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THE EXISTENCE OF FIXED POINTS FOR THE $\cdot/GI/1$ QUEUE

BY JEAN MAIRESSE AND BALAJI PRABHAKAR

CNRS-Université Paris 7 and Stanford University

A celebrated theorem of Burke's asserts that the Poisson process is a fixed point for a stable exponential single server queue; that is, when the arrival process is Poisson, the equilibrium departure process is Poisson of the same rate. This paper considers the following question: Do fixed points exist for queues which dispense i.i.d. services of finite mean, but otherwise of arbitrary distribution (i.e., the so-called $\cdot/GI/1/\infty/FCFS$ queues)? We show that if the service time S is nonconstant and satisfies $\int P\{S \geq u\}^{1/2}\, du < \infty$, then there is an unbounded set $\mathcal{S} \subset (E[S], \infty)$ such that for each $\alpha \in \mathcal{S}$ there exists a unique ergodic fixed point with mean inter-arrival time equal to α. We conjecture that in fact $\mathcal{S} = (E[S], \infty)$.

1. Introduction. Consider a single server First-Come-First-Served queue with infinite waiting room, at which the service times are i.i.d. (a $\cdot/GI/1/\infty/FCFS$ queue). We are interested in the question of whether such queues possess fixed points: an inter-arrival process which has the same distribution as the corresponding inter-departure process.

The question of the existence of fixed points is intimately related to the limiting behavior of the distribution of departure processes from a tandem of queues. Specifically, consider an infinite tandem of $\cdot/GI/1/\infty/FCFS$ queues. The queues are indexed by $k \in \mathbb{N}$ and the customers are indexed by $n \in \mathbb{Z}$. The numbering of each customer is fixed at the first queue and remains the same as he/she passes through the tandem. Each customer leaving queue k immediately enters queue $k+1$. At queue k, write $S(n,k)$ for the service time of customer n and $A(n,k)$ for the inter-arrival time between customers n and $n+1$. We assume that the initial inter-arrival process, $A^0 = (A(n,0), n \in \mathbb{Z})$, is ergodic and independent of $(S(n,k), n \in \mathbb{Z}, k \in \mathbb{N})$. We also assume that the service variables $(S(n,k), n, k)$ are i.i.d. and that $E[S(0,0)] < E[A(0,0)] < \infty$. To avoid trivialities we assume that the service times are nonconstant, that is, $P\{S(0,0) \neq E[S(0,0)]\} > 0$.

By Loynes' results [15], each of the equilibrium departure processes $A^k = (A(n,k), n \in \mathbb{Z})$ for $k \geq 1$ is ergodic of mean $E[A(0,0)]$. The following are natural fixed point problems:

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Existence. For a given service distribution, does there exist a mean α ergodic inter-arrival process such that the corresponding inter-departure process has the same distribution? If yes, call such a distribution an *ergodic fixed point* of mean α.

Uniqueness. If an ergodic fixed point of mean α exists, is it unique?

Convergence. Assume there is a unique ergodic fixed point of mean α. If the inter-arrival process to the first queue, A^0, is ergodic of mean α, then does the law of A^k converge weakly to the ergodic fixed point of mean α as $k \to \infty$? If yes, call the fixed point an *attractor*.

A strand of research in stochastic network theory has pursued these questions for some time. Perhaps the earliest and best-known result is Burke's theorem [7], which shows that the Poisson process of rate $1/\alpha$ is a fixed point for exponential server queues with mean service time $\beta < \alpha$. Anantharam [1] established its uniqueness, and Mountford and Prabhakar [18] established that it is an attractor.

For $\cdot/GI/1/\infty/FCFS$ queues, the subject of this paper, Chang [8] established the uniqueness of an ergodic fixed point, should it exist, assuming that the services have a finite mean and an unbounded support. Prabhakar [19] provides a complete solution to the problems of uniqueness and convergence assuming only a finite mean for the service time and the existence of an ergodic fixed point. However, the existence of such fixed points was only known for exponential and geometric service times.

This paper establishes the existence of fixed points for a large class of service time distributions. We obtain the following result: if the service time S has mean β and if $\int P\{S \geq u\}^{1/2}\,du < \infty$, then there is a set $\mathcal{S}$ closed in (β, ∞), with $\inf\{u \in \mathcal{S}\} = \beta$, $\sup\{u \in \mathcal{S}\} = \infty$ and such that:

(a) For $\alpha \in \mathcal{S}$, there exists a mean α ergodic fixed point for the queue. Given this, [19] implies the attractiveness of the fixed point.

(b) For $\alpha \notin \mathcal{S}$, consider the stationary (but not ergodic) process F of mean α obtained as the convex combination of the ergodic fixed points of means $\underline{\alpha}$ and $\overline{\alpha}$ where $\underline{\alpha} = \sup\{u \in \mathcal{S}, u \leq \alpha\}$ and $\overline{\alpha} = \inf\{u \in \mathcal{S}, \alpha \leq u\}$. (Since $\mathcal{S}$ is closed, $\underline{\alpha}$ and $\overline{\alpha}$ belong to $\mathcal{S}$ and F is a fixed point for the queue.) If the inter-arrival times of the input process have a mean α, then the Cesaro average of the laws of the first k inter-departure processes converges weakly to F as $k \to \infty$.

These results rely heavily on a strong law of large numbers for the total time spent by a customer in a tandem of queues proved in [2]. We conjecture that our results are suboptimal and that in fact $\mathcal{S} = (\beta, \infty)$.

2. Preliminaries. The presence of an underlying probability space $(\Omega, \mathcal{F}, P)$ on which all the r.v.'s are defined is assumed all along. Given a measurable space $(K, \mathcal{K})$, we denote by $\mathcal{L}(K)$ the set of K-valued random variables, and by $\mathcal{M}(K)$ the set of probability measures on $(K, \mathcal{K})$. Throughout the paper, we consider random variables valued in $\mathbb{R}_+^{\mathbb{Z}}$. Equipped with the product topology, or topology of coordinate-wise convergence, $\mathbb{R}_+^{\mathbb{Z}}$ is a Polish space. We shall work on the measurable space $(\mathbb{R}_+^{\mathbb{Z}}, \mathcal{B})$ where $\mathcal{B}$ is the corresponding Borel σ-algebra. With the topology of weak convergence, the space $\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$ is a Polish space. For details see, for instance, [3], [10] or [11]. The weak convergence of $(\mu_n)_n$ to μ is denoted by $\mu_n \xrightarrow{w} \mu$. Furthermore, for $X_n, X \in \mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$, we say that X_n converges weakly to X (and we write $X_n \xrightarrow{w} X$) if the law of X_n converges weakly to the law of X. A process $X \in \mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ is *constant* if $X = (c)^{\mathbb{Z}}$ a.s. for some $c \in \mathbb{R}_+$.

We write $\mathcal{M}_{\mathrm{s}}(\mathbb{R}_+^{\mathbb{Z}})$ for the set of stationary probability measures with finite one-dimensional mean, and $\mathcal{M}_{\mathrm{e}}(\mathbb{R}_+^{\mathbb{Z}})$ for the set of ergodic probability measures with finite one-dimensional mean. For $\alpha \in \mathbb{R}_+$, we denote by $\mathcal{M}_{\mathrm{s}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ and $\mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ the sets of stationary and ergodic probability measures with one-dimensional mean α.

The *strong order* on $\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$, or $\mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$, is defined as follows (see [21] for more on strong orders). Consider $A, B \in \mathcal{L}(\mathbb{R}_+^{\mathbb{Z}})$ with respective distributions μ and ν. We say that A (resp. μ) is *strongly dominated* by B (resp. ν), denoted $A \leq_{\mathrm{st}} B$ (resp. $\mu \leq_{\mathrm{st}} \nu$), if

$$E[f(A)] \leq E[f(B)] \qquad \left(\text{resp.} \int f\,d\mu \leq \int f\,d\nu\right),$$

for any measurable $f : \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}$ which is increasing and such that the expectations are well defined. Here we consider the usual component-wise ordering of $\mathbb{R}_+^{\mathbb{Z}}$.

PROPOSITION 2.1 ([22]). *For μ and ν belonging to $\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$, $\mu \leq_{\mathrm{st}} \nu$ iff $\int f\,d\mu \leq \int f\,d\nu$ for any increasing and* continuous *real function f such that the expectations are well defined. For $\mu_n, \nu_n, n \in \mathbb{N}, \mu$ and ν belonging to $\mathcal{M}(\mathbb{R}_+^{\mathbb{Z}})$, suppose that $\mu_n \xrightarrow{w} \mu$, $\nu_n \xrightarrow{w} \nu$ and that $\mu_n \leq_{\mathrm{st}} \nu_n$. Then $\mu \leq_{\mathrm{st}} \nu$.*

We shall use the following fact a couple of times. Consider two random processes on $\mathbb{R}_+^{\mathbb{Z}}$: A which is ergodic and B which is stationary. Assume that $A \leq_{\mathrm{st}} B$. Let B be compatible with a P-stationary shift $\theta : \Omega \to \Omega$ and denote by $\mathfrak{T}$ the invariant σ-algebra. Then we have

$$E[A(0)] \leq E[B(0) | \mathfrak{T}] \qquad \text{a.s.} \tag{1}$$

Furthermore, if A is independent of B then the conditional law of B on the event $\{E[B(0)|\mathfrak{T}] = E[A(0)]\}$ is equal to the law of A. To prove this, the two ingredients are a representation theorem such as Theorem 1 in [14] and Birkhoff's ergodic theorem.

