2) in infinite volume, as ω has infinitely many events in any bounded time interval; but this can be done by finite-volume approximations, using locality of interactions. Besides, to prove that a certain property holds for η at all times, it is enough to prove it for ω's with finitely many events in bounded time, which allows considering successive events.

The process $(\eta_t,\ t \geq 0)$ defined by (2) is Markov with generator

$$\mathcal{L}f(\eta) = \sum_{x\in\mathbb{Z}} \int_{\mathcal{U}} [f(T^{x,u}\eta) - f(\eta)]\mu(du). \tag{3}$$

Note that different transformations T may lead to the same generator. The transformation T is called *order-preserving* if $\sigma \leq \tau$ implies $T^{x,u}\sigma \leq T^{x,u}\tau$ for every $x \in \mathbb{Z}$ and $u \in \mathcal{U}$; see Section 2.5 for a discussion of this property. A transformation $\tilde{T}$ is said to be *slower* than T, if $\tilde{T}^{x,u}\sigma \leq T^{x,u}\sigma$ for every $x \in \mathbb{Z}$, $u \in \mathcal{U}$ and $\sigma \in \mathcal{E}_u$. A *local perturbation* of T is another transformation $\tilde{T}$ such that $\tilde{T}^{x,u} = T^{x,u}$ for every $u \in \mathcal{U}$ and $x \in \mathbb{Z}$ such that $|x| > D$, where $D \in \mathbb{N}$ is the perturbation range.

2.2. *Main results.* We consider a system defined by some transformation T and a local perturbation $\tilde{T}$ thereof. We make the following assumptions.

ASSUMPTION 2.1. T and $\tilde{T}$ are order-preserving, and $\tilde{T}$ is slower than T.

The second part of the above assumption means that the perturbation tends to slow down *rigthward* motion of particles. We next formulate assumptions on the hydrodynamic behavior of the unperturbed dynamics. To this end, let us first recall standard definitions. Let $N \in \mathbb{N}^*$ denote the scaling parameter, that is, the inverse of the macroscopic intersite distance. To every configuration η, we associate the *empirical measure at scale N* defined as a (locally finite, nonnegative) measure

$$\alpha^N(\eta, dx) = N^{-1} \sum_{y\in\mathbb{Z}} \eta(y)\delta_{y/N}(dx) \tag{4}$$

on $\mathbb{R}$. We say a random sequence of configurations η^N has *density profile* $\rho(\cdot)$ as $N \to \infty$, where $\rho(\cdot)$ is a $[0; K]$-valued Borel function on $\mathbb{R}$, if the random measures α^N converge in probability to the deterministic measure $\rho(\cdot)\,dx$ as $N \to \infty$; here and in the sequel, the space of locally finite measures on $\mathbb{R}$ is equipped with the topology of vague convergence. Density profile on a subinterval of $\mathbb{R}$ is defined by considering the restricted empirical measures. We say the system $[\eta]$ has hydrodynamic profile $\rho(\cdot,\cdot)$ [resp. initial density profile $\rho_0(\cdot)$], where $\rho(\cdot,\cdot)$ [resp. $\rho_0(\cdot)$] is defined on $\mathbb{R}^+ \times \mathbb{R}$ (resp. $\mathbb{R}$), if η^N_{Nt} has density profile $\rho(t,\cdot)$ [resp. $\rho_0(\cdot)$] for every $t \geq 0$ (resp. for $t = 0$).

ASSUMPTION 2.2. The unperturbed hydrodynamic behavior under Euler time scaling is given by the entropy solutions to a scalar conservation law with nonnegative, Lipschitz-continuous flux h defined on $[0; K]$:

$$\partial_t\rho(t,x) + \partial_x h\big(\rho(t,x)\big) = 0. \tag{5}$$

Precisely, for every measurable initial profile ρ_0, every system $[\eta]$ governed by T with initial density profile ρ_0 has hydrodynamic profile $\rho(\cdot,\cdot)$, where $\rho(\cdot,\cdot)$ is the unique entropy solution to (5) with initial datum ρ_0.

REMARK. The above assumption means that, for every ρ_0, the hydrodynamic limit must hold for *any* initial sequence with profile ρ_0. This might seem restrictive. In fact it will be seen (second remark in Section 2.5) that, due to order-preservation, it is equivalent to assume the hydrodynamic limit for *some* initial sequence with profile ρ_0.

$h(\rho)$ is the stationary particle flux for a system with uniform density ρ; as a result we have $h(0) = h(K) = 0$ (see Lemma 5.5). We refer to [17] and [30] for standard definitions and properties of entropy solutions. Lipschitz continuity is the natural regularity assumption for existence and uniqueness of entropy solutions to (5). The assumption $h \geq 0$ means that the *macroscopic* motion of particles is to the right. However, we do not need the latter assumption on microscopic motion, that is, our results also cover systems with both leftward and rightward particle jumps, but positive mean drift.

We set $h^* = \sup h$ and define ϕ^* as follows: $\phi^* = h^*$ if h has no genuine local minimum, otherwise ϕ^* is the infimum of genuine local minima of h. By genuine local minimum we mean a value $h(\rho)$ that is the minimum of h on $[\rho_1; \rho_2]$, with $0 < \rho_1 < \rho < \rho_2 < K$, $h(\rho_1) > h(\rho)$, $h(\rho_2) > h(\rho)$. It is not difficult to show that ϕ^* is itself a genuine local minimum of h. Denote by η^* the configuration that is full up to site 0 and empty to the right of 0. A simple subadditive argument (see Section 5) shows that the *deterministic* limit

$$\phi = \lim_{t \to \infty} t^{-1} \phi_t \in [0; h^*] \tag{6}$$

exists $\mathbb{P}$-almost surely, where ϕ_t denotes the rightward current through 0 up to time t in the perturbed system starting from η^*. This current value ϕ can be interpreted as the maximum current value through $x = 0$ for the system under the perturbation $\tilde{T}$; we shall also call it the *critical current*. It is shown in Corollary 4.4 that every value $\phi \in [0; h^*]$ corresponds to possible perturbations.

Our results assert that the hydrodynamic behavior of the perturbed system is simply determined by ϕ if $\phi \leq \phi^*$ or, equivalently, if $\inf(h; \phi)$ is first nondecreasing and then nonincreasing. First, in order to describe this behavior, we state a new result relative to (5). In Section 3 we shall assign to each maximum current value $0 \leq \phi \leq \phi^*$ a modified entropy condition for weak solutions to (5). Two equivalent forms will be given: one is based on a modified set of admissible discontinuities (Definitions 3.1 and 3.2) and is a rather natural formalization of the idea that current values above ϕ are forbidden. The other definition is more general and relies on modified Kružkov entropies (Definition 3.3). A corresponding weak solution will be called a *ϕ-entropy* solution.

THEOREM 2.1. *For every $\phi \in [0; \phi^*]$, there exists a unique* (*up to a null subset of* $\mathbb{R}^{+*} \times \mathbb{R}$) *$\phi$-entropy solution ρ with given measurable initial datum $\rho_0(\cdot)$.*

For $\phi < h^*$, the selected solution generally differs from the usual entropy solution that describes the unperturbed system. Following is the main result of this paper.

THEOREM 2.2. *Assume ϕ defined by* (6) *lies in* $[0; \phi^*]$. *Then the perturbed hydrodynamic behavior is given by the ϕ-entropy solutions to* (5). *Precisely, for every measurable initial profile ρ_0, every perturbed system $[\eta]$ with initial density profile $\rho_0(\cdot)$ has hydrodynamic profile $\rho(\cdot, \cdot)$, the unique ϕ-entropy solution to* (5) *with initial datum $\rho_0(\cdot)$.*

REMARK. The above result means that the perturbed hydrodynamic limit holds for *any* initial sequence that achieves any given density profile.

When $\phi^* = h^*$, that is, h has no genuine local minimum (which includes, e.g., all concave h), the hydrodynamic limit follows from Theorem 2.2 for *any* local perturbation. If $\phi^* < h^*$, that is, h has genuine local minima, the hydrodynamic limit follows from Theorem 2.2 only for perturbations strong enough to have $\phi \le \phi^*$. For instance, we may consider a family of order-preserving local perturbations T_α indexed by $\alpha \in [0; 1]$, such that the maximum current ϕ_α is continuous and nondecreasing w.r.t. α, with $\phi_0 = 0$ and $\phi_1 = h^*$. This construction is always possible irrespective of the model, see Section 2.4. Define $\alpha_0 = \sup\{\alpha \in [0; 1] : \phi_\alpha \le \phi^*\}$, so that $\phi_{\alpha_0} = \phi^*$, $\alpha_0 = 1$ iff $\phi^* = h^*$ (i.e., h has no genuine local minimum), $\alpha_0 > 0$ if $\phi^* > 0$. By Theorem 2.2, the hydrodynamic behavior of T_α for $0 \le \alpha \le \alpha_0$ is given by the ϕ_α-entropy solution.

Further issues. A natural and challenging question is the existence and characterization of the hydrodynamic limit in the region $\phi^* < \phi < h^*$, when h has at least one genuine local minimum. One may further relax the assumption $h \ge 0$ and consider perturbations that do not act in a definite direction. In these cases knowledge of the maximal current ϕ is no longer sufficient to define a unique entropy condition. Hydrodynamics should be described by a more general class of modified entropy conditions that do not depend on a single parameter. Next, we believe that the order-preservation assumption could be somehow weakened. The point is that a weaker version of it is always true for attractive processes with irreducible jumps, as proved recently by Bramson and Mountford [6]. This would allow, for example, to relax the nearest-neighbor assumption in Example 1 in Section 2.3; Example 3 could also be extended to encompass k-step exclusion processes with nonnearest-neighbor jump kernels (see [11]).

2.3. *A review of examples.* We now discuss some examples of systems that satisfy Assumptions 2.1 and 2.2. Examples of corresponding order-preserving perturbations, that cover the whole range $0 \le \phi \le \phi^*$, will be given in Section 2.4.

In the sequel $b(n,m)$ will be a bounded nonnegative function, defined for $n, m \in \mathbb{N}$, with the following properties:

$$b \text{ is a nondecreasing (resp. nonincreasing) function of } n \text{ (resp. } m), \tag{7}$$

$$b(0,1) = 0. \tag{8}$$

The standard notation $\eta^{x,y}$ for $x, y \in \mathbb{Z}$, $x \neq y$, denotes the configuration obtained from η after a particle has jumped from x to y.

EXAMPLE 1 (Nearest-neighbor attractive processes). Here we make the additional assumptions that b is restricted to $n, m \in \{0, \dots, K\}$, where K is the maximum number of particles per site and

$$b(K,K) = 0. \tag{9}$$

Let $0 \leq q < p$ be two real numbers. We consider the process with infinitesimal generator given by

$$\begin{aligned} \mathcal{L}f(\eta) = &\sum_{x\in\mathbb{Z}} pb\big(\eta(x), \eta(x+1)\big)\{f(\eta^{x,x+1}) - f(\eta)\} \\ &+ \sum_{x\in\mathbb{Z}} qb\big(\eta(x), \eta(x-1)\big)\{f(\eta^{x,x-1}) - f(\eta)\}. \end{aligned} \tag{10}$$

The graphical construction can be obtained as follows. We set $\mathcal{U} = \{-1; 1\} \times (0; \sup b)$ and $\mu = \mu_1 \otimes \mu_2$, with $\mu_1 = p\delta_1 + q\delta_{-1}$ and μ_2 the Lebesgue measure on $(0; \sup b)$ and define $T^{x,u}\eta$ as follows, where $u = (u_1, u_2) \in \mathcal{U}$: $T^{x,u}\eta = \eta^{x,x+u_1}$ if $u_2 \leq b(\eta(x), \eta(x+u_1))$, $T^{x,u}\eta = \eta$ otherwise. Assumption (7) implies that T is attractive, that is, $\eta \leq \zeta$ implies $T^{x,u}\eta \leq T^{x,u}\zeta$. Thus, T is also order-preserving by Proposition 2.1 in Section 2.5. Assumption 2.2 has been verified in the following cases:

1. *Misanthrope processes*. For these systems additional properties of b imply existence of product invariant measures [8], and the validity of Assumption 2.2 follows from [22] (see also [4] for a constructive proof), with an explicit $h \in C^\infty$. Moreover $q < p$ implies that h is positive on $(0; K)$, so $\phi^* > 0$. Several examples are reviewed in [2] and [4]. For instance, here is a natural generalization of ASEP for which the flux has no genuine local minimum and, thus, Theorem 2.2 holds for any value of ϕ: let f be a nondecreasing function defined on $\{0, \dots, K\}$, with $f(0) = 0 < f(K) = 1$, and set $b(n,m) = f(n)(1 - f(m))$. Then we have $h(\rho) = (p-q)F(\rho)(1 - F(\rho))$, where $F(\rho)$ is an increasing, smooth function with $F(0) = 0 < F(K) = 1$. For ASEP we have $f(n) = n \wedge 1$ and $F(\rho) = \rho$. Note that in the latter case h is concave, but this may not hold for more general f.

2. *Nearest-neighbor asymmetric K-exclusion process.* This is the system defined by $b(n,m) = \mathbf{1}_{n>0}\mathbf{1}_{m<K}$, with $K \geq 2$ ($K = 1$ being the exclusion process). Assumption 2.2 was verified in [23]; however, for $q \neq 0$ it is only known that

h is Lipschitz continuous with $h \geq 0$ and, thus, $\phi^* \geq 0$. In the totally asymmetric case $q = 0$, Assumption 2.2 was verified earlier in [26] and the flux proved to be also concave, so Theorem 2.2 holds for any value of ϕ. In this case the original dynamics satisfies a variational coupling and, when the perturbed dynamics still does, Theorem 2.2 should also follow from the approach of [28]; however, our approach allows more general perturbations that do not necessarily admit a variational coupling (see the third remark in Section 2.5).

EXAMPLE 2 (ASEP with speed change). Here we make the assumption that there exists some $r \in \mathbb{N}$ such that

$$\forall n, m \in \mathbb{N}, \qquad b(n, m) = b(n \wedge r, m \wedge r). \tag{11}$$

Site capacity is 1, and the jump rate of a particle depends increasingly (decreasingly) on the distance to the next (previous) particle in the corresponding direction: such "rational behavior" is always implied, in a more general sense, by order-preservation (see Section 2.5). Let

$$d_1(x, \eta) = \inf\{y > x : \eta(y) = 1\} - x - 1,$$
$$d_{-1}(x, \eta) = x - \sup\{y < x : \eta(y) = 1\} - 1.$$

The generator of the process is given by

$$\begin{aligned} \mathcal{L} f(\eta) = {} & \sum_{x \in \mathbb{Z}} p\eta(x) b\big(d_1(x, \eta), d_{-1}(x, \eta)\big)\{f(\eta^{x,x+1}) - f(\eta)\} \\ & + \sum_{x \in \mathbb{Z}} q\eta(x) b\big(d_{-1}(x, \eta), d_1(x, \eta)\big)\{f(\eta^{x,x-1}) - f(\eta)\}. \end{aligned} \tag{12}$$

Here ASEP is recovered as the special case $b(n, m) = (n \wedge 1)$. The graphical construction can be obtained as follows. Let p, q, $\mathcal{U}$ and μ be as in Example 1. We define $T^{x,u}$ as follows: $T^{x,u}\eta = \eta^{x,x+u_1}$, if $\eta(x) = 1$ and $u_2 \leq b(d_{u_1}(x, \eta), d_{-u_1}(x, \eta))$, $T^{x,u}\eta = \eta$ otherwise. T has finite-range r because of (11). Though order-preservation can be shown as a simple consequence of assumption (7), these systems are *not* attractive outside of ASEP. However, a generalization of the particle-hole correspondence used in [18] leads to the following statement: if the inverted version of (10) (i.e., with p and q exchanged) is governed by the entropy solutions to (5), with $-h$ instead of h, then (12) is governed by the entropy solutions to

$$\partial_t \rho + \partial_x \left[\rho h\left(\frac{1}{\rho} - 1\right)\right] = 0. \tag{13}$$

As a result, Assumption 2.2 is valid for (12) whenever it is for (10). However, it is important to note that, despite the correspondence with (10) for the translation-invariant system, local perturbations of (12) *cannot* be reduced to local perturbations of (10), because a *site* in the latter is mapped onto a *particle* in the former. Therefore, in order to deal with perturbations of (12), it is crucial for our approach not to rely on attractiveness.

When b belongs to the "misanthrope" class, we again have $h \in C^{\infty}$ and $h > 0$ on $(0;1)$ for $q < p$, so $\phi^* > 0$. Here are two physically interesting examples. First, b may be *any* nondecreasing function of n only, with $b(0) = 0 < b(1)$; for $p = 1$, this is a stochastic version of the so-called "optimal velocity (OV) models" from traffic-flow theory (see, e.g., [7]), where each car adjusts its speed increasingly from its distance to the one ahead. Next, the "kinetic Ising model" is introduced in [14] and also studied in [21]. This corresponds to $p = 1$ and b given by $b(n,m) = 1 + \delta$ for $n \geq 2$ and $m \geq 1$, $b(n,0) = 1 + \varepsilon$ for $n \geq 2$, $b(1,m) = 1 - \varepsilon$ for $m \geq 1$ and $b(1,0) = 1 - \delta$, where $|\delta| \leq 1$ and $\varepsilon \in [0;1)$. In addition, to satisfy (7), we require $\varepsilon \geq \delta$. δ accounts for the asymmetry of h (h is symmetric w.r.t. $\rho = 1/2$ for $\delta = 0$), while ε measures particle repulsion. For TASEP we have $\varepsilon = \delta = 0$. As ε increases, the flux first exhibits a change of convexity to the right of its maximum, and then a second maximum appears for large densities, with a genuine local minimum ϕ^* inbetween (see [21]). In the latter case, Theorem 2.2 holds only for $\phi \leq \phi^*$.

EXAMPLE 3 (Nearest-available neighbor ASEP). Let $k \in \mathbb{N}^*$ be given. We consider a new generalization of ASEP where a particle chooses a direction and moves to the first empty site in this direction, provided the jump does not exceed k sites. In the context of traffic flow, this may be interpreted as overtaking with prescribed limited visibility. The generator is of the form

$$\begin{aligned} \mathcal{L}f(\eta) = \sum_{x,y \in \mathbb{Z}} & p(y-x)\eta(x)\big(1-\eta(y)\big) \\ & \times \prod_{z=\min(x,y)+1}^{\max(x,y)-1} \eta(z)\{f(\eta^{x,y}) - f(\eta)\} \end{aligned} \tag{14}$$

for some nonnegative function $p(\cdot)$ defined on $\mathbb{Z}$, with the convention that the empty product is equal to 1. We assume that (i) the support of p is finite, (ii) $p(1) + p(-1) > 0$ and (iii) $p(y) \leq p(x)$ if $0 < x \leq y$ or $y \leq x < 0$. This class of dynamics is partly related to k-step exclusion processes introduced in [11]. Namely (14) contains all k-step exclusion processes with nearest-neighbor kernels: this corresponds to the case when $p(x)$ is the probability that a nearest-neighbor random walk starting from 0 reaches x in at most k steps without returning to 0 inbetween. When p is not of the latter form, the system is not a k-step exclusion process; on the other hand k-step exclusion processes with nonnearest-neighbor kernels do not belong to the class (14). Below we give a graphical construction that is attractive under assumption (iii); thus, T is also order-preserving by Proposition 2.1 in Section 2.5. For any p satisfying (i), product Bernoulli measures are stationary for $\mathcal{L}$; if in addition (ii) and (iii) are satisfied, one can show, as in [11], that they are extremal among translation invariant stationary measures. Hence, the validity of Assumption 2.2, follows from [22] or [4], and the flux is given by

$$h(\rho) = \sum_{i=1}^{k} i[p(i) - p(-i)]\rho^i(1-\rho). \tag{15}$$

It is not difficult to show that every $q:\mathbb{Z} \to \mathbb{R}^+$ with finite support is of the form $q(\cdot) = p(\cdot) - p(-\cdot)$ for some $p(\cdot)$ satisfying (i)–(iii) above. Thus, every polynomial flux vanishing at 0 and 1 can be obtained by a proper choice of p. The nearest-neighbor totally asymmetric k-step exclusion process corresponds to $p(x) = 1$ if $1 \le x \le k$, 0 otherwise. It is shown in [4] that the resulting flux has a single inflection point, hence, no genuine local minimum, since $h(0) = h(1) = 0$. Thus, Theorem 2.2 is valid for every value of ϕ. More complex behavior is easy to construct. Allowing maximum jump range 3, one may obtain $h(\rho)$ positive on (0; 1), with degree 4, two inflection points, and either (a) no genuine local minimum or (b) two local maxima and a genuine local minimum $\phi^* > 0$ inbetween. In the latter case, perturbed behavior follows from Theorem 2.2 whenever $\phi \le \phi^*$.

