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LARGE DEVIATIONS AND LAW OF THE ITERATED LOGARITHM FOR PARTIAL SUMS NORMALIZED BY THE LARGEST ABSOLUTE OBSERVATION

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Let $\{X_n,\ 1 \le n < \infty\}$ be a sequence of independent identically distributed random variables in the domain of attraction of a stable law with index $0 < \alpha < 2$. The limit of $x_n^{-1} \log P\{S_n/\max|X_i| \ge x_n\}$ is found when $x_n \to \infty$ and $x_n/n \to 0$. The large deviation result is used to prove the law of the iterated logarithm for the self-normalized partial sums.

1. Introduction and main results. Let $\{X, X_i,\ 1 \le i < \infty\}$ be a sequence of independent, identically distributed nondegenerate random variables (i.i.d. r.v.'s) with distribution function F. Put $S_n = \sum_{1\le i\le n} X_i$ and $M_n = \max_{1\le i\le n}|X_i|$. The influence of the extreme term M_n on the partial sum S_n has attracted considerable attention over the years. Assuming that F is in the domain of attraction of a stable law, Darling (1952) discussed some limiting properties of

$$T_n = S_n/M_n. \tag{1.1}$$

For further results on the effect of M_n on S_n, we refer to Feller (1968), Arov and Bobrov (1960), Hall (1978) and Bingham and Teugels (1981). Maller and Resnick (1984) investigated stability questions for T_n and obtained the necessary and sufficient conditions for the almost sure and "in probability" stability.

The natural generalization of T_n is $T_n^*(p) = S_n/\left(\sum_{1\le i\le n}|X_i|^p\right)^{1/p}$, $0 < p \le \infty$. If $p = 2$, this is just the well-known t-statistic (with an obvious transform) and its distribution under nonstandard conditions was studied by Hotelling (1961) and Efron (1969). Logan, Mallows, Rice and Shepp (1973), Le Page, Woodroofe and Zinn (1981) and Csörgő and Horváth (1988) derived the asymptotic distribution of $T_n^*(p)$, $0 < p < \infty$, when F is in the domain of attraction of a stable law. Griffin and Kuelbs (1989, 1991) showed that the law of the iterated logarithm holds for $T_n^*(2)$ whereas it fails for S_n if F is in the domain of attraction of the normal distribution. Shao (1994) obtained the law of iterated logarithm with precise constant for the lim sup of $T_n^*(p)$, $1 < p < \infty$, when F is in the domain of attraction of a stable or normal law. The main aim of this paper is to prove the law of the iterated logarithm for T_n.

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The proof of the law of the iterated logarithm for T_n is based on the following large deviation result:

THEOREM 1.1. *We assume that there exist* $0 < \alpha < 2$, $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 > 0$ *and* l, *a slowly varying function at* ∞, *such that*

$$P\{X \geq x\} = \frac{c_1 + o(1)}{x^\alpha}\, l(x) \quad \textit{and} \quad P\{X \leq -x\} = \frac{c_2 + o(1)}{x^\alpha}\, l(x) \tag{1.2}$$

as $x \to \infty$. *We also assume that*

$$EX = 0 \quad \textit{if} \quad 1 < \alpha < 2, \tag{1.3}$$

$$X \textit{ is symmetric if} \quad \alpha = 1 \tag{1.4}$$

and

$$c_1 > 0 \quad \textit{if} \quad 0 < \alpha < 1. \tag{1.5}$$

If $\{x_n,\ 1 \leq n < \infty\}$ *is a sequence of positive numbers satisfying*

$$\lim_{n\to\infty} x_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} x_n/n = 0, \tag{1.6}$$

then we have

$$\lim_{n\to\infty} \frac{1}{x_n} \log P\{T_n \geq x_n\} = -\tau(\alpha, c_1, c_2), \tag{1.7}$$

where $\tau = \tau(\alpha, c_1, c_2) > 0$ *is the solution of* $f(\tau) = c_1 + c_2$ *with*