The symbols $\sim$ and $\perp\!\!\!\perp$ stand for "is distributed as" and "is independent of," respectively. We use the notation $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, and $x^+ = \max(x, 0) = x \vee 0$. For u, v in $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{Z}}$, $u \leq v$ denotes $u(n) \leq v(n)$ for all n.

3. The model. We introduce successively the $\cdot/\cdot/1/\infty/FCFS$ queue (Section 3.1), the $G/G/1/\infty/FCFS$ queue (Section 3.2), and the infinite tandem $G/GI/1/\infty/FCFS \to \cdot/GI/1/\infty/FCFS \to \cdots$ (Section 3.3). The presentation is made in an abstract and functional way. However, to help intuition, we use the queueing terminology and notation.

3.1. *The single queue.* Define the mapping

$$\begin{aligned} \Psi : \mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}} &\to \mathbb{R}_+^{\mathbb{Z}} \cup \{(+\infty)^{\mathbb{Z}}\}, \\ (a, s) &\mapsto w = \Psi(a, s), \end{aligned} \tag{2}$$

with

$$\begin{aligned} w(n) &= \Psi(a, s)(n) \\ &= \left[\sup_{j \le n-1} \sum_{i=j}^{n-1} s(i) - a(i) \right]^+ . \end{aligned} \tag{3}$$

A priori, Ψ is valued in $[0, \infty]^{\mathbb{Z}}$, but it is easily checked using (5) below that Ψ actually takes values in $\mathbb{R}_+^{\mathbb{Z}} \cup \{(+\infty)^{\mathbb{Z}}\}$. The map Ψ computes the workloads (w) from the the inter-arrivals (a) and the services (s). Observe that we have, for $m < n$ (Lindley's equations),

$$w(n) = [w(n-1) + s(n-1) - a(n-1)]^+, \tag{4}$$

$$w(n) = \left[\max_{m < j \le n-1} \sum_{i=j}^{n-1} s(i) - a(i) \right] \vee \left[w(m) + \sum_{i=m}^{n-1} s(i) - a(i) \right] \vee 0. \tag{5}$$

Define the mapping

$$\begin{aligned} \Phi : \mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}} &\to \mathbb{R}_+^{\mathbb{Z}}, \\ (a, s) &\mapsto d = \Phi(a, s), \end{aligned} \tag{6}$$

with

$$d(n) = \Phi(a, s)(n) = [a(n) - s(n) - \Psi(a, s)(n)]^+ + s(n+1). \tag{7}$$

Let $L : \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}}$ denote the translation shift: $Lu(n) = u(n+1)$. Equation (7) can be rewritten as $d = [a - s - \Psi(a, s)]^+ + Ls$. Observe that $d = Ls$ when $\Psi(a, s) = (+\infty)^{\mathbb{Z}}$. In particular, d is always finite. The function Φ maps the ordered pair formed by the inter-arrival and service processes into the inter-departure process.

When $w \in \mathbb{R}_+^{\mathbb{Z}}$, the above equations yield

$$\forall n, \qquad d(n) = a(n) + w(n+1) - w(n) + s(n+1) - s(n) \tag{8}$$

or equivalently: $\Phi(a, s) = a + L\Psi(a, s) - \Psi(a, s) + Ls - s$.

The functions Ψ and Φ are, respectively, decreasing and increasing with respect to the first variable:

$$\begin{aligned}&\forall a, b \in \mathbb{R}_+^{\mathbb{Z}}, \forall s \in \mathbb{R}_+^{\mathbb{Z}},\\ &\qquad a \leq b \quad \Longrightarrow \quad \Psi(a, s) \geq \Psi(b, s), \Phi(a, s) \leq \Phi(b, s).\end{aligned} \tag{9}$$

In words, increasing the inter-arrival times increases the inter-departure times and decreases the workloads.

3.2. *The stationary queue and Loynes' results.* Consider a measurable and P-stationary shift $\theta : \Omega \to \Omega$ and denote by $\mathfrak{I}$ the invariant σ-algebra. Consider the random processes $A : \Omega \to \mathbb{R}_+^{\mathbb{Z}}$ and $S : \Omega \to \mathbb{R}_+^{\mathbb{Z}}$. Assume that A and S are compatible with θ (hence stationary), and have a finite and nonzero (one-dimensional) mean. Set $W = \Psi(A, S)$ and $D = \Phi(A, S)$.

This model is called a *stationary queue*. When the shift θ is ergodic, the model is an *ergodic queue*. At last, when the service process S is i.i.d. and nonconstant, the model is called an *i.i.d. queue*. The case of a queue with a constant service process is left out since the fixed point problems are trivial in this case.

The following results are standard and due to Loynes [15]. The processes W and D are clearly compatible with θ, hence stationary. We distinguish three cases.

The stable case. On the event $\{E[S(0)|\mathfrak{I}] < E[A(0)|\mathfrak{I}]\}$, we have $W \in \mathbb{R}_+^{\mathbb{Z}}$ and $E[D(0)|\mathfrak{I}] = E[A(0)|\mathfrak{I}]$. On this event, the queue preserves pathwise means.

The unstable case. On the event $\{E[S(0)|\mathfrak{I}] > E[A(0)|\mathfrak{I}]\}$, we have $W = (\infty)^{\mathbb{Z}}$ and $D = LS$ [i.e., $\forall n$, $D(n) = S(n + 1)$].

The critical case. On the event $\{E[S(0)|\mathfrak{I}] = E[A(0)|\mathfrak{I}]\}$, we have $D = LS$ and anything may happen for W. For instance, if $A = S = (c)^{\mathbb{Z}}$ for $c \in \mathbb{R}_+$, then $W = (0)^{\mathbb{Z}}$. If S is i.i.d. and nonconstant and $A \perp\!\!\!\perp S$, then $W = (\infty)^{\mathbb{Z}}$.

Observe that a consequence of the above is that

$$\big\{E[D(0)|\mathfrak{I}] = E[A(0)|\mathfrak{I}]\big\} = \big\{E[S(0)|\mathfrak{I}] \leq E[A(0)|\mathfrak{I}]\big\}$$

(more rigorously, the symmetric difference of the two events has 0 probability).

When the shift θ is ergodic, we are a.s. in the stable case when $E[S(0)] < E[A(0)]$, respectively, in the unstable case when $E[S(0)] > E[A(0)]$, and in the critical case when $E[S(0)] = E[A(0)]$.

Let σ be the law of S. Define

$$\begin{aligned}\Phi_\sigma : \mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}}) &\to \mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}}),\\ \mu &\mapsto \Phi_\sigma(\mu),\end{aligned} \tag{10}$$

where $\Phi_\sigma(\mu)$ is the law of $\Phi(A, S)$ where $A \sim \mu$, $S \sim \sigma$ and $A \perp\!\!\!\perp S$. The map Φ_σ is called the *queueing map*. A distribution μ such that $\Phi_\sigma(\mu) = \mu$ is called a *fixed point* for the queue. If the inter-arrival process is distributed as a fixed point μ, then so is the inter-departure process. Consider now an ergodic queue. Rephrasing Loynes' results, we get

$$\begin{aligned} \forall\,\alpha > \beta, \qquad & \Phi_\sigma : \mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}}) \to \mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}}), \\ \forall\,\alpha \le \beta, \qquad & \Phi_\sigma : \mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}}) \to \{\sigma\}. \end{aligned}$$

In particular, we have $\Phi_\sigma(\sigma) = \sigma$. We say that σ is a *trivial* fixed point for the ergodic queue.

Below the main objective is to get nontrivial fixed points for Φ_σ in the special case of an i.i.d. queue. More precisely, we want to address the following question: for any $\alpha > \beta$, does there exist a fixed point which is ergodic and of mean α?

3.3. *Stable i.i.d. queues in tandem.* Consider a family $\{S(n,k), n \in \mathbb{Z}, k \in \mathbb{N}\}$ of i.i.d. random variables valued in $\mathbb{R}_+$ with $E[S(0,0)] = \beta \in \mathbb{R}_+^*$. Assume that $S(0,0)$ is nonconstant, that is, $P\{S(0,0) = \beta\} < 1$. For k in $\mathbb{N}$, define $S^k : \Omega \to \mathbb{R}_+^{\mathbb{Z}}$ by $S^k = (S(n,k))_{n\in\mathbb{Z}}$. Let σ be the distribution of S^k. Consider $A^0 = (A(n,0))_{n\in\mathbb{Z}} : \Omega \to \mathbb{R}_+^{\mathbb{Z}}$ and assume that A^0 is stationary, independent of S^k for all k, and satisfies $E[A(0,0)] = \alpha \in \mathbb{R}_+^*$. Let θ be a P-stationary shift such that A^0 and S^k for all k are compatible with θ. Let $\mathfrak{T}$ be the corresponding invariant σ-algebra. We assume that the stability condition $\beta < E[A(0,0)|\mathfrak{T}]$ holds a.s.