An attractive graphical construction of (14) can be defined as follows. We set $\mathcal{U} = \{-1; 1\} \times (0; \sup_x p(x))$, $\mu = \mu_1 \times \mu_2$, with $\mu_1 = \delta_1 + \delta_{-1}$ and μ_2 the Lebesgue measure on $(0; \sup_x p(x))$. Then we define $T^{x,u}\eta$ as follows, where $u = (u_1, u_2)$. Let $y(x, u_1, \eta)$ denote the first η-empty site to the right (left) of x if $u_1 = 1$ ($u_1 = -1$). We set $T^{x,u}\eta = \eta^{x,y(x,u_1,\eta)}$ if $\eta(x) = 1$ and $u_2 \le p(y(x, u_1, \eta) - x)$, otherwise $T^{x,u}\eta = \eta$. We leave the reader to check that attractiveness follows from assumption (iii) for $p(\cdot)$.

2.4. *Particular perturbations.* In this section, we construct a model-independent one-parameter family of order-preserving local perturbations T_α, where $\alpha \in [0; 1]$, with the following properties: for $\alpha = 0$ there is total blockage, that is, particles can never jump over $x = 0$ to the right; for $\alpha = 1$ there is no perturbation; more generally, T_α is slower than T_β for $\alpha \le \beta$. Moreover, T_α reduces to TASEP(α) when the original dynamics is TASEP and is a natural generalization thereof for more general dynamics (see examples below). This construction will also play an important role in the proof of Theorem 2.2 in Section 5.

We first define a conservative transformation T_0 which forbids particles jumping over 0 *to the right*. If there is no label i such that $\sigma(i) \le 0 < T^{x,u}\sigma(i)$, we simply set $T_0^{x,u}\sigma = T^{x,u}\sigma$. Otherwise, let n denote the highest such label; for all labels $i > n$, we set $T_0^{x,u}\sigma(i) = T^{x,u}\sigma(i)$. For labels $i \le n$, we define $T_0^{x,u}\sigma(i)$ from right to left, so that at each step the configuration-admissibility constraint

$$T_0^{x,u}\sigma(i) \le \min\bigl(T_0^{x,u}\sigma(i+1); T_0^{x,u}\sigma(i+K) - 1\bigr)$$

be respected. It is easy to see that this is guaranteed if we set $T_0^{x,u}\sigma(n) = 0$ and then let traffic perturbation propagate inductively to the left according to the rule

$$T_0^{x,u}\sigma(i) = \min\bigl(T^{x,u}\sigma(i); T_0^{x,u}\sigma(i+1); T_0^{x,u}\sigma(i+K) - 1\bigr) \tag{16}$$

for $i < n$. In fact, $T_0^{x,u}\sigma$ is the maximal configuration σ' such that $\sigma' \leq T^{x,u}\sigma$ and $\sigma'(n) = 0$: drivers seek the smallest deceleration compatible with the initial blockage. Though this transformation is defined on distinguishable configurations, it is not affected by a shift of labels. Thus, it can be defined on undistinguishable configurations. T_0 is a local perturbation of T, since for $x \notin (-r; r]$ no particle may cross 0 to the right under $T^{x,u}$. The point of the construction is that it still yields an order-preserving tranformation with uniformly finite range r. The first property follows immediately by induction from (16). The second one may seem less obvious, because a blockage might a priori propagate infinitely to the left. However, another easy induction from (16) reveals that

$$\min(T^{x,u}\sigma; \sigma) \leq T_0^{x,u}\sigma \leq T^{x,u}\sigma. \tag{17}$$

From this one may easily infer that, if $T^{x,u}$ has finite range r, the same holds for $T_0^{x,u}$. To define the *partial blockage* T_α, we enrich $\mathcal{U}$ by setting $\mathcal{U}' = (0; 1) \times \mathcal{U}$. $\mathcal{U}'$ is equipped with the measure $\lambda_{(0;1)} \otimes \mu$. We extend T (and likewise T_0) to the enriched space by setting $T^{x,(\theta,u)} = T^{x,u}$ for $u' = (\theta, u) \in \mathcal{U}'$. Then we set $T_\alpha^{x,u'}\sigma = T_0^{x,u}\sigma$ (resp. $T_\alpha^{x,u'}\sigma = T^{x,u}\sigma$), if $\theta > \alpha$ (resp. $\theta \leq \alpha$). Clearly, T_α is an order-preserving local perturbation of T with range r, T_α is slower than T_β for $\alpha \leq \beta$, and $T_1 = T$. Corollary 4.4 shows that the maximum current ϕ_α for T_α is a nondecreasing, Lipschitz-continuous function of α, with $\phi_0 = 0$ and $\phi_1 = h^*$.

Let $\mathcal{L}_\alpha$ denote the generator associated with T_α via (3). It is easy to see that $\mathcal{L}_\alpha = (1-\alpha)\mathcal{L}_0 + \alpha\mathcal{L}_1$. An interesting property of this construction is that, although a given $\mathcal{L}$ may be obtained from different transformations T, $\mathcal{L}_\alpha$ depends only on $\mathcal{L}$ and not on the choice of the underlying T. This follows from the fact that $T_0^{x,u}\eta = \gamma(\eta, T^{x,u}\eta)$ for some intrinsic mapping γ independent of T, x and u. As an illustration, we may apply the construction to the systems in Section 2.3. In (10) and (12), $\mathcal{L}_\alpha$ is obtained by replacing p with αp for $x = 0$. In (14), whenever $x \leq 0 < y$, one should replace $f(\eta^{x,y}) - f(\eta)$ with $(1-\alpha)[f(\eta^{1,y}) - f(\eta)] + \alpha[f(\eta^{x,y}) - f(\eta)]$.

2.5. *Remarks on order-preservation.* We end this section by discussing some aspects of order-preservation. In particular, we consider its connection with attractiveness and variational coupling, two properties that have been used extensively to verify Assumption 2.2 for a variety of systems.

A physical interpretation. Compare the reaction of a driver at a given site to a given event, in two different environments σ and τ, assuming $\sigma \leq \tau$. This inequality means that the driver sees more space ahead of him and less space behind in environment τ. Order preservation implies that he will move further to the right, that is, with higher (signed) "speed," in the latter environment. We may view such behavior as rational.

The initial configuration problem. Assume two systems $[\eta]$ and $[\zeta]$ are governed by the same order-preserving transformation, and (i) $[\eta]$ and $[\zeta]$ have a common initial density profile (ii) $[\eta]$ has some hydrodynamic profile $\rho(\cdot,\cdot)$. Then it follows easily from Lemma 5.3 that $[\zeta]$ has hydrodynamic profile $\rho(\cdot,\cdot)$. Lemma 5.3 is merely the scaling limit of Corollary 4.1, itself a simple consequence of order-preservation. This explains the remark following Assumption 2.2. Thus, for all examples considered in Section 2.3, the hydrodynamic limit (both unperturbed and perturbed) holds under the assumption of a density profile at time 0, without any further hypothesis on initial configurations.

Order-preservation and variational coupling. The existence of a variational coupling was shown in [24] to be equivalent to the so-called *strong monotonicity* property, which implies order-preservation. In the context of our graphical construction, strong monotonicity means that

$$T^{x,u}\inf(\sigma,\tau)=\inf(T^{x,u}\sigma,T^{x,u}\tau),$$

where the infimum of two labelled configurations is defined labelwise. This property is much more restrictive than order-preservation because it necessarily implies flux concavity, but also appears (as observed in [27]) to rule out partially asymmetric systems and sytems with configuration-dependent jump rates. For instance, there is no known variational coupling for the examples of Section 2.3, when they do not reduce to a totally asymmetric K-exclusion process. Note that the unperturbed system may admit a variational coupling, but the perturbed version may be merely order-preserving: consider, for example, the extension of TASEP(α), where the jump rate from 0 to 1 is modified to $b(d_1(0,\eta),d_{-1}(0,\eta))$, with the notations and assumptions of Section 2.3 (Example 2), and $b\le 1$. Due to monotonicity assumptions on b, the perturbed TASEP is still order-preserving, but does not admit a variational coupling for nonconstant b, though the original system does; thus, the perturbed hydrodynamics cannot be established here by the approach of [28].

Order-preservation and attractiveness. Here we establish a connection between attractiveness and order-preservation for a wide class of transformations; recall from Section 2.3 (Example 1) that the transformation T is said to be attractive if it is nondecreasing w.r.t. *undistinguishable* configurations, that is, $T^{x,u}\eta\le T^{x,u}\zeta$ for every $x\in\mathbb{Z}$ and $u\in\mathcal{U}$ whenever $\eta\le\zeta$. We consider transformations with the following properties in the undistinguishable representation: (i) $T^{x,u}\eta$ differs from η by at most one particle jump. (ii) For given (x,u) there is a single allowed jump direction, that is, right or left (but the jump size may vary with η). (iii) A jumping particle can only move to the nearest *available* site to its right or left. Note in (i) that a single η-jump, in case of overtaking, involves several *labelled* particles' jumps in the distinguishable configuration, as labels must increase to the right. We call *nearest-available neighbor* those transformations that satisfy (i)–(iii). This is a natural extension of the usual nearest-neighbor property: for the latter a particle may not jump more than one step to its right or left. For instance, the nearest-available neighbor ASEP (Example 3 in Section 2.3) with $k \geq 2$, is a nearest-available neighbor, but not nearest-neighbor system. The reader may easily check that all other examples in Section 2.3 are nearest-available neighbor systems.

PROPOSITION 2.1. *Let T be an attractive, nearest-available neighbor transformation. Then T is order-preserving.*

PROOF. Since x and u are fixed, we shall simply write T for $T^{x,u}$, and assume, for example, that particles jump to the right. We consider $\sigma \leq \tau$ and wish to derive $T\sigma \leq T\tau$ from attractiveness. Since $\tau \leq T\tau$, this is immediate if $T\eta = \eta$. We thus assume that $T\eta = \eta^{x,x+y}$, where $x + y$ is the first available site for η to the right of x; this implies that sites $x + 1$ through $x + y - 1$, inclusive, are η-congested. Here is the corresponding distinguishable transformation: letting n denote the label of the σ-particle at x with highest label, $T\sigma$ is defined by moving labels $n + iK$, $0 \leq i < y$, one step to the right.

CASE 1. $x = \sigma(n) < \tau(n)$. It follows easily that σ-particles labelled $n + iK$, where $0 \leq i < y$, are *strictly* behind τ-particle with the same labels. Hence, $T\sigma \leq \tau \leq T\tau$.

CASE 2. $\tau(n) = \sigma(n)$. Since $\sigma \leq \tau$, this implies $\eta(x) \leq \zeta(x)$. We first prove that the undistinguishable transformation then takes a ζ-particle at x to the nearest available site for ζ. We consider $\eta' = \min(\eta, \zeta)$, where the minimum is defined sitewise. By attractiveness, we must have $T\eta' \leq T\eta$. Since $\eta'(x) = \eta(x)$ and an η-particle leaves x, this forces an η' particle to leave x to the nearest available site $x + y'$ for η', where $0 < y' \leq y$. But $\sigma \leq \tau$ implies that $\eta'(z) = \zeta(z)$ for $x + 1 \leq z \leq x + y'$, so $x + y'$ is also the first available site to the right for ζ. Since by attractiveness we must have $T\eta' \leq T\zeta$, it follows that a ζ particle must be taken from some site $z \leq x$ to $x + y'$.

We next claim that this implies $T\sigma \leq T\tau$. Indeed, the only labels affected by T are $n+iK$, where $0 \leq i < y$. But $\sigma(n) \leq \tau(n)$ implies that $\sigma(n+iK) = \tau(n+iK)$ for $0 \leq i < y'$, for which labels both the σ and τ particles are moved; while $\sigma(n + iK) < \tau(n + iK)$ for $y' \leq i < y$, for which labels only the σ-particle is moved, but cannot overtake the τ-particle. $\square$

3. Definition and properties of ϕ-entropy solutions. In this section we recall standard facts about entropy solutions, define the notion of ϕ-entropy solution, and state some related properties useful for the sequel.

3.1. *Entropy solutions.* We recall Kružkov's definition [17] of entropy weak solutions to (5). Defining the Kružkov entropies $\varphi(\rho; c) = |\rho - c|$ and the associated entropy flux $\psi(\rho; c) = \operatorname{sgn}(\rho - c)(h(\rho) - h(c))$, we say a $[0; K]$-valued Borel function ρ on $\mathbb{R}^{+*} \times \mathbb{R}$ is an entropy solution to (5), if and only if Kružkov's entropy inequalities

$$\partial_t \varphi(\rho; c) + \partial_x \psi(\rho; c) \le 0 \tag{18}$$

hold in distribution sense on $\mathbb{R}^{+*} \times \mathbb{R}$ for every $c \in [0; K]$; this implies (5), as can be seen taking $c = 0$ and $c = K$. It is known [17] that there is a unique (up to a null subset of $\mathbb{R}^{+*} \times \mathbb{R}$) entropy solution assuming a given ($[0; K]$-valued, Borel) initial datum $\rho_0(\cdot)$ on $\mathbb{R}$ in the sense

$$\operatorname{ess\,lim}_{t \to 0} \int_a^b |\rho(t, x) - \rho_0(x)| \, dx = 0 \qquad \forall a, b \in \mathbb{R}, \tag{19}$$

where ess lim means that the limit holds along some total subset of $\mathbb{R}^{+*}$. If $\rho(\cdot, \cdot)$ has left/right hand limits $\rho(t, 0^{\pm}) := \lim_{x \to 0^{\pm}} \rho(t, x)$ for a.e. $t > 0$, we may separately consider condition (18) on $\mathbb{R}^{+*} \times \mathbb{R}^*$ (that is away from $x = 0$), and its trace along the line $x = 0$. From differentiation theory in distribution sense, validity of entropy condition (18) on the whole space, (i.e., on $\mathbb{R}^{+*} \times \mathbb{R}$) for every $c \in [0; K]$, is equivalent to the simultaneous two conditions:

(E1) Condition (18) holds away from $x = 0$ (i.e., on $\mathbb{R}^{+*} \times \mathbb{R}^*$).
(E2) For a.e. $t > 0$, the "boundary" pair $(\rho(t, 0^-), \rho(t, 0^+))$ satisfies

$$\psi(\rho^+; c) \le \psi(\rho^-; c) \qquad \forall c \in [0; K] \tag{20}$$

with $\rho^{\pm} = \rho(t, 0^{\pm})$ [the exceptional set in (E2) can be made independent of c, as ψ is continuous w.r.t. c]. Such pairs (ρ^-, ρ^+) will be called *admissible* or *entropic*. A geometric formulation of (20) is

(21) $h(\rho^-) = h(\rho^+)$ and the chord between ρ^- and ρ^+ on the graph of h lies below (resp. above) the graph if $\rho^- \le \rho^+$ (resp. $\rho^- \ge \rho^+$),

where "below" (resp. "above") is meant in the wide sense, that is, the chord and graph may coincide. The first condition $h(\rho^+) = h(\rho^-)$ in (21) expresses mass conservation along the line $x = 0$; if it were satisfied alone, we would only know that the conservation law (5) holds in weak sense on the whole space. The second condition in (21) is the usual Oleinik's entropy condition.

Though the entropy solution is only defined a.e., it has a representative with some regularity properties:

PROPOSITION 3.1. *The entropy solution with Cauchy datum $\rho_0(\cdot) = \rho(0, \cdot)$ has a representative ρ such that*: (i) $\rho \in C^0(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}))$; (ii) *if ρ_0 has locally bounded space variation, so does $\rho(t, \cdot)$ for every $t > 0$.*

In the sequel, we implicitely consider such a representative. One of the main properties of the entropy solution is stability w.r.t. initial datum.

PROPOSITION 3.2. *Assume ρ^1, ρ^2 are entropy solutions with initial data ρ_0^1, ρ_0^2. Define*

$$t \mapsto \Delta(t) := \int_{x+Vt}^{y-Vt} |\rho^1(t,z) - \rho^2(t,z)|\, dz \tag{22}$$

for $0 \le t \le (y-x)/(2V)$, where $\Delta(0)$ is defined by setting $\rho^i(0,x) = \rho_0^i(x)$, and V is the Lipschitz constant of h. Then $\Delta(\cdot)$ is a nonincreasing function. In particular, if ρ_0^1 and ρ_0^2 coincide a.e. on $[x; y]$, then ρ^1 and ρ^2 coincide a.e. on $[x+Vt, y-Vt]$ for every $t \in [0; (y-x)/(2V)]$.

We end up recalling the explicit construction of Riemann solutions, that is, when the initial datum is of the form

$$R_{\rho^-,\rho^+} := \rho^- \mathbf{1}_{\mathbb{R}^{-*}} + \rho^+ \mathbf{1}_{\mathbb{R}^{+*}} \tag{23}$$

(see [5], or [4] for a more general result). We leave alone the obvious case $\rho^+ = \rho^- = \rho$, where the solution has stationary uniform density ρ. Assume, for instance, that $\rho^- > \rho^+$ and denote by $\tilde{h}$ the upper convex envelope of h on the interval determined by $\rho^\pm$. $\tilde{h}$ is a concave function and there exists a nonincreasing function $\tilde{h}'$ such that $\tilde{h}$ has left (resp. right) hand derivative $\tilde{h}'(\rho^-)$ [resp. $\tilde{h}'(\rho^+)$] at every $\rho \in [0; K]$. Denote by v_* and v^* the minimum and maximum of $\tilde{h}'(\rho^\pm)$. Finally, define $\tilde{h}'^{-1}$ as a right-hand inverse of $\tilde{h}'$ on $[v_*, v^*]$; such an inverse is defined uniquely except on an at most countable set of values of v. By extension, set $\tilde{h}'^{-1}(v) = \rho^-$ for $v < v_*$ and $\tilde{h}'^{-1}(v) = \rho^+$ for $v > v^*$. It can be shown that

$$\begin{aligned} \tilde{h}'^{-1}(v^-) &= \sup\Big\{\rho \in [\rho^+; \rho^-] : h(\rho) - v\rho = \sup_{[\rho^+;\rho^-]} h(\cdot) - v.\Big\}, \\ \tilde{h}'^{-1}(v^+) &= \inf\Big\{\rho \in [\rho^+; \rho^-] : h(\rho) - v\rho = \sup_{[\rho^+;\rho^-]} h(\cdot) - v.\Big\}. \end{aligned} \tag{24}$$

For the case $\rho^- < \rho^+$, define $\tilde{h}$ as the lower convex envelope (then $\tilde{h}$ is convex and $\tilde{h}'$ is nondecreasing) and exchange ρ^+ with ρ^- and inf with sup in (24). See [4] for a proof of (24) and an application to hydrodynamics.

