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SELF-NORMALIZED CRAMÉR-TYPE LARGE DEVIATIONS FOR INDEPENDENT RANDOM VARIABLES

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Let $X_1, X_2, \dots$ be independent random variables with zero means and finite variances. It is well known that a finite exponential moment assumption is necessary for a Cramér-type large deviation result for the standardized partial sums. In this paper, we show that a Cramér-type large deviation theorem holds for self-normalized sums only under a finite $(2+\delta)$th moment, $0 < \delta \le 1$. In particular, we show $P(S_n/V_n \ge x) = (1-\Phi(x))(1+O(1)(1+x)^{2+\delta}/d_{n,\delta}^{2+\delta})$ for $0 \le x \le d_{n,\delta}$, where $d_{n,\delta} = (\sum_{i=1}^n EX_i^2)^{1/2}/(\sum_{i=1}^n E|X_i|^{2+\delta})^{1/(2+\delta)}$ and $V_n = (\sum_{i=1}^n X_i^2)^{1/2}$. Applications to the Studentized bootstrap and to the self-normalized law of the iterated logarithm are discussed.

1. Introduction. Let $X_1, X_2, \dots$ be a sequence of independent random variables with $EX_i = 0$ and $0 < EX_i^2 < \infty$ for $i \ge 1$. Set

$$S_n = \sum_{i=1}^n X_i, \qquad B_n^2 = \sum_{i=1}^n EX_i^2, \qquad V_n^2 = \sum_{i=1}^n X_i^2.$$

The classical central limit theorem states that if the Lindeberg condition

$$\frac{1}{B_n^2}\sum_{i=1}^n EX_i^2 I_{\{|X_i|>\varepsilon B_n\}} \to 0 \qquad \text{for every fixed } \varepsilon > 0 \tag{1.1}$$

is satisfied, then

$$\sup_x \big|P(S_n \ge xB_n) - (1-\Phi(x))\big| \to 0 \qquad \text{as } n \to \infty.$$

The central limit theorem is useful when x is not too large or when the error is well estimated. There are two approaches for estimating the error of the normal approximation. One approach is to study the absolute error in the central limit theorem via Berry–Esseen bounds or Edgeworth expansions [see, e.g., Petrov (1975)]. Another approach is to estimate the relative error of $P(S_n \ge xB_n)$ to $1-\Phi(x)$. One of the typical results in this direction is the so-called Cramér

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large deviation. If $X_1, X_2, \dots$ are a sequence of independent and identically distributed (i.i.d.) random variables with zero means and the finite moment-generating function $Ee^{tX_1} < \infty$ for t in a neighborhood of zero, then for $x \ge 0$ and $x = o(n^{1/2})$

$$\frac{P(S_n \ge xB_n)}{1-\Phi(x)} = \exp\left\{x^2\lambda\left(\frac{x}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right], \tag{1.2}$$

$$\frac{P(S_n \le -xB_n)}{\Phi(-x)} = \exp\left\{x^2\lambda\left(\frac{-x}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{1+x}{\sqrt{n}}\right)\right], \tag{1.3}$$

where $\lambda(t)$ is the so-called Cramér's series [see Petrov (1975), Chapter VIII, for details]. In particular, if $Ee^{t|X_1|^{1/2}} < \infty$ for some $t > 0$, then

$$\frac{P(S_n \ge xB_n)}{1-\Phi(x)} \to 1, \qquad \frac{P(S_n \le -xB_n)}{\Phi(-x)} \to 1 \tag{1.4}$$

holds uniformly for $x \in (0, o(n^{1/6}))$. Similar results are also available for independent but not necessarily identically distributed random variables under a finite moment-generating function condition.

It is well known that moment conditions in these classical limit theorems are also necessary. On the other hand, limit theorems for the self-normalized sums S_n/V_n put a totally new countenance on classical limit theorems. When $X_1, X_2, \dots$ are i.i.d. random variables, in contrast to the well-known Hartman–Wintner law of the iterated logarithm (LIL) and its converse by Strassen (1966), Griffin and Kuelbs (1989) obtained a self-normalized LIL for all distributions in the domain of attraction of a normal or stable law. Shao (1997) showed that no moment conditions are needed for a self-normalized large deviation result $P(S_n/V_n \ge x\sqrt{n})$. The tail probability of S_n/V_n is Gaussian like when X_1 is in the domain of attraction of the normal law and sub-Gaussian like when X_1 is in the domain of attraction of a stable law, which in turn enables him to find the precise constant in Griffin and Kuelbs' self-normalized LIL, while Giné, Götze and Mason (1997) proved that the tails of S_n/V_n are uniformly sub-Gaussian when the sequence is stochastically bounded. Shao (1999) established a (1.4) type result for self-normalized sums only under a finite third-moment condition. More precisely, he showed that if $E|X_1|^{2+\delta} < \infty$ for $0 < \delta \le 1$, then

$$\frac{P(S_n \ge xV_n)}{1-\Phi(x)} \to 1, \qquad \frac{P(S_n \le -xV_n)}{\Phi(-x)} \to 1 \tag{1.5}$$

holds uniformly for $x \in (0, o(n^{\delta/(2(2+\delta))}))$. Self-normalized sums have been studied previously in connection with weak convergence by Darling (1952), Logan, Mallows, Rice and Shepp (1973), LePage, Woodroofe and Zinn (1981), Csörgő and Horváth (1988), Csörgő, Haeusler and Mason (1988, 1991), Hahn, Kuelbs and Weiner (1990), Griffin and Mason (1991), Griffin and Pruitt (1989) and Maller (1988, 1991). We refer to Bentkus and Götze (1996) for Berry–Esseen inequalities, Giné, Götze and Mason (1997) for the necessary and sufficient condition for the asymptotic normality and Shao (1998) for a survey on recent developments in this area. Recently, several papers have focused on the self-normalized limit theorems for independent but not necessarily identically distributed random variables. Bentkus, Bloznelis and Götze (1996) obtained the following Berry–Esseen bound:

$$\begin{aligned} &\bigl|P(S_n/V_n \ge x) - (1-\Phi(x))\bigr| \\ &\quad \le A\left(B_n^{-2}\sum_{i=1}^n EX_i^2 I_{\{|X_i|>B_n\}} + B_n^{-3}\sum_{i=1}^n E|X_i|^3 I_{\{|X_i|\le B_n\}}\right), \end{aligned} \tag{1.6}$$

where A is an absolute constant. Assuming only finite third moments, Wang and Jing (1999) derived exponential nonuniform Berry–Esseen bounds. Chistyakov and Götze (1999) refined Wang and Jing's results and obtained the following result, among others: If $X_1, X_2, \dots$ are *symmetric* independent random variables with finite third moments, then

$$P(S_n/V_n \ge x) = (1-\Phi(x))\left(1 + O(1)(1+x)^3 B_n^{-3}\sum_{i=1}^n E|X_i|^3\right) \tag{1.7}$$

for $0 \le x \le B_n/(\sum_{i=1}^n E|X_i|^3)^{1/3}$, where $O(1)$ is bounded by an absolute constant.

Result (1.7) is useful because it provides not only the relative error but also a Berry–Esseen rate of convergence. Although the assumption of symmetry allows us to assert that

$$P(S_n/V_n \ge x) = (1-\Phi(x))\left(1 + O(1)\min\left(1, (1+x)^3 B_n^{-3}\sum_{i=1}^n E|X_i|^3\right)\right)$$

for all $x \ge 0$, it not only takes away the main difficulty in proving a self-normalized limit theorem but also limits its potential applications. The main aim of this paper is to establish a Cramér-type large deviation for general independent random variables. In particular, we show that (1.7) remains valid for nonsymmetric independent random variables. One of the main contributions of this paper is that the exponential moment condition needed for the normalized sum can be considerably reduced to only the finite moment condition of low order, which significantly expands the applicability of such a large deviation result to other fields and especially to statistics.

This paper is organized as follows. The main results are stated in Section 2. Applications to the Studentized bootstrap and to the self-normalized LIL are given in Sections 3 and 4, respectively. Major steps of the proof of Theorem 2.1 as well as proofs of the corollaries of Theorem 2.1 are provided in Section 5. Section 6 presents some preliminary lemmas. Proofs of four propositions used in the main proof are offered in Sections 7–10.

2. Main results. Throughout the paper, we assume that $X_1, X_2, \dots$ are independent random variables with $EX_i = 0$ and $0 < EX_i^2 < \infty$. Further to our earlier notation, we introduce

$$\Delta_{n,x} = \frac{(1+x)^2}{B_n^2} \sum_{i=1}^n EX_i^2 I_{\{|X_i| > B_n/(1+x)\}} + \frac{(1+x)^3}{B_n^3} \sum_{i=1}^n E|X_i|^3 I_{\{|X_i| \le B_n/(1+x)\}}$$

for $x \ge 0$.

THEOREM 2.1. *There is an absolute constant A (> 1) such that*

$$\frac{P(S_n \ge xV_n)}{1 - \Phi(x)} = e^{O(1)\Delta_{n,x}} \quad \text{and} \quad \frac{P(S_n \le -xV_n)}{\Phi(-x)} = e^{O(1)\Delta_{n,x}} \tag{2.1}$$

for all $x \ge 0$ satisfying

$$x^2 \max_{1 \le i \le n} EX_i^2 \le B_n^2 \tag{2.2}$$

and

$$\Delta_{n,x} \le (1+x)^2/A, \tag{2.3}$$

where $|O(1)| \le A$.

Theorem 2.1 provides a very general framework. The following results are direct consequences of the above general theorem.

THEOREM 2.2. *Let $\{a_n, n \ge 1\}$ be a sequence of positive numbers. Assume that*

$$a_n^2 \le B_n^2 / \max_{1 \le i \le n} EX_i^2 \tag{2.4}$$

and

$$\forall \varepsilon > 0, \qquad B_n^{-2} \sum_{i=1}^n EX_i^2 I_{\{|X_i| > \varepsilon B_n/(1+a_n)\}} \to 0 \qquad \text{as } n \to \infty. \tag{2.5}$$

Then

$$\frac{\ln P(S_n/V_n \ge x)}{\ln(1 - \Phi(x))} \to 1, \qquad \frac{\ln P(S_n/V_n \le -x)}{\ln \Phi(-x)} \to 1 \tag{2.6}$$

holds uniformly for $x \in (0, a_n)$.

The next corollary is a special case of Theorem 2.2 and may be of independent interest.

COROLLARY 2.1. *Suppose that* $B_n \geq c\sqrt{n}$ *for some* $c > 0$ *and that* $\{X_i^2, i \geq 1\}$ *is uniformly integrable. Then, for any sequence of real numbers* x_n *satisfying* $x_n \to \infty$ *and* $x_n = o(\sqrt{n}\,)$,

$$\ln P(S_n/V_n \geq x_n) \sim -x_n^2/2. \tag{2.7}$$

When the X_i's have a finite $(2+\delta)$th moment for $0 < \delta \leq 1$, we obtain (1.7) without assuming any symmetric condition.

THEOREM 2.3. *Let* $0 < \delta \leq 1$ *and set*

$$L_{n,\delta} = \sum_{i=1}^{n} E|X_i|^{2+\delta}, \qquad d_{n,\delta} = B_n / L_{n,\delta}^{1/(2+\delta)}.$$

Then

$$\frac{P(S_n/V_n \geq x)}{1 - \Phi(x)} = 1 + O(1)\left(\frac{1+x}{d_{n,\delta}}\right)^{2+\delta} \tag{2.8}$$

and

$$\frac{P(S_n/V_n \leq -x)}{\Phi(-x)} = 1 + O(1)\left(\frac{1+x}{d_{n,\delta}}\right)^{2+\delta} \tag{2.9}$$

for $0 \leq x \leq d_{n,\delta}$, *where* $O(1)$ *is bounded by an absolute constant. In particular, if* $d_{n,\delta} \to \infty$ *as* $n \to \infty$, *we have*

$$\frac{P(S_n \geq xV_n)}{1 - \Phi(x)} \to 1, \qquad \frac{P(S_n \leq -xV_n)}{\Phi(-x)} \to 1 \tag{2.10}$$

uniformly in $0 \leq x \leq o(d_{n,\delta})$.

By the fact that $1 - \Phi(x) \leq 2e^{-x^2/2}/(1+x)$ for $x \geq 0$, it follows from (2.8) that the following exponential nonuniform Berry–Esseen bound

$$\bigl|P(S_n/V_n \geq x) - \bigl(1 - \Phi(x)\bigr)\bigr| \leq A(1+x)^{1+\delta} e^{-x^2/2} / d_{n,\delta}^{2+\delta} \tag{2.11}$$

holds for $0 \leq x \leq d_{n,\delta}$.

The next corollary shows that the range of uniform convergence in Theorem 2.3 can be extended to $[0, O(d_{n,\delta})]$ when $0 < \delta < 1$ under certain circumstances and especially for i.i.d. cases.

COROLLARY 2.2. *Let* $0 < \delta < 1$. *Assume that* $\{|X_i|^{2+\delta}, i \geq 1\}$ *is uniformly integrable and that* $B_n \geq cn^{1/2}$ *for some constant* $c > 0$. *Then* (2.10) *holds uniformly for* $x \in [0, O(n^{\delta/(4+2\delta)})]$.

For i.i.d. random variables, Theorem 2.1 simply reduces to the following corollary:

COROLLARY 2.3. *Let $X_1, X_2, \dots$ be i.i.d. random variables with $EX_i = 0$ and $\sigma^2 = EX_i^2 < \infty$. Then there exists an absolute constant $A > 2$ such that*

$$\frac{P(S_n \ge xV_n)}{1 - \Phi(x)} = e^{O(1)\Delta_{n,x}} \quad \textit{and} \quad \frac{P(S_n \le -xV_n)}{\Phi(-x)} = e^{O(1)\Delta_{n,x}}$$

for all $x \ge 0$ satisfying $\Delta_{n,x} \le (1+x)^2/A$, where $|O(1)| \le A$ and

$$\begin{aligned}\Delta_{n,x} &= (1+x)^2\sigma^{-2}EX_1^2 I_{\{|X_1| > \sqrt{n}\sigma/(1+x)\}} \\ &\quad + (1+x)^3\sigma^{-3}n^{-1/2}E|X_1|^3 I_{\{|X_1| \le \sqrt{n}\sigma/(1+x)\}}.\end{aligned}$$

We end this section with the following remarks:

REMARK 2.1. If the Lindeberg condition (1.1) holds, then $\Delta_{n,x} \to 0$ for every fixed x. Hence, it follows from Theorem 2.1 that

$$\big|P(S_n \ge xV_n) - \big(1 - \Phi(x)\big)\big| \to 0 \qquad \text{as } n \to \infty,$$

which is the central limit theorem for the self-normalized sum.

REMARK 2.2. If $X_1, X_2, \dots$ are i.i.d. random variables with $\sigma^2 = EX_1^2 < \infty$, then condition (2.2) simply reduces to $x \le \sqrt{n}$ and (2.3) to

$$\sigma^{-2}EX_1^2 I_{\{|X_1| > \sqrt{n}\sigma/(1+x)\}} + (1+x)n^{-1/2}\sigma^{-3}E|X_1|^3 I_{\{|X_1| \le \sqrt{n}\sigma/(1+x)\}} \le 1/A,$$

which in turn implies $(1+x) \le \sqrt{n}$. Hence, (2.3) implies (2.2) in the i.i.d. case. However, (2.3) does not imply (2.2) in general. On the other hand, it could be interesting if condition (2.2) in Theorem 2.1 or condition (2.4) in Theorem 2.2 can be removed.

REMARK 2.3. An example given in Shao (1999) shows that in i.i.d. cases, the condition $E|X_1|^{2+\delta} < \infty$ for (1.5) cannot be replaced with $E|X_1|^r < \infty$ for some $r < 2 + \delta$. We believe that condition (2.3) in Theorem 2.1 is the best possible condition.

REMARK 2.4. When $X_1, X_2, \dots$ are i.i.d. random variables, $d_{n,\delta}$ is simply equal to $n^{\delta/(4+2\delta)}(EX_1^2)^{1/2}/(E|X_1|^{2+\delta})^{1/(2+\delta)}$.

3. An application to the Studentized bootstrap. Since Efron (1979) introduced the bootstrap, its properties have been studied extensively. One useful application of the bootstrap is to the construction of the confidence intervals for a population quantity of interest. There are many variants of the bootstrap developed for this purpose, including the percentile method, the percentile-t, the ABC method, the iterated bootstrap. See the monographs of Hall (1992), Efron and Tibshirani (1993) and Davison and Hinkley (1997), for instance.

In this section, we restrict our attention to bootstrapping the Studentized t-statistic (more commonly referred to as the "percentile-t" method) to construct a confidence interval for a population mean. Suppose that $\mathcal{X} = \{X_1, \dots, X_n\}$ is a random sample from some population F. Let $\mathcal{X}^* = \{X_1^*, \dots, X_n^*\}$ be a resample drawn randomly with replacement from $\mathcal{X}$. Write $\bar{X} = n^{-1}\sum_{i=1}^n X_i$ and $\bar{X}^* = n^{-1}\sum_{i=1}^n X_i^*$ for the sample mean and resample mean, respectively. Also, write $\hat{\sigma}^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$ and $\hat{\sigma}^{*2} = (n-1)^{-1}\sum_{i=1}^n (X_i^* - \bar{X}^*)^2$ for the sample variance and resample variance, respectively. Define

$$T_n = \sqrt{n}(\bar{X} - \mu)/\hat{\sigma}, \qquad T_n^* = \sqrt{n}(\bar{X}^* - \bar{X})/\hat{\sigma}^*$$

as the Studentized mean and its bootstrap version. Also, define their corresponding distribution functions by

$$F_n(x) = P(T_n \le x), \qquad \hat{F}_n(x) = P^*(T_n^* \le x),$$

where $P^*(\cdot)$ denotes the conditional probability given the sample $\mathcal{X}$. Note that if the distribution function $F_n(x)$ were known, then a one-sided confidence interval $I_1 = (-\infty, \bar{X} - n^{-1/2}\hat{\sigma}F_n^{-1}(\alpha))$ would have the exact coverage probability $1-\alpha$ in the sense that $P(\mu \in I_1) = 1 - \alpha$ for any prescribed significance level α. The bootstrap percentile-t confidence interval with nominal level $1-\alpha$ is defined to be $\hat{I}_1 = (-\infty, \bar{X} - n^{-1/2}\hat{\sigma}\hat{F}_n^{-1}(\alpha))$.