$$f(t) = \begin{cases} \dfrac{t\alpha(c_2 - c_1)}{\alpha - 1} + c_1\alpha \displaystyle\int_0^1 \frac{e^{tx} - 1 - tx}{x^{\alpha+1}}\,dx & \\ \qquad + c_2\alpha \displaystyle\int_0^1 \frac{e^{-tx} - 1 + tx}{x^{\alpha+1}}\,dx, & \textit{if } 1 < \alpha < 2, \\ c_1 \displaystyle\int_0^1 \frac{e^{tx} + e^{-tx} - 2}{x^2}\,dx, & \textit{if } \alpha = 1, \\ c_1\alpha \displaystyle\int_0^1 \frac{e^{tx} - 1}{x^{\alpha+1}}\,dx + c_2\alpha \displaystyle\int_0^1 \frac{e^{-tx} - 1}{x^{\alpha+1}}\,dx, & \textit{if } 0 < \alpha < 1. \end{cases} \tag{1.8}$$

The following result is an immediate consequence of Theorem 1.1.

THEOREM 1.2. *We assume that* X *is symmetric and there exist* $0 < \alpha < 2$ *and* l, *a slowly varying function at* ∞, *such that*

$$P\{X \geq x\} = \frac{c + o(1)}{x^\alpha} l(x) \quad \textit{as} \quad x \to \infty. \tag{1.9}$$

If $\{x_n,\ 1 \leq n < \infty\}$ *is a sequence of positive numbers satisfying* (1.6), *then we have*

$$\lim_{n\to\infty} \frac{1}{x_n} \log P\{T_n \geq x_n\} = -\tau(\alpha), \tag{1.10}$$

TABLE 1

α	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$\tau(\alpha)$	5.1164	4.2474	3.7366	3.3749	3.0955	2.8682	2.6768	2.5117	2.3666	2.2370
α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.0
$\tau(\alpha)$	2.120	2.0131	1.9147	1.8235	1.7383	1.6583	1.5827	1.5110	1.4426	1.3770
α	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50
$\tau(\alpha)$	1.3140	1.2532	1.1943	1.1369	1.0809	1.0259	0.9718	0.9182	0.8651	0.8120
α	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	
$\tau(\alpha)$	0.7586	0.7047	0.6496	0.5929	0.5338	0.4710	0.4025	0.3244	0.2264	

where $\tau = \tau(\alpha) > 0$ *is the solution of*

$$\int_0^1 \frac{e^{\tau x} + e^{-\tau x} - 2}{x^{\alpha+1}}\, dx = \frac{2}{\alpha}. \tag{1.11}$$

Having the large deviation result in Theorem 1.1 we can prove the law of the iterated logarithm for T_n.

THEOREM 1.3. *If the conditions of Theorem* 1.1 *are satisfied*, *then we have*

$$\limsup_{n\to\infty} \frac{T_n}{\log\log n} = \frac{1}{\tau(\alpha, c_1, c_2)} \quad a.s. \tag{1.12}$$

It is interesting to note that the form of the law of iterated logarithm in (1.12) is different from the related results for S_n in the stable case when nonrandom normalizers are used [cf. Mijnheer (1975) for a review]. Also, Shao (1994) showed that under the conditions of Theorem 1.3,

$$\limsup_{n\to\infty} T_n^*(p)/(2\log\log n)^{(p-1)/p} = \gamma \quad \text{a.s.} \tag{1.13}$$

with some constant $\gamma = \gamma(p, \alpha, c_1, c_2) > 0$ for $p > 1$. We point out that $\log\log n$ is replaced by $(\log\log n)^{(p-1)/p}$ when we compare (1.12) with (1.13).

REMARK 1.1. It is easy to see that $e^x + e^{-x} - 2 \geq x^2$ for every $x \in R$. Therefore, the solution of (1.11) satisfies $\tau(\alpha) \leq \big(2(2-\alpha)/\alpha\big)^{1/2}$. Table 1 provides some numerical values of $\tau(\alpha)$.

REMARK 1.2. For the large deviation result for $T_n^*(p)$ $(p > 1)$ we refer to Shao (1994).

The proofs of Theorems 1.1–1.3 will be presented in the following sections.