Define recursively for all $k \in \mathbb{N}$

$$W^k = \big(W(n,k)\big)_{n\in\mathbb{Z}} = \Psi(A^k, S^k), \tag{11}$$

$$A^{k+1} = \big(A(n,k+1)\big)_{n\in\mathbb{Z}} = \Phi(A^k, S^k). \tag{12}$$

The random processes A^k, S^k and W^k are, respectively, the inter-arrival, service and workload processes at queue k. The random process A^{k+1} is the inter-departure process at queue k and the inter-arrival process at queue $k+1$. Each (A^k, S^k) defines a stable i.i.d. queue according to the terminology of Section 3.2. Globally, this model is called a *tandem of stable i.i.d. queues*.

The sequence $(A^k)_k$ is a Markov chain on the state space $\mathbb{R}_+^{\mathbb{Z}}$. Clearly, μ is a stationary distribution of $(A^k)_k$ if and only if μ is a fixed point for the queue, that is, iff $\Phi_\sigma(\mu) = \mu$. Hence, the problem to be solved can be rephrased as: does the Markov chain $(A^k)_k$ admit nontrivial stationary distributions?

4. Uniqueness of fixed points and convergence. In this section, we recall several results about the uniqueness of fixed points as well as convergence results. Associated with the existence results to be proved in Section 5, the results recalled here complete the picture about fixed point theorems. More importantly, they will be instrumental in several of the later proofs.

THEOREM 4.1 ([2, 17]). *Consider the stable i.i.d. tandem model defined in Section* 3.3 *with an ergodic inter-arrival process of mean* $\alpha > \beta$. *Assume that*

$$\int_{\mathbb{R}_+} P\{S(0,0) \geq u\}^{1/2}\, du < \infty. \tag{13}$$

Then there exists $M(\alpha) \in \mathbb{R}_+$ *such that almost surely* $\lim_{n\to+\infty} n^{-1} \times \sum_{i=0}^{n-1} W(0,i) = M(\alpha)$, *where* $M(\alpha) = \sup_{x\geq 0}(\gamma(x) - \alpha x)$ *and the function* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *depends only on the service process. If we further assume that the initial inter-arrival process satisfies*

$$\begin{aligned} &\exists c, \qquad E\,[S(0,0)] < c < E[A(0,0)], \\ &\qquad\qquad E\left[\sup_{n\in\mathbb{N}^*}\left[\sum_{i=-n}^{-1} c - A(i,0)\right]^+\right] < \infty, \end{aligned} \tag{14}$$

then the convergence to $M(\alpha)$ *also holds in* L_1.

Observe that $M(\alpha)$ depends on the inter-arrival process only via its mean. The function γ in Theorem 4.1 is continuous, strictly increasing, concave and satisfies $\gamma(0) = 0$. For details on γ, refer to [2, 12].

Theorem 4.1 is proved in [2] under the condition: $E[S(0,0)^{3+a}] < \infty$ for some $a > 0$. The above version is proved in [17] (using similar methods as in [2]) and is better since we have

$$\begin{aligned} \left[\exists a > 0, E[S(0,0)^{2+a}] < \infty\right] \quad &\Longrightarrow \quad \int P\{S(0,0) \geq u\}^{1/2}\, du < \infty \\ &\Longrightarrow \quad E[S(0,0)^2] < \infty. \end{aligned}$$

Condition (14) is slightly stronger than $E[W(0,0)] < \infty$. Indeed, recall the following results from [9]. If $E[S(0,0)^2] < \infty$, then, setting $\beta = E[S(0,0)]$,

$$\begin{aligned} &\exists c > \beta, \qquad E\left[\sup_{n\geq 1}\left[\sum_{i=-n}^{-1} c - A(i,0)\right]^+\right] < \infty \\ &\qquad\qquad\qquad \Longrightarrow \quad E[W(0,0)] < \infty, \\ &\qquad\qquad E[W(0,0)] < \infty \\ &\qquad\qquad\qquad \Longrightarrow \quad E\left[\sup_{n\geq 1}\left[\sum_{i=-n}^{-1} \beta - A(i,0)\right]^+\right] < \infty. \end{aligned} \tag{15}$$

Condition (14) is satisfied, for example, by the deterministic process $P\{A^0 = (\alpha)^{\mathbb{Z}}\} = 1$.

The next result requires some preparation. Let $\mathcal{L}_s(\mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}})$ be the set of random processes $((X(n), Y(n))_{n\in\mathbb{Z}}$ which are stationary in n. Consider μ and ν

in $\mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$ and let $\mathcal{D}(\mu,\nu)=\{(X,Y)\in\mathcal{L}_s(\mathbb{R}_+^{\mathbb{Z}}\times\mathbb{R}_+^{\mathbb{Z}})\mid X\sim\mu, Y\sim\nu\}$. That is, $\mathcal{D}(\mu,\nu)$ is the set of jointly stationary processes whose marginals are distributed as μ and ν. The $\bar{\rho}$ distance between μ and ν is given by

$$\bar{\rho}(\mu,\nu)=\inf_{(X,Y)\in\mathcal{D}(\mu,\nu)} E[|X(0)-Y(0)|]. \tag{16}$$

See Gray [13], Chapter 8, for a proof that $\bar{\rho}$ is indeed a distance. Given two r.v.'s A and B with respective laws μ and ν, set $\bar{\rho}(A,B)=\bar{\rho}(\mu,\nu)$. We recall a well-known fact (see also Section 7): convergence in the $\bar{\rho}$ distance implies weak convergence, but not conversely.

THEOREM 4.2 ([8, 19]). *Consider a stationary queue as in Section* 3.2 *with service process S and two inter-arrival processes A and B, possibly of different means. Assume that $A \perp\!\!\!\perp S$ and $B \perp\!\!\!\perp S$. Then,*

$$\bar{\rho}\big(\Phi(A,S),\Phi(B,S)\big)\le\bar{\rho}(A,B). \tag{17}$$

Consider now a stable i.i.d. tandem model as in Section 3.3 *with inter-arrival processes A^0 and B^0 with different laws but such that $E[A(0,0)|\mathfrak{T}]=E[B(0,0)|\mathfrak{T}]$ a.s. Recall that $(A^n)_n$ and $(B^n)_n$ are defined recursively by $A^{n+1}=\Phi(A^n,S^n)$ and $B^{n+1}=\Phi(B^n,S^n)$. Then there exists $k\in\mathbb{N}^*$ such that*

$$\bar{\rho}(A^k,B^k)<\bar{\rho}(A^0,B^0). \tag{18}$$

If we further assume that $B^1=\Phi(B^0,S^0)\sim B^0$, then

$$\lim_{n\to+\infty}\bar{\rho}(A^n,B^0)=0 \quad \textit{and} \quad \textit{hence } A^n \xrightarrow{w} B^0. \tag{19}$$

Chang [8] gives an elegant proof of (17). He also proves (18) for unbounded services. Prabhakar [19] removes this restriction and also establishes (19). As opposed to Theorem 4.1, observe that the convergence result in (19) is proved under the a priori assumption of existence of a fixed point.

Define ("$p:\alpha$" stands for "pathwise means are equal to α")

$$\mathcal{M}_{\mathrm{s}}^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})=\left\{\mu\in\mathcal{M}_{\mathrm{s}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})\mid X\sim\mu\Rightarrow\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}X(i)=\alpha \text{ a.s.}\right\}. \tag{20}$$

Obviously, $\mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})\subset\mathcal{M}_{\mathrm{s}}^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})\subset\mathcal{M}_{\mathrm{s}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$. The ergodic components of $\chi\in\mathcal{M}_{\mathrm{s}}^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ all have one-dimensional mean α. An important consequence of (18) is the following uniqueness result.

COROLLARY 4.3. *Consider an i.i.d. queue as in Section* 3.2. *The corresponding queueing map Φ_σ has at most one fixed point in $\mathcal{M}_{\mathrm{s}}^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ for $\alpha>E[S(0)]$.*

In particular, there is at most one fixed point in $\mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$. In fact, we have the following stronger result.

PROPOSITION 4.4. *Consider an i.i.d. queue as in Section* 3.2 *and* $\alpha > E[S(0)]$. *If* $\zeta \in \mathcal{M}_{\mathrm{s}}^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ *is a fixed point, then it is necessarily ergodic*; *that is*, $\zeta \in \mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$.

PROOF. Suppose that the ergodic decomposition of ζ is given by $\zeta = \int \mu \Gamma(d\mu)$, where Γ is a probability measure on $\mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$. Denote the support of Γ by $\mathrm{supp}(\Gamma) \subset \mathcal{M}_{\mathrm{e}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$. Assume that ζ is nonergodic, meaning that $\mathrm{supp}(\Gamma)$ is not a singleton. Let S be a subset of $\mathrm{supp}(\Gamma)$ such that $0 < \Gamma\{S\} < 1$.