PROPOSITION 3.3. *The Riemann problem starting from $\rho_0 = R_{\rho^-,\rho^+}$ has the self-similar entropy solution $\rho(t,x)$ given by $\rho(t,vt) = \tilde{h}'^{-1}(v)$.*

3.2. *Definition of ϕ-entropy solutions.* We would like to specify how a maximum authorized current, $0 \le \phi \le \phi^*$, through the origin modifies the collection of entropy conditions (18). Because the perturbation is only at the origin, the corresponding solution should still satisfy (18) away from $x = 0$, plus some condition at $x = 0$. The effect of current restriction will be to modify the admissibility condition (20) in (E2). Before defining the new condition, we need to slightly weaken the definition of boundary limits $\rho(t, 0^{\pm})$ to take into account possible flat segments of h.

For two density values $r, \rho \in [0; K]$, it is easy to see that the following properties are equivalent: (i) $\psi(r; c) = \psi(\rho; c)$ for every $c \in [0; K]$ and (ii) h is constant on the closed interval defined by r and ρ. When these equivalent properties are satisfied, we say that r and ρ are *equivalent modulo* h, and we write $r = \rho$ mod. h. Admissibility of the pair (ρ^-, ρ^+) is unchanged if we replace $\rho^{\pm}$ with equivalent densities modulo h. Hence, we may extend the notion of admissibility to a pair (ρ^-, ρ^+) of densities modulo h, where a density value modulo h is an equivalence class modulo h (i.e., a maximal closed interval on which h has constant value). Obviously $h(\cdot)$, $\psi(\cdot; c)$ and ordering relations can be defined on densities modulo h. We say a real-valued function f defined on $\mathbb{R}$ has a right limit $f(x^+) = \rho^+$ modulo h, where ρ^+ is a density value modulo h, if $\lim_{y \to x^+} d(f(y), \rho^+) = 0$, where $d(f(y), \rho^+)$ is the distance from point $f(y) \in \mathbb{R}$ to interval ρ^+; the notion of left limit is defined similarly. With this weakened notion of limit, it is sufficient in condition (E2) above (20) to assume limits $\rho(t, 0^{\pm})$ in modulo h sense. We now define our modified notion of admissibility.

DEFINITION 3.1. We say the modulo h pair $(\rho^-; \rho^+)$ is ϕ-admissible (or ϕ-entropic), if either: (i) (ρ^-, ρ^+) is entropic with $h(\rho^+) = h(\rho^-) \le \phi$ or (ii) $\rho^+ < \rho^-$ and $h(\rho^+) = h(\rho^-) = \phi$.

The above definition is valid because it indeed depends only on the modulo h class of $\rho^{\pm}$. We shall call ϕ-admissible (resp. ϕ-critical) every density value ρ such that $h(\rho) \le \phi$ [resp. $h(\rho) = \phi$]. We, respectively, denote by ρ_ϕ and ρ^ϕ the smallest and greatest ϕ-critical density. The assumption $\phi \le \phi^*$ implies that the graph of h lies above ϕ on the interval $[\rho_\phi; \rho^\phi]$ [as in (21), "above" is meant in the wide sense]. Then condition (21) implies the following: (i) all pairs (ρ^-, ρ^+) such that $h(\rho^+) = h(\rho^-) = \phi$ are ϕ-entropic, and (ii) a ϕ-entropic pair (ρ^-, ρ^+) is nonentropic iff $h(\rho^+) = h(\rho^-) = \phi$, $\rho^+ < \rho^-$ and $\rho^+ \ne \rho^-$ mod. h; such pairs are called *critical shocks*. When $\phi = h^*$, (in which case we must have $\phi^* = h^*$, i.e., h has no genuine local minimum) there is no critical shock, and ϕ-admissibility is equivalent to admissibility. When $\phi < h^*$, (ρ^ϕ, ρ_ϕ) is a critical shock and the maximal one, that is, for every other critical shock (ρ^-, ρ^+) we must have $\rho_\phi \le \rho^+ < \rho^- \le \rho^\phi$. Now, for a $[0; K]$-valued Borel function ρ on $\mathbb{R}^{+*} \times \mathbb{R}$ with limits $\rho(t; 0^{\pm})$ modulo h for a.e. $t > 0$, we set the following definition:

DEFINITION 3.2. We say ρ is a ϕ-entropy solution, if: first, the entropy inequality (18) holds outside $x = 0$ for every $c \in [0; K]$; next, the pair $(\rho(t, 0^-), \rho(t, 0^+))$ modulo h is ϕ-admissible for a.e. $t > 0$.

In order to consider more general solutions without boundary limits, we can turn the admissibility condition at $x = 0$ into a modified Kružkov condition. In the spirit of [2], [3] and [9], we consider entropy inequalities of the form

$$\partial_t \varphi\big(\rho(t, x); r(t, x)\big) + \partial_x \psi\big(\rho(t, x); r(t, x)\big) \leq 0 \tag{25}$$

on $\mathbb{R}^{+*} \times \mathbb{R}$ for a large enough set of admissible stationary solutions $r(t, x) = r(x)$. Here we only assume ρ is a $[0; K]$-valued Borel function.

DEFINITION 3.3. We say ρ is a ϕ-entropy solution, if: (i) entropy condition (18) holds outside $x = 0$ for every $c \in [0; K]$, (ii) (18) holds on the whole space for ϕ-admissible densities c and (iii) (25) holds on the whole space for the critical shock profile $r(x) = R_{\rho^\phi; \rho_\phi}(x)$ [with the notation introduced in (23)].

The idea behind this definition is that the entropy condition is entirely determined by admissible densities *and* the maximal critical shock $(\rho^\phi; \rho_\phi)$. In Section 6 we prove existence and a.e. uniqueness of a ϕ-entropy solution with given Cauchy datum in the sense (19) (Theorem 2.1) and equivalence of both definitions when limits modulo h exist at $x = 0$. Note that $c = 0$ and $c = K$ are always ϕ-admissible for every $\phi \geq 0$ and, thus, eligible in (ii); hence, a ϕ-entropy solution is still a weak solution to the conservation law (5).

3.3. *Properties and particular solutions.* In order to prove the hydrodynamic limit, we shall not directly use the definition of a ϕ-entropy solution, but some of its properties, together with the explicit knowledge of certain solutions. We begin with analogues of Propositions 3.1 and 3.2.

PROPOSITION 3.4. *The ϕ-entropy solution with initial datum ρ_0 has a representative ρ such that*: (i) $\rho \in C^0(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}))$; (ii) *if ρ_0 has locally bounded variation, limits $\rho(t, x^\pm)$ modulo h exist for every $t > 0$ and $x \in \mathbb{R}$.*

Compared to statement (ii) of Proposition 3.1, the above statement (ii) only ensures limits modulo h, but this will be sufficient for our purpose. As already mentioned for entropy solutions, we implicitely consider such a representative.

PROPOSITION 3.5. *Assume ρ^1, ρ^2 are ϕ-entropy solutions with initial data ρ^1_0, ρ^2_0. Define $\Delta(t)$ and V as in Proposition* 3.2. *Then $\Delta(\cdot)$ is a nonincreasing function. In particular, if ρ^1_0 and ρ^2_0 coincide a.e. on $[x; y]$, then ρ^1 and ρ^2 coincide a.e. on $[x + Vt, y - Vt]$ for every $t \in [0; (y - x)/(2V)]$.*

We now describe some particular ϕ-entropy solutions. First, we give an explicit construction for Riemann initial datum $\rho_0 = R_{\rho^-,\rho^+}$. Due to translation-invariance breaking, we restrict to Riemann conditions around 0: only in this case does the solution retain a simple form, though an explicit—but quite involved—construction remains possible for an arbitrarily located step. In order to construct the explicit ϕ-entropy solution for the Riemann problem, we introduce the following notations. Given some density r and initial profile ρ_0, we denote by $\rho(r;t,x)$ [resp. $\rho(t,x;r)$] the entropy solution whose initial datum equals r for $x<0$ ($x>0$) and $\rho_0(x)$ for $x>0$ ($x<0$). Note that if ρ_0 is a Riemann datum, $\rho(r;t,x)$ and $\rho(t,x;r)$ are also Riemann solutions and can still be constructed from Proposition 3.3.

PROPOSITION 3.6. *Assume $\phi \le \phi^*$, $\phi < h^\star$ and*

$$0 \le \rho^+ < \rho^\phi \quad \text{and} \quad \rho_\phi < \rho^- \le K. \tag{26}$$

Then the ϕ-entropy solution ρ with initial datum $\rho_0 = R_{\rho^-,\rho^+}$ exhibits the critical shock $(\rho(t,0^-) = r^-, \rho(t,0^+) = r^+)$ at all times and is given by

$$\rho(t,x) = \rho(r^+;t,x)\mathbf{1}_{\mathbb{R}^{+*}}(x) + \rho(t,x;r^-)\mathbf{1}_{\mathbb{R}^-}(x), \tag{27}$$

where

$$\begin{aligned}
&r^+ = \rho_\phi && \text{if } \rho^+ \le \rho_\phi,\\
&r^+ = \sup\{\rho \le \rho^+; h(\rho) = \phi\} && \text{if } \rho^+ > \rho_\phi,\\
&r^- = \rho^\phi && \text{if } \rho^- \ge \rho^\phi,\\
&r^- = \inf\{\rho \ge \rho^-; h(\rho) = \phi\} && \text{if } \rho^- < \rho^\phi.
\end{aligned} \tag{28}$$

Outside the range (26) *for $\rho^\pm$, the ϕ-entropy solution coincides with the usual entropy solution. In particular, if we start from a constant profile with density $\rho = \rho^- = \rho^+$, the ϕ-entropy solution will differ from the entropy solution and develop a critical shock iff $\rho_\phi < \rho < \rho^\phi$ or, equivalently, $h(\rho) > \phi$.*

We shall define the *fundamental solution* as the one arising from initial datum $R_{K;0}$. This solution plays an important role in the proof of hydrodynamics. By Proposition 3.6, the fundamental solution exhibits the maximal critical shock $(\rho^\phi; \rho_\phi)$ around 0 at all times.

PROOF OF PROPOSITION 3.6.

CASE 1. (26) fails: it is then easy to see from (24), Proposition 3.3 and the condition $\phi \le \phi^*$ that the entropy solution ρ to the Riemann problem has a ϕ-entropic pair $(\rho(t,0^-), \rho(t,0^+))$ at $x=0$ and is thus also the ϕ-entropy solution.

CASE 2. (26) holds. Then $\phi \le \phi^*$ implies that $(r^-; r^+)$ given by (28) is a critical shock. (24), Proposition 3.3 and $\phi \le \phi^*$ imply that $\rho(r^+; t, 0^+) = r^+$ and $\rho(t, 0^-; r^-) = r^-$. Hence, $\rho(t, 0^\pm) = r^\pm$ and ρ has the ϕ-critical shock (r^-, r^+) at the boundary. The entropy solutions $\rho(r^+, \cdot, \cdot)$ and $\rho(\cdot, \cdot, r^-)$ satisfy (18) on $\mathbb{R}^{+*} \times \mathbb{R}$ for every $c \in [0; K]$ and converge in the sense (19) as $t \to 0$ to their Cauchy data; the latter coincide with ρ_0, respectively, for $x > 0$ and $x < 0$. Thus, ρ satisfies (18) away from $x = 0$ for every $c \in [0; K]$ and converges to ρ_0 as $t \to 0$ in the sense (19). □

We end up with stationary solutions useful for the sequel.

LEMMA 3.1. *Let the initial datum $\rho_0(\cdot)$ be piecewise constant modulo h, that is, there is a partition of $\mathbb{R}$ into finitely many intervals, such that the image of the interior of each interval is contained in a class modulo h. We call "boundary point" any point on the boundary of one of the intervals. Then ρ_0 induces the following*:

(i) *a stationary entropy solution iff the pair $(\rho_0(x^-), \rho_0(x^+))$ modulo h is entropic for every boundary point x*;

(ii) *a stationary ϕ-entropy solution iff the pair $(\rho_0(x^-), \rho_0(x^+))$ modulo h is entropic for every boundary point $x \neq 0$, and the pair $(\rho_0(0^-), \rho_0(0^+))$ modulo h is ϕ-entropic.*

PROOF. Set $\rho(t, x) = \rho_0(x)$. By uniqueness of the entropy (or ϕ-entropy) solution with initial datum, it suffices to check that ρ is an entropy (or ϕ-entropy) solution. On any open space interval of the partition, (18) obviously holds as an equality, since the time derivative vanishes and $\psi(\rho; c)$ is constant. Thus, we are reduced to checking admissibility of boundary pairs. □

4. Preliminary material for hydrodynamics. The section is organized as follows. In Section 4.1 we prove comparison lemmas and properties of the particle current. In Section 4.2 we prove existence and properties of the critical macroscopic current.

4.1. *Comparison lemmas.*

Preliminary definitions. r (resp. D) will denote a common interaction (resp. perturbation) range for all transformations T or perturbations $\tilde{T}$, $\hat{T}$ considered in the sequel. We shall say that a system $\eta_.$ (or $\sigma_.$) has the same dynamics as a system $\zeta_.$ (or $\tau_.$), if they are governed by the same local transformation, and that $\eta_.$ (or $\sigma_.$) is slower than $\zeta_.$ (or $\tau_.$), if $\eta_.$ is governed by a slower transformation than $\zeta_.$. We define the cumulative distribution function (c.d.f.) of a particle configuration η

with origin at 0 by

$$F(x;\eta) = \sum_{u=1}^{x} \eta(u) \qquad \text{if } x > 0,$$

$$F(x;\eta) = -\sum_{u=x+1}^{0} \eta(u) \qquad \text{if } x < 0,$$

$$F(0;\eta) = 0.$$

For a configuration with finitely many particles to the left, we define the c.d.f. with origin at $-\infty$ by

$$G(x;\eta) = \sum_{u \le x} \eta(u).$$

Defining $n(x;\sigma)$ as the largest label $n \in \mathbb{Z}$ such that $\sigma(n) \le x$, it is easy to see that $F(x;\eta) = n(x;\sigma) - n(0;\sigma)$. Similarly, denoting by $n_{\min}(\sigma)$ the label of the leftmost σ-particle, we have $G(x;\sigma) = n(x;\sigma) - n_{\min}(\sigma)$. In particular, assume the two configurations σ, τ are labelled in such a way that $n(0;\sigma) = n(0;\tau) = 0$, which we call the labelling with origin at 0. Then it follows easily that, for $k \ge 0$,

$$\sigma(\cdot) \le \tau(\cdot + k) \quad \Longleftrightarrow \quad F(\cdot;\zeta) \le F(\cdot;\eta) + k. \tag{29}$$

The same property holds for G if the leftmost particle is labelled 0 in both configurations. The quantity

$$\Delta(\sigma,\sigma') = n(0,\sigma) - n(0,\sigma') \tag{30}$$

is exactly the current of particles through $x = 0$, that is, the algebraic number of particles newly to the right of 0, when the configuration has changed from σ to σ'. The current may be defined intrinsically as a function of the *undistinguishable* configurations η and η' if the latter have finitely many particles on one side, for example, to the right,

$$\Delta(\eta,\eta') = \sum_{x>0} \eta'(x) - \sum_{x>0} \eta(x). \tag{31}$$

Otherwise, it is not possible, in general, to define a current $\Delta(\eta,\eta')$ intrinsically, for the result depends on the path followed by the distinguishable system in between. However, this can be done if η' results from running a finite-range dynamics, defined by (2), on some time interval. In this case, choosing a distinguishable representation σ of η determines the resulting representation σ' for η', and (30) does not depend on the choice of σ. In particular, if $\eta' = T^{x,u}\eta$ or $\eta' = \tilde{T}^{x,u}\eta$, we have

$$\Delta(\eta,\eta') = \sum_{y=1}^{\max(x+r;0)} \eta'(y) - \sum_{y=1}^{\max(x+r;0)} \eta(y) \tag{32}$$

with the convention $\sum_{y=1}^{0} = 0$. In this case, it is not difficult to see that

$$|\Delta(\eta, \eta')| \le Kr. \tag{33}$$

We end up defining trajectories X_t^x and Y_t^x, $t \ge 0$, that represent the fastest propagation of microscopic discrepancies from $x \in \mathbb{Z}$. This should be viewed as the microscopic analogue of Propositions 3.2 and 3.5. We set $X_0^x = Y_0^x = x$ and define the dynamics as follows. Let $X = X_{t-}^x$ and $Y = Y_{t-}^x$. At time t, X (resp. Y) moves $2r$ steps forward (resp. backward) if an event occurs in $[X - r; X + r)$ (resp. $(Y - r; Y + r]$). Under $\mathbb{P}$, the number of jumps of X_t^x and Y_t^x are Poisson processes with intensity $2r$.

LEMMA 4.1. *Assume $\eta_.$ and $\zeta_.$ share the same dynamics, and their initial configurations η_0 and ζ_0 coincide on the space interval $[x; y]$ for some $x, y \in \mathbb{Z}$ such that $y - x \ge 2r$. Define $\tau_{x,y} = \inf\{t > 0 : Y_t^y - X_t^x < 2r\}$. Then, for every $t \in [0; \tau_{x,y}]$, η_t and ζ_t coincide on the space interval $[X_t^x; Y_t^y]$.*

PROOF. Proving that the property is unchanged by a new event amounts to proving the following: assume η and ζ coincide on $[X; Y]$ with $Y - X \ge 2r$; then, for every $u \in \mathcal{U}$, $\tilde{T}^{z,u}\eta$ and $\tilde{T}^{z,u}\zeta$ coincide on $[X + 2r; Y]$ if $z \in [X - r; X + r)$, on $[X; Y - 2r]$ if $z \in (Y - r; Y + r]$, and on $[X; Y]$ if $z \notin [X - r; X + r) \cup (Y - r; Y + r]$. This follows easily from the fact that $\tilde{T}^{z,u}\eta$ depends and acts only on the restriction of η to $[z - r; z + r]$. $\square$

The following lemma is an obvious consequence of order-preservation.

LEMMA 4.2. *Assume $\sigma_.$ is slower than $\tau_.$, and $\sigma_0(\cdot) \le \tau_0(\cdot + k)$ for some $k \in \mathbb{N}$. Then we have $\sigma_t(\cdot) \le \tau_t(\cdot + k)$ for every $t > 0$.*

The next lemma means that two systems which (in some "weak" sense) are microscopically close at time 0 remain so at later times. This will later imply in the hydrodynamic scaling limit (see Lemma 5.3) that two systems which are macroscopically close at time 0 in the weak topology remain so at later times.