2. Proof of Theorem 1.1. We recall that a positive function $l(x)$ defined on $[a, \infty)$, $a \geq 0$, is called slowly varying (at ∞) if

$$\lim_{x\to\infty} \frac{l(tx)}{l(x)} = 1 \quad \text{for all } t > 0.$$

We need the following properties of slowly varying functions:

$$\text{(H1)} \qquad \lim_{x\to\infty} \sup_{c\le t\le C} \left| \frac{l(tx)}{l(x)} - 1 \right| = 0 \quad \text{for all } 0 < c \le C < \infty.$$

(H2) For all $\varepsilon > 0$, we have that $\lim_{x\to\infty} x^{-\varepsilon} l(x) = 0$ and $\lim_{x\to\infty} x^{\varepsilon} l(x) = \infty$.
(H3) For every $\varepsilon > 0$, there exists x_0 such that for x, t satisfying $xt \ge x_0$,

$$\left(\max\left(t, \frac{1}{t}\right)\right)^{-\varepsilon} \le \frac{l(tx)}{l(x)} \le \left(\max\left(t, \frac{1}{t}\right)\right)^{\varepsilon}.$$

$$\text{(H4)} \qquad \lim_{x\to\infty} \int_a^x y^{\theta} l(y)\, dy \bigg/ \left(\frac{x^{\theta+1} l(x)}{\theta+1}\right) = 1 \quad \text{for any } \theta > -1.$$

$$\text{(H5)} \qquad \lim_{x\to\infty} \int_x^{\infty} y^{\theta} l(y)\, dy / (x^{\theta+1} l(x)) = -\frac{1}{1+\theta} \quad \text{for any } \theta < -1.$$

The proofs of (H1)–(H5) can be found, for example, in Bingham, Goldie and Teugels (1987).

We start with some elementary properties of $f(t)$ defined in (1.8). It is easy to check that

$$f'(t) = \begin{cases} \dfrac{\alpha(c_2 - c_1)}{\alpha - 1} + c_1\alpha \displaystyle\int_0^1 \frac{e^{tx} - 1}{x^{\alpha}}\, dx \\ \qquad + c_2\alpha \displaystyle\int_0^1 \frac{1 - e^{-tx}}{x^{\alpha}}\, dx, & \text{if } 1 < \alpha < 2, \\ c_1 \displaystyle\int_0^1 \frac{e^{tx} - e^{-tx}}{x}\, dx, & \text{if } \alpha = 1, \\ c_1\alpha \displaystyle\int_0^1 \frac{e^{tx}}{x^{\alpha}}\, dx - c_2\alpha \int_0^1 \frac{e^{-tx}}{x^{\alpha}}\, dx, & \text{if } 0 < \alpha < 1, \end{cases} \tag{2.1}$$

and therefore

$$\lim_{t\downarrow 0} f'(t) = \begin{cases} \dfrac{\alpha(c_2 - c_1)}{\alpha - 1}, & \text{if } 1 < \alpha < 2, \\ 0, & \text{if } \alpha = 1, \\ \dfrac{\alpha(c_1 - c_2)}{1 - \alpha}, & \text{if } 0 < \alpha < 1, \end{cases} \tag{2.2}$$

$$\lim_{t\to\infty} f'(t) = \infty \quad \text{for all } 0 < \alpha < 2.$$

It follows from (2.1) that

$$f'(t) \text{ is increasing on } [0, \infty). \tag{2.3}$$

By the definition of $f(t)$, we have

$$f(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} f(t) = \infty \quad \text{for all } 0 < \alpha < 2. \tag{2.4}$$

Now (2.3) and (2.4) imply that there is a unique $\tau = \tau(\alpha, c_1, c_2)$ such that $f(\tau) = c_1 + c_2$ and τ also satisfies $0 < \tau < \infty$. Furthermore, since

$$f(t) \geq \begin{cases} \dfrac{\alpha t(c_2 - c_1)}{\alpha - 1}, & \text{if} \quad c_2 > c_1 \text{ and } 1 < \alpha < 2, \\ \dfrac{(c_1 - c_2)\alpha t}{1 - \alpha}, & \text{if} \quad c_1 > c_2 \text{ and } 0 < \alpha < 1, \end{cases}$$

we immediately obtain

$$\tau < \begin{cases} \dfrac{(c_1 + c_2)(\alpha - 1)}{(c_2 - c_1)\alpha}, & \text{if} \quad c_2 > c_1 \text{ and } 1 < \alpha < 2, \\ \dfrac{(c_1 + c_2)(1 - \alpha)}{\alpha(c_1 - c_2)}, & \text{if} \quad c_1 > c_2 \text{ and } 0 < \alpha < 1. \end{cases} \tag{2.5}$$