The accuracy of the percentile-t method critically depends on how well the bootstrap distribution function $P^*(T_n^* \le x)$ approximates the true distribution function $P(T_n \le x)$. One of the very useful approaches in bootstrap analysis is the Edgeworth expansion [see Singh (1981) and Hall (1988)]. Although this approach reveals many important properties of the bootstrap, it does not give a complete picture. It is well known that Edgeworth expansions focus only on absolute errors and can be very inaccurate when employed to estimate tail probabilities. It is in just those cases, however, that we usually wish to use the bootstrap to approximate distribution functions.

In this section, we are concerned with the relative error properties of the bootstrap. This line of work was initiated by Hall [(1990) and (1992), page 324] who showed that the bootstrap provides an accurate approximation to large deviation probabilities for values of x as large as $o(n^{1/3})$ for the standardized mean under the assumption that the parent distribution has a finite moment-generating function in the neighborhood of the origin and that the characteristic function satisfies Cramér's condition $\limsup_{|t|\to\infty} |Ee^{itX_1}| < 1$. Jing, Feuerverger and Robinson (1994) extended Hall's results to the Studentized t-statistic by applying the saddlepoint approximations for the Student's t-statistic developed by Daniels and Young (1991). However, the moment condition required is

$$E\exp\{tX_1^2\} < \infty$$

for some $t > 0$. Note that this condition is extremely strong since it requires that the tail probability of the underlying distribution drops to zero at least as fast as a normal random variable.

Our next theorem shows that the bootstrap still possesses some large deviation properties under only finite moment conditions. It is worth mentioning that Wood (2000) also recently studied similar issues to those in this section but only for the percentile bootstrap method.

THEOREM 3.1. *If* $E|X_1|^{2+\delta} < \infty$ *for some* $0 < \delta \le 1$, *then*

$$\frac{P^*(T_n^* \ge x)}{P(T_n \ge x)} = 1 + o(1) \quad \text{and} \quad \frac{P^*(T_n^* \le -x)}{P(T_n \le -x)} = 1 + o(1) \qquad \text{a.s.} \tag{3.1}$$

holds uniformly in $0 \le x \le o(n^{\delta/(4+2\delta)})$.

PROOF. We shall only prove the first part of (3.1). Without loss of generality, assume that $\mu = 0$. Note that the distribution functions of T_n and S_n/V_n are closely related via the following identity:

$$\{T_n \ge x\} = \left\{S_n \ge x\left(\frac{n}{n + x^2 - 1}\right)^{1/2} V_n\right\}. \tag{3.2}$$

Applying Theorem 2.3, we see that

$$\frac{P(T_n \ge x)}{1 - \Phi(x)} = 1 + o(1) \tag{3.3}$$

holds uniformly in $0 \le x \le o(n^{\delta/(2(2+\delta))})$.

For the bootstrap distribution, we can apply Theorem 2.3 again [see (3.2) and Remark 2.4] to obtain

$$\left|\frac{P^*(T_n^* \ge x)}{1 - \Phi(x)} - 1\right| \le A(1 + x)^{2+\delta}/d_{n,\delta}^{*\,2+\delta} \tag{3.4}$$

for $0 \le x \le d_{n,\delta}^*$, where

$$\begin{aligned} d_{n,\delta}^* &= n^{\delta/(4+2\delta)}(E^*|X_1^*|^2)^{1/2}/(E^*|X_1^*|^{2+\delta})^{1/(2+\delta)} \\ &= n^{\delta/(4+2\delta)} \frac{(n^{-1}\sum_{i=1}^n X_i^2)^{1/2}}{(n^{-1}\sum_{i=1}^n |X_i|^{2+\delta})^{1/(2+\delta)}}. \end{aligned}$$

By the law of large numbers, we have

$$d_{n,\delta}^*/n^{\delta/(4+2\delta)} \to (EX_1^2)^{1/2}/(E|X_1|^{2+\delta})^{1/(2+\delta)} \qquad \text{a.s. as } n \to \infty. \tag{3.5}$$

The theorem thus follows from the relationships (3.2)–(3.5). □

Theorem 3.1 states that the bootstrap provides an accurate approximation of large deviation probabilities for Studentized t-statistics for values of x as large as $o(n^{\delta/(4+2\delta)})$. In particular, if $\delta = 1$, the region becomes $0 \le x \le o(n^{1/6})$. Note that the region is smaller than $0 \le x \le o(n^{1/3})$ obtained in Jing, Feuerverger and Robinson (1994) since the moment assumption is much weaker here.

4. An application to the self-normalized law of the iterated logarithm. It is known that the law of the iterated logarithm (LIL) is usually a direct consequence of a moderate deviation result. Following Griffin and Kuelbs (1989) and Shao (1997), we have the following self-normalized LIL as another direct application of Theorem 2.1.

THEOREM 4.1. *Let $X_1, X_2, \ldots$ be independent random variables with $EX_i = 0$ and $0 < EX_i^2 < \infty$. Assume that $B_n \to \infty$, $\max_{1 \le k \le n} EX_i^2 \le \frac{1}{4} B_n^2 / \log\log B_n$ for sufficiently large n and that*

$$\forall \varepsilon > 0, \qquad B_n^{-2} \sum_{i=1}^{n} EX_i^2 I_{\{|X_i| > \varepsilon B_n/(\log\log B_n)^{1/2}\}} \to 0 \qquad \text{as } n \to \infty. \tag{4.1}$$

Then

$$\limsup_{n \to \infty} \frac{S_n}{V_n (2 \log\log B_n)^{1/2}} = 1 \qquad a.s. \tag{4.2}$$

In Theorem 4.1 and the remainder of this paper, $\log x$ denotes $\ln \max(x, e)$. One can refer to Wittmann (1987) and Shao (1995) for the LIL for the standardized sums S_n/B_n. Shao (1995) proved that if (4.1) and

$$\forall \varepsilon > 0, \qquad \sum_{n=1}^{\infty} P\big(|X_n| > \varepsilon B_n/(\log\log B_n)^{1/2}\big) < \infty \tag{4.3}$$

are satisfied, then

$$\limsup_{n \to \infty} \frac{S_n}{B_n (2 \log\log B_n)^{1/2}} = 1 \qquad \text{a.s.} \tag{4.4}$$

The following example shows that the self-normalized LIL (4.2) holds but the LIL (4.4) fails. Let $X_1, X_2, \ldots$ be independent random variables satisfying

$$P(X_n = 0) = \frac{3}{4} - \frac{1}{n(\log\log n)^3} + \frac{1}{4 \log\log n},$$
$$P(X_n = \pm 2) = \frac{1}{8} - \frac{1}{8 \log\log n},$$
$$P(X_n = \pm n^{1/2} \log\log n) = \frac{1}{2n(\log\log n)^3}.$$

Then it is easy to see that $EX_n = 0$, $EX_n^2 = 1$ and $\{X_n^2, n \ge 1\}$ are uniformly integrable. Hence, by Theorem 4.1, (4.2) holds. On the other hand, note that

$$\forall \varepsilon > 0, \qquad \sum_{n=1}^{\infty} P\big(|X_n| > \varepsilon B_n (\log\log B_n)^{1/2}\big) = \infty$$

with $B_n = \sqrt{n}$. By the Borel–Cantelli lemma, (4.4) does not hold.

PROOF OF THEOREM 4.1. We follow the proof of Theorem 5.1 in Shao (1997). We first show that

$$\limsup_{n\to\infty} \frac{S_n}{V_n(2\log\log B_n)^{1/2}} \le 1 \qquad \text{a.s.} \tag{4.5}$$

For $\theta > 1$, let $m_k := m_k(\theta) = \min\{n : B_n \ge \theta^k\}$. It follows from condition (4.1) that

$$B_{m_k} \sim \theta^k \qquad \text{as } k \to \infty. \tag{4.6}$$

Let $x_k = (2\log\log B_{m_k})^{1/2}$. Then, for $0 < \varepsilon < 1/2$,

$$\begin{aligned} &P\left(\max_{m_k \le n \le m_{k+1}} \frac{S_n}{V_n} \ge (1+7\varepsilon)x_k\right) \\ &\quad \le P\left(\frac{S_{m_k}}{V_{m_k}} \ge (1+2\varepsilon)x_k\right) + P\left(\max_{m_k \le n \le m_{k+1}} \frac{S_n - S_{m_k}}{V_n} \ge 5\varepsilon x_k\right). \end{aligned} \tag{4.7}$$

By Theorem 2.2, we have

$$P\left(\frac{S_{m_k}}{V_{m_k}} \ge (1+2\varepsilon)x_k\right) \le \exp(-(1+2\varepsilon)x_k^2/2) \le Ck^{-1-\varepsilon} \tag{4.8}$$

for every sufficiently large k.

We estimate the second term in the right-hand side of (4.7) below. Let $\eta = (\theta^2-1)^{1/2}$ and define $z_k = \eta B_{m_k}/x_k$. Set $T_n = \sum_{i=m_k+1}^n X_i I_{\{|X_i| \le z_k\}}$. So,

$$\begin{aligned} &P\left(\max_{m_k \le n \le m_{k+1}} \frac{S_n - S_{m_k}}{V_n} \ge 5\varepsilon x_k\right) \\ &\quad \le P\left(\max_{m_k \le n \le m_{k+1}} T_n \ge 2\varepsilon x_k B_{m_k}\right) \\ &\qquad + P(V_{m_k} \le B_{m_k}/2) + P\left(\sum_{i=1+m_k}^{m_{k+1}} I_{\{|X_i| > z_k\}} \ge (\varepsilon x_k)^2\right). \end{aligned} \tag{4.9}$$

Note that

$$\sum_{i=1+m_k}^{m_{k+1}} EX_i^2 I_{\{|X_i| \le z_k\}} \sim (\theta^2-1)\theta^{2k}$$

and

$$\begin{aligned} \max_{m_k \le n \le m_{k+1}} |ET_n| &\le z_k^{-1} \sum_{i=1+m_k}^{m_{k+1}} EX_i^2 \sim z_k^{-1}(\theta^2-1)\theta^{2k} \\ &\sim (\theta^2-1)^{1/2} x_k B_{m_k} \le \varepsilon x_k B_{m_k}/2 \end{aligned}$$

for $1 < \theta < 1 + \varepsilon^2/8$. By the Bernstein inequality for large k,

$$
\begin{aligned}
\ln P\Big(\max_{m_k \le n \le m_{k+1}} T_n \ge 2\varepsilon x_k B_{m_k}\Big)
&\le -\frac{(\varepsilon x_k B_{m_k})^2}{2((\theta^2-1)\theta^{2k} + \varepsilon x_k B_{m_k} z_k)} \\
&\sim -\frac{\varepsilon^2 x_k^2}{2((\theta^2-1) + \varepsilon(\theta^2-1)^{1/2})} \\
&\le -x_k^2,
\end{aligned}
\tag{4.10}
$$

provided that $\theta (> 1)$ is close enough to 1. By the Bernstein inequality again,

$$
\begin{aligned}
P\big(V_{m_k} \le B_{m_k}/2\big)
&\le P\Bigg(\sum_{i=1}^{m_k} X_i^2 I_{\{|X_i| \le z_k\}} \le B_{m_k}^2/4\Bigg) \\
&\le \exp\bigg(-\frac{(3B_{m_k}^2/4)^2}{2(\sum_{i=1}^{m_k} E X_i^4 I_{\{|X_i| \le z_k\}} + B_{m_k}^2 z_k^2)}\bigg) \\
&\le \exp\bigg(-\frac{B_{m_k}^4}{8B_{m_k}^2 z_k^2}\bigg) \\
&\le \exp(-x_k^2)
\end{aligned}
\tag{4.11}
$$

for $\theta (> 1)$ close to 1. Let $t := t_k = \ln\{(\varepsilon x_k)^2/(\sum_{i=1}^{m_{k+1}} z_k^{-2} E X_i^2 I_{\{|X_i| > z_k\}})\}$. Then $t \to \infty$ by (4.1). From the Chebyshev inequality, it follows that

$$
\begin{aligned}
P\Bigg(\sum_{i=1+m_k}^{m_{k+1}} I_{\{|X_i| > z_k\}} \ge (\varepsilon x_k)^2\Bigg)
&\le e^{-t(\varepsilon x_k)^2} \prod_{i=1+m_k}^{m_{k+1}} \big(1 + (e^t - 1) P(|X_i| > z_k)\big) \\
&\le \exp\Bigg(-t(\varepsilon x_k)^2 + (e^t - 1) \sum_{i=1}^{m_{k+1}} z_k^{-2} E X_i^2 I_{\{|X_i| > z_k\}}\Bigg) \\
&\le \exp\Bigg(-(\varepsilon x_k)^2 \ln\bigg\{\frac{(\varepsilon x_k)^2}{3\sum_{i=1}^{m_{k+1}} z_k^{-2} E X_i^2 I_{\{|X_i| > z_k\}}}\bigg\}\Bigg) \\
&\le \exp(-x_k^2)
\end{aligned}
\tag{4.12}
$$

for sufficiently large k. Combining the above inequalities yields (4.5) by the Borel–Cantelli lemma and the arbitrariness of ε.

Next, we prove that

$$\limsup_{n\to\infty} \frac{S_n}{V_n(2\log\log B_n)} \ge 1 \qquad \text{a.s.} \tag{4.13}$$

Let $n_k = \min\{m : B_m \ge e^{4k\log k}\}$. Then, $B_{n_k} \sim e^{4k\log k}$. Observe that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{S_n}{V_n(2\log\log B_n)^{1/2}} \\
&\quad \ge \limsup_{k\to\infty} \frac{S_{n_k}}{V_{n_k}(2\log\log B_{n_k})^{1/2}} \\
&\quad \ge \limsup_{k\to\infty} \frac{S_{n_k} - S_{n_{k-1}}}{V_{n_k}(2\log\log B_{n_k})^{1/2}} + \liminf_{k\to\infty} \frac{S_{n_{k-1}}}{V_{n_k}(2\log\log B_{n_k})^{1/2}} \qquad (4.14)\\
&\quad = \limsup_{k\to\infty} \frac{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2}}{V_{n_k}} \frac{S_{n_k} - S_{n_{k-1}}}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2}(2\log\log B_{n_k})^{1/2}} \\
&\qquad + \liminf_{k\to\infty} \frac{V_{n_{k-1}}}{V_{n_k}} \frac{S_{n_{k-1}}}{V_{n_{k-1}}(2\log\log B_{n_k})^{1/2}}.
\end{aligned}$$

Since $\{(S_{n_k} - S_{n_{k-1}})/(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2},\ k \ge 1\}$ are independent, it follows from Theorem 2.2 and the Borel–Cantelli lemma that

$$\limsup_{n\to\infty} \frac{S_{n_k} - S_{n_{k-1}}}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2}(2\log\log B_{n_k})^{1/2}} \ge 1 \qquad \text{a.s.} \tag{4.15}$$

Similar to (4.11) and by the Borel–Cantelli lemma, we have

$$\liminf_{k\to\infty} \frac{V_{n_k}}{B_{n_k}} \ge 1/2 \qquad \text{a.s.}$$

Note that

$$P\left(V_{n_{k-1}} \ge \frac{B_{n_k}}{k}\right) \le \frac{k^2 E V_{n_{k-1}}^2}{B_{n_k}^2} = \frac{k^2 B_{n_{k-1}}^2}{B_{n_k}^2} \le k^{-2}.$$

Then by the Borel–Cantelli lemma again,

$$\lim_{k\to\infty} \frac{V_{n_{k-1}}}{V_{n_k}} = 0 \qquad \text{a.s.} \tag{4.16}$$

This proves (4.13) by (4.14)–(4.16). $\square$

5. Proof of the main results. Throughout the remainder of this paper, we use A to denote an absolute constant, which may take different values at each occurrence.