LEMMA 4.3. *If $\eta_.$ and $\zeta_.$ share the same dynamics, then*

$$\sup_{x \le y}\left|\sum_{z=x}^{y} \eta_t(y) - \sum_{z=x}^{y} \zeta_t(y)\right| \le 2 \sup_{x \le y}\left|\sum_{z=x}^{y} \eta_0(y) - \sum_{z=x}^{y} \zeta_0(y)\right|,$$

$$\sup_{x \in \mathbb{Z}}[G(x; \eta_t) - G(x; \zeta_t)] \le \sup_{x \in \mathbb{Z}}[G(x; \eta_0) - G(x; \zeta_0)],$$

$$\sup_{x \in \mathbb{Z}}|G(x; \eta_t) - G(x; \zeta_t)| \le \sup_{x \in \mathbb{Z}}|G(x; \eta_0) - G(x; \zeta_0)|,$$

for every $t \ge 0$, where in the second inequality we assume η_0 and ζ_0 have finitely many particles to the left.

PROOF. Let k denote the right-hand side of the first inequality. If $k=+\infty$, there is nothing to prove. Otherwise, label σ_0 and τ_0 with origin at 0. Since by assumption $|F_0(\cdot;\eta_0)-F_0(\cdot;\zeta_0)|\le k$, (29) and Lemma 4.2 imply $\tau_t(\cdot-k)\le\sigma_t(\cdot)\le\tau_t(\cdot+k)$ for every $t\ge 0$; hence, for every subinterval $I\subset\mathbb{Z}$, $|\sum_{x\in I}(\eta_t(x)-\zeta_t(x))|\le 2k$, and the result follows by definition of the c.d.f. For the second inequality, we give leftmost particles label 0 at time 0. By the G-version of (29) we have $\tau_0(\cdot)\le\sigma_0(\cdot+k)$, and this remains at time t by order preservation. But at time t we still have $n_{\min}=0$ for σ_t and τ_t, so we can use the rightward implication in (29) to recover the inequality for c.d.f.'s. Finally, the third inequality follows from the second one applied to $(\eta_., \zeta_.)$ and $(\zeta_., \eta_.)$. □

With the help of Lemma 4.1, we obtain the following local version of Lemma 4.3.

COROLLARY 4.1. *Let $x\le 0\le y$ in $\mathbb{Z}$, and $\tau_{x,y}$ be as in Lemma 4.1. Then, for every $t\in[0;\tau_{x,y}]$,*

$$\sup_{X_t^x\le x'\le y'\le Y_t^y}\left|\sum_{z=x'}^{y'}\eta_t(z)-\sum_{z=x'}^{y'}\zeta_t(z)\right|\le 2\sup_{x\le x'\le y'\le y}\left|\sum_{z=x'}^{y'}\eta_0(z)-\sum_{z=x'}^{y'}\zeta_0(z)\right|.$$

PROOF. Define the initial configuration ζ_0' as equal to ζ_0 inside $[x;y]$, and η_0 outside. The result follows immediately from applying Lemma 4.3 to η_t and ζ_t' and Lemma 4.1 to ζ_t and ζ_t'. □

A simple reformulation of Lemma 4.2 shows that we can use c.d.f. comparison at time 0 to compare currents through $x=0$ at later times.

LEMMA 4.4. *Assume $\eta_.$ is slower than $\zeta_.$. Then*

$$\sup_{t\ge 0}\big(\Delta(\eta_0,\eta_t)-\Delta(\zeta_0,\zeta_t)\big)\le\max\Big(0;\sup_{x\in\mathbb{Z}}\big(F(x;\zeta_0)-F(x;\eta_0)\big)\Big).$$

PROOF. Let k denote the right-hand side of the inequality. If $k=+\infty$, there is nothing to prove. Otherwise, choose the initial distinguishable representatives σ_0 and τ_0 labelled with origin at 0. By (29), we have $\sigma_0(n)\le\tau_0(n+k)$ for every n; thus, by Lemma 4.2, $\sigma_t(\cdot)\le\tau_t(\cdot+k)$. It follows that $n(0;\sigma_t)\ge n(0;\tau_t)-k$, which implies $\Delta(\eta_0,\eta_t)\le\Delta(\zeta_0,\zeta_t)+k$. □

Exactly as in Corollary 4.1, we can state a local version of the above lemma.

COROLLARY 4.2. *Assume $x\le -2r+1$ and $y\ge 2r$ in $\mathbb{Z}$. Define*

$$\tau'_{x,y}=\inf\{t>0: X_t^x>-2r+1 \text{ or } Y_t^y<2r\}.$$

Then, for every $t \in [0; \tau'_{x,y}]$,

$$\Delta(\eta_0, \eta_t) - \Delta(\zeta_0, \zeta_t) \leq \max\Bigl(0; \sup_{x-1\leq z\leq y} \bigl(F(z;\zeta_0) - F(z;\eta_0)\bigr)\Bigr).$$

PROOF. Define ζ'_0 as in the proof of Corollary 4.1. Applying Lemma 4.4 to η_t and ζ'_t, the problem reduces to proving that

$$\Delta(\zeta_0, \zeta_t) = \Delta(\zeta'_0, \zeta'_t) \qquad \text{for } t < \tau'_{x,y}. \tag{34}$$

By Lemma 4.1, we know that ζ_t and ζ'_t coincide between sites X^x_t and Y^y_t inclusive. Let us show that this implies (34). By induction, the problem reduces to proving that, if ζ_{t^-} and ζ'_{t^-} coincide bewteen sites $-2r-1$ and $2r$ inclusive, then $\Delta(\zeta_{t^-}, \zeta_t) = \Delta(\zeta'_{t^-}, \zeta'_t)$. This is a consequence of the following claim: if η and ζ coincide on the space interval $[-2r+1; 2r]$, $\eta' = \tilde{T}^{x,u}\eta$ and $\zeta' = \tilde{T}^{x,u}\zeta$, then $\Delta(\eta, \eta') = \Delta(\zeta, \zeta')$. Indeed, if $x \leq -r$ or $x \geq r+1$, then $\Delta(\eta, \eta') = \Delta(\eta, \eta') = 0$ by (32). Otherwise, $-r+1 \leq x \leq r$; then, $\tilde{T}^{x,u}$ only acts on the space interval $[x-r; x+r] \subset [-2r+1; 2r]$; since η and ζ coincide on the latter interval, so will η' and ζ'; the result then follows from (32). □

We now turn to the property which is really the core of our approach to proving Theorem 2.2. Let us measure closeness of configurations on the right of $x = 0$ by comparing c.d.f.'s restricted to the right of 0,

$$S(\eta, \zeta) = \sup_{x\in\mathbb{Z}: x\geq 1} \sum_{y=1}^{x} \bigl(\eta(y) - \zeta(y)\bigr). \tag{35}$$

Introduction of a related functional seems to date back to [1]. Now, assume $\eta_.$ and $\zeta_.$ are governed by *any* two local perturbations $\tilde{T}$ and $\hat{T}$ of T. In Lemma 4.5 we prove the following: if $\eta_.$ and $\zeta_.$ are initially close microscopically to the right of 0, and their particle currents through 0 are close, then they remain close to the right of 0. Like Lemma 4.3, this statement will have a macroscopic analogue in the hydrodynamic scaling limit (see Lemma 5.2).

LEMMA 4.5. *There exists a constant* $M = M(K, r, D) > 0$ *such that for every* $t \geq 0$,

$$S(\eta_t, \zeta_t) \leq \max\bigl(M; S(\eta_0, \zeta_0)\bigr) + \sup_{s;0\leq s\leq t} [\Delta(\eta_s, \eta_t) - \Delta(\zeta_s, \zeta_t)]. \tag{36}$$

A similar statement holds on the left of $x = 0$, with an obvious re-definition of S and a possibly different M. As in Corollaries 4.1 and 4.2, we can give a local formulation: to this end, define a local version S_z of S for each $z \in \mathbb{Z}$ such that $z \geq 1$, by restricting the supremum in (35) to $1 \leq x \leq z$.

COROLLARY 4.3. *For every* $0 \le t \le \tau'_{-\infty,y}$ *and* $y \ge 2r$,

$$S_{Y_t^y}(\eta_t, \zeta_t) \le \max\bigl(M; S_y(\eta_0, \zeta_0)\bigr) + \sup_{s; 0 \le s \le t} [\Delta(\eta_s, \eta_t) - \Delta(\zeta_s, \zeta_t)] \tag{37}$$

with the same constant M *as in Lemma* 4.5; $\tau'_{-\infty,y}$ *is defined as* $\tau'_{x,y}$ *in Corollary* 4.2, *formally setting* $x = -\infty$, *that is*, *removing the condition on* X_t^x.

We omit the proof of the corollary, which follows from the lemma in a similar way as Corollaries 4.1 and 4.2. The proof of Lemma 4.5 amounts to studying how S is modified by the transformations $\tilde{T}$ and $\hat{T}$ after each new event.

LEMMA 4.6. *There exists a constant* $M = M(K, r, D) > 0$ *with the following property*: *set* $\eta' = \tilde{T}^{x,u}\eta$, $\zeta' = \hat{T}^{x,u}\zeta$. *Then*

$$S(\eta', \zeta') \le \max\bigl[M; S(\eta, \zeta) + \Delta(\eta, \eta') - \Delta(\zeta, \zeta')\bigr]. \tag{38}$$

LEMMA 4.7. *Let* f *and* δ *be right-continuous*, *piecewise constant functions defined on* $\mathbb{R}^+$. *Assume we have*

$$f(t) \le \max[M; f(t^-) + \delta(t) - \delta(t^-)] \tag{39}$$

for every $t \ge 0$; *then*, *for every* $t \ge 0$,

$$f(t) \le \max[M; f(0)] + \sup_{s: 0 \le s \le t} \bigl(\delta(t) - \delta(s)\bigr). \tag{40}$$

Lemma 4.5 is an immediate consequence of the above two lemmas, where $f(t) = S(\eta_t, \zeta_t)$ and $\delta(t) = \Delta(\eta_0, \eta_t) - \Delta(\zeta_0, \zeta_t)$. The proof of Lemma 4.7 is an elementary induction between t^- and t, where t denotes the latest update time. The main step is to prove Lemma 4.6.

PROOF OF LEMMA 4.6. We prove the statement with $M = 3K(r + \max(r; D))$.

CASE 1. Assume that $x \le -r$. Then η and η' are not modified on the right of 0, so $S(\eta', \zeta') = S(\eta, \zeta)$; on the other hand, (32) implies that $\Delta(\eta, \eta') = \Delta(\zeta, \zeta') = 0$, so (38) follows easily.

CASE 2. Now, assume we have $x \ge \max(r, D) + 1$. In this case, the perturbations are not felt: $\eta' = T^{x,u}\eta$ and $\zeta' = T^{x,u}\zeta$. Again, by (32), we have $\Delta(\eta, \eta') = \Delta(\zeta, \zeta') = 0$; we are going to prove that

$$S(\eta', \zeta') \le \max\bigl(0; S(\eta, \zeta)\bigr). \tag{41}$$

We label σ and τ particles with origin at 0. Because $x \geq r+1$, the following is true for both systems between the initial and final states: labelled particles on the right of 0 are the same, and particles up to site 0 inclusive are untouched. On the other hand, S depends only on particles to the right of 0. Therefore, when evaluating $S(\eta,\zeta)$ and $S(\eta',\zeta')$, we do not modify the result if we consider that σ and τ coincide for labels $n \leq 0$; of course, so will σ' and τ'. This way, we can write

$$S(\eta,\zeta) = \sup_{x>0}\big(F(x,\eta) - F(x,\zeta)\big) = \sup_{x\in\mathbb{Z}}\big(F(x,\eta) - F(x,\zeta)\big)$$

and the same holds for η' and ζ'. Let $k = \max(0; S(\eta,\zeta))$. By (29), we have $\tau(\cdot) \leq \sigma(\cdot + k)$. Because T is order-preserving, we still have $\tau'(\cdot) \leq \sigma'(\cdot + k)$. But σ' and τ' are still labelled with origin at 0; thus, using (29) again, we get

$$S(\eta',\zeta') = \sup_{x>0}\big(F(x,\eta') - F(x,\zeta')\big) = \sup_{x\in\mathbb{Z}}\big(F(x,\eta') - F(x,\zeta')\big) \leq k.$$

CASE 3. Finally, let $-r+1 \leq x \leq \max(r, D)$. In this case, local modifications of η and ζ on the right of 0 can only affect sites 1 to $\max(r, D) + r$ inclusive. It then follows easily from the definition of S that

$$|S(\eta,\zeta) - S(\eta',\zeta')| \leq 2K\big(r + \max(r, D)\big).$$

Hence, (38) is immediate if $\inf[S(\eta,\zeta); S(\eta',\zeta')] \leq K(r + \max(r, D))$. We therefore assume that $\inf[S(\eta,\zeta); S(\eta',\zeta')] > K(r + \max(r, D))$. This implies that the suprema in $S(\eta,\zeta)$ and $S(\eta',\zeta')$ cannot be reached in the interval $[1; r + \max(r, D)]$. Hence, there exists some $y > \max(r, D) + r$ such that

$$S(\eta,\zeta) = \sum_{z=1}^{y}\big(\eta(z) - \zeta(z)\big),$$

$$S(\eta',\zeta') = \sum_{z=1}^{y}\big(\eta'(z) - \zeta'(z)\big).$$

It follows from (32) and $\max(r, D) + r \geq x + r$ that

$$S(\eta',\zeta') - S(\eta,\zeta) = \Delta(\eta,\eta') - \Delta(\zeta,\zeta')$$

which implies (38). $\square$

4.2. *Subadditivity and the critical current.* In this section we prove existence of a critical current by means of a subadditive argument and state some related properties. In the sequel we shall denote by η^* the configuration defined by $\eta^*(x) = K$ if $x \leq 0$, $\eta^*(x) = 0$ if $x > 0$. This configuration has the fundamental property that

$$F(\cdot;\eta^*) \leq F(\cdot;\eta) \qquad \text{for every other configuration } \eta. \tag{42}$$

Lemma 4.4 then implies that the current through $x = 0$ is maximal for the system starting from η^*. In the following lemma this system, as it has a deterministic initial state, is viewed on Ω alone rather than $\Omega \times \Omega'$.

LEMMA 4.8. *Let $\eta_.$ be governed by $\tilde{T}$, and $\eta_0 = \eta^*$. Then there exists $0 \le \phi \le h^*$ such that*

$$\lim_{t\to\infty} t^{-1}\Delta(\eta^*, \eta_t) = \phi, \qquad \mathbb{P}\text{-}a.s. \tag{43}$$

PROOF. Let $\phi_t = \Delta(\eta^*, \eta_t)$. First, we claim that ϕ_t is integrable. To see this, observe that ϕ_t is bounded by a Poisson process, namely

$$\phi_t \le Kr\omega\big((0;t] \times (-r;r] \times \mathcal{U}\big),$$

where the coefficient Kr follows from (33). Next, since the Poisson measure $\mathbb{P}$ is stationary and ergodic under time shift, (43) will follow from Kingman's subadditive theorem [15] if we prove that ϕ is subadditive with respect to time shift θ, that is, $\phi_{t+s}(\omega) \le \phi_t(\omega) + \phi_s(\theta_t\omega)$. Since $\phi_{t+s} = \phi_t + \Delta(\eta_t, \eta_{t+s})$ we must prove that the second term on the right-hand side above is bounded by $\phi_s(\theta_t\omega)$. To this end, we introduce a new system $(\zeta_\tau,\ \tau \ge t)$ starting at time t, with $\zeta_t = \eta^*$, and still governed by the same $\tilde{T}$. It follows from (42) and Lemma 4.4 that $\Delta(\eta_t, \eta_{t+s}) \le \Delta(\zeta_t, \zeta_{t+s}) = \phi_s(\theta_t\omega)$. □

The following lemma will be useful to compare critical currents from different perturbations. For two local perturbations $\tilde{T}_1$, $\tilde{T}_2$ of T, we denote by $U_{\tilde{T}_1,\tilde{T}_2}$ the complement of the set of $u \in \mathcal{U}$ for which $\tilde{T}_i^{x,u}$ coincide for every x.

LEMMA 4.9. *Assume that the local perturbations $\tilde{T}_1, \tilde{T}_2$ satisfy* (43) *with asymptotic currents ϕ_i. Then $|\phi_1 - \phi_2| \le 4r(2D+1)\mu(U_{\tilde{T}_1,\tilde{T}_2})$. Moreover, $\phi_1 \le \phi_2$ if $\tilde{T}_1$ is slower than $\tilde{T}_2$.*

PROOF. The second statement is an immediate consequence of Lemma 4.4, which implies that $\Delta(\eta^*, \eta_t^1) \le \Delta(\eta^*, \eta_t^2)$, where η_t^i evolves according to $\tilde{T}_i$. To prove the first statement, consider

$$N_t(\omega) = \omega\big((0;t] \times [-D;D] \times U_{\tilde{T}_1,\tilde{T}_2}\big), \tag{44}$$

which, under $\mathbb{P}$, is a Poisson process with intensity $(2D+1)\mu(U_{\tilde{T}_1,\tilde{T}_2})$. The result will follow from the law of large numbers for N_t and the inequality

$$|\Delta(\eta^*, \eta_t^1) - \Delta(\eta^*, \eta_t^2)| \le 4rN_t, \tag{45}$$

which we now prove. To this end, considering (30), it is enough to prove that

$$\sigma_t^1(\cdot - 4rN_t) \le \sigma_t^2(\cdot) \le \sigma_t^1(\cdot + 4rN_t). \tag{46}$$

Denote by (t_k, x_k, u_k) the sequence of events in ω up to time t, where $1 \le k \le n$ and $0 < t_1 < \cdots < t_n \le t$. We prove inductively that (46) holds at times t_k, $1 \le k \le n$. Assume it holds at time t_k. Since

$$\sigma^i_{t_{k+1}} = \tilde{T}_i^{x_{k+1},u_{k+1}} \sigma^i_{t_k}, \qquad N_{t_{k+1}} - N_{t_k} = \mathbf{1}_{\{|x_{k+1}| \le D,\, u_{k+1} \in U_{\tilde{T}_1, \tilde{T}_2}\}},$$

the problem reduces to proving that

$$\sigma^1(\cdot - k) \le \sigma^2(\cdot) \le \sigma^1(\cdot + k),$$
$$\sigma'^i = \tilde{T}_i^{x,u} \sigma^i \qquad i \in \{1; 2\},$$

implies

$$\sigma'^1(\cdot - k - 4r\varepsilon) \le \sigma'^2(\cdot) \le \sigma'^1(\cdot + k + 4r\varepsilon) \tag{47}$$

with

$$\varepsilon = \mathbf{1}_{\{|x| \le D, u \in U_{\tilde{T}_1, \tilde{T}_2}\}}.$$

This is proved by considering the following three possible situations. First, we may have $|x| > D$; in this case, the perturbation is not felt, and the same order-preserving transformation $T^{x,u}$ is applied to both σ^i; thus, we still have $\sigma'^1(\cdot - k) \le \sigma'^2(\cdot) \le \sigma'^1(\cdot + k)$ and (47). Next, assume $|x| \le D$ and $u \notin U_{\tilde{T}_1, \tilde{T}_2}$; then both $\tilde{T}_i^{x,u}$ are just the same order-preserving transformation, and the previous argument holds. Finally, assume $|x| \le D$ and $u \in U_{\tilde{T}_1, \tilde{T}_2}$, so that $\varepsilon = 1$. By definition of r, no particle may jump more than $2r$ sites back or ahead. Therefore,

$$\sigma'^2(\cdot) \le \sigma^2(\cdot + 2r) \le \sigma^1(\cdot + k + 2r) \le \sigma'^1(\cdot + k + 4r)$$

and, similarly, $\sigma'^2(\cdot) \ge \sigma'^1(\cdot - k - 4r)$. □

We apply Lemma 4.9 in the following context. To the local perturbation $\tilde{T}$ of T, we can apply the construction of Section 2.4 that yields an increasing family $\tilde{T}_\alpha$ of perturbations of $\tilde{T}$ such that $\tilde{T}_0$ is a total blockage and $\tilde{T}_1 = \tilde{T}$. We denote by $\tilde{\phi}_\alpha$ the value of ϕ arising from $\tilde{T}_\alpha$.