Now we can start working on the proof of Theorem 1.1. Let $\{\xi_n,\ 1 \leq n < \infty\}$ be a sequence of positive numbers such that

$$\lim_{n \to \infty} \frac{l(\xi_n)}{\xi_n^\alpha} \bigg/ \left(\frac{x_n}{n}\right) = 1. \tag{2.6}$$

Also, let

$$a_0 = \begin{cases} \left(\dfrac{\alpha(c_2 - c_1)}{\alpha - 1}\right)^{1/\alpha}, & \text{if} \quad c_2 > c_1,\ 1 < \alpha < 2, \\ 0, & \text{if} \quad c_2 \leq c_1,\ 1 < \alpha < 2, \\ 0, & \text{if} \quad \alpha = 1, \\ \left(\dfrac{\alpha(c_1 - c_2)}{1 - \alpha}\right)^{1/\alpha}, & \text{if} \quad c_1 > c_2,\ 0 < \alpha < 1, \\ 0, & \text{if} \quad c_1 \leq c_2,\ 0 < \alpha < 1. \end{cases} \tag{2.7}$$

For any fixed $0 < \varepsilon < 1$ put $a_\varepsilon = a_0 + \varepsilon$. For $A > 1$, we can write

$$\begin{aligned} P\{T_n \geq x_n\} \leq\ & P\Big\{\max_{1 \leq i \leq n} |X_i| I(|X_i| \leq A\xi_n) < a_\varepsilon \xi_n\Big\} \\ & + P\bigg\{\frac{1}{M_n} \sum_{1 \leq i \leq n} X_i I(|X_i| > A\xi_n) \geq \varepsilon x_n\bigg\} \\ & + P\bigg\{\frac{1}{M_n} \sum_{1 \leq i \leq n} X_i I(|X_i| \leq A\xi_n) \geq (1 - \varepsilon) x_n, \\ & \qquad \max\nolimits_{1 \leq i \leq n} |X_i| I(|X_i| \leq A\xi_n) \geq a_\varepsilon \xi_n\bigg\} \\ :=\ & I_1 + I_2 + I_3. \end{aligned} \tag{2.8}$$

By (1.2) and (H3), we have

$$\begin{aligned}
I_1 &= (P\{|X|I(|X| \le A\xi_n) < a_\varepsilon \xi_n\})^n \\
&= (P\{|X| < a_\varepsilon \xi_n\} + P\{|X| > A\xi_n\})^n \\
&= \left(1 - \frac{(c_1+c_2)(1+o(1))l(a_\varepsilon \xi_n)}{(a_\varepsilon \xi_n)^\alpha} + \frac{(c_1+c_2)(1+o(1))l(A\xi_n)}{(A\xi_n)^\alpha}\right)^n \\
&\le \left(1 - \frac{(c_1+c_2)(1+o(1))l(\xi_n)}{(a_\varepsilon \xi_n)^\alpha} + \frac{(c_1+c_2)(1+o(1))l(\xi_n)}{A^{\alpha/2}\xi_n^\alpha}\right)^n \\
&\le \exp\left(-\frac{(c_1+c_2)(1+o(1))l(\xi_n)n}{a_\varepsilon^\alpha \xi_n^\alpha} + \frac{(c_1+c_2)(1+o(1))l(\xi_n)n}{A^{\alpha/2}\xi_n^\alpha}\right) \\
&\le \exp\left(-\frac{(c_1+c_2)(1+o(1))x_n}{a_\varepsilon^\alpha} + \frac{(c_1+c_2)(1+o(1))x_n}{A^{\alpha/2}}\right).
\end{aligned}$$

