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FINITARY CODING FOR THE ONE-DIMENSIONAL T, T^{-1} PROCESS WITH DRIFT[1]

BY MICHAEL KEANE AND JEFFREY E. STEIF[2]

Wesleyan University and Chalmers University of Technology

We show that there is a finitary isomorphism from a finite state independent and identically distributed (i.i.d.) process to the T, T^{-1} process associated to one-dimensional random walk with positive drift. This contrasts with the situation for simple symmetric random walk in any dimension, where it cannot be a finitary factor of any i.i.d. process, including in $d \geq 5$, where it becomes weak Bernoulli.

Kalikow (1982) proved that the T, T^{-1} process associated to simple symmetric random walk in one dimension (to be defined below) is not a Bernoulli shift, solving a problem that had been open for over 10 years. A relatively extensive study of the T, T^{-1} processes associated to arbitrary random walks on $\mathbb{Z}^d$ was done by den Hollander and Steif (1997), who studied how the properties of Bernoulli and weak Bernoulli are reflected in the behavior of the underlying random walk. It turns out that for simple symmetric random walk, the behavior changes between two and three dimensions (changing from not being Bernoulli to being Bernoulli) and between four and five dimensions (changing from not being weak Bernoulli to being weak Bernoulli) as well. In Steif (2001), it was shown that for any $d \geq 1$, if the random walk has mean 0 or is symmetric, then the color process $\{C_{S_i}\}$ (defined below) is not a finitary factor of any independent and identically distributed (i.i.d.) process. Here we show that for simple random walk with positive drift in one dimension, there is a finitary isomorphism from a finite state i.i.d. process to the corresponding T, T^{-1} process.

We first give the definition of the general T, T^{-1} process. We are slightly terse; the reader may refer to den Hollander and Steif (1997) for full details. For a fixed integer $d \geq 1$, let $\{X_i\}_{i\in\mathbb{Z}}$ be an i.i.d. process taking values in $\mathbb{Z}^d$. Let $\{S_n\}_{n\in\mathbb{Z}}$ be

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the corresponding *random walk* on $\mathbb{Z}^d$ defined by

$$S_0 = 0, \qquad S_n = \begin{cases} \sum_{i=1}^{n} X_i, & n \geq 1, \\ -\sum_{i=n+1}^{0} X_i, & n \leq -1. \end{cases}$$

Next, let $\{C_z\}_{z\in\mathbb{Z}^d}$ be i.i.d. random variables taking values 1 and 0 each with probability $1/2$ and independent of $\{X_i\}_{i\in\mathbb{Z}}$.

Now consider the process

$$Z := \{Z_i\}_{i\in\mathbb{Z}} \qquad \text{where } Z_i = (X_i, C_{S_i}),$$

which we call the *T, T^{-1} process associated with* $\{X_i\}_{i\in\mathbb{Z}}$. It is easy to see that $\{Z_i\}_{i\in\mathbb{Z}}$ is a stationary process; it is essentially a so-called *skew product* in ergodic theory. Note, importantly, that even if the random walk is living in $\mathbb{Z}^d$, the T, T^{-1} process is always a process indexed by $\mathbb{Z}$. The name "T, T^{-1} process" comes from the case where the random walk is simple symmetric random walk in one dimension and where T refers to the shift of the color process; then from the point of view of the walker, the color sequence is moving by T or T^{-1}. This particular process is the simplest example of a process which is a K automorphism but not a Bernoulli shift; the existence of such processes had been a major open question (the T, T^{-1} process for simple random walk in one dimension was not the first such example, but is the simplest).

We assume that the reader is familiar with the notions of (a) Bernoulli shifts, (b) factor maps and (c) finitary factor maps. All of these can be found in the union of den Hollander and Steif (1997) and Keane and Smorodinsky (1977). However, we give a very concise description of these concepts which possibly suffices in itself for the reader who is not familiar with the above works. Given two stationary processes, a *factor map* from one to the other is a mapping which takes realizations of the first process to realizations of the second process in a translation invariant way. If the factor map is invertible, then it is called an *isomorphism* and the two processes are called *isomorphic*. A *Bernoulli shift* is a process which is isomorphic to an i.i.d. process. A factor map or isomorphism is *finitary* if for almost every realization of the first process, there is an n depending on the realization such that the 0th bit of the image realization is already determined by the bits between times $-n$ and n of the first realization. These are the types of codings that are "implementable" and are equivalent to the factor map being continuous a.s. See Meshalkin (1959) for the first beautiful example of a nontrivial finitary isomorphism (whose inverse is also finitary). By a *finite-valued process*, we mean a process which takes on only a finite number of values.

