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BOOK REVIEW

GREGORY F. LAWLER, *Intersections of Random Walks. Probability and Its Applications.* Birkhauser, Boston, 1991, 219 pages

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The contents of the book are as follows:

Random walks have fascinated and perplexed the mathematical community for about a century. Although there are a variety of complications and variations by means of which the basic model can be generalized, the behavior in the simplest case is already complex and surprising.

Consider a symmetric nearest-neighbor random walk on the integer lattice $\mathbf{Z}^d$. To what extent does the behavior of the walker depend upon the dimension d? On one hand, the mean-squared displacement is independent of dimension and $E(|S_n|^2) = n$ for every natural number n, where S_n is the walker's position after n steps. On the other, Polya proved in 1921 that if $d \le 2$, such a walk is recurrent, whereas if $d \ge 3$, then the walk is transient.

The intersection properties considered by Gregory Lawler in *Intersections of Random Walks* are invariably dimension-dependent. The starting point for his investigations are the probabilities $p_n(x)$ that a walk beginning at the origin reaches the node $x \in \mathbf{Z}^d$ at the completion of its nth step. The first observation is that this probability can only be positive if the parity of n matches that of the sum of the components of x, in which case we write $n \leftrightarrow x$. The next observation is that the central limit theorem implies that $n^{-1/2}S_n$ converges in distribution to a normally distributed random variable in R^d.

A heuristic argument suggests that for large n, $p_n(x)$ should be approximately equal to

$$\overline{p}_n(x) = 2\left(\frac{d}{2\pi n}\right)^{1/2} \exp\left(\frac{-d|x|^2}{2n}\right).$$

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This is made precise by letting

$$E(n,x)=\begin{cases} p_n(x)-\overline{p}_n(x), & \text{if } n\leftrightarrow x,\\ 0, & \text{otherwise,}\end{cases}$$

and deriving estimates on $|E(n,x)|$, $|\nabla_y E(n,x)|$ and $|\nabla_y^2 E(n,x)|$. Here

$$\nabla_y f(x)=f(x+y)-f(x)$$

and

$$\nabla_y^2 f(x)=f(x+y)-2f(x)+f(x-y).$$

The *local central limit theorem* establishes the following: if $y\leftrightarrow 0$, then

$$|\nabla_y^i E(n,x)|\le c_y O(n^{-(d+i+2)/2})$$

and

$$|\nabla_y^i E(n,x)|\le c_y |x|^{-2} O(n^{-(d+i)/2})$$

for $i=0,1,2$, where c_y depends only on y.

Lawler gratefully acknowledges his debt to Spitzer's book [11] for his exposition of the local central limit theorem, as well as for other standard results which are proved in the opening chapter. The intended audience of this book would appear to be upper level graduate students and working research mathematicians because so many standard results from measure theory and a year of graduate level probability are assumed without proof: for example, martingales, stopping times and Brownian motion. Beyond this, the text is almost entirely self-contained, with the exception of certain routine extensions which are included among the exercises (there are some 15 exercises included in the first two chapters).

The most important tool needed for the corpus of theorems in Chapters 3–5 is harmonic measure; this is the topic of Chapter 2. The central question is as follows: given a subset A of $\mathbf{Z}^d$ and a walker who starts a random walk from infinity, what is the probability of a particular node in A being the first one which the walker hits, conditioned on the event that the set A is eventually encountered? The existence of such a measure is derived in such a way that an upper bound on the rate of convergence (which is highly dependent on dimension) is given. The notion of capacity is developed in a way which is independent of the usual interpretation in terms of electrical circuits.

Those familiar with topics from continuous harmonic functions including the Dirichlet problem, Green's function and the Harnack inequality will probably enjoy seeing this material adapted to the discrete case in Chapters 1 and 2, even if they have no interest in subsequent portions of the text.

Chapters 3–5 are devoted to questions involving the probability that the paths of independent random walks will intersect, including lower bounds on the probability that any intersection occurs and expected number of intersections. The primary goal is to analyze the probability that two independent random walks with a common starting point will intersect. The techniques are very different for four dimensions, where the probability goes to zero like the reciprocal of the logarithm of the number of steps, than they are for three dimensions, where the probability goes to zero as the number of steps raised to a negative exponent.

The most important applications of the results in Chapters 1–5 are to self-avoiding random walks. A self-avoiding walk (SAW) is a one-to-one $\mathbf{Z}^d$-valued function ω whose domain is either the natural numbers $\mathbf{N}$ or some initial segment thereof: $\{0, 1, 2, \ldots, n\}$. Such a walk is clearly not Markovian, since the extension of an n-step walk to an $n+1$-step walk clearly depends on the entire history and not simply the node $\omega(n)$. Indeed, an n-step self-avoiding walk may be at a "dead end" in the sense that all of the nearest neighbours of the node $\omega(n)$ could be occupied at earlier times as $\omega(i)$ for some $0 \leq i < n$. For this reason, the SAW cannot be analyzed as a sum of i.i.d. random variables. Although the SAW has been studied by chemists and physicists since the 1960s (as a model for polymer growth [3] and the $N \to 0$ limit of the N-vector model [2], respectively), few rigorous mathematical results are known.