COROLLARY 4.4. *$\alpha \mapsto \tilde{\phi}_\alpha$ is a nondecreasing, Lipschitz-continuous function of $\alpha \in [0; 1]$, with $\tilde{\phi}_0 = 0$ and $\tilde{\phi}_1 = \phi$.*

PROOF. Nondecreasingness follows from the first part of Lemma 4.9, since $\tilde{T}_\alpha$ is increasing in α. Lipschitz continuity follows from the second part of the same lemma and definition of $\tilde{T}_\alpha$. Finally, $\tilde{\phi}_0 = 0$ because under $\tilde{T}_0$ no particle ever jumps to the right of 0; and $\tilde{\phi}_1 = \phi$ because $\tilde{T}_1 = \tilde{T}$. □

5. Proof of the hydrodynamic limit. The section is organized as follows. First, in Section 5.1 we state scaling limit versions of the comparison results from Section 4.1. These will serve as the basis for the sequel. Then the hydrodynamic limit is established in three main steps: in Section 5.2, for the "fundamental" system starting from configuration η^* defined in Section 4.2; in Section 5.3, starting from a Riemann profile; eventually in Section 5.4, from an arbitrary initial profile.

5.1. *Comparison results in scaling limit form.* In the sequel, notations $\varphi_t^{N,[\eta]} := N^{-1}\Delta(\eta_0^N, \eta_{Nt}^N)$ and $\alpha_t^{N,[\eta]} := \alpha^N(\eta_{Nt}^N, dx)$, respectively, denote the rescaled current and empirical measure for a system $[\eta]$, with α^N and Δ defined in (4) and (30). To begin with, here is a preliminary technical result.

LEMMA 5.1. (i) *The sequence of processes $\varphi^{N,[\eta]}$ is tight w.r.t. the topology of local uniform convergence, and any limit in law is a random process $\varphi^{[\eta]}$ with a.s. Lipschitz-continuous paths.* (ii) *The sequence of processes $\alpha^{N,[\eta]}$ is tight w.r.t. the topology of local uniform convergence, and any limit in law is a random process $\alpha^{[\eta]}$ with a.s. continuous paths.*

PROOF. Let $J_t = \omega((0;t] \times (-r;r] \times \mathcal{U})$. $(J_t)_{t>0}$ is a Poisson process, and (33) implies

$$|\varphi_t^{N,[\eta]} - \varphi_s^{N,[\eta]}| \le KrN^{-1}(J_{Nt} - J_{Ns}) \tag{48}$$

for $0 < s < t$, from which (i) follows easily. To prove (ii) we observe that

$$\alpha_t^{N,[\eta]}((0;x]) - \alpha_0^{N,[\eta]}((0;x]) = \varphi_t^{N,[\eta]}(0) - \varphi_t^{N,[\eta]}(x), \tag{49}$$

where $\varphi_t^{N,[\eta]}(x)$ is defined as $\varphi_t^{N,[\eta]}$, but considering the current through site $[Nx]$ instead of 0. Hence, by (i) the sequence of real-valued processes $\alpha_.^{N,[\eta]}((0;x])$ is tight w.r.t. local uniform convergence for every subinterval $(0;x] \subset \mathbb{R}$. This implies that the sequence $\alpha^{N,[\eta]}$ is tight w.r.t. uniform local convergence, as the topology generated by intervals $(0;x]$ for measures is stronger than the vague topology. By (i) the limiting process $\alpha^{[\eta]}$ is such that $\alpha_t^{[\eta]}((0;x])$ is uniformly Lipschitz continuous in t for every $x \in \mathbb{R}$, which implies continuity in the vague topology. □

Conventions used in the sequel. From now on random processes $\varphi^{[\eta]}$ and $\alpha^{[\eta]}$ will always denote arbitrary subsequential limits in law for the processes $\varphi^{N,[\eta]}$ and $\alpha^{N,[\eta]}$ as $N \to \infty$. Statements like "Let $\alpha^{[\eta]}$ (or $\varphi^{[\eta]}$) be an arbitrary limit..." will not be repeated systematically. When *simultaneously* considering such limits for different systems $[\eta], [\zeta], \ldots$, these limits must be understood as joint limits, that is, $(\alpha^{[\eta]}, \varphi^{[\eta]}, \alpha^{[\zeta]}, \varphi^{[\zeta]}, \ldots)$ is an arbitrary subsequential limit in law for the sequence $(\alpha^{N,[\eta]}, \varphi^{N,[\eta]}, \alpha^{N,[\zeta]}, \varphi^{N,[\zeta]}, \ldots)$. When *successively* introducing such limits, it will be implicit that we each time consider a further subsequence along which the latest limit exists jointly with all previous ones. This is possible thanks to Lemma 5.1 and will not be repeated in the sequel.

Lemmas 5.2–5.4 will be the key arguments in the proof of Theorem 2.2. As they are easy scaling limits of Corollary 4.3, Corollaries 4.1 and 4.2 (combined with Lemma 5.1), we omit their proofs. In these lemmas, r is the constant defined at the beginning of Section 4.1. "For every t" means a.s. with an exceptional set independent of t. This will always be true in the sequel for α and φ limiting processes, and we shall omit to mention "a.s."

LEMMA 5.2. *Let $[\eta]$ and $[\zeta]$ be governed by two (possibly different) local perturbations of the same transformation, and $y \in \mathbb{R}^{+*}$. Then, for every $0 \le t \le y/(2r)$,*

$$\begin{aligned}
&\sup_{x:\,0\le x\le y-2rt}\big(\alpha_t^{[\eta]}([0;x]) - \alpha_t^{[\zeta]}([0;x])\big)\\
&\quad\le \sup_{x:\,0\le x\le y}\big(\alpha_0^{[\eta]}([0;x]) - \alpha_0^{[\zeta]}([0;x])\big)\\
&\qquad+ \sup_{s:\,0\le s\le t}\big[\big(\varphi_t^{[\eta]} - \varphi_s^{[\eta]}\big) - \big(\varphi_t^{[\zeta]} - \varphi_s^{[\zeta]}\big)\big],
\end{aligned} \tag{50}$$

and a similar statement on the negative half-line, with $y < 0$ and suprema over $y \le x \le 0$ and $y + 2rt \le x \le 0$ in (50).

LEMMA 5.3. *Let $[\eta]$ and $[\zeta]$ have the same dynamics, and $x, y \in \mathbb{R}$ with $x < y$. Then, for every $0 \le t \le (y-x)/4r$,*

$$\begin{aligned}
&\sup_{x+2rt\le x'\le y'\le y-2rt}\big|\alpha_t^{[\eta]}([x';y']) - \alpha_t^{[\zeta]}([x';y'])\big|\\
&\quad\le 2\sup_{x\le x'\le y'\le y}\big|\alpha_0^{[\eta]}([x';y']) - \alpha_0^{[\zeta]}([x';y'])\big|.
\end{aligned} \tag{51}$$

If, in addition, there exists a deterministic constant $C > 0$ such that

$$\sum_{x\in\mathbb{Z}}\big(\eta_0^N(x) + \zeta_0^N(x)\big) \le CN \qquad \forall N \in \mathbb{N}^*,$$

then, for every $t \ge 0$,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\big|\alpha_t^{[\eta]}\big((-\infty;x]\big) - \alpha_t^{[\zeta]}\big((-\infty;x]\big)\big|\\
&\quad\le \sup_{x\in\mathbb{R}}\big|\alpha_0^{[\eta]}\big((-\infty;x]\big) - \alpha_0^{[\zeta]}\big((-\infty;x]\big)\big|.
\end{aligned} \tag{52}$$

LEMMA 5.4. *Let $[\eta]$ and $[\zeta]$ have the same dynamics, and $x, y \in \mathbb{R}$ with $x < 0 < y$. Then, for every $0 \le t \le \tau'(x,y) := \inf(-x,y)/(2r)$,*

$$\big|\varphi_t^{[\eta]} - \varphi_t^{[\zeta]}\big| \le \sup_{x\le z\le y}\big|\alpha_0^{[\eta]}([0;z]) - \alpha_0^{[\zeta]}([0;z])\big|. \tag{53}$$

REMARK. In practical use of the above lemmas, $[\eta]$ will be a system whose hydrodynamic limit we want to establish, and $[\zeta]$ will be a system with known hydrodynamic limit. Hence, $\alpha^{[\zeta]}$ and $\varphi^{[\zeta]}$ will be uniquely determined limits, namely the hydrodynamic profile and current of $[\zeta]$; combinations of the three lemmas with different suitably chosen $[\zeta]$'s will show that $\alpha^{[\eta]}$ is uniquely determined, thus, establishing hydrodynamic limit for $[\eta]$. In most cases only profile properties of $[\zeta]$ will be used: then by the second remark in Section 2.5, we only need to specify the initial density profiles of $[\zeta]$; the sequence of initial configurations can be chosen in any arbitray way (e.g., deterministic) achieving the desired profile.

We end up with a result showing that knowledge of the hydrodynamic limit for the unperturbed system implies that of the macroscopic particle current.

LEMMA 5.5. *Let $[\eta]$ be governed by T, with initial density profile ρ_0. Then, for every $t > 0$, $\varphi^{N,[\eta]}$ converges in probability to*

$$\int_0^t h(\rho(s,0^-))\,ds = \int_0^t h(\rho(s,0^+))\,ds \tag{54}$$

as $N \to \infty$, where ρ is the entropy solution to (5) *with initial datum ρ_0.*

Note that the limits in (54) always make sense because the entropy solution has locally bounded space variation at positive times. An immediate byproduct of Lemma 5.5 is that $h(K) = 0$. Indeed, consider the unperturbed system governed by T, starting from a totally congested configuration. By Assumption 2.2, the hydrodynamic profile is uniformy equal to K. Since particles do not move, the current is always 0, hence, the result.

PROOF OF LEMMA 5.5.

Step 1. First, we assume η_0^N has finitely many particles to the right; then this remains true at time t. (31) and Assumption 2.2 imply that $\varphi_t^{N,[\eta]}$ converges in probability to $\int_{x>0} \rho(s,x)\,dx - \int_{x>0} \rho(0,x)\,dx$ as $N \to \infty$. That this is equal to (54) follows from integrating (5) over $x > 0$.

Step 2. The assumption on η_0^N is relaxed. We consider another system $[\eta']$ with the same dynamics, but initial configurations $\eta_0'^N$ defined by removing all η_0^N-particles for $x > aN$, where a is a large constant. By Assumption 2.2, η' has density profile ρ', the entropy solution to (5) with initial datum $\rho_0' = \rho_0 \mathbf{1}_{(-\infty;a]}$. By Lemma 4.1, η_{Nt}^N and $\eta_{Nt}'^N$ coincide up to site Y_{Nt}^{Na}, and thus, their density profiles ρ and ρ' coincide for $x \le a - 2rt$. By Step 1, $\varphi_t^{N,[\eta']}$ converges as $N \to \infty$ to (54) with ρ' instead of ρ, but this does not change the result if $t < a/(2r)$. Lemma 5.4, with arbitrary x and $y = a$, now implies that (54) holds for $t < a/(2r)$, and the result follows from choosing arbitrarily large a. $\square$

5.2. *Hydrodynamic limit for the fundamental system.* In the sequel we denote by $[\eta^*]$ the "fundamental" system with deterministic initial configurations $\eta^N = \eta^*$ independent of N, where η^* is the fundamental configuration defined in Section 4.2. We prove the following particular case of Theorem 2.2.

LEMMA 5.6. *The system $[\eta^*]$ has hydrodynamic profile $\rho^*(\cdot,\cdot)$, where ρ^* is the fundamental ϕ-entropy solution defined below Proposition* 3.6.

PROOF. Let $\alpha^{[\eta*]}$ be a subsequential limit. We must show that $\alpha_t^{[\eta^*]} = \rho^*(t,\cdot)\,dx$ for every $t > 0$. We know by Proposition 3.6 that $\rho^*(t,x) = \rho(\rho_\phi, t, x)$ for $x > 0$; recall from Proposition 3.6 that $\rho(\rho_\phi,\cdot,\cdot)$ denotes the *entropy* solution with initial profile $\rho_0(\cdot)\mathbf{1}_{\mathbb{R}^{+*}} + \rho_\phi \mathbf{1}_{\mathbb{R}^{-*}}$, with here $\rho_0 = R_{K,0}$. Let $[\eta^r]$ be a system with initial density profile $\rho_0(\cdot)\mathbf{1}_{\mathbb{R}^{+*}} + \rho_\phi \mathbf{1}_{\mathbb{R}^{-*}}$; by Assumption 2.2, $[\eta^r]$ has hydrodynamic profile $\rho(\rho_\phi,\cdot,\cdot)$. By Proposition 3.6, we have $\rho^*(t,0^+) = \rho(\rho_\phi, t, 0^+) = \rho_\phi$. Hence, by Lemma 5.5, $\varphi_t^{[\eta^r]} = th(\rho_\phi) = t\phi$ for every $t > 0$. We also know by Lemma 4.8 that $\varphi_t^{[\eta^*]} = t\phi$ for every $t > 0$. Since initial density profiles of $[\eta^*]$ and $[\eta^r]$ coincide on $\mathbb{R}^{+*}$, we conclude from Lemma 5.2 with a sequence of y's growing to ∞ (and an exceptional set valid for the whole sequence) that $\alpha_t^{[\eta^*]}$ coincides with $\alpha_t^{[\eta^r]} = \rho(\rho_\phi, t,\cdot)\,dx$ on $\mathbb{R}^{+*}$ for every $t > 0$; hence, $\alpha_t^{[\eta^*]} = \rho^*(t,\cdot)\,dx$ on $\mathbb{R}^{+*}$. The same argument can be used on $\mathbb{R}^{-*}$, by comparing $[\eta^*]$ to an unperturbed system $[\eta^l]$ with hydrodynamic profile $\rho(\cdot,\cdot,\rho^\phi)$. $\square$

5.3. *Hydrodynamic limit for initial admissible Riemann profiles.* For every pair $(\rho^-;\rho^+)$ modulo h, let $\mathcal{R}_{\rho^-;\rho^+}$ denote the set of profiles $\rho(\cdot)$ such that $\rho(x) \in \rho^-$ for a.e. $x < 0$ and $\rho(x) \in \rho^+$ for a.e. $x > 0$. Such profiles are called Riemann profiles modulo h. By Lemma 3.1, any profile $\rho(\cdot) \in \mathcal{R}_{\rho^-;\rho^+}$, with $(\rho^-;\rho^+)$ an ϕ-admissible pair modulo h, is a stationary ϕ-entropy solution. We now prove Theorem 2.2 for such initial profiles.

LEMMA 5.7. *Assume $[\eta]$ has initial density profile $\rho_0 \in \mathcal{R}_{\rho^-;\rho^+}$, with $(\rho^-;\rho^+)$ an ϕ-admissible pair modulo h. Then* (i) *$[\eta]$ has stationary hydrodynamic profile $\rho_0(\cdot)$.* (ii) *The rescaled current $\varphi_t^{N,[\eta]}$ converges in probability to $th(\rho^\pm)$ for every $t > 0$.*

The proof will be split into different cases in Sections 5.3.1–5.3.3, according to the values of ρ^- and ρ^+. In each case we prove that $\alpha_t^{[\eta]} = \rho_0(\cdot)\,dx$ and $\varphi_t^{[\eta]} = th(\rho^\pm)$ for every $t > 0$, where $(\alpha^{[\eta]}, \varphi^{[\eta]})$ is an arbitrary subsequential limit.

5.3.1. *The maximal critical shock profile.* We assume $\rho_0 = R_{\rho^\phi;\rho_\phi}$. Fix some $A > 0$ and $\varepsilon > 0$. Since ρ^* is self-similar, that is, $\rho^*(t,x) = \rho^*(1, x/t)$, with $\rho^*(\cdot, 0^-) = \rho^\phi$ and $\rho^*(\cdot, 0^+) = \rho_\phi$, we can choose $\tau > 0$ large enough, so that

$$\sup_{t>0,|x|\le A} |\rho^*(\tau + t, x) - \rho_0(x)| \le \varepsilon. \tag{55}$$

Let $[\zeta]$ denote a system such that ζ_0^N has the same distribution as $\eta^*_{N\tau}$. By Markov property ζ^N_{Nt} has the same distribution as $\eta^*_{N(\tau+t)}$ for every $t \ge 0$. In particular, ζ^N_{Nt} has density profile $\rho^*(\tau + t, \cdot)$ by Lemma 5.6. Now apply (51) with $x = -A$ and $y = A$. Using $\alpha_0^{[\eta]} = \rho_0(\cdot)\,dx$, $\alpha_t^{[\zeta]} = \rho^*(\tau + t, \cdot)\,dx$ and (55), we find

$$\sup_{|x|,|y|\le A-2rt}\left|\alpha_t^{[\eta]}([x;y]) - \int_x^y \rho_0(z)\,dz\right| \le 6A\varepsilon$$

for every $t < A/(2r)$; letting $\varepsilon \to 0$ and then $A \to +\infty$ (i.e., taking a common exceptional set for sequences of ε's and A's), we conclude that $\alpha_t^{[\eta]} = \rho_0(\cdot)\,dx$ for every $t > 0$. Now we apply (53). To this end observe that $\varphi_t^{[\zeta]} = t\phi$, because, by Markov property, $\varphi_t^{N,[\zeta]}$ has the same distribution as $\varphi^{N,[\eta^*]}_{\tau+t} - \varphi^{N,[\eta^*]}_\tau$, and the latter converges in probability to $(\tau + t)\phi - \tau\phi = t\phi$ by Lemma 4.8. Thus, (53) yields $|\varphi_t^{[\eta]} - t\phi| \le A\varepsilon$ for $t < A/(2r)$. Again, letting $\varepsilon \to 0$ and then $A \to +\infty$, we find that $\varphi_t^{[\eta]} = t\phi$ for every $t > 0$.

5.3.2. *ϕ-admissible pairs with critical current value.* We assume that $\rho^\pm$ are critical densities modulo h. Consider the approximating initial density profile

$$\rho_0^\varepsilon(x) = R_{\rho^\phi;\rho_\phi}(x)\mathbf{1}_{(-\varepsilon;\varepsilon)}(x) + \rho_0(x)\mathbf{1}_{(-\varepsilon;\varepsilon)^c}(x).$$

By Lemma 3.1, ρ_0^ε is a stationary ϕ-entropy solution because it is piecewise constant modulo h and exhibits three successive boundary pairs modulo h: $(\rho^-;\rho^\phi)$ at $x = -\varepsilon$, $(\rho^\phi;\rho_\phi)$ at $x = 0$, $(\rho_\phi;\rho^+)$ at $x = \varepsilon$. The first pair is entropic: indeed, $\rho^- \le \rho^\phi$ because ρ^ϕ is the largest critical density; and if $\rho^- < \rho^\phi$, the chord between ρ^- and ρ^ϕ must lie below the graph of h, as $h(\rho^-) = h(\rho^\phi) = \phi$ and h has no local minimum with value smaller than ϕ. The third pair is entropic for similar reasons. Finally, the second one is ϕ-entropic, since it is the maximal critical shock.