According to (2.5) and (2.7), we can choose ε small enough so that $(c_1+c_2)a_\varepsilon^{-\alpha} > \tau$. Therefore,

$$I_1 \le \exp(-(1+o(1))x_n\tau + (c_1+c_2)(1+o(1))x_n A^{-\alpha/2}). \tag{2.9}$$

Applying the Chernoff large deviation theorem to the binomial random variable $B(n, p)$, we get

$$P\{B(n, p) \ge an\} \le \left(\frac{ep}{a}\right)^{an} \tag{2.10}$$

for all $a > 0$. By (2.10) we have

$$I_2 \le P\left\{\sum_{1\le i\le n} I(|X_i| \ge A\xi_n) \ge \varepsilon x_n\right\} \le \left(\frac{3nP\{|X| \ge A\xi_n\}}{\varepsilon x_n}\right)^{\varepsilon x_n}, \tag{2.11}$$

and (1.2) and (H3) yield

$$\begin{aligned}
I_2 &\le \left(\frac{6n(c_1+c_2)l(A\xi_n)}{\varepsilon x_n (A\xi_n)^\alpha}\right)^{\varepsilon x_n} \\
&\le \left(\frac{12n(c_1+c_2)l(\xi_n)}{\varepsilon x_n A^{\alpha/2}\xi_n^\alpha}\right)^{\varepsilon x_n} \\
&\le \left(\frac{24n(c_1+c_2)x_n}{\varepsilon x_n A^{\alpha/2} n}\right)^{\varepsilon x_n} \qquad (2.12)\\
&= \exp\left(-\varepsilon x_n \log \frac{\varepsilon A^{\alpha/2}}{24(c_1+c_2)}\right) \\
&\le \exp\left(-\frac{x_n}{\varepsilon}\right),
\end{aligned}$$

provided that $A^{\alpha/2} \ge \varepsilon^{-1} 24(c_1+c_2)\exp(1/\varepsilon^2)$.

To estimate I_3, we introduce $Y = X\,I\{|X| \le A\xi_n\}$ and $Y_i = X_i\,I\{|X_i| \le A\xi_n\}$. Elementary arguments give

$$\begin{aligned}
I_3 &\le P\bigg\{\sum_{1\le i\le n} X_i I(|X_i| \le A\xi_n) \ge (1-\varepsilon)x_n \max_{1\le i\le n}|X_i|I(|X_i| \le A\xi_n),\\
&\qquad\qquad \max_{1\le i\le n}|X_i|I(|X_i|\le A\xi_n) \ge a_\varepsilon \xi_n\bigg\}\\
&= P\bigg\{\sum_{1\le i\le n} Y_i \ge (1-\varepsilon)x_n \max_{1\le i\le n}|Y_i|,\ \max_{1\le i\le n}|Y_i| \ge a_\varepsilon\xi_n\bigg\}\\
&= nP\bigg\{\sum_{1\le i\le n} Y_i \ge (1-\varepsilon)x_n|Y_1|,\ \max_{1\le i\le n}|Y_i| \le |Y_1|,\ |Y_1| \ge a_\varepsilon\xi_n\bigg\}\\
&\le n\int_{a_\varepsilon\xi_n}^{A\xi_n} P\bigg\{\sum_{2\le i\le n} Y_i \ge (1-2\varepsilon)x_n y,\ \max_{2\le i\le n}|Y_i| \le y\bigg\}\,dP\{|X|\le y\}\\
&\le n\int_{a_\varepsilon\xi_n}^{A\xi_n} \inf_{0\le t<\infty}\Big(\exp(-t(1-2\varepsilon)x_n y)\big(E\exp(tY)I(|Y|\le y)\big)^{n-1}\Big)\,dP\{|X|\le y\}\\
&= n\int_{a_\varepsilon}^{A}\bigg(\inf_{0\le t<\infty}\exp\bigg(\frac{-t(1-2\varepsilon)x_n y\xi_n}{n-1}\bigg)\\
&\qquad\qquad \times E\exp(tY)I(|Y|\le y\xi_n)\bigg)^{n-1} dP\{|X|\le y\xi_n\}\\
&= n\int_{a_\varepsilon}^{A}\bigg(\inf_{0\le t<\infty}\exp\bigg(-\frac{t(1-2\varepsilon)x_n}{n-1}\bigg)\\
&\qquad\qquad \times E\exp\bigg(\frac{tY}{\xi_n y}\bigg)I\{|Y|\le y\xi_n\}\bigg)^{n-1} dP\{|X|\le y\xi_n\}\\
&\le n\sup_{a_\varepsilon\le y\le A}\bigg(\inf_{0\le t<\infty}\exp\bigg(-\frac{t(1-2\varepsilon)x_n}{n-1}\bigg)\\
&\qquad\qquad \times E\exp\bigg(\frac{tY}{\xi_n y}\bigg)I\{|Y|\le y\xi_n\}\bigg)^{n-1} P\{|X|\ge a_\varepsilon\xi_n\}.
\end{aligned}$$