The central problems for SAWs involve counting and asymptotic growth rates. Let C_n be the number of n-step SAWs. It is conjectured that $C_n \sim A\mu^n n^{\gamma-1}$ in all dimensions except $d = 4$, where it is believed that $C_n \sim A\mu^n (\log n)^{1/4}$; the constants A, μ and γ depend only on the dimension d. More importantly for the subject at hand, it is conjectured that

$$\langle |\omega(n)|^2 \rangle \sim \begin{cases} Dn^{2\nu}, & \text{if } d \neq 4, \\ Dn\,(\log n)^{1/4}, & \text{otherwise,} \end{cases}$$

where $\langle \cdot \rangle$ is the expected value over the set of all C_n n-step SAWs equipped with uniform measure. The result clearly holds for $d = 1$ with $D = \nu = 1$. Hara and Slade [5, 6] have recently shown that the result also holds for $d \geq 5$ with $\nu = 1/2$. The question remains open for $d = 2$, 3, and 4. It is conjectured that $\nu = 3/4$ in two dimensions and $\nu = 0.59$ in three dimensions. These conjectures come from Monte Carlo simulation, renormalization group theory and analogy with other models in statistical mechanics. For more details, see the book by Madras and Slade [9] in the same series.

How might one go about generating self-avoiding walks on a computer? Most of the popular methods involve transformations of walks of a fixed length; see [10], for example. An entirely different approach would be as follows: generate a simple random walk of a given length. When this is done, scan the list of nodes visited and look for repeat visits. When this occurs, erase the intervening "loop" in the walk. Continue until all self-intersections have been removed. The result will be a walk, shorter than the original with an unpredictable length, which is self-avoiding. Clearly, all self-avoiding walks can be obtained in such a fashion, but what measure is assigned to the set of all n-step walks by this process?

It was Lawler himself who first proposed such a process in 1980 [7]; he dubbed the walk so obtained a loop-erased self-avoiding walk (LESAW). In Chapter 6 of the book we are considering here, he gives a persuasive heuristic aaccount of why a LESAW should grow at a faster rate than an ordinary SAW, this after introducing the SAW and its associated problems and machinery and describing a couple of other related models: the Joyce–Domb model, the Edwards model and the myopic SAW.

In the final chapter of *Intersections of Random Walks*, Lawler exploits his results of Chapters 3–5 in order to prove that the LESAW is qualitatively the same as the SAW, but is quantitatively different. Specifically, he derives upper bounds for the number of nodes erased during the loop-erasing procedure and uses these to show that the mean-squared displacement $\langle|\lambda(n)|^2\rangle$ has asymptotic lower bounds of the same form as the power laws quoted above for SAWs, but with larger exponents. Perhaps the most important lower bound is for three dimensions: in this case $\langle|\lambda(n)|^2\rangle$ is asymptotically bounded below by $Dn^{6/5}$. If the conjecture concerning SAWs is correct, then this clearly distinguishes LESAWs from SAWs. In two dimensions, Lawler shows that the asymptotic lower bound for $\langle|\lambda(n)|^2\rangle$ is $Dn^{3/2}$. This does not distinguish the behavior of the LESAW from the conjectured behavior of the SAW, but Lawler hypothesizes that the true growth rate for the three-dimensional LESAW is more like $n^{8/5}$. This conjecture is borne out by Monte Carlo results in the literature [4, 1]. In four dimensions, $\langle|\lambda(n)|^2\rangle/n$ is shown to be asymptotically bounded between $(\log n)^{1/3}$ and $(\log n)^{1/2}$, again outstripping the growth of the SAW. (More recently, Lawler has shown $\langle|\lambda(n)|^2\rangle/n$ to be asymptotically bounded by $(\log n)^{1/3}$; see [8].) Finally, Lawler shows that the LESAW, like the SAW, behaves like Brownian motion in dimensions five and higher.

It would only be a modest exaggeration to call this book a gem. It is relatively short and has a unity of purpose, both of which serve to give it a manageable scale. It is very technical, especially in Chapters 2–5, but the organization is efficient without descending into slickness. Most of the results are Lawler's own, drawn from a series of at least eight articles published between 1980 and 1990; it is far more satisfying to review these results as a coherent and well-organized whole than in a piecemeal fashion. This book should be considered almost indispensable for those with an interest in self-avoiding walks and will appeal to anyone interested in applications of probability theory and harmonic measure.

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