Consider a stable i.i.d. tandem model as in Section 3.3. Let A^0 and B^0 be two inter-arrival processes, independent of the services, and such that $A^0 \sim \zeta$, $B^0 \sim \zeta$, $A^0 \perp\!\!\!\perp B^0$. Define $(A^k)_k$ and $(B^k)_k$ as in (12). Let $C_{\mathrm{b}}(\mathbb{R}_+^{\mathbb{Z}})$ be the set of continous and bounded functions from $\mathbb{R}_+^{\mathbb{Z}}$ to $\mathbb{R}$. Recall that L is the left translation shift of $\mathbb{R}_+^{\mathbb{Z}}$ and define recursively $L^{i+1} = L \circ L^i$. Define the θ-invariant events

$$\mathcal{A} = \left\{ \exists \mu \in S, \forall f \in C_{\mathrm{b}}(\mathbb{R}_+^{\mathbb{Z}}), \lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(L^i A^0) = \int f \, d\mu \right\},$$

$$\mathcal{B} = \left\{ \exists \mu \in \mathrm{supp}(\Gamma) \setminus S, \forall f \in C_{\mathrm{b}}(\mathbb{R}_+^{\mathbb{Z}}), \lim_n \frac{1}{n} \sum_{i=0}^{n-1} f(L^i B^0) = \int f \, d\mu \right\}.$$

Roughly speaking, on the event $\mathcal{A} \cap \mathcal{B}$, the processes A^0 and B^0 are distributed according to different components of the ergodic decomposition of ζ. Using the independence of A^0 and B^0, we have

$$P\{\mathcal{A} \cap \mathcal{B}\} = P\{\mathcal{A}\} P\{\mathcal{B}\} = \Gamma\{S\}(1 - \Gamma\{S\}) > 0.$$

Define the processes

$$\widetilde{A}^0 = A^0 \mathbb{1}_{\mathcal{A} \cap \mathcal{B}} + (\alpha)^{\mathbb{Z}} \mathbb{1}_{(\mathcal{A} \cap \mathcal{B})^c}, \qquad \widetilde{B}^0 = B^0 \mathbb{1}_{\mathcal{A} \cap \mathcal{B}} + (\alpha)^{\mathbb{Z}} \mathbb{1}_{(\mathcal{A} \cap \mathcal{B})^c}.$$

By construction, the laws of $\widetilde{A}^0$ and $\widetilde{B}^0$ are different and we have $E[\widetilde{A}(0,0)|\mathfrak{T}] = E[\widetilde{B}(0,0)|\mathfrak{T}] = \alpha$ almost surely. Hence we can apply (18) in Theorem 4.2: there exists $k \in \mathbb{N}^*$ such that $\bar{\rho}(\widetilde{A}^k, \widetilde{B}^k) < \bar{\rho}(\widetilde{A}^0, \widetilde{B}^0)$. We deduce easily that $\bar{\rho}(A^k, B^k) < \bar{\rho}(A^0, B^0)$. This is in obvious contradiction with $\bar{\rho}(A^0, B^0) = 0$ which follows from $A^0 \sim B^0$. We conclude that the support of Γ is a singleton. $\square$

5. Existence of fixed points. Consider the stable i.i.d. tandem model of Section 3.3. The objective is to prove Theorem 5.1, that is, to obtain nontrivial stationary distributions for $(A^k)_k$, or equivalently nontrivial fixed points for Φ_σ.

The first step is classical and consists of considering Cesaro averages of the laws of A^k. Consider the quadruple (A^k, S^k, W^k, A^{k+1}) and denote its law by

$\nu_k \in \mathcal{M}(\mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}} \times [0,\infty]^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}})$. For $n \in \mathbb{N}^*$, define $\mu_n \in \mathcal{M}(\mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}} \times [0,\infty]^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}})$ by

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu_k.$$

The following interpretation may be useful: μ_n is the law of (A^N, S^N, W^N, A^{N+1}) where N is a r.v. uniformly distributed over $\{0,\dots,n-1\}$ and independent of all the other r.v.'s of the problem.

For all $n \in \mathbb{N}^*$, consider a quadruple of random processes $(\widehat{A}^n, \widehat{S}^n, \widehat{W}^n, \widehat{D}^n)$ distributed according to μ_n. We have

$$\widehat{S}^n \sim \sigma, \qquad \widehat{S}^n \perp\!\!\!\perp \widehat{A}^n, \qquad \widehat{W}^n = \Psi(\widehat{A}^n, \widehat{S}^n), \qquad \widehat{D}^n = \Phi(\widehat{A}^n, \widehat{S}^n). \tag{21}$$

First of all, we argue that the sequence $(\mu_n)_n$ is tight. Denote by $\mu_n^{(1)}, \mu_n^{(2)}, \mu_n^{(3)}$ and $\mu_n^{(4)}$ the marginals of μ_n corresponding respectively to the laws of $\widehat{A}^n, \widehat{S}^n, \widehat{W}^n$ and $\widehat{D}^n$. Since $\mu_n^{(3)}$ is defined on the compact space $[0,\infty]^{\mathbb{Z}}$ and since $\mu_n^{(2)} = \sigma$, the only point to be argued is that $(\mu_n^{(1)})_n$ and $(\mu_n^{(4)})_n$ are tight. According to Loynes' results, we have $\mu_n^{(1)}, \mu_n^{(4)} \in \mathcal{M}_{\mathrm{s}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ [we even have $\mu_n^{(1)}, \mu_n^{(4)} \in \mathcal{M}_{\mathrm{s}}^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})$]. For $\varepsilon > 0$, the set $K = \prod_{i\in\mathbb{Z}}[0, 2^{|i|+2}/\varepsilon]$ is compact in the product topology according to Tychonoff's theorem. It is immediate to check that for $\mu \in \mathcal{M}_{\mathrm{s}}^{\alpha}(\mathbb{R}_+^{\mathbb{Z}})$, we have $\mu\{K\} \geq 1 - \alpha\varepsilon$. We conclude that $(\mu_n^{(1)})_n$ and $(\mu_n^{(4)})_n$ are tight.

Consequently, by Prohorov's theorem, $(\mu_n)_n$ admits weakly converging subsequences. Let μ be a subsequential limit of $(\mu_n)_n$. Consider a quadruple of random processes

$$(\widehat{A}, \widehat{S}, \widetilde{W}, \widetilde{D}) \sim \mu. \tag{22}$$

It follows immediately from (21) that

$$\widehat{S} \sim \sigma, \qquad \widehat{S} \perp\!\!\!\perp \widehat{A}. \tag{23}$$

Recall that we have $\widehat{D}^n = [\widehat{A}^n - \widehat{S}^n - \widehat{W}^n]^+ + L\widehat{S}^n$. By the continuous mapping theorem, we deduce that

$$\widetilde{D} = [\widehat{A} - \widehat{S} - \widetilde{W}]^+ + L\widehat{S}. \tag{24}$$

On the other hand, it is not a priori true that $\widetilde{W} = \Psi(\widehat{A}, \widehat{S})$ and $\widetilde{D} = \Phi(\widehat{A}, \widehat{S})$ (which is the reason for the notation $\widehat{A}, \widehat{S}$ on the one side and $\widetilde{W}, \widetilde{D}$ on the other). Using (5) we have, for all $k < l-1$,

$$\begin{aligned}&\left[\max_{k<j\leq l-1}\sum_{i=j}^{l-1}\widehat{S}^n(i) - \widehat{A}^n(i)\right]^+\\ &\qquad \leq \widehat{W}^n(l) = \left[\max_{k<j\leq l-1}\sum_{i=j}^{l-1}\widehat{S}^n(i) - \widehat{A}^n(i)\right]^+ \vee \left[\widehat{W}^n(k) + \sum_{i=k}^{l-1}\widehat{S}^n(i) - \widehat{A}^n(i)\right].\end{aligned}$$

By the continuous mapping theorem, we get

$$\begin{aligned}
&\left[\max_{k<j\le l-1}\sum_{i=j}^{l-1}\widehat{S}(i)-\widehat{A}(i)\right]^+\\
&\qquad\le \widetilde{W}(l)=\left[\max_{k<j\le l-1}\sum_{i=j}^{l-1}\widehat{S}(i)-\widehat{A}(i)\right]^+\vee\left[\widetilde{W}(k)+\sum_{i=k}^{l-1}\widehat{S}(i)-\widehat{A}(i)\right].
\end{aligned}\tag{25}$$

By letting k go to $-\infty$, and using (24), we conclude that

$$\Psi(\widehat{A},\widehat{S})\le\widetilde{W},\qquad L\widehat{S}\le\widetilde{D}\le\Phi(\widehat{A},\widehat{S}).\tag{26}$$

The right-hand side equality in (25) also shows that $\widetilde{W}\in\mathbb{R}_+^{\mathbb{Z}}\cup\{(\infty)^{\mathbb{Z}}\}$ (a priori the definition only implied $\widetilde{W}\in[0,\infty]^{\mathbb{Z}}$).

The next argument which uses properties of Cesaro averages to show that $\widehat{A}\sim\widetilde{D}$ is standard. Let ζ be the distribution of A^0. We have by definition $A^n\sim\Phi_\sigma^n(\zeta)$ and $\widehat{A}^n\sim n^{-1}\sum_{i=0}^{n-1}\Phi_\sigma^i(\zeta)=\zeta_n$. We have

$$\widehat{D}^n=\Phi(\widehat{A}^n,\widehat{S}^n)\sim\Phi_\sigma(\zeta_n)=\zeta_n+\frac{1}{n}\big(\Phi_\sigma^n(\zeta)-\zeta\big),$$

where the left-hand side equality makes sense as an equality between signed measures. We deduce that $\Phi_\sigma(\zeta_n)-\zeta_n$ converges in total variation, hence also weakly, to the zero measure (here we consider weak and total variation convergence of signed measures). There is a subsequence along which ζ_n, respectively $\Phi_\sigma(\zeta_n)$, converges to the law of $\widehat{A}$, respectively $\widetilde{D}$. We conclude that

$$\widehat{A}\sim\widetilde{D}.\tag{27}$$

Now if we manage to prove that $\widetilde{D}=\Phi(\widehat{A},\widehat{S})$, we can conclude that the law of $\widehat{A}$ is a fixed point for the queue. We now turn our attention to proving this last and tricky point.