Step 1. Let us first consider a system $[\zeta]$ with initial density profile $\rho_0^\varepsilon(\cdot)$. We want to prove that $[\zeta]$ has stationary hydrodynamic profile ρ_0^ε, and $\varphi^{N,[\zeta]}$ converges in probability to $th(\rho_\phi) = th(\rho^\phi) = t\phi$ for every $t > 0$. By Markov property, it is enough to show it for $t \le \tau$ with some $\tau > 0$. We consider $\tau := \varepsilon/(2r)$ and prove that (i) $\alpha_t^{[\zeta]} = \rho_0^\varepsilon(\cdot)\,dx$, (ii) $\varphi_t^{[\zeta]} = t\phi$, for any $t \le \tau$ and for any subsequential limit. Consider the initial profile $\rho_0^{\varepsilon,r} := \rho_0^\varepsilon(\cdot)\mathbf{1}_{\mathbb{R}^{+*}} + \rho_\phi\mathbf{1}_{\mathbb{R}^{-*}}$. ρ_0^ε is now a stationary entropy solution because, as compared to ρ_0^ε, only the entropic pair (ρ_ϕ, ρ^+) at $x = \varepsilon$ remains. Let $[\zeta^r]$ be an unperturbed system with initial density profile $\rho_0^{\varepsilon,r}(\cdot)$; by Assumption 2.2, $[\zeta^r]$ has stationary hydrodynamic profile $\rho_0^{\varepsilon,r}(\cdot)$, and by Lemma 5.5 we have $\varphi_t^{[\zeta^r]} = th(\rho_\phi) = t\phi$ for every $t > 0$. Let also $[\zeta']$ be a system with initial profile $R_{\rho^\phi;\rho_\phi}$. By Section 5.3.1 applied to $[\zeta']$ and Lemma 5.4 applied to $[\zeta]$ and $[\zeta']$ with a sequence of y's growing to ∞, we have $\varphi_t^{[\zeta]} = \varphi_t^{[\zeta']} = t\phi$ for $t \le \tau'(-\varepsilon, +\infty) = \varepsilon/(2r) = \tau$. Since initial density profiles of $[\zeta]$ and $[\zeta^r]$ coincide on $\mathbb{R}^{+*}$, Lemma 5.2 implies that $\alpha_t^{[\zeta]}$ coincides with $\alpha_t^{[\zeta^r]}$ on $\mathbb{R}^{+*}$ for $t \le \tau$, so $\alpha_t^{[\zeta]} = \rho_0^\varepsilon(\cdot)\,dx$ on $\mathbb{R}^{+*}$ for $t \le \tau$. For the negative half-line, we use similar arguments with $\rho_0^{\varepsilon,l} := \rho_0^\varepsilon(\cdot)\mathbf{1}_{\mathbb{R}^{-*}} + \rho^\phi \mathbf{1}_{\mathbb{R}^{+*}}$.

Step 2. Since $\alpha_0^{[\eta]} = \rho_0(\cdot)\,dx$, and by Step 1 $\alpha_t^{[\zeta]} = \rho_0^\varepsilon(\cdot)\,dx$ for every $t > 0$, Lemma 5.3 yields

$$\sup_{x+2rt \le x' \le y' \le y-2rt} \left| \alpha_t^{[\eta]}([x'; y']) - \int_{x'}^{y'} \rho_0^\varepsilon(z)\,dz \right| \le 4K\varepsilon$$

for any $x < y$ and $t \le (y - x)/4r$; letting $x \to -\infty$, $y \to +\infty$ and $\varepsilon \to 0$, we obtain $\alpha_t^{[\eta]} = \rho_0(\cdot)\,dx$ for every $t \ge 0$. Finally, we apply Lemma 5.4 to $[\eta]$ and $[\zeta]$ with arbitrarily small x and large y. By Step 1 we have $\varphi_t^{[\zeta]} = t\phi$ for every $t > 0$, so we get $|\varphi_t^{[\eta]} - t\phi| \le K\varepsilon$ for every $t > 0$, and letting $\varepsilon \to 0$, we find $\varphi_t^{[\eta]} = t\phi$ for every $t > 0$.

5.3.3. *ϕ-admissible pairs below critical current.* We assume $(\rho^-; \rho^+)$ is an entropic pair such that $h(\rho^+) = h(\rho^-) < \phi$. Note that in this case, by Lemma 3.1, $\rho_0(\cdot)$ is both a stationary entropy solution and a stationary ϕ-entropy solution. By Corollary 4.4, there exists some $\beta \in [0; 1)$ such that $\tilde{\phi}_\beta = h(\rho^\pm)$. We introduce two systems $[\eta^1]$ and $[\eta^\beta]$, with the same initial configurations as $[\eta]$, repectively, unperturbed and governed by $\tilde{T}_\beta$.

$[\eta^1]$ has stationary hydrodynamic profile $\rho_0(\cdot)$ by Assumption 2.2; moreover, by Lemma 5.5, $\varphi_t^{N,[\eta^1]}$ converges in probability to $th(\rho^\pm)$ for every $t > 0$. On the other hand, since $h(\rho^\pm) = \tilde{\phi}_\beta$, $\rho^\pm$ are critical densities for the perturbation $\tilde{T}_\beta$. Thus, by Section 5.3.2, $[\eta^\beta]$ has stationary hydrodynamic profile $\rho_0(\cdot)$ and $\varphi_t^{N,[\eta^\beta]}$ converges in probability to $th(\rho^\pm)$ for every $t > 0$. Observe that η^β is slower than η and η is slower than η^1; and all three have the same initial configurations. Thus, by Lemma 4.4,

$$\varphi_t^{N,[\eta^\beta]} \le \varphi_t^{N,[\eta]} \le \varphi_t^{N,[\eta^1]}$$

for every $t > 0$. It follows that

$$\varphi_t^{[\eta]} = \varphi_t^{[\eta^\beta]} = \varphi_t^{[\eta^1]} = th(\rho^\pm) \tag{56}$$

for every $t > 0$; thus, $\varphi_t^{N,[\eta]}$ converges in probability to $th(\rho^{\pm})$. Now Lemma 5.2 and (56) imply that $\alpha_t^{[\eta]}$ and $\alpha_t^{[\eta^1]}$ coincide both on $\mathbb{R}^{+*}$ and $\mathbb{R}^{-*}$ for every $t > 0$, hence, $\alpha_t^{[\eta]} = \rho_0(\cdot)\,dx$ for every $t > 0$; thus, $[\eta]$ has stationary hydrodynamic profile $\rho_0(\cdot)$.

5.4. *Hydrodynamic limit for general initial profiles.* The proof is divided into two steps. First, we prove that Theorem 2.2 holds for small times when ρ_0 is locally as in Lemma 5.7.

LEMMA 5.8. *Assume ρ_0 has the following form: there exist $\varepsilon > 0$, and $(\rho^-; \rho^+)$ a ϕ-admissible pair modulo h, such that $\rho_0(x) \in \rho^-$ for $-\varepsilon < x < 0$ and $\rho_0(x) \in \rho^+$ for $0 < x < \varepsilon$. Let $[\eta]$ be a system governed by $\tilde{T}$ with initial density profile ρ_0. Then for every $t < \min(\varepsilon/V, \varepsilon/(2r))$ (with V defined in Proposition 3.2), η^N_{Nt} has density profile $\rho(t,\cdot)$ as $N \to \infty$, where ρ is the ϕ-entropy solution with intial datum ρ_0.*

PROOF. Consider initial profiles ρ_0^r and ρ_0^l such that ρ_0^r (resp. ρ_0^l) coincides with ρ_0 for $x > 0$ ($x < 0$) and $\rho_0^r(x) \in \rho^+$ ($\rho_0^l(x) \in \rho^-$) for $x < 0$ ($x > 0$). We denote the corresponding entropy solutions by $\rho^r(t,x)$ and $\rho^l(t,x)$. Finally, for $t < \varepsilon/V$, define $\rho(t,x)$ as $\rho^r(t,x)$ for $x > 0$ and $\rho^l(t,x)$ for $x < 0$. We now prove that, on the time interval $[0; \varepsilon/V)$, ρ is the ϕ-entropy solution starting from ρ_0. First, observe that (i) $\rho_0^r = \rho^+$ ($\rho_0^l = \rho^-$) modulo h on $(-\varepsilon; \varepsilon)$, (ii) a modulo h-constant profile is a stationary entropy solution by Lemma 3.1, and thus, (iii) by Proposition 3.2, $\rho^r(t,x) = \rho^+$ ($\rho^l(t,x) = \rho^-$) modulo h for $t < \varepsilon/V$ and $|x| < \varepsilon - Vt$; hence, the modulo h-pair $(\rho(t,0^-); \rho(t,0^+))$ is ϕ-entropic. Next, on either side of 0, ρ coincides with an entropy solution, which implies that it satisfies (18) away from 0 for every $c \in [0; K]$. Finally, $\rho(0,\cdot) = \rho_0(\cdot)$, and $\rho(t,\cdot) \to \rho(0,\cdot)$ locally in L^1 for $t \to 0$ because the entropy solutions $\rho^{r/l}$ converge to their initial data in the sense (19). The conclusion then follows from uniqueness and Definition 3.2.

Consider a subsequential limit $(\alpha^{[\eta]}, \varphi^{[\eta]})$. Let $[\zeta]$ be a system with the same dynamics as $[\eta]$ and a modulo h-Riemann initial density profile $\rho_0' \in \mathcal{R}_{\rho^-,\rho^+}$ such that ρ_0' coincides with ρ_0 on $(-\varepsilon; \varepsilon)$. Then Lemma 5.4 shows that $\varphi_t^{[\eta]} = \varphi_t^{[\zeta]}$ for every $t < \varepsilon/(2r)$, while by Lemma 5.7 we have $\varphi_t^{[\zeta]} = th(\rho^{\pm})$ for every $t > 0$. Next, let $[\eta^r]$ be an unperturbed system with initial density profile ρ_0^r. By Assumption 2.2 and Lemma 5.5 we have $\varphi_t^{[\eta^r]} = th(\rho^r(t, 0^{\pm})) = th(\rho^{\pm})$ for $t \le \varepsilon/V$. Lemma 5.2 then implies that $\alpha_t^{[\eta]}$ coincides with $\alpha_t^{[\eta^r]}$ on $\mathbb{R}^{+*}$ for every $t < \varepsilon/(2r)$. Hence, the restriction of $\alpha_t^{[\eta]}$ to $\mathbb{R}^{+*}$ is $\rho(t,\cdot)\,dx$ for every $t < \min(\varepsilon/V, \varepsilon/(2r))$. The same result can be proved on $\mathbb{R}^{-*}$ by considering $[\eta^l]$ with initial density profile ρ_0^l. $\square$

We are now ready to conclude the proof of Theorem 2.2.

Step 1. We first restrict to the following situation: $[\eta]$ is a system with initial density profile $\rho_0(\cdot)$, where ρ_0 has locally bounded variation and finite integral, and

$$\sum_{x\in\mathbb{Z}} \eta_0^N(x) \le CN \qquad \forall N \in \mathbb{N}^* \tag{57}$$

for some deterministic constant $C > 0$. The system being conservative, (57) still holds at later times. Let $\alpha^{[\eta]}$ be a subsequential limit. Because of (57), $\alpha_t^{[\eta]}$ is a finite nonnegative measure for every $t \ge 0$. Set

$$E(t) := \sup_{x\in\mathbb{R}} \left| \alpha_t^{[\eta]}\big((-\infty; x]\big) - \int_{-\infty}^x \rho(t, z)\, dz \right|,$$

where $\rho(\cdot, \cdot)$ denotes the ϕ-entropy solution with initial datum $\rho_0(\cdot)$; note that, by mass conservation in (5), $\rho(t, \cdot)$ has finite integral, and thus, $E(\cdot)$ is a.s. finite. We want to prove that a.e. path of $E(\cdot)$ is identically zero. Note that $E(\cdot)$ has a version with a.s. Lipschitz-continuous paths. It follows on the one hand, from the obvious relation

$$\alpha_t^{N,[\eta]}\big((-\infty; x]\big) = \alpha_0^{N,[\eta]}\big((-\infty; x]\big) - \varphi_t^{N,[\eta]}(x),$$

combined with Lemma 5.1, where $\varphi_t^{[N,\eta]}(x)$ is as in (49); on the other hand, from the relation

$$\int_{-\infty}^x \rho(t, z)\, dz = \int_{-\infty}^x \rho_0(z)\, dz - \int_0^t h\big(\rho(s, x^\pm)\big)\, ds,$$

which holds because ρ is a weak solution to (5); recall that, since $\rho_0(\cdot)$ has locally bounded variation, ρ has limits modulo h at x by Proposition 3.4. Since $E(0) = 0$ a.s. by construction, it is enough to prove that with probability one we have $E' \le 0$ a.e. To each $s > 0$ and $\varepsilon > 0$, we associate the following: (i) A modulo h-Riemann profile R^s defined as follows: for $x > 0$ (resp. $x < 0$), $R^s(x)$ is the projection of $\rho(s, x)$ on the modulo h class $\rho(s, 0^+)$ [resp. $\rho(s, 0^-)$]; here, projection on a closed interval I is meant in usual sense, that is, the closest point in I. (ii) A *delayed* system $[\eta^{s,\varepsilon}]$ whose evolution begins only at macroscopic time s; this means that $[\eta^{s,\varepsilon}]$ is a collection of processes $(\eta^{s,\varepsilon})_.^N = ((\eta^{s,\varepsilon})_\tau^N, \tau \ge Ns)$; the initial configuration $(\eta^{s,\varepsilon})_{Ns}^N(\omega, \omega')$ is fixed at time Ns, and the evolution (2) restricted to events ω occuring after this time. We impose that the sequence $(\eta^{s,\varepsilon})_{Ns}^N$ has density profile

$$\rho_0^{s,\varepsilon}(\cdot) = \rho(s, \cdot)\mathbf{1}_{(-\varepsilon;\varepsilon)^c} + R^s(\cdot)\mathbf{1}_{(-\varepsilon;\varepsilon)}$$

as $N \to \infty$, and, moreover, that the initial configurations $(\eta^{s,\varepsilon})_{Ns}^N$ satisfy (57) with a new constant C; this is possible because $\rho(s, \cdot)$ has finite integral equal to that of $\rho_0(\cdot)$, and thus, $\rho_0^{s,\varepsilon}(\cdot)$ has finite integral.

Denote by $(\rho^{s,\varepsilon}(t,x),\ t \geq s)$, the ϕ-entropy solution to (5) with datum $\rho_0^{s,\varepsilon}$ at time s. For $t \geq s$, the triangle inequality yields

$$E(t) \leq E_1^{s,\varepsilon}(t) + E_2^{s,\varepsilon}(t) + E^{s,\varepsilon}(t), \tag{58}$$

where $E^{s,\varepsilon}$ is defined as E, but with $\alpha^{[\eta^{s,\varepsilon}]}$ (resp. $\rho^{s,\varepsilon}$) instead of $\alpha^{[\eta]}$ (resp. ρ), and

$$E_1^{s,\varepsilon}(t) = \sup_{x\in\mathbb{R}} \bigl|\alpha_t^{[\eta]}((-\infty,x]) - \alpha_t^{[\eta^{s,\varepsilon}]}((-\infty,x])\bigr|,$$
$$E_2^{s,\varepsilon}(t) = \sup_{x\in\mathbb{R}} \left|\int_{-\infty}^x \rho(t,z)\,dz - \int_{-\infty}^x \rho^{s,\varepsilon}(t,z)\,dz\right|.$$

By Lemma 5.8, there is a constant $C = C(V,r) > 0$ [different from the one in (57)] such that $\alpha_t^{[\eta^{s,\varepsilon}]} = \rho^{s,\varepsilon}(t,\cdot)\,dx$ for $s \leq t \leq s + C\varepsilon$. Thus, for such times t we have $E^{s,\varepsilon}(t) = 0$. On the other hand,

$$\begin{aligned} E_2^{s,\varepsilon}(s + C\varepsilon) &\leq \int_{\mathbb{R}} |\rho(s + C\varepsilon, x) - \rho^{s,\varepsilon}(s + C\varepsilon, x)|\,dx \\ &\leq \int_{\mathbb{R}} |\rho(s,x) - \rho_0^{s,\varepsilon}(x)|\,dx := e(s;\varepsilon), \end{aligned} \tag{59}$$

note that $e(s;\varepsilon)$ is nonrandom and, by construction of ρ_0^s, $\lim_{\varepsilon\to 0} e(s;\varepsilon)/\varepsilon = 0$ for each $s > 0$; the second inequality follows from Proposition 3.5. Finally,

$$\begin{aligned} E_1^{s,\varepsilon}(s + C\varepsilon) &\leq E_1^{s,\varepsilon}(s) \leq E(s) + E^{s,\varepsilon}(s) + E_2^{s,\varepsilon}(s) \\ &\leq E(s) + e(s;\varepsilon), \end{aligned} \tag{60}$$

where the first inequality follows from Lemma 5.3 (with an exceptional set independent of s) because η_{Ns}^N and $(\eta^{s,\varepsilon})_{Ns}^N$ satisfy (57), and the second one follows from the triangle inequality. We conclude from (58)–(60) that, for every $\varepsilon > 0$, a.e. realization of $E(\cdot)$ satisfies $E(s+C\varepsilon) \leq E(s) + 2e(s;\varepsilon)$ for every $s > 0$. We can choose a common exceptional set for all ε's in a vanishing sequence. Thus, almost every realization of E satisfies $E' \leq 0$ almost everywhere.

Step 2. Now $[\eta]$ has initial density profile $\rho_0(\cdot)$, assumed only measurable. Let $\alpha^{[\eta]}$ be a subsequential limit. For $\varepsilon > 0$, let ρ_0^ε satisfy the assumptions of Step 1 and approximate ρ_0 in the sense $\int_{|x|\leq 1/\varepsilon} |\rho_0(x) - \rho_0^\varepsilon(x)|\,dx \leq \varepsilon$; denote by ρ^ε the corresponding ϕ-entropy solution. Consider a system $[\zeta]$ satisfying (57) with initial density profile ρ_0^ε. It is now easy to conclude that $\alpha_t^{[\eta]} = \rho(t,\cdot)\,dx$ for every $t > 0$, by using (i) (51) in Lemma 5.3, (ii) $\alpha_t^{[\zeta]} = \rho^\varepsilon(t,\cdot)\,dx$ from step one and (iii) Proposition 3.5 for $\rho(t,\cdot)$ and $\rho^\varepsilon(t,\cdot)$.

6. Existence and uniqueness of the ϕ-entropy solution. This section is devoted to the proofs of Theorem 2.1 and Propositions 3.4 and 3.5. We shall need the more general framework of *measure-valued* (m.v.) solutions to (1); see, for example, [10]. m.v. solutions are Young measures; we recall that a ($[0; K]$-valued) Young measure ν on $\mathbb{R}^{+*} \times \mathbb{R}$ is a weakly measurable mapping $(t, x) \mapsto \nu_{t,x}(d\rho)$ whose values are probability measures on $[0; K]$. The Young measure ν reduces to a Borel function $\rho(t, x)$ iff $\nu_{t,x} = \delta_{\rho(t,x)}$ a.e.; we then say it is "of Dirac form."