In this article, we simply consider the case of one-dimensional simple random walk with drift. We prove the following theorem.

THEOREM 1. *For the case where $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$ with $p > 1/2$, there exists a finitary isomorphism from a finite-valued i.i.d. process to $\{Z_n\}$.*

REMARKS. (i) The inverse of the isomorphism that we construct will not be finitary. In particular, our definition of a *finitary isomorphism* is simply an isomorphism which is also finitary; there is no demand that the inverse is also finitary.

(ii) The existence of such a finitary isomorphism does not immediately follow from the fact that $\{Z_n\}$ is Bernoulli or even weak Bernoulli. In fact, restricting to simple symmetric random walk in d dimensions, it is known [see den Hollander and Steif (1997)] that $\{Z_n\}$ is Bernoulli if $d \geq 3$ and weak Bernoulli if $d \geq 5$, but nonetheless, in these cases [see Steif (2001)], even the second coordinate of $\{Z_n\}$ is not a finitary factor of any i.i.d. process.

The proof consists of two parts. The first part is to construct a countable state Markov chain and an isomorphism from this Markov chain to $\{Z_n\}$ which is finitary (although the inverse will not be finitary). The key idea here is to exploit the excursion structure of our transient random walk. The second half of the proof is to construct a finitary isomorphism (whose inverse is also finitary) from an i.i.d. process to the Markov chain constructed in the first step. This second part is done, in part, by using the main result in Rudolph (1982). Then we simply compose these isomorphisms.

1. Proof of main result.

First step. We now construct a countable state Markov chain. Rather than defining it by specifying its transition probabilities and then checking that there is a stationary distribution, we instead define it as a (nonfinitary) factor of an i.i.d. process.

Formally, the state space is as follows. Let $\mathcal{S}$ denote the set of finite ± 1 sequences $\{U_i\}_{-k \leq i \leq l}$ indexed by $\{-k, \ldots, l\}$ for some integers $k, l \geq 0$ and satisfying

$$U(-k) = 1, \qquad \sum_{i=-k}^{r} U_i \geq 1 \qquad \text{for all } r \in \{-k, \ldots, l\} \quad \text{and} \quad \sum_{i=-k}^{l} U_i = 1.$$

The state space for the Markov chain will be $\mathcal{S} \times \{0, 1\}$. A heuristic description of this chain is given below.

Let $X := \{X_i\}_{i \in \mathbb{Z}}$ be the steps for our random walk [so $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$] and let $C := \{C_i\}_{i \in \mathbb{Z}}$ be i.i.d. random variables taking values 1 and 0 each with probability $1/2$ and independent of the X process. We now define a Markov chain $Y := \{Y_i\}_{i\in\mathbb{Z}}$ with state space $\mathcal{S} \times \{0, 1\}$ which is a factor of $\{(X_i, C_i)\}_{i\in\mathbb{Z}}$. The first coordinate of the Y process depends only on the X process.

For $r < s \in \mathbb{Z}$, write $r \sim s$ if $\sum_{r+1}^{s} X_i = 0$. (This sort of means that the walk is at the same location at times r and s.) Let

$$G := \left\{n : \sum_{k=n+1}^{l} X_k \geq 1 \text{ for all } l \geq n + 1\right\}$$

or, equivalently,

$$G := \{n : \nexists\, k > n \text{ with } k \sim n\}.$$

(These are just the times at which the walk is about to leave and never return to its present position.) Note that if $r < s$ are two successive elements in G, then we must have $\sum_{r+1}^{s} X_i = 1$.