Stationarity is preserved by weak convergence. Hence the law of $(\widehat{A}(n),\widehat{S}(n),\widetilde{W}(n),\widetilde{D}(n))_n$ is stationary in n. Let θ be a stationary shift on the underlying probability space such that $(\widehat{A},\widehat{S},\widetilde{W},\widetilde{D})$ is compatible with θ. Let $\mathfrak{I}$ be the corresponding invariant σ-algebra.

Using (26) and (27), we deduce that $\widehat{A}\ge_{\mathrm{st}}\widehat{S}$. In particular, $E[\widehat{A}(0)|\mathfrak{I}]\ge\beta$ a.s. using (1). Define the events

$$\mathcal{A}=\{E[\widehat{A}(0)|\mathfrak{I}]=\beta\},\qquad\mathcal{A}^c=\{E[\widehat{A}(0)|\mathfrak{I}]>\beta\}.$$

Using Loynes' results for the critical case, we have $\Phi(\widehat{A},\widehat{S})=L\widehat{S}$ on the event $\mathcal{A}$. Now using (26), we deduce that $\widetilde{D}=\Phi(\widehat{A},\widehat{S})=L\widehat{S}$ on the event $\mathcal{A}$.

Since $\widehat{A}\ge_{\mathrm{st}}\widehat{S}$ and $\widehat{A}\perp\!\!\!\perp\widehat{S}$, we have, according to (1),

$$\widehat{A}=\overline{S}\mathbb{1}_{\mathcal{A}}+\widehat{A}\mathbb{1}_{\mathcal{A}^c},$$

where $\overline{S} \sim \widehat{S}$. Furthermore, we have just proved that

$$\widetilde{D} = L\widehat{S}\mathbb{1}_{\mathcal{A}} + \widetilde{D}\mathbb{1}_{\mathcal{A}^c}.$$

Since $\widehat{A} \sim \widetilde{D}$, we deduce readily that $\widehat{A}\mathbb{1}_{\mathcal{A}^c} \sim \widetilde{D}\mathbb{1}_{\mathcal{A}^c}$. On the event $\mathcal{A}^c$, we have, using Birkhoff's ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} \widehat{A}(i) = E[\widehat{A}(0)|\mathfrak{T}] > \beta \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{-1} \widetilde{D}(i) > \beta.$$

In view of $\widetilde{D} = [\widehat{A} + \widehat{S} - \widetilde{W}]^+ + L\widehat{S}$, we deduce that on $\mathcal{A}^c$, we have $\widetilde{W} \in \mathbb{R}_+^{\mathbb{Z}}$ a.s. For $k < l-1$, set $Z_k = [\widetilde{W}(k) + \sum_{i=k}^{l-1} \widehat{S}(i) - \widehat{A}(i)]$. Using Birkhoff's ergodic theorem, on the event $\mathcal{A}^c$, Z_k converges in probability to $-\infty$ as k goes to $-\infty$. Going back to the inequalities in (25), it follows that on the event $\mathcal{A}^c$,

$$\widetilde{W}(l) = \left[\sup_{j \le l-1} \sum_{i=j}^{l-1} \widehat{S}(i) - \widehat{A}(i) \right]^+ = \Psi(\widehat{A}, \widehat{S}).$$

It implies that on the event $\mathcal{A}^c$, we have $\widetilde{D} = \Phi(\widehat{A}, \widehat{S})$. Summarizing all of the above, we have proved that

$$\widetilde{D} = \Phi(\widehat{A}, \widehat{S}) \qquad \text{a.s.} \tag{28}$$

Let ζ be the law of $\widehat{A}$ and $\widetilde{D}$. We have just proved that $\Phi_\sigma(\zeta) = \zeta$. The only point left out is to find conditions ensuring that ζ is not equal to the trivial fixed point σ.

REMARK (Ergodic queues in tandem). Up to this point in the proof, the assumption that the service processes are i.i.d. has not been used. All of the above remains valid if we assume only that $\sigma \in \mathcal{M}_e^{\beta}(\mathbb{R}_+^{\mathbb{Z}})$ (still assuming that the service processes $S^k \sim \sigma$ are independent of one another and independent of A^0). From now on, the i.i.d. assumption becomes central.

First, we need to show that

$$\widetilde{W} = \Psi(\widehat{A}, \widehat{S}) \qquad \text{a.s.} \tag{29}$$

We have just proved the equality on $\mathcal{A}^c$, it remains to prove it on $\mathcal{A}$. Denote by $\mathfrak{T}_{\widehat{A}}$ and $\mathfrak{T}_{\widehat{S}}$ the σ-algebras generated respectively by $\widehat{A}$ and $\widehat{S}$. Clearly $\mathcal{A} \in \mathfrak{T}_{\widehat{A}}$ which implies that $\widehat{A}\mathbb{1}_{\mathcal{A}} = \overline{S}\mathbb{1}_{\mathcal{A}}$ is measurable with respect to $\mathfrak{T}_{\widehat{A}}$. We conclude that we have: $\widehat{A} \perp\!\!\!\perp \widehat{S} \Rightarrow \mathfrak{T}_{\widehat{A}} \perp\!\!\!\perp \mathfrak{T}_{\widehat{S}} \Rightarrow \overline{S}\mathbb{1}_{\mathcal{A}} \perp\!\!\!\perp \widehat{S}$. On the event $\mathcal{A}$, we have for all n,

$$\widetilde{W}(n) \ge \Psi(\widehat{A}, \widehat{S})(n) = \left[\sup_{j \le n-1} \sum_{i=j}^{n-1} \widehat{S}(i) - \overline{S}(i) \right]^+.$$

Using that the services are i.i.d. and nonconstant and that $\overline{S}$ and $\widehat{S}$ are independent, we have on the event $\mathcal{A}$: $\widetilde{W} = \Psi(\widehat{A}, \widehat{S}) = (\infty)^{\mathbb{Z}}$. In addition to (29), we have proved the following:

$$\begin{aligned} &\widetilde{W} = (\infty)^{\mathbb{Z}} \qquad \text{on } \{E[\widehat{A}(0)|\mathfrak{T}] = \beta\}, \\ &\widetilde{W} \in \mathbb{R}_+^{\mathbb{Z}} \qquad \text{on } \{E[\widehat{A}(0)|\mathfrak{T}] > \beta\}. \end{aligned} \tag{30}$$

Consequently, if $\widetilde{W} = (\infty)^{\mathbb{Z}}$ a.s. then $\zeta = \sigma$, and if $P\{\widetilde{W} \in \mathbb{R}_+^{\mathbb{Z}}\} > 0$ then ζ is a nontrivial fixed point for the queue.

Assume now that the moment condition $\int P\{S(0,0) \geq u\}^{1/2}\, du < \infty$ is satisfied. This is the condition needed in Theorem 4.1 to obtain that $\lim_n n^{-1} \times \sum_{i=0}^{n-1} W(0,i) = M(\alpha)$ a.s. for a finite constant $M(\alpha)$. Let us prove that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} W(0,i) = M(\alpha) \text{ a.s.} \quad \Longrightarrow \quad \widetilde{W}(0) \in \mathbb{R}_+ \text{ a.s.} \tag{31}$$

We argue by contradiction; hence, suppose that $P\{\widetilde{W}(0) = +\infty\} = a > 0$. Fix $K > 0$. Let f be a strictly increasing function of $\mathbb{N}$ such that $\mu_{f(n)} \xrightarrow{w} \mu$. We have $\widehat{W}^{f(n)}(0) \xrightarrow{w} \widetilde{W}(0)$. Recall that $P\{\widehat{W}^n(0) \geq K\} = n^{-1}\sum_{i=0}^{n-1} P\{W(0,i) \geq K\}$. We deduce that

$$\forall\, b \in (0,a), \exists\, N, \forall\, n = f(k) \geq N, \qquad \frac{1}{n}\sum_{i=0}^{n-1} P\{W(0,i) \geq K\} \geq b.$$

Fix $b \in (0,a), c \in (0,b)$ and $n = f(k) \geq N$. Define the event $\mathcal{E} = \{n^{-1} \times \sum_{i=0}^{n-1} \mathbb{1}_{\{W(0,i)\geq K\}} \geq c\}$ and set $q = P\{\mathcal{E}\}$. We have

$$\begin{aligned} &\sum_{i=0}^{n-1} \mathbb{1}_{\{W(0,i)\geq K\}} \\ &\qquad = \left(\sum_{i=0}^{n-1} \mathbb{1}_{\{W(0,i)\geq K\}}\right)\mathbb{1}_{\mathcal{E}} + \left(\sum_{i=0}^{n-1} \mathbb{1}_{\{W(0,i)\geq K\}}\right)\mathbb{1}_{\mathcal{E}^c} \leq n\mathbb{1}_{\mathcal{E}} + nc\mathbb{1}_{\mathcal{E}^c}. \end{aligned}$$

Taking expectations, we get

$$nb \leq \sum_{i=0}^{n-1} P\{W(0,i) \geq K\} \leq nq + n(1-q)c.$$

We conclude that $q \geq (b-c)/(1-c) > 0$. Since this last inequality is valid for any K, we clearly have a contradiction with the a.s. convergence of $n^{-1}\sum_{i=0}^{n-1} W(0,i)$ to a finite constant.