PROOF OF THEOREM 2.2. Noting that $\forall\, 0<\varepsilon\le 1$ and $0\le x\le a_n$

$$\begin{aligned}
&B_n^{-2}\sum_{i=1}^n EX_i^2 I_{\{|X_i|>B_n/(1+x)\}}+(1+x)B_n^{-3}\sum_{i=1}^n E|X_i|^3 I_{\{|X_i|\le B_n/(1+x)\}}\\
&\quad=B_n^{-2}\sum_{i=1}^n EX_i^2 I_{\{|X_i|>B_n/(1+x)\}}+(1+x)B_n^{-3}\sum_{i=1}^n E|X_i|^3 I_{\{|X_i|\le\varepsilon B_n/(1+a_n)\}}\\
&\qquad+(1+x)B_n^{-3}\sum_{i=1}^n E|X_i|^3 I_{\{\varepsilon B_n/(1+a_n)<|X_i|\le B_n/(1+x)\}}\\
&\quad\le B_n^{-2}\sum_{i=1}^n EX_i^2 I_{\{|X_i|>B_n/(1+x)\}}+\varepsilon(1+x)B_n^{-2}/(1+a_n)\sum_{i=1}^n E|X_i|^2\\
&\qquad+B_n^{-2}\sum_{i=1}^n E|X_i|^2 I_{\{\varepsilon B_n/(1+a_n)<|X_i|\le B_n/(1+x)\}}\\
&\quad\le\varepsilon+B_n^{-2}\sum_{i=1}^n E|X_i|^2 I_{\{|X_i|>\varepsilon B_n/(1+a_n)\}},
\end{aligned}$$

we have by (2.5)

$$\Delta_{n,x}=o\big((1+x)^2\big)\qquad\text{as } n\to\infty$$

uniformly for $0\le x\le a_n$. Now Theorem 2.2 follows from Theorem 2.1. □

PROOF OF COROLLARY 2.1. For any a_n satisfying $a_n\to\infty$ and $a_n=o(B_n)$, the uniform integrability implies that (2.4) and (2.5) are satisfied and hence the corollary follows. □

PROOF OF THEOREM 2.3. (2.8) and (2.9) follow from the Berry–Esseen bound (1.6) for $0\le x\le 8A$. When $x>8A$, it is easy to see that

$$\Delta_{n,x}\le(1+x)^{2+\delta}L_{n,\delta}/B_n^{2+\delta}=\left(\frac{1+x}{d_{n,\delta}}\right)^{2+\delta}\le(1+x)^2/A$$

and that $d_{n,\delta}^2\max_{i\le n}EX_i^2\le B_n^2$. Thus, conditions (2.2) and (2.3) are satisfied. Now, the result follows from Theorem 2.1. □

PROOF OF COROLLARY 2.2. Let $d>0$ and $x_n=d\,n^{\delta/(4+2\delta)}$. It suffices to show that

$$\Delta_{n,x_n}=o(1)\qquad\text{as } n\to\infty.\tag{5.1}$$

Similarly to the proof of Theorem 2.2, we have, for any $0 < \varepsilon < 1$,

$$\begin{aligned}\Delta_{n,x_n} &\le (1+x_n)^2 B_n^{-2} \sum_{i=1}^n E X_i^2 I_{\{|X_i| > B_n/(1+x_n)\}} \\ &\quad + \varepsilon^{1-\delta}(1+x_n)^{2+\delta} B_n^{-(2+\delta)} \sum_{i=1}^n E|X_i|^{2+\delta} I_{\{|X_i| \le \varepsilon B_n/(1+x_n)\}} \\ &\quad + (1+x_n)^{2+\delta} B_n^{-(2+\delta)} \sum_{i=1}^n E|X_i|^{2+\delta} I_{\{\varepsilon B_n/(1+x_n) < |X_i| \le B_n/(1+x_n)\}} \\ &\le (1+x_n)^{2+\delta} B_n^{-(2+\delta)} \sum_{i=1}^n E|X_i|^{2+\delta} I_{\{|X_i| > \varepsilon B_n/(1+x_n)\}} + O(1)\varepsilon^{1-\delta} \\ &= o(1) + O(1)\varepsilon^{1-\delta},\end{aligned}$$

since $\{|X_i|^{2+\delta},\ i \ge 1\}$ is uniformly integrable. This proves (5.1) by the arbitrariness of ε and hence the corollary. □

PROOF OF THEOREM 2.1. We use the same notation as before. We shall only prove the first part in (2.1) since the second part can be easily obtained by changing x to $-x$ in the first part. The main idea of the proof is to reduce the problem to that of a one-dimensional large deviation. It suffices to show that

$$P(S_n \ge x V_n) \ge \bigl(1 - \Phi(x)\bigr) e^{-A\Delta_{n,x}} \tag{5.2}$$

and

$$P(S_n \ge x V_n) \le \bigl(1 - \Phi(x)\bigr) e^{A\Delta_{n,x}} \tag{5.3}$$

for all $x > 0$ satisfying (2.2) and (2.3).

Let

$$b := b_x = x/B_n. \tag{5.4}$$

Observe that, by the Cauchy inequality,

$$x V_n \le (x^2 + b^2 V_n^2)/(2b).$$

Thus, we have

$$P(S_n \ge x V_n) \ge P\bigl(S_n \ge (x^2 + b^2 V_n^2)/(2b)\bigr) = P(2bS_n - b^2 V_n^2 \ge x^2).$$

Therefore, the lower bound (5.2) follows from the following proposition immediately.

PROPOSITION 5.1. *There exists an absolute constant $A > 1$ such that*

$$P(2bS_n - b^2 V_n^2 \ge x^2) = \bigl(1 - \Phi(x)\bigr) e^{O(1)\Delta_{n,x}} \tag{5.5}$$

for all $x > 0$ satisfying (2.2) *and* (2.3), *where $|O(1)| \le A$.*

As for the upper bound (5.3): when $0 < x \le 2$, this bound is a direct consequence of the Berry–Esseen bound given by (1.6). For $x > 2$, let

$$\tau := \tau_{n,x} = B_n/(1+x) \tag{5.6}$$

and define

$$\bar{X}_i = X_i I_{\{|X_i| \le \tau\}}, \qquad \bar{S}_n = \sum_{i=1}^n \bar{X}_i, \qquad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_i^2,$$

$$S_n^{(i)} = S_n - X_i, \qquad V_n^{(i)} = (V_n^2 - X_i^2)^{1/2}, \qquad \bar{B}_n^2 = \sum_{i=1}^n E\bar{X}_i^2.$$

Noting that for any $s, t \in R^1$, $c \ge 0$ and $x \ge 1$,

$$\begin{aligned} x\sqrt{c+t^2} &= \sqrt{(x^2-1)c + t^2 + c + (x^2-1)t^2} \\ &\ge \sqrt{(x^2-1)c + t^2 + 2t\sqrt{(x^2-1)c}} \\ &= t + \sqrt{(x^2-1)c}, \end{aligned}$$

we have

$$\{s + t \ge x\sqrt{c+t^2}\} \subset \{s \ge (x^2-1)^{1/2}\sqrt{c}\}. \tag{5.7}$$

Hence,

$$\begin{aligned} &P(S_n \ge xV_n) \\ &\quad \le P(\bar{S}_n \ge x\bar{V}_n) + P\Big(S_n \ge xV_n, \max_{1\le i\le n} |X_i| > \tau\Big) \\ &\quad \le P(\bar{S}_n \ge x\bar{V}_n) + \sum_{i=1}^n P(S_n \ge xV_n, |X_i| > \tau) \\ &\quad \le P(\bar{S}_n \ge x\bar{V}_n) + \sum_{i=1}^n P\big(S_n^{(i)} \ge (x^2-1)^{1/2} V_n^{(i)}, |X_i| > \tau\big) \\ &\quad \le P(\bar{S}_n \ge x\bar{V}_n) + \sum_{i=1}^n P\big(S_n^{(i)} \ge (x^2-1)^{1/2} V_n^{(i)}\big) P(|X_i| > \tau). \end{aligned} \tag{5.8}$$

By the inequality $(1+y)^{1/2} \ge 1 + y/2 - y^2$ for any $y \ge -1$, we have

$$\begin{aligned} P(\bar{S}_n \ge x\bar{V}_n) &= P\left(\bar{S}_n \ge x\left(\bar{B}_n^2 + \sum_{i=1}^n (\bar{X}_i^2 - E\bar{X}_i^2)\right)^{1/2}\right) \\ &\le P\left(\bar{S}_n \ge x\bar{B}_n\left\{1 + \frac{1}{2\bar{B}_n^2}\sum_{i=1}^n (\bar{X}_i^2 - E\bar{X}_i^2)\right.\right. \\ &\qquad\qquad \left.\left. - \frac{1}{\bar{B}_n^4}\left(\sum_{i=1}^n (\bar{X}_i^2 - E\bar{X}_i^2)\right)^2\right\}\right) \\ &:= K_n. \end{aligned} \tag{5.9}$$

Thus, the upper bound (5.3) follows from the next three propositions.

PROPOSITION 5.2. *There is an absolute constant A such that*

$$P\big(S_n^{(i)} \geq xV_n^{(i)}\big) \leq (1+x^{-1})\frac{1}{\sqrt{2\pi}x}\exp(-x^2/2 + A\Delta_{n,x}) \tag{5.10}$$

for any $x > 2$ satisfying (2.2) *and* (2.3).

PROPOSITION 5.3. *There exists an absolute constant A such that*

$$K_n \leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}} + Ae^{-3x^2} \tag{5.11}$$

for all $x > 2$ satisfying (2.2) *and* (2.3).

PROPOSITION 5.4. *There exists an absolute constant A such that*

$$K_n \leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}} + A\big(\Delta_{n,x}/(1+x)^2\big)^{4/3} \tag{5.12}$$

for $x > 2$ with $\Delta_{n,x}/(1+x)^2 \leq 1/128$.

In fact, Propositions 5.3 and 5.4 imply that

$$K_n \leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}} \tag{5.13}$$

for all $x > 2$ satisfying conditions (2.2) and (2.3). To see this, consider two cases. If $\Delta_{n,x}/(1+x)^2 \leq (1-\Phi(x))^3/128$, then by (5.12)

$$\begin{aligned}
K_n &\leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}\Big(1 + (1+x)^{-2}\Delta_{n,x}\big(\Delta_{n,x}/(1+x)^2\big)^{1/3}\big/\big(1-\Phi(x)\big)\Big) \\
&\leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}\big(1+\Delta_{n,x}/(1+x)^2\big) \\
&\leq \big(1-\Phi(x)\big)e^{2A\Delta_{n,x}}.
\end{aligned}$$

When $\Delta_{n,x}/(1+x)^2 > (1-\Phi(x))^3/128$, by (5.11)

$$\begin{aligned}
K_n &\leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}\big(1 + Ae^{-3x^2}\big/\big(1-\Phi(x)\big)\big) \\
&\leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}\big(1 + A\big(1-\Phi(x)\big)^3\big) \\
&\leq \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}\big(1 + 128A\Delta_{n,x}/(1+x)^2\big) \\
&\leq \big(1-\Phi(x)\big)e^{129A\Delta_{n,x}}
\end{aligned}$$

as desired.

On the other hand, for $x > 2$, we can use Proposition 5.2 and the fact that $(2\pi)^{-1/2}(x^{-1} - x^{-3})e^{-x^2/2} \le 1 - \Phi(x)$ for $x > 0$ to get

$$
\begin{aligned}
&P\big(S_n^{(i)} \ge (x^2-1)^{1/2} V_n^{(i)}\big)\\
&\quad\le \big(1 + (x^2-1)^{-1/2}\big)\frac{1}{\sqrt{2\pi}(x^2-1)^{1/2}}\exp(-x^2/2 + A\Delta_{n,x})\\
&\quad\le \frac{A}{\sqrt{2\pi}x}\exp(-x^2/2 + A\Delta_{n,x})\\
&\quad\le \frac{A}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp(-x^2/2 + A\Delta_{n,x})\\
&\quad\le A\big(1-\Phi(x)\big)\exp(A\Delta_{n,x}).
\end{aligned}
\tag{5.14}
$$

It follows from (5.8), (5.9), (5.13) and (5.14) that

$$
\begin{aligned}
P(S_n \ge xV_n) &\le P(\bar{S}_n \ge x\bar{V}_n) + \sum_{i=1}^n P\big(S_n^{(i)} \ge (x^2-1)^{1/2}V_n^{(i)}\big)P(|X_i| > \tau)\\
&\le \big(1-\Phi(x)\big)e^{A\Delta_{n,x}} + \sum_{i=1}^n A\big(1-\Phi(x)\big)\exp(A\Delta_{n,x})P(|X_i| > \tau)\\
&\le \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}\left(1 + A\sum_{i=1}^n P(|X_i| > \tau)\right)\\
&\le \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}\left(1 + A\sum_{i=1}^n \tau^{-2}EX_i^2 I(|X_i| > \tau)\right)\\
&\le \big(1-\Phi(x)\big)e^{A\Delta_{n,x}}(1 + A\Delta_{n,x})\\
&\le \big(1-\Phi(x)\big)e^{2A\Delta_{n,x}}.
\end{aligned}
$$

This completes the proof for the upper bound (5.3). □

6. Preliminary lemmas. We first provide some lemmas that will be used in the proofs of the propositions.

LEMMA 6.1. *Let X be a random variable with $EX = 0$ and $EX^2 < \infty$. Then, for any $0 < b < \infty$, $\lambda > 0$ and $\theta > 0$,*

$$
Ee^{\lambda bX - \theta(bX)^2} = 1 + (\lambda^2/2 - \theta)b^2EX^2 + O_{\lambda,\theta}\delta_b, \tag{6.1}
$$

where $\delta_b = b^2EX^2I_{\{|bX|>1\}} + b^3E|X|^3I_{\{|bX|\le 1\}}$ and $O_{\lambda,\theta}$ denotes a quantity that is bounded by a finite constant depending only on λ and θ. In (6.1), $|O_{\lambda,\theta}| \le \max(\lambda + |\lambda^2/2 - \theta| + e^{\lambda^2/(4\theta)},\ \lambda\theta + \theta^2/2 + (\lambda+\theta)^3e^{\lambda}/6)$.

This is Lemma 2.1 in Shao (1999). See the Appendix for a detailed proof.

LEMMA 6.2. *Let X be a random variable with $EX = 0$ and $EX^2 < \infty$. For $0 < b < \infty$, let $\xi := \xi_b = 2bX - (bX)^2$. Then, for $\lambda > 0$,*

$$Ee^{\lambda\xi} = 1 + (2\lambda^2 - \lambda)b^2EX^2 + O_{\lambda,0}\delta_b, \tag{6.2}$$

$$E\xi e^{\lambda\xi} = (4\lambda - 1)b^2EX^2 + O_{\lambda,1}\delta_b, \tag{6.3}$$

$$E\xi^2 e^{\lambda\xi} = 4b^2EX^2 + O_{\lambda,2}, \delta_b, \tag{6.4}$$

$$E|\xi|^3 e^{\lambda\xi} = O_{\lambda,3}\delta_b, \tag{6.5}$$

$$(E\xi e^{\lambda\xi})^2 = O_{\lambda,4}\delta_b, \tag{6.6}$$

where δ_b is defined as in Lemma 6.1,

$$|O_{\lambda,0}| \le \max\bigl(2\lambda + |2\lambda^2 - \lambda| + e^\lambda, 2.5\lambda^2 + 4\lambda^3 e^\lambda/3\bigr),$$
$$|O_{\lambda,1}| \le \max\bigl(2 + |4\lambda - 1| + \max(e^\lambda, e/\lambda), 5\lambda + 13.5\lambda^2 e^\lambda\bigr),$$
$$|O_{\lambda,2}| \le \max\bigl(4 + \max\bigl(e^\lambda, (e/(2\lambda))^2\bigr), 5 + 27\lambda e^\lambda\bigr),$$
$$|O_{\lambda,3}| \le 27e^\lambda,$$
$$|O_{\lambda,4}| \le 2\max\bigl(\bigl(\max(e^\lambda, e/\lambda) + 2\bigr)^2, (1 + 9\lambda e^\lambda)^2\bigr).$$

In particular, when $\lambda = 1/2$, $|O_{\lambda,0}| \le 2.65$, $|O_{\lambda,1}| \le 8.1$, $|O_{\lambda,2}| \le 28$, $|O_{\lambda,3}| \le 45$, $|O_{\lambda,4}| \le 150$, and

$$Ee^{\xi/2} = e^{O_5\delta_b}, \qquad \text{where } |O_5| \le 5.5. \tag{6.7}$$

PROOF. Proofs of (6.2)–(6.5) are similar to those in Lemma 2.2 in Shao (1999). See the Appendix. As for (6.6), we have

$$\begin{aligned} &|E\xi e^{\lambda\xi}|^2 \\ &\quad \le 2\bigl(\max(e^\lambda, e/\lambda) + 2\bigr)^2 b^2EX^2 I_{\{|bX|>1\}} + 2(1 + 9\lambda e^\lambda)^2\bigl(b^2EX^2 I_{\{|bX|\le 1\}}\bigr)^2 \\ &\quad \le 2\bigl(\max(e^\lambda, e/\lambda) + 2\bigr)^2 b^2EX^2 I_{\{|bX|>1\}} + 2(1 + 9\lambda e^\lambda)^2 b^3E|X|^3 I_{\{|bX|\le 1\}}. \end{aligned}$$

Next, we prove (6.7). Since (6.2) implies

$$Ee^{\xi/2} \le e^{2.65\delta_b},$$

it suffices to show that

$$Ee^{\xi/2} \ge e^{-5.5\delta_b}. \tag{6.8}$$

When $2.65\delta_b \le 0.8$ by (6.2), again,

$$Ee^{\xi/2} \ge 1 - 2.65\delta_b \ge \exp\bigl(2.65\delta_b\bigl(\ln(1 - 0.8)\bigr)/0.8\bigr) \ge e^{-5.4\delta_b}.$$

When $\delta_b > 0.8/2.65$ and if $E(bX)^2 I_{\{|bX|>1\}} > E(bX)^2/11$, then by the Jensen inequality,

$$Ee^{\xi/2} \ge e^{E\xi/2} = e^{-E(bX)^2/2} \ge e^{-5.5\delta_b}.$$

If $E(bX)^2 I_{\{|bX|>1\}} \le E(bX)^2/11$, by the fact that $EY^2 I_{\{Y\le 1\}} \le EY^2 P(Y \le 1)$ for any non-negative random variable Y, we get

$$\begin{aligned}
Ee^{\xi/2} &\ge Ee^{\xi/2} I_{\{|bX|\le 1\}} \ge e^{-1.5} P(|bX| \le 1) \\
&\ge e^{-1.5}\big(E(bX)^2 I_{\{|bX|\le 1\}}/E(bX)^2\big) \ge e^{-1.5} 10/11 \\
&\ge e^{-5.5(0.8/2.65)} \ge e^{-5.5\delta_b}.
\end{aligned}$$

This proves (6.8). $\square$

LEMMA 6.3. *Let $\{\xi_i, 1 \le i \le n\}$ be a sequence of independent random variables with $Ee^{h\xi_i} < \infty$ for $0 < h < H$, where $H > 0$. For $0 < \lambda < H$, put*

$$m(\lambda) = \sum_{i=1}^n E\xi_i e^{\lambda\xi_i}/Ee^{\lambda\xi_i}, \qquad \sigma^2(\lambda) = \sum_{i=1}^n \big(E\xi_i^2 e^{\lambda\xi_i}/Ee^{\lambda\xi_i} - (E\xi_i e^{\lambda\xi_i}/Ee^{\lambda\xi_i})^2\big).$$

Then

$$P\left(\sum_{i=1}^n \xi_i \ge y\right) \ge \frac{3}{4}\left(\prod_{i=1}^n Ee^{\lambda\xi_i}\right) e^{-\lambda m(\lambda) - 2\lambda\sigma(\lambda)} \tag{6.9}$$

provided that

$$0 < \lambda < H \quad \text{and} \quad m(\lambda) \ge y + 2\sigma(\lambda). \tag{6.10}$$

The proof follows from Lemma 4.1 of Shao (1997) with a simple modification. See the Appendix.