Fix $n \in \mathbb{Z}$. The first coordinate of Y_n, denoted by Y_n^1, is defined as follows. Heuristically, Y_n^1 describes that piece of the random walk between the last time in G before time n and the first time in G after time n. First, let

$$A_n := \sup\{k \leq n - 1 : k \in G\}$$

and

$$B_n := \inf\{k \geq n : k \in G\}.$$

Next let $k_n = n - A_n - 1$ and $l_n = B_n - n$, noting that $k_n, l_n \geq 0$. For $i \in \{-k_n, \dots, l_n\}$, let $(Y_n^1)_i = X_{n+i}$. It is easy to see that $\{(Y_n^1)_i\}_{-k_n \leq i \leq l_n} \in \mathcal{S}$. The second coordinate, denoted by Y_n^2, is defined to be C_{B_n}. This completes the definition of Y. It is clear that it is a factor of $\{(X_i, C_i)\}_{i\in\mathbb{Z}}$. (Note that the factor map is not invertible since, from the Y process, we can determine the C_i's only at the locations in G; it is also easy to see that this factor map is not finitary.) Some thought reveals the crucial fact that Y is a (countable state) Markov chain.

We now construct a finitary coding from Y to Z. Fix $m \in \mathbb{Z}$. Let

$$G_0^{(m)} = \inf\{r \geq m : l_r = 0\}$$

and, for $i \geq 1$, let

$$G_i^{(m)} = \inf\{r \geq G_{i-1}^{(m)} + 1 : l_r = 0\}.$$

(Of course the $G_i^{(m)}$'s are just the successive elements of G after or at time m, but the point is that we are expressing them finitarily in terms of the Y process.) Next, we let $R_m := -\sum_{j=1}^{l_m} (Y_m^1)_j$ (when $l_m = 0$, the sum is taken to be 0); it is easy to see that this is ≥ 0. (In words, R_m is the number of steps "down" the walker has to go from time m until the first time he or she is in G.) Finally we let f map Y to Z via

$$f(Y)_m = \Big((Y_m^1)_0, Y^2_{G^{(m)}_{R_m}}\Big).$$

Some thought reveals that this mapping is a finitary isomorphism from Y to Z (although the inverse is not finitary).

Second step. In this step, we apply the following theorem from Rudolph (1982) to our countable state Markov chain Y.

THEOREM 2 (Rudolph). *If a countable state Markov chain is mixing, has finite entropy and there exists a state such that the return probabilities to this state decay exponentially, then there is a finitary isomorphism that has an inverse that is also finitary from a finite-valued i.i.d. process with the same entropy to this Markov chain.*

We now verify that Y satisfies the three properties in Theorem 2. Once we do this, Theorem 2 together with what we did in step one proves Theorem 1 (simply by composing the two finitary isomorphisms).

First, Y was obtained as a (nonfinitary) factor of an i.i.d. process and hence is mixing. For the finite entropy condition, we show the stronger fact that the one-dimensional marginal has finite entropy. Since the second coordinate of Y_0 has just two states, we need only show that Y_0^1 has finite entropy. The following lemma follows immediately from standard large deviation results for i.i.d. processes.

LEMMA 1. *Let $\{X_i\}_{i\geq 1}$ be i.i.d. random variables with $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$ with $p > 1/2$. Let $N := \max\{n : S_n = 0\}$. Then*

$$P(N \geq k) \leq C_1 e^{-C_2 k}$$

for all k and some constants C_1 and C_2 in $(0, \infty)$.

It is easy to check that Lemma 1 immediately implies that

$$P(\{l_0 \geq n\}) \leq C_3 e^{-C_4 n}$$

for all n and some constants C_3 and C_4 in $(0, \infty)$. An only slightly more involved argument gives a similar bound for $P(\{k_0 \geq n\})$. (There is an asymmetry between k_0 and l_0 due to the asymmetry in the definition of G which "looks into the future" rather than "into the past.")

Let $R := k_0 + l_0$. The above analysis implies that R has an exponential tail and hence has finite entropy. Next, given $R = r$, there are at most $(r + 1)2^r$ possibilities for Y_0^1. This together with the fact that R has an exponential tail gives us that the conditional entropy of Y_0^1 given R is finite and hence Y_0^1 itself has finite entropy, as desired.