M.v. entropy solutions in the Kružkov sense and m.v. ϕ-entropy solutions in the sense of Definition 3.3 are defined by applying the following rule to (18), (19) and (25): for a Young measure $\nu_{t,x}$, instead of a function $\rho(t, x)$, replace any expression of the form $F(t, x, \rho(t, x))$ by the mean value $\int F(t, x, \rho)\, d\nu_{t,x}(\rho)$. We also extend the definition of limits modulo h to a Young measure as follows: we say the (Dirac) limit $\nu_{t,x^+} = \delta_{\rho^+}$ exist, where ρ^+ is a density modulo h, if $\lim_{y\to x^+} \int d(\rho, \rho^+)\, d\nu_{t,y}(\rho) = 0$, where $d(\rho, \rho^+)$ is the distance from point $\rho \in [0; K]$ to interval ρ^+; we similarly define the left limit $\nu_{t,x^-} = \delta_{\rho^-}$. We say the Young measure ν is a m.v. ϕ-entropy solution in the sense of Definition 3.2, iff it satisfies (18) in the m.v. sense outside $x = 0$ for every $c \in [0; K]$, and has limits $\nu_{t,0^\pm} = \delta_{\rho(t,0^\pm)}$ modulo h for a.e. $t > 0$, such that the pair $(\rho(t, 0^-), \rho(t, 0^+))$ modulo h is ϕ-admissible for every $t > 0$.

6.1. *Restricted uniqueness and equivalence of definitions.* Here we want to prove the following result.

PROPOSITION 6.1. *For m.v. ϕ-entropy solutions with boundary limits $\nu_{t,0^\pm} = \delta_{\rho(t,0^\pm)}$ modulo h for a.e. $t > 0$*: (i) *Definitions* 3.2 *and* 3.3 *are equivalent.* (ii) *A solution with given initial datum* [*in m.v. sense of* (19)] *is unique and of Dirac form.* (iii) *For two such solutions $\nu^i_{t,x} = \delta_{\rho^i(t,x)}$, $\Delta(t)$ defined in* (22) *is nonincreasing on a total subset of* $(0; (y - x)/(2V))$. (iv) *In particular, if the ν^i have initial data ρ^i_0 in the sense of* (19) *that coincide a.e. on* $[x; y]$, *then ρ^i coincide a.e. on the set* $\{(t, z) : 0 < t < (y - x)/2V; x + Vt < z < y - Vt\}$.

In the title we speak of "restricted" uniqueness because one has to assume existence of boundary limits. General uniqueness in the sense of Definition 3.3 will be proved in Section 6.3. The standard uniqueness proof for entropy solutions, due to Kružkov [17], and extended to m.v. solutions by Szepessy [29], relies on two lemmas. The first one states that two Young measures satisfying entropy inequalities (18) satisfy a "coupling" entropy inequality. The second one shows that the coupling entropy inequality implies a L^1 contraction inequality, which immediately yields uniqueness. We recall these two results below; the former is given in a slightly more general form needed in the sequel (but which follows from exactly the same proof as the original lemma).

LEMMA 6.1. *Assume that for each* $i \in \{1,2\}$, $\mathcal{C}_i$ *is a subset of* $[0;K]$ *and* ν^i *a Young measure supported a.e. on* $\mathcal{C}_i$. *Assume further that* Ω *is an open subset of* $\mathbb{R}^{+*} \times \mathbb{R}$ *such that* ν^1 (*resp.* ν^2) *satisfies* (18) *on* Ω *for every* $c \in \mathcal{C}_2$ (*resp.* $c \in \mathcal{C}_1$). *Then the coupling entropy inequality*

$$\begin{aligned}\partial_t\left[\int \varphi(\rho^1;\rho^2)\nu^1_{t,x}(d\rho^1)\nu^2_{t,x}(d\rho^2)\right]\\ + \partial_x\left[\int \psi(\rho^1;\rho^2)\nu^1_{t,x}(d\rho^1)\nu^2_{t,x}(d\rho^2)\right] \le 0\end{aligned} \tag{61}$$

holds in distribution sense on Ω.

Inequality (61) is the m.v. form of (25) with ν^1 and ν^2 instead of ρ and r.

LEMMA 6.2. *Assume* (61) *holds on* $(0;\varepsilon) \times \mathbb{R}$ *for some* $\varepsilon > 0$. *Given* $x < y$, *define* $\Delta(t)$ *as in* (22), *but in m.v. form*, *that is*,

$$\Delta(t) = \int_{x+Vt}^{y-Vt} \int |\rho^1 - \rho^2| \nu^1_{t,x}(d\rho^1)\nu^2_{t,x}(d\rho^2)\, dx.$$

Then $\Delta(t)$ *is nonincreasing on a total subset of* $(0;(y-x)/2V) \cap (0;\varepsilon)$. *If, in addition*, ν^i *have initial data* ρ^i_0 *in m.v. sense of* (19), *then* 0 *can be added to the former subset*, *with* $\Delta(0)$ *evaluated by setting* $\nu^i_{0,x} = \delta_{\rho^i_0(x)}$. *In particular, if* ρ^i_0 *coincide on* $[x;y]$ *then*, *a.e. on the set* $\{(t,z): 0 < t < \min(\varepsilon;(y-x)/2V); x + Vt < z < y - Vt\}$, ν^i *coincide and are of Dirac form.*

In the ϕ-entropy case, (18) is only available for all c on $\mathbb{R}^{+*} \times \mathbb{R}^*$. Therefore, Lemma 6.1 yields (61) on $\mathbb{R}^{+*} \times \mathbb{R}^*$ for two m.v. ϕ-entropy solutions. To establish (61) on $\mathbb{R}^{+*} \times \mathbb{R}$ we need to prove its trace along $x = 0$. Define

$$J(\rho_1^-, \rho_1^+; \rho_2^-, \rho_2^+) = \psi(\rho_1^+;\rho_2^+) - \psi(\rho_1^-;\rho_2^-).$$

Note that this function can be defined on densities modulo h, as it does not depend on particular representatives. From differentiation theory in distribution sense, we have the following:

LEMMA 6.3. *Assume* ν^1 *and* ν^2 *satisfy* (61) *on* $\mathbb{R}^{+*} \times \mathbb{R}^*$ *and have limits* $\nu^i_{t,0^\pm} = \delta_{\rho^i(t,0^\pm)}$ *modulo* h *for a.e.* $t > 0$. *Then* (61) *holds on* $\mathbb{R}^{+*} \times \mathbb{R}$ *iff* $J(\rho^1(t,0^-), \rho^1(t,0^+); \rho^2(t,0^-), \rho^2(t,0^+)) \le 0$ *for a.e.* $t > 0$.

The next lemma shows (statement 1) that the above trace condition is, indeed, satisfied. Statement 2 will be used to establish equivalence of definitions in Proposition 6.1. Note that, if we seek to extend Definition 3.1 to $\phi > \phi^*$, statement 1 remains valid but statement 2 fails. The interpretation is that specifying a maximum current no longer determines a unique entropy condition: when

$\phi > \phi^*$, admissibility in the sense of Definition 3.1 is only one among other admissibility criteria producing solutions with maximal current ϕ, whereas it is the only one when $\phi \le \phi^*$.

LEMMA 6.4.

1. $J(\rho^-, \rho^+; r^-, r^+) \le 0$ *whenever* (ρ^-, ρ^+) *and* (r^-, r^+) *are* ϕ*-admissible.*
2. *The pair* (ρ^-, ρ^+) *is* ϕ*-admissible if and only if*: (i) $J(\rho^-, \rho^+; c, c) \le 0$ *for every* ϕ*-admissible density* c, *and* (ii) $J(\rho^-, \rho^+; \rho^\phi, \rho_\phi) \le 0$.

PROOF. 1. Since $J(\rho^-, \rho^+; r^-, r^+) = J(\rho^-, \rho^+; r^-, r^-) + J(r^-, r^+; \rho^+, \rho^+)$, it is enough to show $J(\rho^-, \rho^+; c, c) \le 0$ for a ϕ-admissible pair (ρ^-, ρ^+) and a ϕ-admissible density c. Assume first that (ρ^-, ρ^+) is entropic; then, by (20), $J(\rho^-, \rho^+; c, c) \le 0$ holds even for every $c \in [0; K]$. Now, if (ρ^-, ρ^+) is ϕ-entropic, but not entropic, we must have $h(\rho^\pm) = \phi$ and $\rho^+ < \rho^-$; thus, $J(\rho^-, \rho^+; c, c) \le 0$ if c is an admissible density because then $h(c) \le \phi = h(\rho^\pm)$.

2. That ϕ-admissibility implies (i) and (ii) follows from the first part of the proof, since $(\rho^\phi; \rho_\phi)$ and $(c; c)$ are ϕ-admissible pairs. We now prove the converse implication. First, taking $c = 0$ and $c = K$ in (i) shows that $h(\rho^+) = h(\rho^-)$. Next, (ii) implies that $h(\rho^\pm) \le \phi$. Indeed, assume we have $h(\rho^\pm) > \phi$. Then, $\rho^\pm$ both lie in the interval $(\rho_\phi; \rho^\phi)$, which implies $J(\rho^-, \rho^+; \rho^\phi, \rho_\phi) > 0$, in contradiction with (ii). We are left with the following cases. Either $h(\rho^\pm) = \phi$, in which case $(\rho^-; \rho^+)$ is ϕ-entropic. Or, $h(\rho^\pm) < \phi$. Then (ii) imposes $\mathrm{sgn}(\rho^+ - \rho_\phi) \ge \mathrm{sgn}(\rho^- - \rho^\phi)$. Combined with $h(\rho^\pm) < \phi$, this implies either $\rho^+ = \rho^- < \rho_\phi$, or $\rho^+ = \rho^- > \rho^\phi$, or $\rho^- < \rho_\phi < \rho^\phi < \rho^+$. $(\rho^-; \rho^+)$ is obviously ϕ-admissible in the first two cases, and it is in the third one because $\phi \le \phi^*$ implies that the chord determined by $\rho^\pm$ lies below the graph of h. □

PROOF OF PROPOSITION 6.1. (i) Let ν be a m.v. ϕ-entropy solution with limits $\nu_{t,0^\pm} = \delta_{\rho(t,0^\pm)}$ modulo h. Both Definitions 3.2 and 3.3 imply that ν satisfies entropy inequalities (18) on $\mathbb{R}^{+*} \times \mathbb{R}^*$ for every $c \in [0; K]$, and, thus, also (25) with $r(\cdot) = R_{\rho^\phi,\rho_\phi}(\cdot)$, as $R_{\rho^\phi,\rho_\phi}(\cdot)$ is constant on $\mathbb{R}^{+*}$ and $\mathbb{R}^{-*}$. It remains to show that (18) with ϕ-admissible c and (25) with $r(\cdot) = R_{\rho^\phi,\rho_\phi}(\cdot)$, extend to $\mathbb{R}^{+*} \times \mathbb{R}$ iff the modulo h boundary pair $(\rho(t, 0^-); \rho(t, 0^+))$ is ϕ-admissible. This follows from Lemma 6.3 [with $\nu^1 = \nu$, and $\nu^2_{t,x} = \delta_{R_{\rho^\phi,\rho_\phi}(x)}$ or $\nu^2_{t,x} = c$], and statement (ii) of Lemma 6.4.

(ii), (iii) and (iv): Let ν^i, $i \in \{1; 2\}$, be m.v. ϕ-entropy solutions with limits $\nu^i_{t,0^\pm} = \delta_{\rho^i(t,0^\pm)}$ modulo h. By Lemma 6.1, ν^i satisfy the coupling entropy inequality (61) on $\mathbb{R}^{+*} \times \mathbb{R}^*$. This inequality extends to $\mathbb{R}^{+*} \times \mathbb{R}$ by Lemma 6.3 and statement (i) of Lemma 6.4. The rest then follows from Lemma 6.2. □

6.2. *Existence of a ϕ-entropy solution with limits modulo h at $x=0$.* We now prove existence of a ϕ-entropy solution satisfying Propositions 3.4 and 3.5, when the initial datum ρ_0 has locally bounded variation. To this end we construct the ϕ-entropy solution as a limit by smoothing out the singularity at $x=0$ as follows. Let $\alpha(\cdot)$ denote a smooth function strictly monotonous on either side of 0, with minimum value $\alpha(0)=\phi/h^*$ and maximum value $\alpha(x)=1$ for $|x|\geq 1$, and $\alpha^\varepsilon(x)=\alpha(x/\varepsilon)$. We consider the spatially heterogeneous conservation law

$$\partial_t\rho+\partial_x[\alpha^\varepsilon(x)h(\rho)]=0. \tag{62}$$

Note that the maximum current at $x=0$ is $\alpha^\varepsilon(0)h^*=\phi$. We denote by $\rho^\varepsilon(t,x)$ the unique entropy solution to (62) with initial datum $\rho_0(\cdot)$.

REMARK ON SPATIALLY HETEROGENEOUS CONSERVATION LAWS. In Section 3.1 we considered conservation laws without space dependence. Kružkov's theory also incorporates such dependence. In the case of (62), entropy inequalities write

$$\begin{aligned}\partial_t\varphi\bigl(\rho^\varepsilon(t,x);c\bigr)&+\partial_x\bigl[\alpha^\varepsilon(x)\psi\bigl(\rho^\varepsilon(t,x);c\bigr)\bigr]\\&+\operatorname{sgn}\bigl(\rho^\varepsilon(t,x)-c\bigr)\partial_x\alpha^\varepsilon(x)h(c)\leq 0\end{aligned} \tag{63}$$

and the coupling entropy inequality (61) for two entropy solutions ρ^ε, r^ε writes

$$\partial_t\varphi(\rho^\varepsilon;r^\varepsilon)+\partial_x[\alpha^\varepsilon(x)\psi(\rho^\varepsilon;r^\varepsilon)]\leq 0. \tag{64}$$

Proposition 3.1 is still valid, and Proposition 3.2 holds with the same V independent of ε, as this is a uniform Lipschitz constant for $\alpha^\varepsilon(x)h(\cdot)$.

We want to prove the following result:

PROPOSITION 6.2. (i) *As $\varepsilon\to 0$, ρ^ε converges in $C^0(\mathbb{R}^+;L^1_{\mathrm{loc}}(\mathbb{R}))$ to a ϕ-entropy solution $\rho\in C^0(\mathbb{R}^+;L^1_{\mathrm{loc}}(\mathbb{R}))$ of* (5) *with initial datum $\rho_0(\cdot)$ and limits $\rho(t,0^\pm)$ modulo h for every $t>0$.* (ii) *Two solutions constructed in this way from initial data ρ_0^1, ρ_0^2 with locally bounded variation satisfy Proposition* 3.5.

The proof of Proposition 6.2 relies on two lemmas. To begin with we must show that the collection $(\rho^\varepsilon,\varepsilon>0)$ is sequentially relatively compact in some sufficient topology. We shall have to consider convergence in the sense of Young measures. We recall that a sequence $(\rho^n,\ n\in\mathbb{N})$ of Borel functions on $\mathbb{R}^{+*}\times\mathbb{R}$ is said to converge to Young measure ν as $n\to\infty$ iff

$$\lim_{n\to\infty}\int F\bigl(t,x,\rho^n(t,x)\bigr)\,dt\,dx=\int F(t,x,\rho)\,d\nu_{t,x}(\rho)\,dt\,dx \tag{65}$$

for every smooth, compactly supported function F on $\mathbb{R}^{+*}\times\mathbb{R}\times\mathbb{R}$. This more generally implies (65) if F is compactly supported, and there is a total subset $\mathcal{S}$

of $\mathbb{R}^{+*} \times \mathbb{R}$ such that F is continuous at (t, x, ρ) for every $(t, x) \in \mathcal{S}$ and $\rho \in [0; K]$. If the ρ^n are uniformly bounded, convergence to a Young measure $\nu_{t,x} = \delta_{\rho(t,x)}$ of Dirac form is equivalent to convergence to ρ locally in $L^1(\mathbb{R}^{+*} \times \mathbb{R})$. The following lemma relies on uniform estimates for time and space continuity modulus; it will be proved in Appendix.

LEMMA 6.5. *The collection* $(\rho^\varepsilon, \varepsilon > 0)$ *is sequentially relatively compact w.r.t. convergence in the sense* (65), *and any limiting Young measure* ν *has a representative with limits* $\nu_{t,x^\pm} = \delta_{\rho(t,x^\pm)}$ *modulo* h *for every* $t > 0$ *and* $x \in \mathbb{R}$, *where* $\rho(t, x^\pm)$ *are densities modulo* h. *Moreover, we have the uniform estimate*

$$\int_I |\rho^\varepsilon(t + \delta; x) - \rho^\varepsilon(t, x)|\, dx \le C\delta \qquad \forall t \ge 0, \delta > 0, \varepsilon > 0 \tag{66}$$

for every bounded interval $I \subset \mathbb{R}$, *where the constant* C *depends only on* I, h *and* ρ_0, *and we set* $\rho^\varepsilon(0, x) = \rho_0(x)$.

Estimating (66) will be necessary to show a posteriori convergence in the stronger topology of Proposition 6.2. The next lemma is conceptually crucial: we show that the maximal critical shock profile arises as a limit of (62). Namely,

LEMMA 6.6. *There exists a sequence of continuous functions* $r^\varepsilon(x)$ *such that* r^ε *is a stationary entropy solution to* (62), *and* $r^\varepsilon \to R_{\rho^\phi;\rho_\phi}$ *locally in* $L^1(\mathbb{R})$ *as* $\varepsilon \to 0$.

PROOF. For $\lambda \in [0; h^*]$, we define the following inverse functions for h:

$$\begin{aligned} r^-(\lambda) &= \sup\{\rho \in [0; K] : h(\rho) = \lambda\},\\ r^+(\lambda) &= \inf\{\rho \in [0; K] : h(\rho) = \lambda\}, \end{aligned} \tag{67}$$

and define $r(x)$ for $x \in [-1; 1]$ by

$$\begin{aligned} r(x) &= r^+\left(\frac{\phi}{\alpha(x)}\right) \qquad \text{if } x > 0,\\ r(x) &= r^-\left(\frac{\phi}{\alpha(x)}\right) \qquad \text{if } x < 0. \end{aligned} \tag{68}$$

By construction, r is a decreasing function with $r(-1) = \rho^\phi$ and $r(1) = \rho_\phi$, which we extend outside $[-1; 1]$ by setting $r(x) = \rho^\phi$ (resp. ρ_ϕ) for $x < -1$ (resp. $x > 1$). This way we have $\alpha(x)h(r(x)) = \phi$ for every $x \in \mathbb{R}$. Therefore, $r^\varepsilon(x) = r(x/\varepsilon)$ is a stationary weak solution of (62) converging to $R_{\rho^\phi;\rho_\phi}$ locally in L^1. The crucial point is that our construction indeed provides a stationary *entropy* solution to (62). This follows from the following property of r: at every point of discontinuity $x \in (-1; 1)$ of r, the density values $r(x^-) > r(x^+)$ determine a horizontal chord entirely above the graph of h. This implies that all decreasing discontinuities in r^ε are entropic. □

PROOF OF PROPOSITION 6.2. Let ν be some subsequential limiting Young measure for ρ^ε as $\varepsilon \to 0$. By Lemma 6.5, ν has boundary limits modulo h at $x = 0$. We shall prove that ν is a m.v. ϕ-entropy solution with initial datum $\rho_0(\cdot)$. Proposition 6.1 will then imply the following: (i) ν is of Dirac form: $\nu_{t,x} = \delta_{\rho(t,x)}$, thus, proving existence of a ϕ-entropy solution ρ; (ii) ρ is independent of the subsequence, and thus, the whole sequence ρ^ε converges to ρ locally in $L^1(\mathbb{R}^{+*} \times \mathbb{R})$; the sharper local convergence in $C^0(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}))$ then follows from (66).