LEMMA 6.4. *Let $\{\xi_i, 1 \le i \le n\}$ be a sequence of independent random variables with $E\xi_i = 0$ and $E\xi_i^2 < \infty$. Then, for $a > 0$,*

$$P\left(\left|\sum_{i=1}^n \xi_i\right| \ge a\left(4D_n + \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}\right)\right) \le 8e^{-a^2/2}, \tag{6.11}$$

where $D_n = (\sum_{i=1}^n E\xi_i^2)^{1/2}$.

The proof is a special case of Lemma 3.1 in He and Shao (2000), but with some specified constants. See the Appendix for a detailed proof.

7. Proof of Proposition 5.1. Throughout the rest of the paper, $O(1)$ will denote a quantity bounded by an absolute constant.

We first consider the case of $0 < x \le 2$. Let $\tau := \tau_{n,x} = B_n/(1+x)$ and $Y_i = (2bX_i - (bX_i)^2)I_{\{|X_i|\le\tau\}}$ with $b = x/B_n$. Then,

$$\begin{aligned}
EY_i &= -b^2EX_i^2 + b^2EX_i^2I_{\{|X_i|>\tau\}} - 2bEX_iI_{\{|X_i|>\tau\}}\\
&= -b^2EX_i^2 + O(1)b\tau^{-1}EX_i^2I_{\{|X_i|>\tau\}},\\
EY_i^2 &= 4b^2EX_i^2 - 4b^2EX_i^2I_{\{|X_i|>\tau\}} - 4b^3EX_i^3I_{\{|X_i|\le\tau\}} + b^4EX_i^4I_{\{|X_i|\le\tau\}}\\
&= 4b^2EX_i^2 + O(1)b^2\big(EX_i^2I_{\{|X_i|>\tau\}} + \tau^{-1}E|X_i|^3I_{\{|X_i|\le\tau\}}\big),\\
E|Y_i|^3 &= O(1)b^3E|X_i|^3I_{\{|X_i|\le\tau\}},\\
\mathrm{Var}(Y_i) &= 4b^2EX_i^2 + O(1)b^2\big(EX_i^2I_{\{|X_i|>\tau\}} + \tau^{-1}E|X_i|^3I_{\{|X_i|\le\tau\}}\big)\\
&\quad - \big(-b^2EX_i^2I_{\{|X_i|\le\tau\}} - 2bEX_iI_{\{|X_i|>\tau\}}\big)^2\\
&= 4b^2EX_i^2 + O(1)b^2\big(EX_i^2I_{\{|X_i|>\tau\}} + \tau^{-1}E|X_i|^3I_{\{|X_i|\le\tau\}}\big)\\
&\quad + O(1)\big((b^2EX_i^2I_{\{|X_i|\le\tau\}})^2 + 4b^2EX_i^2I_{\{|X_i|>\tau\}}\big)\\
&= 4b^2EX_i^2 + O(1)b^2\big(EX_i^2I_{\{|X_i|>\tau\}} + \tau^{-1}E|X_i|^3I_{\{|X_i|\le\tau\}}\big).
\end{aligned}$$

Thus

$$x^2 - \sum_{i=1}^n EY_i = 2x^2 + O(1)b\tau^{-1}\sum_{i=1}^n EX_i^2I_{\{|X_i|>\tau\}} \tag{7.1}$$

and, by (2.3),

$$\begin{aligned}
\sum_{i=1}^n \mathrm{Var}(Y_i) &= 4x^2 + O(1)b^2\sum_{i=1}^n\big(EX_i^2I_{\{|X_i|>\tau\}} + \tau^{-1}E|X_i|^3I_{\{|X_i|\le\tau\}}\big)\\
&= 4x^2\big(1 + O(1)(1+x)^{-2}\Delta_{n,x}\big) \ge 3x^2.
\end{aligned}\tag{7.2}$$

It is easy to see that

$$\begin{aligned}
&\left|P(2bS_n - b^2V_n^2 \ge x^2) - P\left(\sum_{i=1}^n Y_i \ge x^2\right)\right|\\
&\qquad \le P\left(\max_{1\le i\le n}|X_i| > \tau\right) \le \sum_{i=1}^n \tau^{-2}EX_i^2I_{\{|bX_i|>\tau\}}.
\end{aligned}\tag{7.3}$$

Let $a = \sum_{i=1}^n E(Y_i)$ and $\sigma^2 = \sum_{i=1}^n \mathrm{Var}(Y_i)$. By (7.1) and (7.2), we have

$$(x^2 - a)/\sigma = x + O(1)(1+x)^{-2}\Delta_{n,x}$$

and hence by the Berry–Esseen inequality we have

$$\begin{aligned} P\left(\sum_{i=1}^n Y_i \ge x^2\right) &= 1 - \Phi\big((x^2 - a)/\sigma\big) + O(1)\sigma^{-3}\sum_{i=1}^n E|Y_i|^3 \\ &= 1 - \Phi(x) + O(1)(1+x)^{-2}\Delta_{n,x}, \end{aligned}$$

which, together with (7.3), yields

$$P(2bS_n - b^2V_n^2 \ge x^2) = 1 - \Phi(x) + O(1)(1+x)^{-2}\Delta_{n,x} = \big(1-\Phi(x)\big)e^{O(1)\Delta_{n,x}}$$

as desired.

We next consider the case of $x > 2$. It suffices to show that

$$P(2bS_n - b^2V_n^2 \ge x^2) \le \big(1-\Phi(x)\big)e^{A\Delta_{n,x}} \tag{7.4}$$

and

$$P(2bS_n - b^2V_n^2 \ge x^2) \ge \big(1-\Phi(x)\big)e^{-A\Delta_{n,x}}. \tag{7.5}$$

Let $\xi_i = 2bX_i - (bX_i)^2$ and define

$$V_i(u) = Ee^{\xi_i/2}I_{\{\xi_i\le u\}}/Ee^{\xi_i/2}.$$

Let $\eta_1, \eta_2, \dots, \eta_n$ be independent random variables with η_i having the distribution function $V_i(u)$. Set $m_n = \sum_{i=1}^n E\eta_i$, $\sigma_n = (\sum_{i=1}^n \mathrm{Var}(\eta_i))^{1/2}$ and

$$G_n(t) = P\left(\frac{\sum_{i=1}^n(\eta_i - E\eta_i)}{\sigma_n} \le t\right), \qquad \varepsilon_n = \frac{x^2 - m_n}{\sigma_n}.$$

It follows from Lemma 6.2 that

$$Ee^{\xi_i/2} = e^{O(1)\delta_{b,i}}, \tag{7.6}$$

$$E\xi_i e^{\xi_i/2} = b^2EX_i^2 + O(1)\delta_{b,i}, \tag{7.7}$$

$$E\xi_i^2 e^{\xi_i/2} = 4b^2EX_i^2 + O(1)\delta_{b,i}, \tag{7.8}$$

$$E|\xi_i|^3 e^{\xi_i/2} = O(1)\delta_{b,i}, \tag{7.9}$$

$$\big(E\xi_i e^{\xi_i/2}\big)^2 = O(1)\delta_{b,i}, \tag{7.10}$$

where $\delta_{b,i} = b^2EX_i^2I_{\{|bX_i|>1\}} + b^3E|X_i|^3I_{\{|bX_i|\le 1\}}$. Noting that (2.2) implies $b^2EX_i^2 \le 1$ for $1 \le i \le n$ and that $\sum_{i=1}^n b^2EX_i^2I_{\{|bX_i|>1\}} + b^3E|X_i|^3I_{\{|bX_i|\le 1\}} \le \Delta_{n,x}$, we have

$$m_n = \sum_{i=1}^n E\eta_i = \sum_{i=1}^n E\big(\xi_i e^{\xi_i/2}\big)/Ee^{\xi_i/2} = x^2 + O(1)\Delta_{n,x}, \tag{7.11}$$

$$\begin{aligned} \sigma_n^2 &= \sum_{i=1}^n \mathrm{Var}(\eta_i) = \sum_{i=1}^n \big\{E\big(\xi_i^2 e^{\xi_i/2}\big)/Ee^{\xi_i/2} - (E\eta_i)^2\big\} \\ &= 4x^2 + O(1)\Delta_{n,x} \ge x^2 \end{aligned} \tag{7.12}$$

and

$$v_n := \sum_{i=1}^n E|\eta_i|^3 = \sum_{i=1}^n E\big(|\xi_i|^3 e^{\xi_i/2}\big)/Ee^{\xi_i/2} = O(1)\Delta_{n,x}, \tag{7.13}$$

where the $O(1)$'s above are bounded by an absolute constant, say c_1. The inequality in (7.12) holds provided that A in (2.3) is greater than $4c_1$.

By the conjugate method [cf. (4.9) of Petrov (1965)], we have

$$\begin{aligned}
&P(2bS_n - b^2V_n^2 \ge x^2)\\
&\quad = P\left(\sum_{i=1}^n \xi_i \ge x^2\right)\\
&\quad = \left(\prod_{i=1}^n Ee^{\xi_i/2}\right)\int_{x^2}^{\infty} e^{-u/2}\,dP\left(\sum_{i=1}^n \eta_i \le u\right)\\
&\quad = \left(\prod_{i=1}^n Ee^{\xi_i/2}\right)e^{-m_n/2}\int_{\varepsilon_n}^{\infty} e^{-t\sigma_n/2}\,dG_n(t) \qquad (7.14)\\
&\quad \le \left(\prod_{i=1}^n Ee^{\xi_i/2}\right)e^{-m_n/2}\int_{-c_1\Delta_{n,x}/\sigma_n}^{\infty} e^{-t\sigma_n/2}\,dG_n(t)\\
&\quad := \left(\prod_{i=1}^n Ee^{\xi_i/2}\right)e^{-m_n/2}\{L_{n,1} + R_{n,,1}\}\\
&\quad := K_{n,1},
\end{aligned}$$

where

$$\begin{aligned}
L_{n,1} &= \int_{-c_1\Delta_{n,x}/\sigma_n}^{\infty} e^{-t\sigma_n/2}\,d\Phi(t),\\
R_{n,1} &= \int_{-c_1\Delta_{n,x}/\sigma_n}^{\infty} e^{-t\sigma_n/2}\,d\big(G_n(t) - \Phi(t)\big).
\end{aligned}$$

By (7.11) and (7.12),

$$-t\sigma_n/2 \le c_1\Delta_{n,x}/2$$

for $t > -c_1\Delta_{n,x}/\sigma_n$. Thus, by integration by parts, the Berry–Esseen inequality and (7.12) and (7.13),

$$|R_{n,1}| \le 16(v_n/\sigma_n^3)\exp(c_1\Delta_{n,x}/2) \le c_2\Delta_{n,x}x^{-3}\exp(c_1\Delta_{n,x}/2). \tag{7.15}$$

As for $L_{n,1}$, we have

$$
\begin{aligned}
L_{n,1} &= \frac{\exp(\sigma_n^2/8)}{\sqrt{2\pi}} \int_{-c_1\Delta_{n,x}/\sigma_n+\sigma_n/2}^{\infty} e^{-t^2/2}\,dt \\
&= \exp(\sigma_n^2/8)\bigl(1-\Phi(x)\bigr) + \frac{\exp(\sigma_n^2/8)}{\sqrt{2\pi}} \int_{-c_1\Delta_{n,x}/\sigma_n+\sigma_n/2}^{x} e^{-t^2/2}\,dt \\
&\le \exp(\sigma_n^2/8)\bigl(1-\Phi(x)\bigr) \\
&\quad + \exp(\sigma_n^2/8)\exp\bigl(-(\sigma_n/2-c_1\Delta_{n,x}/\sigma_n)^2/2\bigr) \\
&\qquad \times |\sigma_n/2 - x - c_1\Delta_{n,x}/\sigma_n| \\
&\le \exp(\sigma_n^2/8)\bigl(1-\Phi(x)\bigr) + c_3(\Delta_{n,x}/x)\exp(c_1\Delta_{n,x}/2).
\end{aligned}
\tag{7.16}
$$

Therefore, by (7.6), (7.11), (7.15) and (7.16)

$$
\begin{aligned}
K_{n,1} &\le \exp(1.5c_1\Delta_{n,x} - x^2/2)(L_{n,1}+R_{n,1}) \\
&\le \exp(1.5c_1\Delta_{n,x} - x^2/2) \\
&\quad \times \bigl(\exp(x^2/2 + c_1\Delta_{n,x})\bigl(1-\Phi(x)\bigr) \\
&\qquad + c_4(\Delta_{n,x}/x)\exp(c_1\Delta_{n,x}/2)\bigr) \\
&= \bigl(1-\Phi(x)\bigr)\exp(2c_1\Delta_{n,x})\left(1 + c_4\Delta_{n,x}\frac{e^{-x^2/2}}{x(1-\Phi(x))}\right) \\
&\le \bigl(1-\Phi(x)\bigr)\exp(c_5\Delta_{n,x}).
\end{aligned}
\tag{7.17}
$$

This proves (7.4).

Next we prove (7.5) by considering the following two cases.

CASE 1. Assume

$$\Delta_{n,x} \le x. \tag{7.18}$$

Similar to (7.14), we have

$$P(2bS_n - b^2V_n^2 \ge x^2) \ge \left(\prod_{i=1}^n Ee^{\xi_i/2}\right) e^{-m_n/2}\{L_{n,2}+R_{n,2}\} := K_{n,2}, \tag{7.19}$$

where

$$L_{n,2} = \int_{c_1\Delta_{n,x}/\sigma_n}^{\infty} e^{-t\sigma_n/2}\,d\Phi(t)$$

and

$$R_{n,2} = \int_{c_1\Delta_{n,x}/\sigma_n}^{\infty} e^{-t\sigma_n/2}\,d\bigl(G_n(t)-\Phi(t)\bigr).$$

Following the proof of (7.15), we have

$$|R_{n,2}| \le c_2 \Delta_{n,x} x^{-3} \exp(-c_1 \Delta_{n,x}/2). \tag{7.20}$$

Observe that

$$\begin{aligned} L_{n,2} &= \frac{\exp(\sigma_n^2/8)}{\sqrt{2\pi}} \int_{c_1\Delta_{n,x}/\sigma_n + \sigma_n/2}^{\infty} e^{-t^2/2}\,dt \\ &\ge \frac{\exp(\sigma_n^2/8)}{\sqrt{2\pi}} \frac{\exp(-(c_1\Delta_{n,x}/\sigma_n + \sigma_n/2)^2/2)}{1 + c_1\Delta_{n,x}/\sigma_n + \sigma_n/2} \\ &\ge c_5 \exp\bigl(-c_1\Delta_{n,x}/2 - (c_1\Delta_{n,x}/\sigma_n)^2/2\bigr)/x \\ &\ge c_6 \exp(-c_1\Delta_{n,x}/2)/x \end{aligned} \tag{7.21}$$

by (7.18). Similarly to (7.16),

$$\begin{aligned} L_{n,2} &= \exp(\sigma_n^2/8)\bigl(1-\Phi(x)\bigr) \\ &\quad - \frac{\exp(\sigma_n^2/8)}{\sqrt{2\pi}} \int_x^{c_1\Delta_{n,x}/\sigma_n + \sigma_n/2} e^{-t^2/2}\,dt \\ &\ge \exp(\sigma_n^2/8)\bigl(1-\Phi(x)\bigr) \\ &\quad - \exp(\sigma_n^2/8)\exp(-x^2/2)|\sigma_n/2 - x - c_1\Delta_{n,x}/\sigma_n| \\ &\ge \exp(\sigma_n^2/8)\bigl(1-\Phi(x)\bigr)\biggl(1 - c_3\Delta_{n,x}\frac{\exp(-x^2/2)}{x(1-\Phi(x))}\biggr) \\ &\ge \exp(\sigma_n^2/8)\bigl(1-\Phi(x)\bigr)(1 - 2c_3\Delta_{n,x}). \end{aligned} \tag{7.22}$$

It follows from (7.21) for $c_3\Delta_{n,x} > 1/4$ and from (7.22) for $c_3\Delta_{n,x} \le 1/4$ that

$$L_{n,2} \ge \exp(x^2/2 - c_5\Delta_{n,x})\bigl(1-\Phi(x)\bigr). \tag{7.23}$$

Now (7.20), (7.21) and (7.23) yield

$$\begin{aligned} K_{n,2} &\ge \exp(-1.5c_1\Delta_{n,x} - x^2/2)L_{n,2}(1 - |R_{n,2}|/L_{n,2}) \\ &\ge \exp(-c_6\Delta_{n,x})\bigl(1-\Phi(x)\bigr)(1 - c_7\Delta_{n,x}/x^2) \\ &\ge \exp(-c_8\Delta_{n,x})\bigl(1-\Phi(x)\bigr) \end{aligned} \tag{7.24}$$

provided $\Delta_{n,x} \le x^2/(2c_7)$. This proves (7.5) under condition (7.18).