We conclude that under the assumptions of Theorem 4.1, the fixed point ζ is nontrivial. Summarizing all of the above, we obtain the following result.

THEOREM 5.1. *Consider a single server infinite buffer FCFS queue with an i.i.d. service process S satisfying*: $E[S(0)] \in \mathbb{R}_+^*$, $P\{S(0) = E[S(0)]\} < 1$ *and* $\int P\{S(0) \geq u\}^{1/2}\, du < \infty$. *Then there exists an ergodic inter-arrival process A with $A \perp\!\!\!\perp S$ and $E[S(0)] < E[A(0)] < \infty$, and such that the corresponding inter-departure process D has the same distribution as A.*

PROOF. Consider a tandem of queues as in Section 3.3 where the service processes S^k are distributed as S with law σ. Consider the process $\widehat{A}$ with law ζ as defined in (22). By the ergodic decomposition theorem and the linearity of Φ_σ, we have

$$\zeta = \int_{\mathcal{M}_{\mathrm{e}}(\mathbb{R}_+^{\mathbb{Z}})} \chi\,\Gamma(d\chi), \qquad \Phi_\sigma(\zeta) = \int_{\mathcal{M}_{\mathrm{e}}(\mathbb{R}_+^{\mathbb{Z}})} \Phi_\sigma(\chi)\Gamma(d\chi).$$

But $\zeta = \Phi_\sigma(\zeta)$. Therefore, the uniqueness of ergodic decompositions and the mean preservation property of stable queues imply that

$$\zeta_\alpha = \int_{\mathcal{M}_{\mathrm{e}}^\alpha(\mathbb{R}_+^{\mathbb{Z}})} \chi\,\Gamma(d\chi) = \int_{\mathcal{M}_{\mathrm{e}}^\alpha(\mathbb{R}_+^{\mathbb{Z}})} \Phi_\sigma(\chi)\Gamma(d\chi) = \Phi_\sigma(\zeta_\alpha)$$

for every α in the support of $E[\widehat{A}(0)|\mathfrak{I}]$. By Proposition 4.4, the distributions ζ_α are ergodic. According to (31), which holds since $\int P\{S(0) \geq u\}^{1/2}\, du < \infty$, we have $P\{\widetilde{W} \in \mathbb{R}_+^{\mathbb{Z}}\} = 1$ and $E[\widehat{A}(0)|\mathfrak{I}] > E[S(0)]$ according to (30). Hence any α in the support of $E[\widehat{A}(0)|\mathfrak{I}]$ is such that $\alpha > E[S(0)]$ and we conclude that the corresponding distribution $\zeta_\alpha \in \mathcal{M}_{\mathrm{e}}^\alpha(\mathbb{R}_+^{\mathbb{Z}})$ is such that $\Phi_\sigma(\zeta_\alpha) = \zeta_\alpha$. $\square$

To the best of our knowledge, this provides the first positive answer (apart from the cases of exponential and geometric service times) to the intriguing question of the existence of nontrivial ergodic fixed points for a $\cdot/GI/1/\infty/FCFS$ queue.

6. Values of the means for which a fixed point exists. Consider a tandem of stable i.i.d. queues as in Section 3.3 and let Φ_σ be the corresponding queueing operator. Assume also that the condition (13) holds. Define

$$\mathcal{S} = \{\alpha \in (\beta, +\infty) \mid \exists\, \mu \in \mathcal{M}_{\mathrm{e}}^\alpha(\mathbb{R}_+^{\mathbb{Z}}), \Phi_\sigma(\mu) = \mu\}. \tag{32}$$

According to Theorem 5.1, the set $\mathcal{S}$ is nonempty. We establish in Theorem 6.4 that $\mathcal{S}$ is unbounded, and closed in (β, ∞). We believe that $\mathcal{S} = (\beta, +\infty)$ but we have not been able to prove this last point (see Conjecture 6.6). Proposition 6.5 also describes the limiting behavior from inputing in the tandem an ergodic inter-arrival process whose mean α does not belong to $\mathcal{S}$ (the case $\alpha \in \mathcal{S}$ is settled by Theorem 4.2).

From now on, for $\alpha \in \mathcal{S}$, denote by ζ_α the unique ergodic fixed point of mean α and by A_α an inter-arrival process distributed as ζ_α. Let S be distributed as σ and independent of all other r.v.'s. Also it is convenient to denote by $\mathcal{L}(A)$ the law of a r.v. A, and by $\operatorname{supp} A$ its support.

The following argument is used several times. Consider $\alpha \in \mathcal{S}$ and let $(A^n)_n$ be defined as in (12) starting from an ergodic process A^0 of mean α. According to (19), we have $A^n \xrightarrow{w} A_\alpha$. It implies that $n^{-1}\sum_{i=0}^{n-1} \mathcal{L}(A^i) \xrightarrow{w} \mathcal{L}(A_\alpha)$. According to (28), we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}\big(\Psi(A^i, S^i)\big) \xrightarrow{w} \mathcal{L}\big(\Psi(A_\alpha, S)\big). \tag{33}$$

We now prove a series of preliminary lemmas.

LEMMA 6.1. *For any $\alpha > \beta$, $\mathcal{S} \cap (\beta, \alpha) \neq \varnothing$.*

PROOF. Fix $\alpha > \beta$. Let $(A^n)_n$ be defined as in (12) starting from an ergodic process A^0 of mean α. Let $\widehat{A}$ be distributed as a weak subsequential limit of the Cesaro averages of the laws of $(A^k)_k$. Recall from the proof of Theorem 5.1 that

$$\operatorname{supp} E[\widehat{A}(0)|\mathfrak{T}] \subset \mathcal{S} \subset (\beta, \infty). \tag{34}$$

By Fatou's lemma, $E[\widehat{A}(0)] \leq \alpha$. Since $E[\widehat{A}(0)] = E[E[\widehat{A}(0)|\mathfrak{T}]]$, we conclude that $\mathcal{S} \cap (\beta, \alpha] \neq \varnothing$. $\square$

LEMMA 6.2. *Consider an ergodic inter-arrival process A^0 of mean $\alpha > \beta$. Let $\widehat{A}$ be distributed as a weak subsequential limit of the Cesaro averages of the laws of $(A^k)_k$. Consider $\delta \in \mathcal{S} \cap (\beta, \alpha]$ (resp. $\delta \in \mathcal{S} \cap [\alpha, \infty)$, assuming $\mathcal{S} \cap [\alpha, \infty) \neq \varnothing$), then $A_\delta \leq_{\mathrm{st}} \widehat{A}$ and $\Psi(\widehat{A}, S) \geq_{\mathrm{st}} \Psi(A_\delta, S)$ [resp., $A_\delta \geq_{\mathrm{st}} \widehat{A}$ and $\Psi(\widehat{A}, S) \leq_{\mathrm{st}} \Psi(A_\delta, S)$]. Further, if $\mathcal{S} \cap [\alpha, \infty) \neq \varnothing$, then $E[\widehat{A}(0)] = \alpha$.*

PROOF. Consider the case $\delta \in \mathcal{S} \cap [\alpha, \infty)$. The other case can be treated similarly. Define the process $B^0 = \delta\alpha^{-1}A^0$, that is,

$$\forall n, \qquad B(n,0) = \frac{\delta}{\alpha} A(n,0).$$

The process B^0 is ergodic and of mean δ. At mean δ, Φ_σ admits the fixed point ζ_δ. By (19), we have $B^k \xrightarrow{w} A_\delta$. By construction, we have $A^0 \leq B^0$ almost surely. Using the monotonicity property (9), we get that, for all $k \in \mathbb{N}$,

$$A^k \leq B^k \quad \text{and} \quad \Psi(A^k, S^k) \geq \Psi(B^k, S^k).$$

It implies that for all $k \in \mathbb{N}^*$,

$$\frac{1}{k}\sum_{i=0}^{k-1} \mathcal{L}(A^i) \leq_{\mathrm{st}} \frac{1}{k}\sum_{i=0}^{k-1} \mathcal{L}(B^i)$$

and

$$\frac{1}{k}\sum_{i=0}^{k-1}\mathcal{L}\big(\Psi(A^k,S^k)\big) \geq_{\mathrm{st}} \frac{1}{k}\sum_{i=0}^{k-1}\mathcal{L}\big(\Psi(B^k,S^k)\big).$$

Going to the limit along an appropriate subsequence and applying (33), we obtain

$$\widehat{A} \leq_{\mathrm{st}} A_\delta \quad \text{and} \quad \Psi(\widehat{A},S) \geq_{\mathrm{st}} \Psi(A_\delta,S).$$