First, we check that ν satisfies (19) in m.v. sense. The following technical point is left to the reader: because of (66), ρ^ε converges to ν in a stronger sense than (65), that is, without time integration,

$$\lim_{\varepsilon \to 0} \int F\big(x, \rho^\varepsilon(t,x)\big)\, dx = \int F(x,\rho)\, d\nu_{t,x}(\rho)\, dx \tag{69}$$

for every $t > 0$, if F is compactly supported, and there exists a total subset $\mathcal{S}$ of $\mathbb{R}$ such that F is continuous at (x, ρ) for every $x \in \mathcal{S}$ and $\rho \in [0; K]$. Hence, we may pass to the limit in (66) with $t = 0$ to obtain

$$\int_I |\rho - \rho_0(x)|\, d\nu_{\delta,x}\, dx \le C\delta,$$

for every bounded interval I, which yields the desired result. Next, we establish the ϕ-entropy conditions of Definition 3.3 for ν. As an entropy solution to (62), ρ^ε satisfies entropy inequality (63) in distribution sense on $\mathbb{R}^{+*} \times \mathbb{R}$ for every $c \in [0; K]$. Because of convergence in the sense (65) and the fact that $\alpha^\varepsilon(x) = 1$ for $|x| \ge \varepsilon$, in the limit $\varepsilon \to 0$ we get (18) in m.v. form for ν away from $x = 0$ (i.e., on $\mathbb{R}^{+*} \times \mathbb{R}^*$); this is easily seen taking test functions supported away from $x = 0$. Next, we consider the sequence of entropy solutions $r^\varepsilon(t,x) = r^\varepsilon(x)$ to (62) constructed in Lemma 6.6. Since ρ^ε and r^ε are entropy solutions for the conservation law (62), they satisfy the coupling entropy inequality (64) in distribution sense on $\mathbb{R}^{+*} \times \mathbb{R}$. For the same reasons as above, combined with convergence of r^ε to $R_{\rho^\phi;\rho_\phi}$, this implies the m.v. form of (25) for ν and $r(x) = R_{\rho^\phi;\rho_\phi}$, which establishes the entropy condition. Finally, two solutions ρ^i obtained in this way from initial data ρ_0^i, $i \in \{1; 2\}$ satisfy Proposition 3.5. This follows from Proposition 3.2 applied to the corresponding solutions $\rho^{i,\varepsilon}$ of (62) and passing to the limit. □

6.3. *General existence and uniqueness.* So far we have proved Theorem 2.1, Propositions 3.4 and 3.5 in the particular case where $\rho_0(\cdot)$ has locally bounded variation. We now extend Theorem 2.1, Proposition 3.5 and statement (i) of Proposition 3.4 to the case of bounded measurable $\rho_0(\cdot)$. Existence of a ϕ-entropy solution in the sense of Definition 3.3, with the properties of Propositions 3.4 and 3.5, follows from a simple approximation procedure:

LEMMA 6.7. *Assume* $\rho_0(\cdot)$ *is any measurable* $[0; K]$*-valued initial datum. Let* $(\rho_0^n, n \in \mathbb{N})$ *be a sequence of approximations with locally bounded variation, converging to* $\rho_0(\cdot)$ *locally in* L^1. *Denote by* ρ^n *the corresponding* ϕ*-entropy solutions from Proposition* 6.2. *Then*, (i) *as* $n \to \infty$, ρ^n *converges in* $C^0(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}))$ *to a* ϕ*-entropy solution* ρ *in the sense of Definition* 3.3 *with initial datum* ρ_0 *and* (ii) *solutions constructed in this way from two initial data* ρ_0^1 *and* ρ_0^2 *satisfy Proposition* 3.5.

PROOF. Statement (ii) of Proposition 6.2 implies that (ρ^n) is a Cauchy sequence in $C^0(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}))$ and, thus, converges to some $\rho \in C^0(\mathbb{R}^+; L^1_{\text{loc}}(\mathbb{R}))$. It is easy to see that Definition 3.3 is stable for such convergence; hence, ρ is a ϕ-entropy solution with Cauchy datum ρ_0; this establishes (i). For two initial data ρ_0^1 and ρ_0^2, we may apply statement (ii) of Proposition 6.2 to the corresponding sequences $\rho^{1,n}$ and $\rho^{2,n}$; passing to the limit, we obtain Proposition 3.5 for the ϕ-entropy solutions ρ^1 and ρ^2, that is, statement (ii). □

It remains to prove that the solution constructed above is the unique one in the sense of Definition 3.3. Compared to Proposition 6.1, the difficulty is that we cannot assume existence of boundary limits. The key idea will be first to compare a general ϕ-entropy solution with one that has boundary limits. This is done in the following lemma.

LEMMA 6.8. *Assume* ρ^1 *is a* ϕ*-entropy solution in the sense of Definition* 3.3 *with Cauchy datum* ρ_0^1, *and* ρ^2 *is the* ϕ*-entropy solution constructed in Section* 6.2 *with Cauchy datum* ρ_0^2, *where* ρ_0^2 *has locally bounded variation. Then* $\Delta(t) \le \Delta(0)$ *for a.e.* $0 < t < (y - x)/(2V)$, *where* $\Delta(t)$ *is defined as in* (22).

This lemma implies that any ϕ-entropy solution in the sense of Definition 3.3 is the limit of ρ_0^n from Lemma 6.7, hence, establishing uniqueness. Before proving Lemma 6.8, we need the following preliminary lemma, which yields additional entropy inequalities for a ϕ-entropy solution.

LEMMA 6.9. *A* ϕ*-entropy solution* ρ *in the sense of Definition* 3.3 *satisfies* (25) *on* $\mathbb{R}^{+*} \times \mathbb{R}$ *with every stationary profile* $r(t, x) = r(x)$ *in* $\mathcal{R}_{r^-;r^+}$, *where* $(r^-; r^+)$ *is a* ϕ*-admissible pair modulo* h.

PROOF.

CASE 1. Let us first assume that the pair (r^-, r^+) is entropic. Then $r(t, x)$ satifies the usual Kružkov inequalities (18) for every $c \in [0; K]$. On the other hand $r^\pm$ are both admissible densities. Thus we may apply Lemma 6.1 with $\Omega = \mathbb{R}^{+*} \times \mathbb{R}$, $\nu^1_{t,x} = \delta_{\rho(t,x)}$, $\nu^2_{t,x} = \delta_{r(x)}$, $\mathcal{C}_1 = [0; K]$, $\mathcal{C}_2$ the set of ϕ-admissible densities.

CASE 2. Assume (r^-, r^+) is not entropic, so it must be a critical shock with $\rho_\phi \le r^+ \le r^- \le \rho^\phi$. For $\varepsilon > 0$, we define $r^\varepsilon(t,x) = r(t,x)\mathbf{1}_{(-\varepsilon,\varepsilon)}(x) + R_{\rho^\phi,\rho_\phi}(x)\mathbf{1}_{(-\varepsilon,\varepsilon)^c}(x)$. Below we argue that ρ and r^ε satisfy (25). Then $\varepsilon \to 0$ will produce the same for ρ and r, because $r^\varepsilon \to r$ in L^1 sense, which allows to pass to the limit in the distributional inequality.

First, ρ and r^ε satisfy (25) on $\mathbb{R}^{+*} \times (-\varepsilon, \varepsilon)$ by Definition 3.3. Let us show they also do on $\mathbb{R}^{+*} \times \mathbb{R}^*$ and we shall be done. To this end observe that compared to r, r^ε has two additional increasing discontinuities: the pairs (r^-, ρ^ϕ) and (ρ_ϕ, r^+). Both are entropic, because the condition $\phi \le \phi^*$ implies that the chord joining any two critical densities lies below the graph of h. Thus, by the same arguments as in the proof of Lemma 3.1, r^ε satisfies (18) for all c outside $x = 0$, because it is piecewise constant with entropic pairs. We may now apply Lemma 6.1 with $\Omega = \mathbb{R}^{+*} \times \mathbb{R}^*$, $\nu^1_{t,x} = \delta_{\rho(t,x)}$, $\nu^2_{t,x} = \delta_{r^\varepsilon(x)}$, and $\mathcal{C}_1$ and $\mathcal{C}_2$ as in Case 1. □

PROOF OF LEMMA 6.8. For every $s > 0$, limits $\rho^2(s, 0^\pm)$ modulo h exist by Proposition 6.2, and we consider the following: (i) A modulo h-Riemann profile $R_s(\cdot)$ defined as follows: for $z > 0$ (resp. $z < 0$), $R_s(z)$ is the projection of $\rho^2(s,z)$ on the modulo h class $\rho^2(s, 0^+)$ [resp. $\rho^2(s, 0^-)$]. (ii) A family $(\rho^{2,\varepsilon}(s,\cdot), \varepsilon > 0)$ of density profiles with locally bounded variation, such that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_x^y |\rho^2(s,z) - \rho^{2,\varepsilon}(s,z)|\, dz = 0,$$

with x and y taken from definition of Δ in (22). (iii) A new Cauchy datum

$$\rho_s^{2,\varepsilon}(0,\cdot) = \mathbf{1}_{(-2V\varepsilon;2V\varepsilon)} R_s + \mathbf{1}_{(-2V\varepsilon;2V\varepsilon)^c} \rho^{2,\varepsilon}(s,\cdot). \tag{70}$$

Set $E(s;\varepsilon) = \int_x^y |\rho^2(s,z) - \rho_s^{2,\varepsilon}(0,z)|\, dz$. We shall prove that, for every $\varepsilon > 0$, there exists a total subset $\mathcal{T}_\varepsilon$ of $\{(s,t) \in (0;+\infty) \times (0;\varepsilon) : s + t < (y-x)/(2V)\}$ such that

$$\Delta(s+t) - \Delta(s) \le 2E(s;\varepsilon) \qquad \forall (s,t) \in \mathcal{T}_\varepsilon. \tag{71}$$

By existence of limits modulo h, the contribution of $(-2V\varepsilon; 2V\varepsilon) \cap (x;y)$ to $E(s;\varepsilon)$ is $o(\varepsilon)$ for every s, thus, $E(s;\varepsilon) = o(\varepsilon)$ as $\varepsilon \to 0$ for any fixed s. Thus, (71) implies the derivative Δ' in distribution sense is nonpositive: to see this take a test function f on $(0;+\infty)$, set $t = \varepsilon u$, integrate (71) against $f(s)$ over $(s,u) \in (0;+\infty) \times (0;1)$, and let $\varepsilon \to 0$. Hence, there is a total subset $\mathcal{T}$ of $(0;+\infty)$ on which $\Delta(\cdot)$ is nonincreasing. Let $t \in \mathcal{T}$; since (19) holds for ρ^1 and ρ^2, we can find a sequence $t_n \to 0$ as $n \to \infty$, with $t_n \in \mathcal{T}$, such that $\Delta(t_n) \to \Delta(0)$; thus, $\Delta(t) \le \Delta(0)$ for every $t \in \mathcal{T}$. This completes the proof of the lemma.

We now turn to the proof of (71). Denote by $\mathcal{S} \subset (0; (y-x)/(2V))$ the set (with total Lebesgue measure) of Lebesgue points of Δ. We take $s \in \mathcal{S}$ as the new time origin and set $\rho^i_s(t,\cdot) = \rho^i(s+t,\cdot)$ for $i \in \{1;2\}$. Since $\rho_s^{2,\varepsilon}(0,\cdot)$ has locally bounded variation, we may consider the ϕ-entropy solution $\rho_s^{2,\varepsilon}(t,\cdot)$ with initial datum $\rho_s^{2,\varepsilon}(0,\cdot)$, as constructed in Proposition 6.2. We write $\Delta(s+t) \le \Delta'_\varepsilon(s;t) + \Delta''_\varepsilon(s;t)$, where

$$\Delta'_\varepsilon(s;t) = \int_{x+V(s+t)}^{y-V(s+t)} |\rho_s^1(t,z) - \rho_s^{2,\varepsilon}(t,z)|\,dx,$$
$$\Delta''_\varepsilon(s;t) = \int_{x+V(s+t)}^{y-V(s+t)} |\rho_s^2(t,z) - \rho_s^{2,\varepsilon}(t,z)|\,dx.$$

Note that ρ_s^2 and $\rho_s^{2,\varepsilon}$ belong to $C^0(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}))$ by Proposition 6.2, which implies continuity of Δ''. This and statement (iii) of Proposition 6.1 imply

$$\Delta''_\varepsilon(s;t) \le \Delta''_\varepsilon(s;0) \le E(s;\varepsilon) \tag{72}$$

for every $t > 0$. We cannot apply statement (iii) of Proposition 6.1 to ρ_s^1 and $\rho_s^{2,\varepsilon}$ in Δ'_ε because ρ^1 is not assumed to have boundary limits modulo h. Nevertheless, we can prove the coupling inequality (61) for ρ_s^1 and $\rho_s^{2,\varepsilon}$ as follows: $\rho_s^{2,\varepsilon}(0,\cdot)$ coincides with R_s for $|x| < 2V\varepsilon$; R_s is a stationary ϕ-entropy solution by Lemma 3.1. Since $\rho_s^{2,\varepsilon}(0,\cdot)$ and R_s are ϕ-entropy solutions with boundary limits modulo h, by statement (iv) of Proposition 6.1, they coincide a.e. on the set $\{t < 2\varepsilon, |x| < V(2\varepsilon - t)\}$. Thus, by Lemma 6.9, ρ^1 and $\rho_s^{2,\varepsilon}$ satisfy (61) on the domain $\{t < \varepsilon, |x| < V\varepsilon\}$. On the other hand, by Lemma 6.1 they also satisfy (61) on $\mathbb{R}^{+*} \times \mathbb{R}^*$; thus, they finally do on $(0;\varepsilon) \times \mathbb{R}$. Lemma 6.1 then implies

$$\delta^{-1}\int_t^{t+\delta} \Delta'_\varepsilon(s;u)\,du \le \delta^{-1}\int_0^\delta \Delta'_\varepsilon(s;u)\,du$$
$$\le \delta^{-1}\int_s^{s+\delta} \Delta(u)\,du + \delta^{-1}\int_0^\delta \Delta''_\varepsilon(s;u)\,du$$

for every $\delta \in (0;\varepsilon)$. Define $\mathcal{S}_{s,\varepsilon}$ as the set (with total Lebesgue measure) of $t \in (0;\varepsilon)$ that are Lebesgue points of $\Delta'_\varepsilon(s;\cdot)$, and $\mathcal{T}_\varepsilon = \{(s,t) \in \mathcal{S} \times (0;\varepsilon) : t \in \mathcal{S}_{s,\varepsilon}\}$. Since $\Delta''_\varepsilon(s;\cdot)$ is continuous, letting $\delta \to 0$ produces

$$\Delta'_\varepsilon(s;t) \le \Delta(s) + \Delta''_\varepsilon(s;0) \le \Delta(s) + E(s;\varepsilon) \qquad \forall\,(s,t) \in \mathcal{T}_{s,\varepsilon} \tag{73}$$

and (71) follows from (72) and (73). □

APPENDIX

Proof of Lemma 6.5. We shall need the following elementary lemma.

LEMMA A.1. *Assume $u \in L^\infty(\mathbb{R})$ and $\partial_x u \le v$ in distribution sense for some measure v with locally finite variation. Then,* $\mathrm{TV}_I \le 2(\|u\|_\infty + |v|(I))$, *where* TV_I *denotes total variation on interval I.*

PROOF. By assumption, $v - \partial_x u$ is a nonnegative distribution, and, thus, a nonnegative, locally finite measure. Thus, $\partial_x u$ is a locally finite measure. We have the unique decomposition $\partial_x u = (\partial_x u)^+ - (\partial_x u)^-$ with $(\partial_x u)^+$ and $(\partial_x u)^-$ nonnegative measures, and

$$|\partial_x u| = (\partial_x u)^+ + (\partial_x u)^- = 2(\partial_x u)^+ - \partial_x u \le 2|v| - \partial_x u.$$

The result follows from integration of the above inequality. □

PROOF OF LEMMA 6.5. We start proving (66). By semigroup property, $\rho^\varepsilon(\delta + \cdot, \cdot)$ is the entropy solution to (62) with initial datum $\rho^\varepsilon(\delta, \cdot)$. Hence, by Proposition 3.2,

$$\int_I |\rho^\varepsilon(t+\delta, x) - \rho^\varepsilon(t, x)|\, dx \le \int_{I^{Vt}} |\rho^\varepsilon(\delta, x) - \rho^\varepsilon(0, x)|\, dx, \tag{74}$$

where $I^{Vt} := \{x \in \mathbb{R} : d(x, I) \le Vt\}$. Since ρ_0 has locally bounded variation and ρ solves (62), we have the formal computation

$$\begin{aligned}\lim_{\delta\to 0} \delta^{-1} \int_{I^{Vt}} |\rho^\varepsilon(\delta, x) - \rho(0, x)|\, dx &= \int_{I^{Vt}} |\partial_t \rho(0, x)|\, dx \\ &= \int_{I^{Vt}} \big|\partial_x[\alpha^\varepsilon(x) h(\rho_0(x))]\big|\, dx,\end{aligned}$$

which can be justified for the entropy solution by smooth approximation of the initial datum because then the solution is locally strong at small times. (66) follows because α^ε has total variation bounded by 2. Next, denoting by TV_I total variation on an interval I, we have the estimate

$$\mathrm{TV}_I\big[\alpha^\varepsilon(\cdot)\psi\big(\rho^\varepsilon(t; \cdot); c\big)\big] \le C \tag{75}$$

for every $t > 0$ and $c \in [0; K]$, with again a uniform constant C that depends only on I, h and ρ_0. This follows from (63) for ρ^ε and Lemma A.1 with $u(\cdot) = \partial_x[\alpha^\varepsilon(\cdot)\psi(\rho^\varepsilon(t, \cdot); c)]$; variation of the measure v can be controlled using estimate (66) and the fact that total variation of α^ε is bounded by 2.

$(\rho^\varepsilon, \varepsilon > 0)$ is sequentially relatively compact in the sense (65) because ρ^ε is uniformly bounded, see [10]. Let ν be a subsequential limiting Young measure. (66) implies the stronger convergence (69). Set $u_c^\varepsilon = \psi(\rho^\varepsilon; c)$. Since ψ is uniformly Lipschitz continuous, (66) also holds with u_c^ε instead of ρ^ε and another uniform constant C independent of c and ε. By standard criteria (66) and (75) imply relative compacity of $(u_c^\varepsilon, \varepsilon > 0)$ in $C^0(\mathbb{R}^+; L^1_{\mathrm{loc}}(\mathbb{R}))$ for each $c \in [0; K]$, and that limits have locally bounded space variation at all times. Since $c \mapsto u_c^\varepsilon(t, x)$ is uniformy Lipschitz continuous, one may extract a further subsequence along which u_c^ε converges to some u_c simultaneously for all c. It is easy to see (we leave this as a technical exercise for the reader) that this implies the following: (i) existence of densities $\rho(t, x^\pm)$ modulo h such that $u_c(t, x^\pm) = \psi(\rho(t, x^\pm); c)$

for every $t > 0$, $x \in \mathbb{R}$ and $c \in [0; K]$, and (ii) there is a version of ν with limits $\nu_{t,x^\pm} = \rho(t, x^\pm)$ modulo h for every $t > 0$ and $x \in \mathbb{R}$. This follows from the fact that a density ρ modulo h is entirely determined by the collection of values $(\psi(\rho; c), c \in [0; K])$. $\square$

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