CASE 2. Assume

$$\Delta_{n,x} > x. \tag{7.25}$$

By Lemma 6.2, for any $0.5 \le \lambda \le 0.55$,

$$Ee^{\lambda\xi_i} = 1 + (2\lambda^2 - \lambda)b^2 EX_i^2 + O(1)\delta_{b,i}, \tag{7.26}$$

$$E\xi_i e^{\xi_i/2} = (4\lambda - 1)b^2 EX_i^2 + O(1)\delta_{b,i}, \tag{7.27}$$

$$E\xi_i^2 e^{\xi_i/2} = 4b^2 EX_i^2 + O(1)\delta_{b,i}, \tag{7.28}$$

where $|O(1)| \le c_8 = 35$ for all $0.5 \le \lambda \le 0.55$. Let

$$\lambda = 1/2 + c_9\Delta_{n,x}/x^2,$$

where $c_9 = 6 + 7c_8$. Choose $A > 40c_9$ in (2.3) so that $0.5 \le \lambda \le 0.55$. We now apply Lemma 6.3 to estimate the lower bound (7.5). Let $m(\lambda)$ and $\sigma(\lambda)$ be as in Lemma 6.3. Define

$$G_1 = \{1 \le i \le n : (4\lambda - 1)b^2 EX_i^2 \ge c_8\delta_{b,i}\}$$

and

$$G_2 = \{1 \le i \le n : (4\lambda - 1)b^2 EX_i^2 < c_8\delta_{b,i}\}.$$

By (7.26)–(7.28), (7.6) and the fact that $Ee^{\lambda\xi_i} \ge \max(e^{-\lambda b^2 EX_i^2}, (Ee^{\xi_i/2})^{\lambda/2})$, we have

$$\begin{aligned}
m(\lambda) &\ge \sum_{i=1}^n \frac{(4\lambda-1)b^2EX_i^2 - c_8\delta_{b,i}}{Ee^{\lambda\xi_i}} \\
&= \sum_{i\in G_1} \bigl((4\lambda-1)b^2EX_i^2 - c_8\delta_{b,i}\bigr) \\
&\quad + \sum_{i\in G_1} \bigl((4\lambda-1)b^2EX_i^2 - c_8\delta_{b,i}\bigr)\frac{(1 - Ee^{\lambda\xi_i})}{Ee^{\lambda\xi_i}} \\
&\quad + \sum_{i\in G_2} \frac{(4\lambda-1)b^2EX_i^2 - c_8\delta_{b,i}}{Ee^{\lambda\xi_i}} \\
&\ge \sum_{i\in G_1} \bigl((4\lambda-1)b^2EX_i^2 - c_8\delta_{b,i}\bigr) \\
&\quad + \sum_{i\in G_2} \bigl((4\lambda-1)b^2EX_i^2 - 2c_8\delta_{b,i}\bigr) \\
&\quad - \sum_{i\in G_1} (4\lambda-1)b^2EX_i^2\bigl((2\lambda^2-\lambda)b^2EX_i^2 + c_8\delta_{b,i}\bigr)e^{\lambda} \\
&\quad - \sum_{i\in G_2} c_8\delta_{b,i}e^{\lambda} \\
&\ge (4\lambda-1)x^2 - (4\lambda-1)(2\lambda^2-\lambda)e^{\lambda}x^2 - 7c_8\Delta_{n,x} \\
&\ge (1 + c_9\Delta_{n,x}/x^2)x^2 - 7c_8\Delta_{n,x} \\
&= x^2 + (c_9 - 7c_8)\Delta_{n,x}.
\end{aligned} \tag{7.29}$$

Similarly, we obtain from (7.26) and (7.28) that

$$
\begin{aligned}
m(\lambda) &\le \sum_{i=1}^{n} \frac{(4\lambda-1)b^2EX_i^2 + c_8\delta_{b,i}}{(Ee^{\xi_i/2})^{\lambda/2}} \\
&\le (4\lambda-1)x^2 + c_{10}\Delta_{n,x} = x^2 + c_{11}\Delta_{n,x}
\end{aligned}
\tag{7.30}
$$

and

$$
\begin{aligned}
\sigma^2(\lambda) &\le \sum_{i=1}^{n} \frac{E\xi_i^2 e^{\lambda\xi_i}}{e^{\lambda E\xi_i}} \\
&\le \sum_{i=1}^{n} (4b^2EX_i^2 + c_8\delta_{b,i})e^{\lambda} \le 8x^2 + 2c_8\Delta_{n,x} \le 9x^2
\end{aligned}
\tag{7.31}
$$

for $A \ge 40c_9$ in (2.3). Therefore, (6.10) is satisfied by (7.29) and (7.31). Thus, by Lemma 6.3, (7.6), (7.30), (7.31) and (7.25)

$$
\begin{aligned}
P(2bS_n - b^2V_n^2 \ge x^2) &\ge \tfrac{3}{4}\left(\prod_{i=1}^{n}(Ee^{\xi_i/2})^{\lambda/2}\right)e^{-\lambda m(\lambda)-2\lambda\sigma(\lambda)} \\
&\ge \tfrac{3}{4}\exp(-x^2/2 - c_{12}\Delta_{n,x} - 5x) \\
&\ge \bigl(1-\Phi(x)\bigr)\exp(-c_{12}\Delta_{n,x} - 8x) \\
&\ge \bigl(1-\Phi(x)\bigr)\exp(-c_{13}\Delta_{n,x}).
\end{aligned}
\tag{7.32}
$$

This proves (7.5) for the case of (7.25).

8. Proof of Proposition 5.2. The proof of Proposition 5.2 follows from the next three lemmas.

LEMMA 8.1. *For $x \ge 2$, we have*

$$P(S_n \ge xV_n,\, V_n^2 \ge 9B_n^2) \le 2\exp(-x^2 + A\Delta_{n,x}).$$

PROOF. Let $\hat{S}_n = \sum_{i=1}^{n} X_i I_{\{bX_i \le A_0\}}$, where $b = x/B_n$ and A_0 is an absolute constant to be determined later. Observe that

$$
\begin{aligned}
&P(S_n \ge xV_n,\, V_n^2 \ge 9B_n^2) \\
&\quad\le P(\hat{S}_n \ge xV_n/2,\, V_n^2 \ge 9B_n^2) + P\left(\sum_{i=1}^{n} X_i I_{\{bX_i > A_0\}} \ge xV_n/2\right) \\
&\quad\le P(\hat{S}_n \ge 3xB_n/2) + P\left(\sum_{i=1}^{n} I_{\{|bX_i| > A_0\}} \ge x^2/4\right) \\
&\quad:= J_1(n) + J_2(n).
\end{aligned}
\tag{8.1}
$$

Since $E(bX_i I_{\{bX_i \le A_0\}}) = -E(bX_i I_{\{bX_i > A_0\}}) \le 0$ for every i and

$$e^s \le 1 + s + s^2/2 + \max(s^3, 0)e^s/6,$$

we have

$$\begin{aligned}
J_1(n) &= P\biggl(\biggl(\frac{3x}{2B_n}\biggr)\hat{S}_n \ge \frac{9x^2}{4}\biggr)\\
&\le \exp\biggl(-\frac{9x^2}{4}\biggr) E\exp\biggl\{\frac{3}{2}b\hat{S}_n\biggr\}\\
&\le \exp\biggl(-\frac{9x^2}{4}\biggr)\prod_{i=1}^n \biggl\{1 + \frac{3}{2}E\bigl(bX_i I_{\{bX_i \le A_0\}}\bigr)\\
&\qquad\qquad + \frac{9}{8}E(bX_i)^2 + \frac{27e^{3A_0/2}}{48}E|bX_i|^3 I_{\{|bX_i| \le A_0\}}\biggr\}\\
&\le \exp\biggl(-\frac{9x^2}{4}\biggr)\prod_{i=1}^n \biggl\{1 + \frac{9}{8}E(bX_i)^2 + O(1)E|bX_i|^3 I_{\{|bX_i| \le A_0\}}\biggr\}\\
&\le \exp\biggl(-\frac{9x^2}{4}\biggr)\exp\biggl\{\frac{9}{8}\sum_{i=1}^n E(bX_i)^2 + O(1)\sum_{i=1}^n E|bX_i|^3 I_{\{|bX_i| \le A_0\}}\biggr\} \qquad (8.2)\\
&= \exp\biggl(-\frac{9x^2}{8}\biggr)\exp\biggl\{O(1)\sum_{i=1}^n E|bX_i|^3 I_{\{|bX_i| \le x/(1+x)\}}\\
&\qquad\qquad + O(1)\sum_{i=1}^n E|bX_i|^3 I_{\{x/(1+x) < |bX_i| \le A_0\}}\biggr\}\\
&\le \exp\biggl(-\frac{9x^2}{8}\biggr)\exp\biggl\{\frac{O(1)x^3}{B_n^3}\sum_{i=1}^n E|X_i|^3 I_{\{|X_i| \le B_n/(1+x)\}}\\
&\qquad\qquad + \frac{O(1)x^2}{B_n^2}\sum_{i=1}^n E|X_i|^2 I_{\{|X_i| > B_n/(1+x)\}}\biggr\}\\
&\le \exp\biggl(-\frac{9x^2}{8} + O(1)\Delta_{n,x}\biggr).
\end{aligned}$$

As for $J_2(n)$, let $Y_i = I_{\{|bX_i| > A_0\}}$. For $t > 0$ we have, with the help of $e^s \le 1 + \max(0, s)e^s$ and the Chebyshev inequality,

$$\begin{aligned}
J_2(n) &= P\biggl(\sum_{i=1}^n tY_i \ge tx^2/4\biggr)\\
&\le e^{-tx^2/4}\prod_{i=1}^n Ee^{tY_i} \qquad (8.3)
\end{aligned}$$

$$
\begin{aligned}
&\le e^{-tx^2/4} \prod_{i=1}^{n} [1 + e^t P(|bX_i| > A_0)] \\
&\le e^{-tx^2/4} \exp\left\{ e^t \sum_{i=1}^{n} b^2 E X_i^2 / A_0^2 \right\} \\
&= \exp(-tx^2/4 + x^2 e^t / A_0^2) \\
&\le \exp(-x^2)
\end{aligned}
$$

if we choose $t = 5$ and $A_0 = 30$, for instance.

The lemma is thus proved by combining (8.1)–(8.3). $\square$

LEMMA 8.2. *There is an absolute constant A such that*

$$
P(S_n \ge x V_n) \le (1 + x^{-1}) \frac{1}{\sqrt{2\pi} x} \exp(-x^2/2 + A\Delta_{n,x}) \tag{8.4}
$$

for any $x > 2$ satisfying (2.2) *and* (2.3).

PROOF. For $0 < \epsilon < 1/2$, note that

$$
\begin{aligned}
\{S_n \ge x V_n\} &\subset \left\{ S_n \ge \frac{1}{2}\left(b V_n^2 + \frac{x^2 - \epsilon^2}{b} \right) \right\} \\
&\qquad \cup \left\{ S_n \ge x V_n,\ S_n < \frac{1}{2}\left(b V_n^2 + \frac{x^2 - \epsilon^2}{b} \right) \right\} \\
&\subset \{2bS_n - b^2 V_n^2 \ge x^2 - \epsilon^2\} \\
&\qquad \cup \{S_n \ge x V_n,\ 2xbV_n < b^2 V_n^2 + x^2 - \epsilon^2\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\{S_n \ge x V_n,\ 2xbV_n < b^2 V_n^2 + x^2 - \epsilon^2\} \\
&\quad = \{S_n \ge x V_n,\ |bV_n - x| > \epsilon\} \\
&\quad \subset \{S_n \ge x V_n,\ |b^2 V_n^2 - x^2| > \epsilon x\} \\
&\quad \subset \{S_n \ge x V_n,\ V_n^2 \ge 9 B_n^2\} \\
&\qquad \cup \{S_n \ge x V_n,\ x^2 + \epsilon x < b^2 V_n^2 < 9x^2\} \\
&\qquad \cup \{S_n \ge x V_n,\ b^2 V_n^2 < x^2 - \epsilon x\}.
\end{aligned}
$$

Using Proposition (5.1) and the well-known inequality $1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x > 0$, we get

$$
P(2bS_n - b^2 V_n^2 \ge x^2 - \epsilon^2) \le \frac{1}{\sqrt{2\pi} x} \exp(-x^2/2 + \epsilon^2/2 + A\Delta_{n,x}) \tag{8.5}
$$

for $x \geq 2$.

Set $B_1 = \{(s,t) : s \geq x\sqrt{t}, x^2 + \epsilon x < t < 9x^2\}$. Then,

$$\begin{aligned}
P(S_n &\geq xV_n,\ x^2 + \epsilon x < b^2V_n^2 < 9x^2) \\
&= P\big((bS_n, b^2V_n^2) \in B_1\big) \\
&\leq E\exp\bigg(x^{-1}(x^2 + \epsilon x)^{1/2}(bS_n - b^2V_n^2/9) \\
&\qquad\qquad - \inf_{(s,t)\in B_1} x^{-1}(x^2 + \epsilon x)^{1/2}(s - t/9)\bigg) \\
&= \prod_{i=1}^n E\exp\big(x^{-1}(x^2 + \epsilon x)^{1/2}(bX_i - b^2X_i^2/9)\big) \\
&\quad \times \exp\Big(-x^{-1}(x^2 + \epsilon x)^{1/2}\{x(x^2 + \epsilon x)^{1/2} - (x^2 + \epsilon x)/9\}\Big) \\
&\leq \exp\left(\left\{\frac{x^2 + \epsilon x}{2x^2} - \frac{(x^2 + \epsilon x)^{1/2}}{9x}\right\}b^2B_n^2 + O(1)\sum_{i=1}^n \delta_{b,i}\right) \qquad \text{[by (6.1)]} \\
&\quad \times \exp\Big(-x^{-1}(x^2 + \epsilon x)^{1/2}\{x(x^2 + \epsilon x)^{1/2} - (x^2 + \epsilon x)/9\}\Big) \\
&\leq \exp\big(-x^2/2 - \epsilon x/4 + O(1)\Delta_{n,x}\big).
\end{aligned}$$

Similarly, letting $B_2 = \{(s,t) : s \geq x\sqrt{t}, 0 \leq t < x^2 - \epsilon x\}$ yields

$$\begin{aligned}
P(S_n &\geq xV_n,\ V_n^2 \leq x^2 - \epsilon x) \\
&= P\big((bS_n, b^2V_n^2) \in B_2\big) \\
&\leq E\exp\bigg(x^{-1}(x^2 - \epsilon x)^{1/2}(bS_n - 2b^2V_n^2) \\
&\qquad\qquad - \inf_{(s,t)\in B_2} x^{-1}(x^2 + \epsilon x)^{1/2}(s - 2t)\bigg) \\
&= \prod_{i=1}^n E\exp\big(x^{-1}(x^2 - \epsilon x)^{1/2}(bX_i - 2b^2X_i^2)\big) \\
&\quad \times \exp\Big(-x^{-1}(x^2 - \epsilon x)^{1/2}\{x(x^2 - \epsilon x)^{1/2} - 2(x^2 - \epsilon x)\}\Big) \\
&\leq \exp\left(\left\{\frac{x^2 - \epsilon x}{2x^2} - \frac{2(x^2 - \epsilon x)^{1/2}}{x}\right\}b^2B_n^2 + O(1)\sum_{i=1}^n \delta_{b,i}\right) \qquad \text{[by (6.1)]} \\
&\quad \times \exp\Big(-x^{-1}(x^2 - \epsilon x)^{1/2}\{x(x^2 - \epsilon x)^{1/2} - 2(x^2 - \epsilon x)\}\Big) \\
&\leq \exp\big(-x^2/2 - \epsilon x/4 + O(1)\Delta_{n,x}\big).
\end{aligned}$$

Now letting $\epsilon = 1/x^{1/2}$, the term $e^{\epsilon^2/2}$ in (8.5) can be bounded by

$$e^{\epsilon^2/2} = e^{1/(2x)} \le 1 + \frac{1}{2x} e^{1/(2x)} < 1 + \frac{1}{x} \qquad \text{for } x \ge 2.$$

Then inequality (8.4) follows from all the above inequalities. $\square$

LEMMA 8.3. *Let $B_{n,k}^2 = B_n^2 - EX_k^2$ and*

$$\Delta_{n,x}^{(k)} = \frac{(1+x)^2}{B_{n,k}^2} \sum_{i=1, i\ne k}^n EX_i^2 I_{\{|X_i| > B_{n,k}/(1+x)\}} + \frac{(x+1)^3}{B_{n,k}^3} \sum_{i=1, i\ne k}^n E|X_i|^3 I_{\{|X_i| \le B_{n,k}/(1+x)\}}.$$

Then

$$\max_{1 \le i \le n} \Delta_{n,x}^{(i)} \le 8 \Delta_{n,x}$$

for any $x > 2$ satisfying (2.2).