We are left with having to show that $E[\widehat{A}(0)] = \alpha$. Observe that $k^{-1} \times \sum_{i=0}^{k-1}\mathcal{L}(B^k) \xrightarrow{w} \zeta_\delta$, and that the one-dimensional marginals converge in expectation since $k^{-1}\sum_{i=0}^{k-1} E[B(0,i)] = \delta = E[A_\delta(0)]$. It follows by Theorem 5.4 of [3] that the sequence $(k^{-1}\sum_{i=0}^{k-1}\mathcal{L}(B^k))_k$ is uniformly integrable. It implies that the dominated sequence $(k^{-1}\sum_{i=0}^{k-1}\mathcal{L}(A^i))_k$ is also uniformly integrable. Along an appropriate subsequence, this last sequence converges weakly to the law of $\widehat{A}$ and we conclude (Theorem 5.4 of [3]) that it also converges in expectation. Since $k^{-1}\sum_{i=0}^{k-1} E[A(0,k)] = \alpha$ for all k, we deduce that $E[\widehat{A}(0)] = \alpha$. $\square$

LEMMA 6.3. *The following statements are true*:

(a) *for* $\alpha,\delta \in \mathcal{S}$ *and* $\alpha < \delta$, $A_\alpha \leq_{\mathrm{st}} A_\delta$ *and* $\Psi(A_\alpha,S) \geq_{\mathrm{st}} \Psi(A_\delta,S)$;
(b) *for* $\alpha \in \mathcal{S}$, $E[\Psi(A_\alpha,S)(0)] = M(\alpha)$, *where* $M(\alpha)$ *is defined in Theorem* 4.1.

PROOF. Part (a) is a direct consequence of Lemma 6.2. Consider part (b). Fix $\alpha \in \mathcal{S}$. Consider A^0 an ergodic inter-arrival process of mean α satisfying condition (14). From Theorem 4.1, we have

$$\lim_n \frac{1}{n} E[\Psi(A^i,S^i)(0)] = M(\alpha).$$

Starting from (33) and applying Fatou's lemma, we get

$$E[\Psi(A_\alpha,S)(0)] \leq \lim_n \frac{1}{n} E[\Psi(A^i,S^i)(0)] = M(\alpha).$$

Now let us prove that $M(\alpha) \leq E[\Psi(A_\alpha,S)(0)]$. By Lemma 6.1, there exists $\delta \in \mathcal{S} \cap (\beta,\alpha)$. Define the process $B^0 = \alpha\delta^{-1}A_\delta$ and let $(B^n)_n$ be defined as in (12). The process B^0 is ergodic of mean α. We also have $B^0 \geq A_\delta$ a.s. Using (9), this implies

$$\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{L}\big(\Psi\big(B^i,S^i(0)\big)\big) \leq_{\mathrm{st}} \mathcal{L}\big(\Psi(A_\delta,S)(0)\big) \qquad \text{for all } n.$$

Since $E[\Psi(A_\delta,S)(0)] \leq M(\delta) < \infty$, the sequence $\{n^{-1}\sum_{i=0}^{n-1}\mathcal{L}(\Psi(B^i, S^i)(0)), n \in \mathbb{N}^*\}$ is uniformly integrable. Furthermore, we have from (33) that $n^{-1}\sum_{i=0}^{n-1}\mathcal{L}(\Psi(B^i,S^i)(0))\xrightarrow{w}\mathcal{L}(\Psi(A_\alpha,S)(0))$. Applying Theorem 5.4 of [3], weak convergence plus uniform integrability implies convergence in expectation:

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} E[\Psi(B^i,S^i)(0)] = E[\Psi(A_\alpha,S)(0)].$$

Now recall from Theorem 4.1 that we have $n^{-1}\sum_{i=0}^{n-1}\Psi(B^i,S^i)(0)\to M(\alpha)$ almost surely. Applying Fatou's lemma, we get

$$M(\alpha)\le \lim_n \frac{1}{n}\sum_{i=0}^{n-1} E[\Psi(B^i,S^i)(0)].$$

Summarizing, we have $M(\alpha)\le E[\Psi(A_\alpha,S)(0)]$. This completes the proof. $\square$

THEOREM 6.4. *The set* $\mathcal{S}$ *is closed in* (β,∞) *and* $\inf\{u\in\mathcal{S}\}=\beta$, $\sup\{u\in\mathcal{S}\}=+\infty$.

PROOF. A direct consequence of Lemma 6.1 is that $\inf\{u\in\mathcal{S}\}=\beta$. We prove that $\sup\{u\in\mathcal{S}\}=+\infty$ by contradiction. Thus, suppose $\sup\{u\in\mathcal{S}\}<\infty$ and consider $\alpha>\sup\{u\in\mathcal{S}\}$. Let A^0 be an ergodic inter-arrival process of mean α satisfying condition (14). Let $\widehat{A}$ be distributed as a weak subsequential limit of the Cesaro averages of the laws of $(A^k)_k$. By Lemma 6.2, $A_\delta\le_{\mathrm{st}}\widehat{A}$ for any $\delta\in\mathcal{S}$. According to (1), this implies that $\delta\le E[\widehat{A}(0)|\mathfrak{T}]$ a.s. Since $\operatorname{supp} E[\widehat{A}(0)|\mathfrak{T}]\subset\mathcal{S}$, see (34), we conclude that almost surely

$$E[\widehat{A}(0)|\mathfrak{T}]=\sup\{u\in\mathcal{S}\}\in\mathcal{S}.$$

Set $\eta=\sup\{u\in\mathcal{S}\}$. Since $\widehat{A}$ is a fixed point, we must have $\widehat{A}\sim A_\eta$. In particular, along an appropriate subsequence, we have that $n^{-1}\sum_{i=0}^{n-1}\mathcal{L}(\Psi(A^i,S^i))$ converges weakly to $\mathcal{L}(\Psi(A_\eta,S))$. Now, a sequential use of Lemma 6.3, Fatou's lemma and Theorem 4.1 gives us

$$M(\eta)=E[\Psi(A_\eta,S)(0)]\le \lim_n \frac{1}{n}\sum_{i=0}^{n-1}E[\Psi(A^i,S^i)(0)]=M(\alpha).$$

It follows from the properties of γ recalled after the statement of Theorem 4.1 that $M(x)$ is a positive and decreasing function that is *strictly decreasing* on the interval $\{x\mid M(x)>0\}$. Since $\alpha<\eta$ and $M(\alpha)\le M(\eta)$, we conclude that $M(\alpha)=M(\eta)=0$. Thus, $E[\Psi(A_\eta,S)(0)]=0$, that is, $P\{\Psi(A_\eta,S)=(0)^{\mathbb{Z}}\}=1$. Let us input the process A_η into the tandem of queues. Using (8) recursively, we obtain

$$\begin{aligned}A_\eta^k(0)&=A_\eta(0)+\sum_{i=0}^{k-1}[S(1,i)-S(0,i)]+\sum_{i=0}^{k-1}\big[\Psi(A_\eta^i,S^i)(1)-\Psi(A_\eta^i,S^i)(0)\big]\\&=A_\eta(0)+\sum_{i=0}^{k-1}[S(1,i)-S(0,i)].\end{aligned}$$

Since the service times are i.i.d. and nonconstant, the partial sums $\sum_{i=0}^{k-1}[S(1,i) - S(0,i)]$ form a null-recurrent random walk. Thus there is a k for which $A_\eta^k(0) < 0$ with strictly positive probability, which is impossible. Or, we cannot have $M(\eta) = 0$. In turn, this implies $\sup\{u \in \mathcal{S}\} = \infty$, and via Lemma 6.2 we get that $E[\widehat{A}(0)] = \alpha$.

We now prove that $\mathcal{S}$ is closed in (β, ∞). Consider a sequence α_k of elements of $\mathcal{S}$ that increases to $\alpha \in (\beta, \infty)$. Let A^0 and $\widehat{A}$ be defined as above (for the mean α). Using Lemma 6.2, we have $A_{\alpha_k} \leq_{\mathrm{st}} \widehat{A}$ and using (1), we have $\alpha_k \leq E[\widehat{A}(0)|\mathfrak{T}]$ a.s. Passing to the limit, we get $\alpha \leq E[\widehat{A}(0)|\mathfrak{T}]$ a.s. Since $E[\widehat{A}(0)] = E[E[\widehat{A}(0)|\mathfrak{T}]] = \alpha$, we conclude that $\operatorname{supp} E[\widehat{A}(0)|\mathfrak{T}] = \{\alpha\}$. It implies that $\alpha \in \mathcal{S}$. The proof works similarly when α_k is a decreasing sequence. $\square$

PROPOSITION 6.5. *Consider an ergodic inter-arrival process A^0 of mean α. There are two possibilities*:

1. *if $\alpha \in \mathcal{S}$, then $\bar{\rho}(A^k, A_\alpha) \xrightarrow{k} 0$ and hence $A^k \xrightarrow{w} A_\alpha$*;
2. *if $\alpha \notin \mathcal{S}$, then $k^{-1}\sum_{i=0}^{k-1}\mathcal{L}(A^i) \xrightarrow{w} p\mathcal{L}(A_{\underline{\alpha}}) + (1-p)\mathcal{L}(A_{\overline{\alpha}})$, where*

$$\underline{\alpha} = \sup\{u \in \mathcal{S}; u \leq \alpha\}, \qquad \overline{\alpha} = \inf\{u \in \mathcal{S}; \alpha \leq u\} \quad \textit{and} \quad p = \frac{\overline{\alpha} - \alpha}{\overline{\alpha} - \underline{\alpha}}. \tag{35}$$

In words, the weak Cesaro limit is a linear combination of the largest ergodic fixed point of mean less than α and of the smallest ergodic fixed point of mean more than α. The weak Cesaro limit always has mean α.