Last, we find a state for Y which has exponentially decaying return time probabilities. The state we take is $s := (\{1\}, 0)$, noting that the corresponding k and l are both 0. (Note also that $Y_n^1 = \{1\}$ if and only if n and $n-1$ are both in G.)

The following elementary fact, the proof of which is left to the reader, is needed.

LEMMA 2. *Let $\{Y_i\}_{i\geq 1}$ be i.i.d. positive-valued random variables such that $P(Y_1 = 1) > 0$ and $P(Y_1 \geq n) \leq e^{-cn}$ for all n and some $c > 0$. Let $T := \inf\{n : Y_n = 1\}$. Then for some $c' > 0$,*

$$P\left(\sum_{i=1}^{T} Y_i \geq n\right) \leq e^{-c'n}$$

for all n.

We now apply Lemma 2 with $Y_i := G_i^{(0)} - G_{i-1}^{(0)}$ for $i \geq 1$ which is easily checked to be i.i.d. Lemma 1 can be used to show that these random variables satisfy the main hypothesis of Lemma 2. The conclusion of Lemma 2 to these random variables then allows us to conclude the exponential decay of the return probabilities to the state s. This completes the proof of Theorem 1.

REMARK. The proof of step one, which shows that there is a finitary isomorphism (although the inverse is not finitary) from the countable state Markov chain which we constructed to the T, T^{-1} process, is direct. The second part was based on Theorem 2, which in turn is based on a fairly long and involved article [Rudolph (1981)]. A more direct proof of Theorem 2 along the lines of Keane and Smorodinsky (1977, 1979a, b) might also be possible. The motivation for Rudolph (1981) came, in any case, from the desire to generalize Keane and Smorodinsky (1979a).

REMARK. With some more work, the argument of Theorem 1 can be generalized to show that there is a finitary homomorphism from a finite state i.i.d. process to the T, T^{-1} process associated to any nearest neighbor random walk in $\mathbb{Z}^d$ with drift. In such a case, there will be a direction, say the x axis, which has positive drift, in which case we can then exploit the renewal structure which is present for the x coordinate of the random walk exactly as we did in the proof of Theorem 1.

REFERENCES

DEN HOLLANDER, F. and STEIF, J. E. (1997). Mixing properties of the generalized T, T^{-1}-process. *J. Anal. Math.* **72** 165–202.

KALIKOW, S. (1982). T, T^{-1} transformation is not loosely Bernoulli. *Ann. of Math.* (2) **115** 393–409.

KEANE, M. and SMORODINSKY, M. (1977). A class of finitary codes. *Israel J. Math.* **26** 352–371.

KEANE, M. and SMORODINSKY, M. (1979a). Bernoulli schemes of the same entropy are finitarily isomorphic. *Ann. of Math.* (2) **109** 397–406.

KEANE, M. and SMORODINSKY, M. (1979b). Finitary isomorphisms of irreducible Markov shifts. *Israel J. Math.* **34** 281–286.

MESHALKIN, L. D. (1959). A case of isomorphism of Bernoulli schemes. *Dokl. Akad. Nauk. SSSR* **128** 41–44.

RUDOLPH, D. J. (1981). A characterization of those processes finitarily isomorphic to a Bernoulli shift. *Progr. Math.* **10** 1–64.

RUDOLPH, D. J. (1982). A mixing Markov chain with exponentially decaying return times is finitarily Bernoulli. *Ergodic Theory Dynam. Systems* **2** 85–97.

STEIF, J. E. (2001). The T, T^{-1}-process, finitary codings and weak Bernoulli. *Israel J. Math.* **125** 29–43.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KEIO UNIVERSITY
YOKOHAMA
JAPAN
AND
DEPARTMENT OF MATHEMATICS
WESLEYAN UNIVERSITY
MIDDLETOWN, CONNECTICUT
AND
KORTEWEG–DE VRIES INSTITUTE
FOR MATHEMATICS
UNIVERSITY OF AMSTERDAM
AMSTERDAM
THE NETHERLANDS
E-MAIL: mkeane@wesleyan.edu

DEPARTMENT OF MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY
S–41296 GÖTEBORG
SWEDEN
AND
DEPARTMENT OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332-1060
E-MAIL: steif@math.chalmers.se
URL: www.math.chalmers.se/~steif/