PROOF. It follows from (2.2) that

$$\max_{1 \le i \le n} \left\{ \frac{EX_k^2}{B_n^2} \right\} \le \frac{1}{x^2} < \frac{1}{4}.$$

Therefore, for any $1 \le k \le n$,

$$\frac{1}{B_{n,k}^2} = \frac{1}{B_n^2} \frac{1}{(1 - EX_k^2/B_n^2)} < \frac{4}{3B_n^2}.$$

As a result, we get

$$\begin{aligned}
\Delta_{n,x}^{(k)} &= \frac{(1+x)^2}{B_{n,k}^2} \sum_{i=1, i\ne k}^n EX_i^2 I_{\{|X_i| > B_{n,k}/(1+x)\}} \\
&\quad + \frac{(1+x)^3}{B_{n,k}^3} \sum_{i=1, i\ne k}^n E|X_i|^3 I_{\{|X_i| \le B_{n,k}/(1+x)\}} \\
&\le \frac{4(1+x)^2}{3B_n^2} \sum_{i=1}^n EX_i^2 I_{\{|X_i| > B_{n,k}/(1+x)\}} \\
&\quad + \left(\frac{4}{3}\right)^{3/2} \frac{(1+x)^3}{B_n^3} \sum_{i=1}^n E|X_i|^3 I_{\{|X_i| \le B_n/(1+x)\}} \\
&\quad + \frac{4(1+x)^2}{3B_n^2} \sum_{i=1}^n EX_i^2 I_{\{B_{n,k}/(1+x) < |X_i| \le B_n/(1+x)\}}
\end{aligned}$$

$$\leq \frac{4(1+x)^2}{3B_n^2} \sum_{i=1}^n EX_i^2 I_{\{|X_i|>B_n/(1+x)\}}$$

$$+ 2\left(\frac{4}{3}\right)^{3/2} \frac{(x+1)^3}{B_n^3} \sum_{i=1}^n E|X_i|^3 I_{\{|X_i|\leq B_n/(1+x)\}}$$

$$\leq 8\Delta_{n,x}. \qquad \square$$

9. Proof of Proposition 5.3. To prove (5.11), let

$$D_n^2 = \sum_{i=1}^n E\bar{X}_i^4, \qquad \eta_i = \bar{X}_i^2 - E\bar{X}_i^2, \qquad \bar{b} := \bar{b}_{n,x} = x/\bar{B}_n.$$

We have

$$K_n \leq P\left(\left|\sum_{i=1}^n \eta_i\right| \geq \sqrt{6}x\left(4D_n + \left(\sum_{i=1}^n \eta_i^2\right)^{1/2}\right)\right)$$

$$+ P\Biggl(\bar{S}_n \geq x\bar{B}_n\left\{1 + \frac{1}{2\bar{B}_n^2}\sum_{i=1}^n \eta_i - \frac{1}{\bar{B}_n^4}\left(\sum_{i=1}^n \eta_i\right)^2\right\},$$

$$\left|\sum_{i=1}^n \eta_i\right| \leq \sqrt{6}x\left(4D_n + \left(\sum_{i=1}^n \eta_i^2\right)^{1/2}\right)\Biggr)$$

$$\leq 8e^{-3x^2} + K_{n,1}$$

by (6.11), where

$$K_{n,1} \leq P\left(\bar{S}_n \geq x\bar{B}_n\left\{1 + \frac{1}{2\bar{B}_n^2}\sum_{i=1}^n \eta_i - \frac{12x^2(16D_n^2 + \sum_{i=1}^n \eta_i^2)}{\bar{B}_n^4}\right\}\right)$$

$$\leq P\left(\bar{S}_n \geq \frac{x^2}{2\bar{b}}\left\{1 + \bar{B}_n^{-2}\sum_{i=1}^n \bar{X}_i^2 - \frac{24x^2(16D_n^2 + \sum_{i=1}^n \bar{X}_i^4)}{\bar{B}_n^4}\right\}\right)$$

$$= P\left(2\bar{b}\bar{S}_n - \bar{b}^2\bar{V}_n^2 + 24\bar{b}^4\sum_{i=1}^n(\bar{X}_i^4 + 16E\bar{X}_i^4) \geq x^2\right).$$

Let $\bar{\xi}_i = 2\bar{b}\bar{X}_i - \bar{b}^2\bar{X}_i^2 + 24\bar{b}^4(\bar{X}_i^4 + 16E\bar{X}_i^4)$. Then it is easy to see that (6.3)–(6.7) in Lemma 6.2 remain valid for $\bar{\xi}_i$ when $\lambda = 1/2$ with $O_{\lambda,l}$ bounded by an absolute constant for $l = 1, \ldots, 5$. Hence, it follows from the proof of (7.4) that

$$P\left(2\bar{b}\bar{S}_n - \bar{b}^2\bar{V}_n^2 + 24\bar{b}^4\sum_{i=1}^n(\bar{X}_i^4 + 16E\bar{X}_i^4) \geq x^2\right) \leq (1 - \Phi(x))e^{A\Delta_{n,x}}.$$

This proves (5.11).

10. Proof of Proposition 5.4. Let

$$\begin{aligned}\widetilde{U}_j(x) &= \bar{X}_j - \frac{x}{2\bar{B}_n}(\bar{X}_j^2 - E\bar{X}_j^2) + \frac{x}{\bar{B}_n^3}(\bar{X}_j^2 - E\bar{X}_j^2)^2,\\ U_j(x) &= \widetilde{U}_j(x) - E\widetilde{U}_j(x),\\ W_{j,k}(x) &= 2x(\bar{X}_j^2 - E\bar{X}_j^2)(\bar{X}_k^2 - E\bar{X}_k^2).\end{aligned}$$

We can rewrite

$$K_n = P\left(\frac{1}{\bar{B}_n}\sum_{j=1}^n U_j(x) + \frac{1}{\bar{B}_n^4}\sum_{j<k} W_{j,k}(x) \ge x - \frac{1}{\bar{B}_n}\sum_{j=1}^n E\widetilde{U}_j(x)\right).$$

Set $\tau = B_n/(1+x)$ and

$$\begin{aligned}\Delta &= \Delta_{n,x}/(1+x)^2\\ &= B_n^{-2}\sum_{i=1}^n EX_i^2 I_{\{|X_i|>\tau\}} + (1+x)B_n^{-3}\sum_{i=1}^n E|X_i|^3 I_{\{|X_i|\le\tau\}}.\end{aligned}$$

Note that the condition $\Delta \le 1/128$ yields

$$\tfrac{8}{9}B_n^2 \le \bar{B}_n^2 \le B_n^2, \tag{10.1}$$

$$|E\widetilde{U}_j(x)| \le \frac{1+x}{B_n}EX_j^2 I_{\{|X_j|>\tau\}} + \frac{3}{B_n^2}E|X_j|^3 I_{\{|X_j|\le\tau\}}, \tag{10.2}$$

$$|EU_j^2(x) - E\bar{X}_j^2| \le EX_j^2 I_{\{|X_j|>\tau\}} + \frac{14x}{B_n}E|X_j|^3 I_{\{|X_j|\le\tau\}}, \tag{10.3}$$

$$E|U_j(x)|^3 \le 64E|X_j|^3 I_{\{|X_j|\le\tau\}}, \tag{10.4}$$

$$\begin{aligned}\Delta &\ge \left(\frac{1}{B_n^2}EX_j^2 I_{\{|X_j|>\tau\}}\right)^{3/2} + \left(\frac{1}{B_n^2}EX_j^2 I_{\{|X_j|\le\tau\}}\right)^{3/2}\\ &\ge \frac{1}{2}\left(\frac{EX_j^2}{B_n^2}\right)^{3/2}\end{aligned} \tag{10.5}$$

for every $1 \le j \le n$.

Throughout the remainder of this section, we use the following notation:

$$g_j(t,x) = Ee^{itU_j(x)/\bar{B}_n}, \qquad T_n = \frac{1}{\bar{B}_n}\sum_{j=1}^n U_j(x), \qquad \Lambda_n = \frac{1}{\bar{B}_n^4}\sum_{1\le i<j\le n} W_{i,j}(x).$$

The proof of (5.12) is based on the following lemmas.

LEMMA 10.1. *If* $|t| \le \Delta^{-1}/64$, *then for all* $1 \le i, k \le n$,

$$\left|\prod_{j\ne i,k} g_j(t,x)\right| \le e^{-t^2/8}. \tag{10.6}$$

If $|t| \le \Delta^{-1/3}/64$, *then*

$$\left|\prod_{j=1}^{n} g_j(t,x) - e^{-t^2/2}\right| \le A\Delta(t^2 + |t|^6)e^{-t^2/2} \tag{10.7}$$

and

$$\left|\prod_{j=1}^{n} g_j(t,x) - e^{-t^2/2}\left\{1 + \sum_{j=1}^{n}\big(g_j(t,x) - 1\big) + \frac{t^2}{2}\right\}\right| \le A\Delta^{4/3}(t^2 + t^6)e^{-t^2/2}. \tag{10.8}$$

PROOF. By using Taylor's expansion of e^{ix}, we have

$$\left|g_j(t,x) - 1 + \frac{t^2 EU_j^2(x)}{2\bar{B}_n^2}\right| \le \frac{|t|^3 E|U_j(x)|^3}{6\bar{B}_n^3}. \tag{10.9}$$

It can be easily shown from (10.3), (10.4) and (10.9) that

$$\begin{aligned}|g_j(t,x)| \le 1 &- \frac{t^2}{2\bar{B}_n^2}\big(E\bar{X}_j^2 - EX_j^2 I_{\{|X_j|>\tau\}}\big)\\ &+ \frac{(9t^2 + 16|t|^3)(1+x)}{B_n^3} E|X_j|^3 I_{\{|X_j|\le\tau\}}.\end{aligned}$$

Recalling $\Delta \le 1/128$, we get $E\bar{X}_i^2 \le \{E|X_i|^3 I_{\{|X_i|\le\tau\}}\}^{2/3} \le B_n^2/(16 \cdot 4^{1/3})$. Therefore, for $|t| \le \Delta^{-1}/64$,

$$\begin{aligned}&\left|\prod_{j\ne i,k} g_j(t,x)\right|\\ &\quad \le \exp\left\{-\frac{t^2}{2}\left(1 - \frac{1}{2\bar{B}_n^2}(E\bar{X}_i^2 + E\bar{X}_k^2)\right) + (9t^2 + 16|t|^3)\Delta\right\} \le e^{-t^2/8}.\end{aligned}$$

This proves (10.6).

Next we prove (10.7) and (10.8). Assume $|t| \le \Delta^{-1/3}/64$. By (10.9), we can write

$$g_j(t,x) = 1 - r_j(t,x),$$

where

$$r_j(t,x) = \frac{t^2}{2\bar{B}_n^2} EU_j^2(x) + \theta\frac{|t|^3}{6\bar{B}_n^3} E|U_j(x)|^3 \quad \text{and} \quad |\theta| \le 1.$$

It follows from (10.3)–(10.5) that $|r_j(t,x)| \le 1/4$ and

$$\begin{aligned} |r_j(t,x)|^2 &\le \frac{t^4}{\bar{B}_n^4}\big(E|U_j(x)|^2\big)^2 + \frac{|t|^6}{18\bar{B}_n^6}\big(E|U_j(x)|^3\big)^2 \\ &\le \frac{A|t|^4}{B_n^4}\bigg((E\bar{X}_j^2)^2 + \bigg(E|X_j|^2 I_{\{|X_j|>\tau\}} \\ &\qquad + \frac{1+x}{B_n}E|X_j|^3 I_{\{|X_j|\le\tau\}}\bigg)^2\bigg) \\ &\le \frac{At^4}{B_n^4}(EX_j^2)^2. \end{aligned} \tag{10.10}$$

Writing $\ln(1+z) = z + \theta z^2$ for $|z| < 1/2$, where $|\theta| \le 1$, we have for $|t| \le \Delta^{-1/3}/64$

$$\begin{aligned} \ln\prod_{j=1}^n g_j(t,x) &= \sum_{j=1}^n \ln\big(1 + (g_j(t,x)-1)\big) \\ &= \sum_{j=1}^n \big(g_j(t,x)-1\big) + \theta\sum_{j=1}^n |r_j(t,x)|^2 \\ &= \sum_{j=1}^n \big(g_j(t,x)-1\big) + O(1)\bigg(\frac{1}{B_n^4}\sum_{j=1}^n (EX_j^2)^2\bigg)t^4, \end{aligned} \tag{10.11}$$

where $|O(1)| \le A$. By (10.3), (10.4) and (10.9)

$$\begin{aligned} &\bigg|\sum_{j=1}^n \big(g_j(t,x)-1\big) + \frac{t^2}{2}\bigg| \\ &\quad\le \frac{t^2}{2\bar{B}_n^2}\sum_{j=1}^n |EU_j^2(x) - E\bar{X}_j^2| + \frac{|t|^3}{6\bar{B}_n^3}E|U_j(x)|^3 \\ &\quad\le A(t^2+|t|^3)\Delta, \end{aligned} \tag{10.12}$$

which, together with (10.11), yields

$$\begin{aligned} &\bigg|\ln\prod_{j=1}^n g_j(t,x) + \frac{t^2}{2}\bigg| \\ &\quad\le \bigg|\sum_{j=1}^n \big(g_j(t,x)-1\big) + \frac{t^2}{2}\bigg| + O(1)t^4\bigg(\frac{1}{B_n^4}\sum_{j=1}^n (E|X_j|^2)^2\bigg) \\ &\quad\le A(t^2+|t|^3)\Delta + At^4\Delta^{4/3} \le A(t^2+|t|^3)\Delta \end{aligned} \tag{10.13}$$

for $|t| \le \Delta^{-1/3}/64$. In terms of (10.11)–(10.13), by using $|e^z - 1 - z| \le \frac{z^2}{2}e^{|z|}$, we find that if $|t| \le \Delta^{-1/3}/64$, then

$$\prod_{j=1}^{n} g_j(t,x) - e^{-t^2/2} = e^{-t^2/2}\left[\exp\left\{\ln \prod_{j=1}^{n} g_j(t,x) + \frac{t^2}{2}\right\} - 1\right]$$
$$= e^{-t^2/2}\left\{\sum_{j=1}^{n}\big(g_j(t,x)-1\big) + \frac{t^2}{2} + r^*(t,x)\right\},$$

where

$$|r^*(t,x)| \le At^4\Delta^{4/3} + \frac{1}{2}\left|\ln \prod_{j=1}^{n} g_j(t,x) + \frac{t^2}{2}\right|^2 \exp\left\{\left|\ln \prod_{j=1}^{n} g_j(t,x) + \frac{t^2}{2}\right|\right\}$$
$$\le A(t^4+t^6)\Delta^{4/3}.$$

This proves (10.8). (10.7) follows directly from (10.8) and (10.12). □

LEMMA 10.2. *If* $|t| \le \Delta^{-1}/64$, *then*

$$|E\Lambda_n e^{itT_n}| \le A\Delta^2 t^2 e^{-t^2/8}, \tag{10.14}$$

$$\left|Ee^{it(T_n+\Lambda_n)} - \prod_{j=1}^{n} g_j(t,x)\right| \le A\Delta^2\big(|t|^{3/2} + t^2 e^{-t^2/8}\big). \tag{10.15}$$

PROOF. Observe that $E(W_{j,l}(x)|X_j) = 0$ for $j \ne l$. It follows from (10.4), Hölder's inequality and the independence of U_j and U_l that

$$\begin{aligned}
&\big|EW_{j,l}(x)e^{it(U_j(x)+U_l(x))/\bar{B}_n}\big| \\
&\quad = \big|EW_{j,l}(x)\big(e^{itU_j(x)/\bar{B}_n} - 1\big)\big(e^{itU_l(x)/\bar{B}_n} - 1\big)\big| \\
&\quad \le 2\left(\frac{|t|}{\bar{B}_n}\right)^2 E\big\{|W_{j,l}(x)|U_j(x)||U_l(x)|\big\} \\
&\quad \le \frac{2t^2}{\bar{B}_n^2}\big\{E|W_{j,l}(x)|^{3/2}\big\}^{2/3}\big\{E|U_j(x)|^3\big\}^{1/3}\big\{E|U_l(x)|^3\big\}^{1/3} \\
&\quad \le \frac{Axt^2}{B_n^2}\big(E|X_j|^3 I_{\{|X_j|\le\tau\}}\big)\big(E|X_l|^3 I_{\{|X_l|\le\tau\}}\big).
\end{aligned}$$

Thus by independence of $U_j(x)'s$ and (10.6),

$$\begin{aligned}
|E\Lambda_n e^{itT_n}| &\le \frac{1}{\bar{B}_n^4}\sum_{j<l}\big|EW_{j,l}(x)e^{it(U_j(x)+U_l(x))/\bar{B}_n}\big| \prod_{k\ne j,l}|g_k(t,x)| \\
&\le A\Delta^2 t^2 e^{-t^2/8}.
\end{aligned}$$

This proves (10.14).

As for (10.15), since $|e^{iz} - 1 - iz| \le 2|z|^{3/2}$ for every real z, we have

$$\begin{aligned}
&\left|Ee^{it(T_n+\Lambda_n)} - Ee^{itT_n} - itE\Lambda_n e^{itT_n}\right| \\
&\quad \le 2|t|^{3/2} E|\Lambda_n|^{3/2} \\
&\quad \le \frac{6|t|^{3/2}}{B_n^6} E\left|\sum_{i=1}^{n-1}(\bar{X}_i^2 - E\bar{X}_i^2)\sum_{j=i+1}^{n}(\bar{X}_j^2 - E\bar{X}_j^2)\right|^{3/2} x^{3/2} \qquad (10.16)\\
&\quad \le \frac{A|t|^{3/2}}{B_n^6}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(E|X_i|^3 I_{\{|X_i|\le\tau\}}\right)\left(E|X_j|^3 I_{\{|X_j|\le\tau\}}\right)x^{3/2} \\
&\quad \le A|t|^{3/2}\Delta^2,
\end{aligned}$$

where, in the second to last inequality, we used a moment inequality for the U-statistics [see, e.g., Dharmadhikari, Fabian and Jogdeo (1968)].