PROOF. The case $\alpha \in \mathcal{S}$ is a restatement of (19). Consider $\alpha \notin \mathcal{S}$. Denote by $\widehat{A}$ a process whose law is a weak subsequential limit of the Cesaro averages of the laws of $(A^k)_k$. By Lemma 6.2, we have $A_u \leq_{\mathrm{st}} \widehat{A} \leq_{\mathrm{st}} A_v$ for any $u, v \in \mathcal{S}$ such that $u < \alpha < v$. Therefore, using (1), we get that $u \leq E[\widehat{A}(0)|\mathfrak{T}] \leq v$ a.s. Since $\operatorname{supp} E[\widehat{A}(0)|\mathfrak{T}] \subset \mathcal{S}$ [see (34)] and $E[\widehat{A}(0)] = \alpha$ (Lemma 6.2) we conclude that $\operatorname{supp} E[\widehat{A}(0)|\mathfrak{T}] = \{\underline{\alpha}, \overline{\alpha}\}$, where $\underline{\alpha}$ and $\overline{\alpha}$ are defined as in (35).

We know from Section 5 that the law of $\widehat{A}$ is a fixed point. Given that $\operatorname{supp} E[\widehat{A}(0)|\mathfrak{T}] = \{\underline{\alpha}, \overline{\alpha}\}$, Proposition 4.4 tells us that $\widehat{A} \sim pA_{\underline{\alpha}} + (1-p)A_{\overline{\alpha}}$ for some p. Therefore $E[\widehat{A}(0)] = p\underline{\alpha} + (1-p)\overline{\alpha}$ and from $E[\widehat{A}(0)] = \alpha$, we conclude that $p = (\overline{\alpha} - \alpha)/(\overline{\alpha} - \underline{\alpha})$.

A consequence of the above argument is that any convergent subsequence of $k^{-1}\sum_{i=0}^{k-1}\mathcal{L}(A^i)$ must converge weakly to $p\mathcal{L}(A_{\underline{\alpha}}) + (1-p)\mathcal{L}(A_{\overline{\alpha}})$. Recalling an argument of Section 5, the sequence $(k^{-1}\sum_{i=0}^{k-1}\mathcal{L}(A^i), k \in \mathbb{N}^*)$ is tight, hence sequentially compact. This implies that $k^{-1}\sum_{i=0}^{k-1}\mathcal{L}(A^i) \xrightarrow{w} p\mathcal{L}(A_{\underline{\alpha}}) + (1-p)\mathcal{L}(A_{\overline{\alpha}})$. $\square$

The previous results characterize $\mathcal{S}$ to a certain extent. We believe that more is true.

CONJECTURE 6.6. For any $\alpha > \beta = E[S(0,0)]$, there exists an ergodic fixed point of mean α. That is, $\mathcal{S} = (\beta, +\infty)$.

It is possible to show that $\mathcal{S}$ is equal to the image of the derivative of γ defined in Theorem 4.1. (Since γ is concave, its derivative γ' is continuous except at a countable number of points. At the points of discontinuity, we consider that both the left and the right-hand limits belong to the image.) Hence the conjecture is true if the function γ has a continuous derivative. However, we have not been able to prove this. The function γ defines the limit shape of an oriented last-passage time percolation model on $\mathbb{N}^2$ with weights $(S(i,j))_{i,j}$ on the lattice points; see [2, 12, 17]. Establishing the smoothness of the limit shape in percolation models is usually a difficult question.

7. Complements. In proving Theorem 5.1, an essential step was to establish the identity (28): $\widetilde{D} = \Phi(\widehat{A}, \widehat{S})$. This can be rephrased as the weak continuity of the operator Φ_σ of an i.i.d. queue on the converging subsequences of the Cesaro averages of the laws of A^k. In fact a much stronger result holds:

THEOREM 7.1. *For a stationary queue defined as in Section* 3.2, *the operator Φ_σ is weakly continuous on $\mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$.*

Theorem 7.1 is a generalization of a result due to Borovkov ([4], Chapter 11 or [5], Chapter 4); see also [6]. Borovkov proves that for an ergodic queue, Φ_σ is weakly continuous on $\bigcup_{\beta<x} \mathcal{M}_e^x(\mathbb{R}_+^{\mathbb{Z}})$. The proof of Theorem 7.1, which follows closely the arguments in [4, 5], appears in the preprint version [16] of the present article.

We have quoted Theorem 7.1 since we believe it to be of independent interest. However, we have not included the proof since Theorem 7.1 does not provide any shortcut to the proof of Theorem 5.1. Let us explain this last point in more detail.

Considering Theorem 7.1, a natural approach to the existence of fixed points for Φ_σ is the following. Consider the $\mathbb{R}$-vector space $\mathbb{M}$ of finite signed measures on $\mathbb{R}^{\mathbb{Z}}$, and observe that $\mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$ is a convex subset of $\mathbb{M}$. Equipped with the topology of weak convergence, recall that $\mathbb{M}$ is a locally convex and Hausdorff space, and that $\mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$ is closed in $\mathbb{M}$. Now, find a convex and compact subset $\mathcal{C}$ of $\mathcal{M}_s(\mathbb{R}_+^{\mathbb{Z}})$ such that Φ_σ maps $\mathcal{C}$ into itself. Since Φ_σ is continuous, the existence of a fixed point in $\mathcal{C}$ then follows from the Schauder–Tychonoff fixed point theorem ([20], Chapter 5).

A suitable candidate for the set $\mathcal{C}$ is dicted by Loynes' results. Indeed, assume that σ is ergodic and consider $\alpha > E[S(0)]$. The set $\mathcal{M}_e^\alpha(\mathbb{R}_+^{\mathbb{Z}})$ is mapped into itself by Φ_σ. However, it is not convex. Its convexification is the set $\mathcal{M}_s^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ defined in (20). The set $\mathcal{M}_s^{p:\alpha}(\mathbb{R}_+^{\mathbb{Z}})$ is not weakly closed [as can be seen by considering $(\xi_n)_n$ defined in (36)]. Its closure is the set $\bigcup_{x\le\alpha} \mathcal{M}_s^x(\mathbb{R}_+^{\mathbb{Z}})$.

Since $\Phi_\sigma(\mu) \geq_{\mathrm{st}} \sigma$ for all μ, we deduce the following natural and "minimal" candidate for $\mathcal{C}$:

$$\mathcal{C} = \bigcup_{x \leq \alpha} \mathcal{M}_{\mathrm{s}}^x(\mathbb{R}_+^{\mathbb{Z}}) \cap \{\mu \mid \mu \geq_{\mathrm{st}} \sigma\}.$$

It is easily checked that $\mathcal{C}$ is compact, convex, and mapped into itself by Φ_σ. We therefore conclude that there exists a fixed point in $\mathcal{C}$. The problem is that $\mathcal{C}$ is too large: it contains the trivial fixed point σ, and we have no way to assert the existence of a nontrivial fixed point.

Building on the above idea, one could try the same approach with another topology on $\mathcal{M}_{\mathrm{s}}(\mathbb{R}_+^{\mathbb{Z}})$: the one induced by the $\bar{\rho}$ distance defined in (16). According to Theorem 4.2, the map Φ_σ is 1-Lipschitz on $\mathcal{M}_{\mathrm{s}}(\mathbb{R}_+^{\mathbb{Z}})$, hence continuous. However, there is no clear way to build a compact and convex set on which to work. Indeed, let $\xi_n \in \mathcal{M}_{\mathrm{e}}^1(\mathbb{R}_+^{\mathbb{Z}})$ be the distribution of the periodic process whose period is given by

$$(\underbrace{0, \ldots, 0}_{n}, \underbrace{2, \ldots, 2}_{n}). \tag{36}$$

It is easy to see that $(\xi_n)_n$ is not sequentially compact in $\mathcal{M}_{\mathrm{s}}(\mathbb{R}_+^{\mathbb{Z}})$ for the $\bar{\rho}$ topology. Indeed, we have $\xi_n \xrightarrow{w} \xi$, where ξ is defined by $P\{\xi = (0)^{\mathbb{Z}}\} = P\{\xi = (2)^{\mathbb{Z}}\} = 1/2$. Since convergence in the $\bar{\rho}$ topology implies weak convergence, if $(\xi_n)_n$ admits a subsequential limit in the $\bar{\rho}$ topology, then it has to be ξ. However, it is easy to check that $\bar{\rho}(\xi_n, \xi) = 1$ for all n.

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LIAFA
UNIVERSITY DENIS DIDEROT
CASE 7014
2 PLACE JUSSIEU
F-75251 PARIS CEDEX 05
FRANCE
E-MAIL: jean.mairesse@liafa.jussieu.fr

DEPARTMENTS OF ELECTRICAL ENGINEERING
AND COMPUTER SCIENCE
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305-9510
E-MAIL: balaji@stanford.edu