Now (10.15) follows easily from (10.16) and (10.14). □

LEMMA 10.3. *If* $|t| \le \Delta^{-1}/64$, *then for any* $0 \le m(t) \le 1$,

$$\begin{aligned}
&\left|Ee^{it(T_n+\Lambda_n)}\right| \\
&\quad \le (1+2|t|)e^{-m(t)t^2/8} + A\Delta^{4/3}|t|m^{4/3}(t) + A\Delta^2|t|^{3/2}m(t). \qquad (10.17)
\end{aligned}$$

PROOF. We follow the randomization method used in Bentkus, Bloznelis and Götze (1996). See Alberink (2000) as well. Let $X_1^*, \dots, X_n^*$ be independent copies of $X_1, \dots, X_n$ and let $\epsilon_1, \dots, \epsilon_n$ be i.i.d. random variables with

$$P(\epsilon_i = 1) = 1 - P(\epsilon_i = 0) = m(t)$$

independent of all other random variables. Let $\bar{X}_j^*$, $U_j^*(x)$ and $W_{j,k}^*(x)$ denote the random variables $\bar{X}_j$, $U_j(x)$ and $W_{j,k}(x)$ with X_j and X_k replaced by independent copies X_j^* and X_k^*. Furthermore, we let $\widetilde{W}_{j,k}^*(x) = 2x(\bar{X}_j^2 - E\bar{X}_j^2)(\bar{X}_k^{*2} - E\bar{X}_k^{*2})$ and define

$$T_{1n}^* = \frac{1}{\bar{B}_n}\sum_{j=1}^n \epsilon_j U_j(x), \qquad T_{2n}^* = \frac{1}{\bar{B}_n}\sum_{j=1}^n (1-\epsilon_j)U_j^*(x),$$

$$\Lambda_{1n}^* = \frac{1}{\bar{B}_n^4}\sum_{i<j}\epsilon_i\epsilon_j W_{i,j}(x), \qquad \Lambda_{2n}^* = \frac{1}{\bar{B}_n^4}\sum_{i<j}\epsilon_i(1-\epsilon_j)\widetilde{W}_{i,j}^*(x),$$

$$\Lambda_{3n}^* = \frac{1}{\bar{B}_n^4}\sum_{i<j}(1-\epsilon_i)(1-\epsilon_j)W_{i,j}^*(x).$$

It is easy to see that [cf. Alberink (2000)]

$$T_n + \Lambda_n \stackrel{d}{=} T_{1n}^* + T_{2n}^* + \Lambda_{1n}^* + \Lambda_{2n}^* + \Lambda_{3n}^*, \qquad (10.18)$$

where $\stackrel{d}{=}$ denotes the same in distribution. It follows from (10.18), $|e^{it}-1| \le |t|$ and $|e^{it}-1-it| \le 2|t|^{3/2}$ that

$$
\begin{aligned}
&\bigl|Ee^{it(T_n+\Lambda_n)}\bigr| \\
&\quad = \bigl|Ee^{it(T_{1n}^*+T_{2n}^*+\Lambda_{1n}^*+\Lambda_{2n}^*+\Lambda_{3n}^*)}\bigr| \\
&\quad \le \bigl|Ee^{it(T_{1n}^*+T_{2n}^*+\Lambda_{2n}^*+\Lambda_{3n}^*)}\bigr| + |t|E|\Lambda_{1n}^*| \\
&\quad \le E\bigl|E\bigl(e^{it(T_{1n}^*+\Lambda_{2n}^*)}|X^*,\epsilon\bigr)\bigr| + |t|E|\Lambda_{1n}^*| \qquad (10.19)\\
&\quad \le E\bigl|E\bigl(e^{itT_{1n}^*}|X^*,\epsilon\bigr)\bigr| + |t|E\bigl|E\bigl(\Lambda_{2n}^*e^{itT_{1n}^*}|X^*,\epsilon\bigr)\bigr| \\
&\qquad\quad + 2|t|^{3/2}E|\Lambda_{2n}^*|^{3/2} + |t|E|\Lambda_{1n}^*| \\
&\quad := \Xi_1(t,x)+\Xi_2(t,x)+\Xi_3(t,x)+\Xi_4(t,x),
\end{aligned}
$$

where $X^* = (X_1^*,\dots,X_n^*)$ and $\epsilon = (\epsilon_1,\dots,\epsilon_n)$.

We first estimate $\Xi_3(t,x)$ and $\Xi_4(t,x)$. Recalling that ϵ_j's are i.i.d. random variables independent of all other random variables, we get

$$
\begin{aligned}
E|\Lambda_{2n}^*|^{3/2} &\le \frac{8x^{3/2}}{B_n^6}E\left|\sum_{i=1}^{n-1}\epsilon_i(\bar{X}_i^2-E\bar{X}_i^2)\sum_{j=i+1}^{n}(1-\epsilon_j)(\bar{X}_j^{*2}-E\bar{X}_j^{*2})\right|^{3/2} \\
&\le \frac{Ax^{3/2}E|\epsilon_1|^{3/2}}{B_n^6}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\bigl(E|X_i|^3 I_{\{|X_i|\le\tau\}}\bigr)\bigl(E|X_j|^3 I_{\{|X_j|\le\tau\}}\bigr) \\
&\le A\Delta^2 m(t).
\end{aligned}
$$

Similarly,

$$
E|\Lambda_{1n}^*| \le (E|\Lambda_{1n}^*|^{3/2})^{2/3} \le A\Delta^{4/3}m^{4/3}(t).
$$

These facts imply that

$$
\Xi_3(t,x)+\Xi_4(t,x) \le A\Delta^{4/3}|t|m^{4/3}(t) + A\Delta^2|t|^{3/2}m(t). \tag{10.20}
$$

Next we estimate $\Xi_1(t,x)$. Note that

$$
\begin{aligned}
&E\bigl|E\bigl(e^{it\epsilon_j U_j(x)/\bar{B}_n}|\epsilon_j\bigr)\bigr|^2 \\
&\quad = E\bigl\{E\bigl(e^{it\epsilon_j(U_j(x)-U_j^*(x))/\bar{B}_n}|\epsilon_j\bigr)\bigr\} \\
&\quad = Ee^{it\epsilon_j(U_j(x)-U_j^*(x))/\bar{B}_n},
\end{aligned}
$$

because $U_j^*(x)$ is an independent copy of $U_j(x)$. Using (10.3), (10.4) and Taylor's expansion of e^{iz} yields

$$
\begin{aligned}
&Ee^{it\epsilon_j(U_j(x)-U_j^*(x))/\bar{B}_n}\\
&\quad \le 1-\frac{t^2}{\bar{B}_n^2}E\epsilon_1^2EU_j^2(x)+\frac{|t|^3}{3\bar{B}_n^3}E|\epsilon_1|^3E|U_j(x)|^3\\
&\quad \le 1-\frac{m(t)t^2}{\bar{B}_n^2}\big(E\bar{X}_j^2-EX_j^2I_{\{|X_j|>\tau\}}\big)\\
&\qquad +\frac{m(t)(18t^2+32|t|^3)(1+x)}{B_n^3}E|X_j|^3I_{\{|X_j|\le\tau\}}.
\end{aligned}
$$

Therefore, by Hölder's inequality,

$$
\begin{aligned}
&E\big|E\big(e^{it\epsilon_jU_j(x)/\bar{B}_n}|\epsilon_j\big)\big|\\
&\quad \le \big(E\big|E\big(e^{it\epsilon_jU_j(x)/\bar{B}_n}|\epsilon_j\big)\big|^2\big)^{1/2}\\
&\quad \le \exp\bigg\{-\frac{m(t)t^2}{2\bar{B}_n^2}\big(E\bar{X}_j^2-EX_j^2I_{\{|X_j|>\tau\}}\big)\\
&\qquad +\frac{m(t)(9t^2+16|t|^3)(1+x)}{B_n^3}E|X_j|^3I_{\{|X_j|\le\tau\}}\bigg\},
\end{aligned}
\tag{10.21}
$$

which, together with the independence of $U_j(x)$ and ϵ_j, implies that if $|t|\le\Delta^{-1}/64$, then

$$
\begin{aligned}
\Xi_1(t,x)=E\big|E\big(e^{itT_{1n}^*}|\epsilon\big)\big| &\le \prod_{j=1}^n E\big|E\big(e^{it\epsilon_jU_j(x)/\bar{B}_n}|\epsilon_j\big)\big|\\
&\le \exp\bigg\{-\frac{m(t)t^2}{2}+m(t)(9t^2+16|t|^3)\Delta\bigg\}\\
&\le e^{-m(t)t^2/8}.
\end{aligned}
\tag{10.22}
$$

Finally, we estimate $\Xi_2(t,x)$. Since $\Delta\le 1/128$, we get $E\bar{X}_i^2\le\{E|X_i|^3\times I_{\{|X_i|\le\tau\}}\}^{2/3}\le B_n^2/(16\cdot 4^{1/3})$. Hence, similarly to (10.22), it follows from (10.21) that, for all $1\le j\le n$,

$$
\begin{aligned}
\Omega_{1j} &:= E\big|E\big(e^{it(T_{1n}^*-\epsilon_jU_j(x))/\bar{B}_n}|\epsilon_1,\dots,\epsilon_{j-1},\epsilon_{j+1},\dots,\epsilon_n\big)\big|\\
&\le \prod_{k\ne j}E\big|E\big(e^{it\epsilon_kU_k(x)/\bar{B}_n}|\epsilon_k\big)\big|\\
&\le \exp\bigg\{-\frac{m(t)t^2}{2}(1-\bar{B}_n^{-2}E\bar{X}_j^2)+m(t)(9t^2+16|t|^3)\Delta\bigg\}\\
&\le e^{-m(t)t^2/8}.
\end{aligned}
\tag{10.23}
$$

On the other hand, we have

$$
\begin{aligned}
\Omega_{2j} &:= E\big|E\{(\bar{X}_j^2 - E\bar{X}_j^2)e^{it\epsilon_j U_j(x)/\bar{B}_n}|\epsilon_j\}\big| \\
&= E\big|E\{(\bar{X}_j^2 - E\bar{X}_j^2)(e^{it\epsilon_j U_j(x)/\bar{B}_n} - 1)|\epsilon_j\}\big| \\
&\le \frac{2|t|}{\bar{B}_n} E\bar{X}_j^2 |U_j(x)| \le \frac{3|t|}{B_n}(E|\bar{X}_j|^3)^{2/3}\big(E|U_j(x)|^3\big)^{1/3} \qquad (10.24)\\
&\le \frac{12|t|}{B_n} E|X_j|^3 I_{\{|X_j|\le\tau\}}.
\end{aligned}
$$

In terms of (10.23), (10.24) and the independence of $U_j(x)$, $U_j^*(x)$ and ϵ_j, we get

$$
\begin{aligned}
\Xi_2(t,x) &= |t|E\big|E\big(\Lambda_{2n}^* e^{itT_{1n}^*}|X^*, \epsilon\big)\big| \\
&\le \frac{4|tx|}{B_n^4}\sum_{j\ne k} E|\bar{X}_k^{*2} - E\bar{X}_k^{*2}|E\big|E\{(\bar{X}_j^2 - E\bar{X}_j^2)e^{itT_{1n}^*}|\epsilon\}\big| \\
&\le \frac{8|tx|}{B_n^4}\sum_{j\ne k} \Omega_{1j}\Omega_{2j} E\bar{X}_k^2 \qquad (10.25)\\
&\le 96\Delta t^2 e^{-m(t)t^2/8} \\
&\le 2|t|e^{-m(t)t^2/8}.
\end{aligned}
$$

Putting the above estimates (10.19), (10.20), (10.22) and (10.25) together, we have (10.17) immediately. The proof of Lemma 10.3 is complete. □

LEMMA 10.4. *Let F be a distribution function with the characteristic function f. Then, for all $y \in R$ and $M > 0$, it holds that*

$$\lim_{z\downarrow y} F(z) \le \frac{1}{2} + V.P.\int_{-M}^{M} e^{-iyt}\frac{1}{M}K\left(\frac{t}{M}\right)f(t)\,dt, \qquad (10.26)$$

$$\lim_{z\uparrow y} F(z) \ge \frac{1}{2} - V.P.\int_{-M}^{M} e^{-iyt}\frac{1}{M}K\left(-\frac{t}{M}\right)f(t)\,dt, \qquad (10.27)$$

where

$$V.P.\int_{-M}^{M} = \lim_{h\downarrow 0}\left(\int_{-M}^{-h} + \int_{h}^{M}\right),$$

$K(s) = K_1(s)/2 + iK_2(s)/(2\pi s)$ *and*

$$K_1(s) = 1 - |s|, \qquad K_2(s) = \pi s(1-|s|)\cot(\pi s) + |s| \qquad \textit{for } |s| < 1$$

and $K(s) \equiv 0$ *for* $|s| \ge 1$.

This lemma can be found in Prawitz (1972). The result was also used by Bentkus, Götze and van Zwet (1997) for an Edgeworth expansion for symmetric statistics.

LEMMA 10.5. *There exists an absolute constant A such that*

$$P\left(\frac{1}{\bar{B}_n}\sum_{j=1}^n U_j(x) + \frac{1}{\bar{B}_n^4}\sum_{j<k} W_{j,k}(x) \ge y\right) \le \bigl(1-\Phi(y)\bigr) + A(1+x)\Delta e^{-y^2/2} + A\Delta^{4/3} \tag{10.28}$$

for $|y| \le 4(1+x)$.

PROOF. By using (10.27) in Lemma 10.4, we have, for any $y \in R$ and $M = \Delta^{-1}/64$,

$$P\left(\frac{1}{\bar{B}_n}\sum_{j=1}^n U_j(x) + \frac{1}{\bar{B}_n^4}\sum_{j<k} W_{j,k}(x) \ge y\right) \le \frac{1}{2}(|I| + |J-1|),$$

where $K_1(s)$ and $K_2(s)$ are defined as in Lemma 10.4,

$$I = \frac{1}{M}\int_{-M}^{M} e^{-iyt} K_1\left(-\frac{t}{M}\right) E e^{it(T_n+\Lambda_n)}\,dt$$

and

$$J = \frac{i}{\pi} V.P.\int_{-M}^{M} e^{-iyt} K_2\left(-\frac{t}{M}\right) E e^{it(T_n+\Lambda_n)}\,\frac{dt}{t}.$$

It suffices to show that

$$|I| \le A\Delta e^{-y^2/2} + A\Delta^{4/3} \tag{10.29}$$

for $y \in R$ and

$$|J-1| \le 2\bigl(1-\Phi(y)\bigr) + A(1+x)e^{-y^2/2}\Delta + A\Delta^{4/3} \tag{10.30}$$

for $|y| \le 4(1+x)$.

Without loss of generality, we assume that $\Delta \le 1/(6\cdot 64)^4$. Let $M_1 = \Delta^{-1/3}/64$. Rewrite $I = I_1 + I_2$, where

$$I_1 = \frac{1}{M}\int_{-M_1}^{M_1} e^{-iyt} K_1\left(-\frac{t}{M}\right) E e^{it(T_n+\Lambda_n)}\,dt,$$

$$I_2 = \frac{1}{M}\int_{M_1\le|t|\le M} e^{-iyt} K_1\left(-\frac{t}{M}\right) E e^{it(T_n+\Lambda_n)}\,dt.$$

It is easy to see that $M_1 \ge 6$ and $0 \le [36t^{-2}\ln|t|] \le 1$ for $|t| \ge M_1$. Hence, by Lemma 10.3 with $m(t) = [36t^{-2}\ln|t|]$,

$$|I_2| \le \frac{1}{M}\int_{M_1\le|t|\le M} \bigl|E e^{it(T_n+\Lambda_n)}\bigr|\,dt \le A\Delta^{4/3}. \tag{10.31}$$

Noting $K_1(s) = 1 - |s|$, for $|s| < 1$, we obtain $|I_1| \le |I_{11}| + |I_{12}|$, where

$$I_{11} = \frac{1}{M} \int_{-M_1}^{M_1} e^{-iyt} E e^{it(T_n+\Lambda_n)}\, dt,$$

$$I_{12} = \frac{2}{M^2} \int_0^{M_1} t \big| E e^{it(T_n+\Lambda_n)} \big|\, dt.$$

It is obvious that $|I_{12}| \le \frac{2}{M^2} \int_0^{M_1} t\, dt \le A\Delta^{4/3}$. Note that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyt-t^2/2}\, dt = e^{-y^2/2}.$$

It follows from (10.7) and (10.15) that

$$\begin{aligned}
|I_{11}| &\le \frac{1}{M} \left| \int_{-\infty}^{\infty} e^{-iyt-t^2/2}\, dt \right| + \frac{1}{M} \int_{|t|>M_1} e^{-t^2/2}\, dt \\
&\quad + \frac{1}{M} \int_{-M_1}^{M_1} \big| E e^{it(T_n+\Lambda_n)} - e^{-t^2/2} \big|\, dt \\
&\le A\Delta e^{-y^2/2} + A\Delta^2.
\end{aligned}$$

This proves (10.29) by the above inequalities.

To prove (10.30), let

$$f_n(t, x) = \left(1 + \sum_{j=1}^{n} \big(g_j(t, x) - 1\big) + \frac{t^2}{2} \right) e^{-t^2/2}.$$

We may write

$$J = J_{11} + J_{12} + J_{13} + J_2,$$

where

$$\begin{aligned}
J_{11} &= \frac{i}{\pi} V.P. \int_{-M_1}^{M_1} e^{-iyt} f_n(t, x)\, \frac{dt}{t}, \\
J_{12} &= \frac{i}{\pi} V.P. \int_{-M_1}^{M_1} e^{-iyt} \big(E e^{it(T_n+\Lambda_n)} - f_n(t, x) \big)\, \frac{dt}{t}, \\
J_{13} &= \frac{i}{\pi} V.P. \int_{-M_1}^{M_1} e^{-iyt} \left(K_2\left(-\frac{t}{M}\right) - 1 \right) E e^{it(T_n+\Lambda_n)}\, \frac{dt}{t}, \\
J_2 &= \frac{i}{\pi} V.P. \int_{M_1<|t|\le M} e^{-iyt} K_2\left(-\frac{t}{M}\right) E e^{it(T_n+\Lambda_n)}\, \frac{dt}{t}
\end{aligned}$$

and $M_1 = \Delta^{-1/3}/64$. Similarly to (10.31), it follows that $|J_2| \le A\Delta^{4/3}$. By using (10.8), (10.15) and the fact that $\frac{1}{\bar{B}_n^4}\sum_{j=1}^n (E\bar{X}_j^2)^2 \le \Delta^{4/3}$, we have

$$\begin{aligned}
|J_{12}| &\le \int_{-M_1}^{M_1} \frac{1}{|t|}\big|Ee^{it(T_n+\Lambda_n)} - f_n(t,x)\big|\,dt \\
&\le \int_{-M_1}^{M_1} \frac{1}{|t|}\left|Ee^{it(T_n+\Lambda_n)} - \prod_{j=1}^n g_j(t,x)\right|dt \\
&\quad + \int_{-M_1}^{M_1} \frac{1}{|t|}\left|\prod_{j=1}^n g_j(t,x) - f_n(t,x)\right|dt \\
&\le A\Delta^{4/3}.
\end{aligned}$$

It is easy to check that $|K_2(s) - 1| \le 4s^2$ for $|s| \le 1/2$ [cf., e.g., Lemma 2.1 in Bentkus (1994)]. Hence,

$$\begin{aligned}
|J_{13}| &\le A\Delta^2 \int_{-M_1}^{M_1} |t|\big|Ee^{it(T_n+\Lambda_n)}\big|\,dt \\
&\le A\Delta^2 \int_{-M_1}^{M_1} |t|\,dt \le A\Delta^{4/3}.
\end{aligned}$$

On the other hand, simple calculation shows that

$$\frac{i}{2\pi} V.P. \int_{-\infty}^{\infty} e^{-iyt} f_n(t,x)\,\frac{dt}{t} = -\frac{1}{2} + \Phi(y) + \mathcal{L}_n(y),$$

where

$$\mathcal{L}_n(y) = \sum_{j=1}^n \left\{ E\Phi\left(y - \frac{U_j(x)}{\bar{B}_n}\right) - \Phi(y)\right\} - \frac{1}{2}\Phi''(y).$$

Therefore,

$$\begin{aligned}
|J-1| &\le |J_{11} - 1| + |J_{12}| + |J_{13}| + |J_2| \\
&\le \left|\frac{i}{\pi} V.P. \int_{-\infty}^{\infty} e^{-iyt} f_n(t,x)\,\frac{dt}{t} - 1\right| \\
&\quad + A\int_{|t|\ge M_1} |f_n(t,x)|\,\frac{dt}{|t|} + A\Delta^{4/3} \\
&\le 2\big(1 - \Phi(y)\big) + 2|\mathcal{L}_n(y)| + A\Delta^{4/3}
\end{aligned}$$

and (10.30) follows if we prove

$$\mathcal{L}_n(y) \le A(1+x)e^{-y^2/2}\Delta \tag{10.32}$$

for $|y| \le 4(1+x)$.

By using Taylor's expansion of $\Phi(y)$ and noting $|U_j(x)| \le 9\bar{B}_n/(1+x)$, we have

$$\begin{aligned}
\mathcal{L}_{jn}(y) &:= \left| E\Phi\left(y - \frac{U_j(x)}{\bar{B}_n}\right) - \Phi(y) - \frac{EU_j^2(x)}{2\bar{B}_n^2}\Phi^{(2)}(y)\right| \\
&\le \frac{1}{6\bar{B}_n^3} E|U_j(x)|^3 \Phi'''\left(y + \theta\frac{|U_j(x)|}{\bar{B}_n}\right) \qquad [\text{where } |\theta| \le 1] \\
&\le \frac{A(1+y^2)e^{-y^2/2}}{B_n^3} E|X_j|^3 I_{\{|X_j| \le \tau\}} \exp\bigl(y|U_j(x)|/\bar{B}_n\bigr).
\end{aligned}$$

This estimate, together with $|y| \le 4(1+x)$ and (10.3), implies that

$$\begin{aligned}
|\mathcal{L}_n(y)| &\le \sum_{j=1}^n \mathcal{L}_{jn}(y) + \frac{\Phi''(y)}{2\bar{B}_n^2}\sum_{j=1}^n |EU_j^2(x) - E\bar{X}_j^2| \\
&\le A(1+x)e^{-y^2/2}\Delta,
\end{aligned}$$

as desired. $\square$

We are now ready to prove Proposition 5.4. Let $y_0 = x - \bar{B}_n^{-1}\sum_{j=1}^n E\widetilde{U}_j(x)$. From (10.2), we get $\bar{B}_n^{-1}\sum_{j=1}^n |E\widetilde{U}_j(x)| \le 4(1+x)\Delta$. Hence, by (10.28) with $y = y_0$,

$$\begin{aligned}
K_n &= P\left(\frac{1}{\bar{B}_n}\sum_{j=1}^n U_j(x) + \frac{1}{\bar{B}_n^4}\sum_{j<k} W_{j,k}(x) \ge y_0\right) \\
&\le \bigl(1 - \Phi(y_0)\bigr) + A(1+x)e^{-y_0^2/2}\Delta + A\Delta^{4/3} \\
&\le \bigl(1 - \Phi(x)\bigr) + A(1+x)\Delta e^{-x^2/2+3(1+x)^2\Delta} + A\Delta^{4/3} \\
&\le \bigl(1 - \Phi(x)\bigr)\bigl(1 + A\Delta_{n,x}e^{3\Delta_{n,x}}\bigr) + A\Delta^{4/3} \\
&\le \bigl(1 - \Phi(x)\bigr)e^{A\Delta_{n,x}} + A\Delta^{4/3},
\end{aligned}$$

where we have used the following facts:

$$\begin{aligned}
-y_0^2 &\le -x^2 + 2x|y_0 - x| \ \le\ -x^2 + 6(1+x)^2\Delta \\
|\Phi(y_0) - \Phi(x)| &\le |y_0 - x|\Phi'(x + \theta|y_0 - x|) \qquad [\text{for some } |\theta| \le 1] \\
&\le A(1+x)\Delta e^{-(x-|y_0-x|)^2/2} \\
&\le A(1+x)\Delta e^{-x^2/2+3\Delta_{n,x}}.
\end{aligned}$$

The proof of Proposition 5.4 is now complete.

APPENDIX

PROOF OF LEMMA 6.1. Define

$$I_1(b) = E\big(e^{\lambda bX-\theta(bX)^2} - 1\big)I_{\{|bX|>1\}},$$

$$I_2(b) = E\big(e^{\lambda bX-\theta(bX)^2} - 1\big)I_{\{|bX|\le 1\}}.$$

Noting that $\lambda(bs) - \theta(bs)^2 \le \lambda^2/(4\theta)$ for $s \in R^1$, we get

$$|I_1(b)| \le e^{\lambda^2/(4\theta)} P(|bX| > 1) \le e^{\lambda^2/(4\theta)} b^2 EX^2 I_{\{|bX|>1\}}.$$

From the inequality

$$|e^s - 1 - s - s^2/2| \le |s|^3 e^{s\vee 0}/6$$

for any $s \in R^1$ and $EX = 0$, it follows that

$$\begin{aligned}
I_2(b) &= E\big(\lambda bX - \theta(bX)^2\big)I_{\{|bX|\le 1\}} + \tfrac{1}{2}E\big(\lambda bX - \theta(bX)^2\big)^2 I_{\{|bX|\le 1\}}\\
&\quad + \tfrac{1}{6}O(1)e^{\lambda}E|\lambda bX - \theta(bX)^2|^3 I_{\{|bX|\le 1\}}\\
&= -\lambda bEXI_{\{|bX|>1\}} + (\lambda^2/2 - \theta)b^2EX^2 I_{\{|bX|\le 1\}} - \lambda\theta b^3 EX^3 I_{\{|bX|\le 1\}}\\
&\quad + (\theta^2/2)b^4EX^4 I_{\{|bX|\le 1\}} + O(1)(\lambda+\theta)^3 e^{\lambda} b^3 E|X|^3 I_{\{|bX|\le 1\}}\\
&= (\lambda^2/2 - \theta)b^2EX^2 + O(1)(\lambda + |\lambda^2/2 - \theta|)b^2EX^2 I_{\{|bX|>1\}}\\
&\quad + O(1)\big(\lambda\theta + \theta^2/2 + (\lambda+\theta)^3 e^{\lambda}/6\big)b^3E|X|^3 I_{\{|bX|\le 1\}},
\end{aligned}$$

where $|O(1)| \le 1$. This proves (6.1) by the above bounds on $I_1(b)$ and $I_2(b)$. $\square$

PROOF OF LEMMA 6.2. (6.2) is a direct consequence of (6.1). Write

$$E\xi^k e^{\lambda\xi} = E\xi^k e^{\lambda\xi} I_{\{|bX|>1\}} + E\xi^k e^{\lambda\xi} I_{\{|bX|\le 1\}}, \qquad k = 1, 2, 3.$$

Noting that $\xi \le 1$, $|\xi|^k e^{\lambda\xi} \le \lambda^{-k}\sup_{s\le\lambda}|s|^k e^s = \lambda^{-k}\max(\lambda^k e^{\lambda}, (k/e)^{-k})$ for $k = 1, 2, 3$, $|e^s - 1 - s| \le 0.5 s^2 e^{s\vee 0}$, and $|e^s - 1| \le |s|e^{s\vee 0}$ for $s \in R^1$, we have

$$\begin{aligned}
&E\xi e^{\lambda\xi} I_{\{|bX|\le 1\}}\\
&\quad = E\xi(1+\lambda\xi)I_{\{|bX|\le 1\}} + E\xi(e^{\lambda\xi} - 1 - \lambda\xi)I_{\{|bX|\le 1\}}\\
&\quad = -2bEXI_{\{|bX|>1\}} + (4\lambda - 1)b^2EX^2 I_{\{|bX|\le 1\}}\\
&\qquad + \lambda E\big(-4(bX)^3 + (bX)^4\big)I_{\{|bX|\le 1\}} + 0.5O(1)\lambda^2 e^{\lambda}E|\xi|^3 I_{\{|bX|\le 1\}}\\
&\quad = (4\lambda - 1)b^2EX^2 + O(1)(2 + |1 - 4\lambda|)b^2EX^2 I_{\{|bX|>1\}}\\
&\qquad + O(1)(5\lambda + 13.5\lambda^2 e^{\lambda})b^3E|X|^3 I_{\{|bX|\le 1\}},
\end{aligned}$$

$$
\begin{aligned}
&E\xi^2 e^{\lambda\xi} I_{\{|bX|\le 1\}} \\
&\quad = E\xi^2 I_{\{|bX|\le 1\}} + E\xi^2(e^{\lambda\xi}-1) I_{\{|bX|\le 1\}} \\
&\quad = 4b^2 EX^2 I_{\{|bX|\le 1\}} + E\bigl(-4(bX)^3 + (bX)^4\bigr) I_{\{|bX|\le 1\}} \\
&\qquad + O(1)\lambda e^{\lambda} E|\xi|^3 I_{\{|bX|\le 1\}} \\
&\quad = 4b^2 EX^2 + O(1) b^2 EX^2 I_{\{|bX|>1\}} \\
&\qquad + O(1)(5 + 27\lambda e^{\lambda}) b^3 E|X|^3 I_{\{|bX|\le 1\}}, \\
&E|\xi|^3 e^{\lambda\xi} I_{\{|bX|\le 1\}} \\
&\quad = O(1) e^{\lambda} E|\xi|^3 I_{\{|bX|\le 1\}} \\
&\quad = O(1) 27 e^{\lambda} b^3 E|X|^3 I_{\{|bX|\le 1\}}, \\
&|E\xi e^{\lambda\xi}| \\
&\quad \le \max(e^{\lambda}, e/\lambda) P(|bX|>1) + \bigl|E\xi I_{\{|bX|\le 1\}}\bigr| + \bigl|E\xi(e^{\lambda\xi}-1) I_{\{|bX|\le 1\}}\bigr| \\
&\quad \le \bigl(\max(e^{\lambda}, e/\lambda) + 2\bigr) bE|X| I_{\{|bX|>1\}} + b^2 EX^2 I_{\{|bX|\le 1\}} + \lambda e^{\lambda} E\xi^2 I_{\{|bX|\le 1\}} \\
&\quad \le \bigl(\max(e^{\lambda}, e/\lambda) + 2\bigr) bE|X| I_{\{|bX|>1\}} + (1 + 9\lambda e^{\lambda}) b^2 EX^2 I_{\{|bX|\le 1\}}, \\
&|E\xi e^{\lambda\xi}|^2 \\
&\quad \le 2\bigl(\max(e^{\lambda}, e/\lambda) + 2\bigr)^2 b^2 EX^2 I_{\{|bX|>1\}} + 2(1 + 9\lambda e^{\lambda})^2 \bigl(b^2 EX^2 I_{\{|bX|\le 1\}}\bigr)^2 \\
&\quad \le 2\bigl(\max(e^{\lambda}, e/\lambda) + 2\bigr)^2 b^2 EX^2 I_{\{|bX|>1\}} + 2(1 + 9\lambda e^{\lambda})^2 b^3 E|X|^3 I_{\{|bX|\le 1\}}.
\end{aligned}
$$

□

PROOF OF LEMMA 6.3. Let

$$V_i(u) = Ee^{\lambda\xi_i} I_{\{\xi_i \le u\}} / Ee^{\lambda\xi_i}.$$

Consider the sequence of independent random variables $\{\eta_i, 1 \le i \le n\}$ with η_i having the distribution function $V_i(u)$. Denote by $F_n(x)$ the distribution function of the random variable $(\sum_{i=1}^n (\eta_i - E\eta_i))/(\sum_{i=1}^n \mathrm{Var}(\eta_i))^{1/2}$. In terms of the conjugate method [cf. (4.9) of Petrov (1965)], we have

$$P\left(\sum_{i=1}^n \xi_i \ge y\right) = \left(\prod_{i=1}^n Ee^{\lambda\xi_i}\right) e^{-\lambda m(\lambda)} \int_{-(m(\lambda)-y)/\sigma(\lambda)}^{\infty} e^{-\lambda\sigma(\lambda)t}\, dF_n(t).$$

By (6.10) and the Chebyshev inequality,

$$
\begin{aligned}
&\int_{-(m(\lambda)-y)/\sigma(\lambda)}^{\infty} e^{-\lambda\sigma(\lambda)t}\, dF_n(t) \\
&\qquad \ge \int_{-2}^{2} e^{-\lambda\sigma(\lambda)t}\, dF_n(t)
\end{aligned}
$$

$$\ge e^{-2\lambda\sigma(\lambda)} P\left(\left|\sum_{i=1}^n (\eta_i - E\eta_i)\right| \le 2\left(\sum_{i=1}^n \mathrm{Var}(\eta_i)\right)^{1/2}\right)$$
$$\ge \tfrac{3}{4} e^{-2\lambda\sigma(\lambda)}.$$

This reduces to (6.9). □

PROOF OF LEMMA 6.4. When $a \le 1$, (6.11) is trivial. When $a > 1$, let $\{\eta_i, 1 \le i \le n\}$ be an independent copy of $\{\xi_i, 1 \le i \le n\}$. Then by the Chebyshev inequality,

$$\begin{aligned} P&\left(\left|\sum_{i=1}^n \eta_i\right| \le 2D_n, \sum_{i=1}^n \eta_i^2 \le 4D_n^2\right) \\ &\ge 1 - P\left(\left|\sum_{i=1}^n \eta_i\right| > 2D_n\right) - P\left(\sum_{i=1}^n \eta_i^2 > 4D_n^2\right) \\ &\ge 1 - \tfrac{1}{4} - \tfrac{1}{4} = \tfrac{1}{2}. \end{aligned}$$

Let $\{\varepsilon_i, 1 \le i \le n\}$ be a Rademacher sequence independent of $\{\xi_i, 1 \le i \le n\}$ and $\{\eta_i, 1 \le i \le n\}$. Noting that

$$\begin{aligned} &\left\{\left|\sum_{i=1}^n \xi_i\right| \ge a\left(4D_n + \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}\right), \left|\sum_{i=1}^n \eta_i\right| \le 2D_n, \sum_{i=1}^n \eta_i^2 \le 4D_n^2\right\} \\ &\quad\subset \left\{\left|\sum_{i=1}^n (\xi_i - \eta_i)\right| \ge a\left(4D_n + \left(\sum_{i=1}^n (\xi_i - \eta_i)^2\right)^{1/2} - \left(\sum_{i=1}^n \eta_i^2\right)^{1/2}\right) - 2D_n,\right. \\ &\qquad\qquad \left.\sum_{i=1}^n \eta_i^2 \le 4D_n^2\right\} \\ &\quad\subset \left\{\left|\sum_{i=1}^n (\xi_i - \eta_i)\right| \ge a\left(2D_n + \left(\sum_{i=1}^n (\xi_i - \eta_i)^2\right)^{1/2}\right) - 2D_n\right\} \\ &\quad\subset \left\{\left|\sum_{i=1}^n (\xi_i - \eta_i)\right| \ge a\left(\sum_{i=1}^n (\xi_i - \eta_i)^2\right)^{1/2}\right\} \end{aligned}$$

and that $\{\xi_i - \eta_i, 1 \le i \le n\}$ is a sequence of independent symmetric random variables, we have

$$
\begin{aligned}
P&\left(\left|\sum_{i=1}^n \xi_i\right| \ge a\left(4D_n + \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}\right)\right) \\
&= P\left(\left|\sum_{i=1}^n \xi_i\right| \ge a\left(4D_n + \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}\right), \left|\sum_{i=1}^n \eta_i\right| \le 2D_n, \sum_{i=1}^n \eta_i^2 \le 4D_n^2\right) \\
&\qquad \times \left(P\left(\left|\sum_{i=1}^n \eta_i\right| \le 2D_n, \sum_{i=1}^n \eta_i^2 \le 4D_n^2\right)\right)^{-1} \\
&\le 4P\left(\left|\sum_{i=1}^n (\xi_i - \eta_i)\right| \ge a\left(\sum_{i=1}^n (\xi_i - \eta_i)^2\right)^{1/2}\right) \\
&= 4P\left(\left|\sum_{i=1}^n \varepsilon_i(\xi_i - \eta_i)\right| \ge a\left(\sum_{i=1}^n (\xi_i - \eta_i)^2\right)^{1/2}\right) \\
&\le 8e^{-a^2/2}
\end{aligned}
$$

as desired, where in the last inequality, we used the following inequality

$$
P\left(\left|\sum_{i=1}^n \varepsilon_i x_i\right| \ge a\left(\sum_{i=1}^n x_i^2\right)^{1/2}\right) \le 2e^{-a^2/2}
$$

for any real numbers x_i, $1 \le i \le n$ and $a > 0$; see, for example, Ledoux and Talagrand [(1991), page 90]. □

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