Next we use (1.2) and then (2.6) to get

$$\begin{aligned}
I_3 &\le \frac{n(c_1+c_2+1)l(a_\varepsilon\xi_n)}{(a_\varepsilon\xi_n)^\alpha}\\
&\quad\times \sup_{a_\varepsilon\le y\le A}\bigg(\inf_{0\le t<\infty}\exp\bigg(-\frac{t(1-2\varepsilon)x_n}{n-1}\bigg)E\exp\bigg(\frac{tY}{\xi_n y}\bigg)I\{|Y|\le y\xi_n\}\bigg)^{n-1}\\
&\le \frac{2x_n(c_1+c_2+1)}{a_\varepsilon^\alpha}\\
&\quad\times \sup_{a_\varepsilon\le y\le A}\bigg(\inf_{0\le t<\infty}\exp\bigg(-\frac{t(1-2\varepsilon)x_n}{n-1}\bigg)E\exp\bigg(\frac{tY}{y\xi_n}\bigg)I(|Y|\le y\xi_n)\bigg)^{n-1}.
\end{aligned} \tag{2.13}$$

The following lemma gives an estimate for $E\exp(tY/(y\xi_n))I\{|Y| \le y\xi_n\}$.

LEMMA 2.1. *If the conditions of Theorem* 1.1 *are satisfied, then*

$$\begin{aligned}
1 - E\exp\left(\frac{tY}{y\xi_n}\right)&I\{|Y| \le y\xi_n\}\\
&= \frac{x_n}{n-1}\left(\frac{c_1+c_2}{y^\alpha} - \frac{c_1+c_2}{A^\alpha} - \frac{f(t)}{y^\alpha} + r_n(t, y)\right),
\end{aligned}$$

where $f(t)$ *is defined in* (1.8) *and* $\lim_{n\to\infty}\sup_{d\le t\le D}\sup_{a_\varepsilon\le y\le A}|r_n(t, y)| = 0$ *for all* $0 < d < D < \infty$.

PROOF. In what follows we use $r_{n,i}(y)$ or $r_{n,i}(t, y)$ ($i = 1, 2, \dots, 10$) to denote functions satisfying

$$\lim_{n\to\infty}\sup_{d\le y\le D}|r_{n,i}(y)| = 0 \quad \text{or} \quad \lim_{n\to\infty}\sup_{a_\varepsilon\le t\le A}\sup_{d\le y\le D}|r_{n,i}(t, y)| = 0.$$

First we note that

$$\begin{aligned}
&1 - E\exp(tY/(y\xi_n))I\{|Y| \le y\xi_n\}\\
&\qquad = E(1-\exp(tY/(y\xi_n)))I(|Y| \le y\xi_n) + P\{|Y| \ge y\xi_n\}\\
&\qquad = P\{y\xi_n \le |X| \le A\xi_n\} + E(1-\exp(tY/(y\xi_n)))I(|Y| \le y\xi_n).
\end{aligned}\tag{2.14}$$

By (1.2), (2.6) and (H1), we have

$$P\{y\xi_n \le |X| \le A\xi_n\} = \frac{x_n}{n}\left(\frac{c_1+c_2}{y^\alpha} - \frac{c_1+c_2}{A^\alpha} + r_{n,1}(y)\right).$$

It is easy to see that

$$\begin{aligned}
&E\left(1-\exp\left(\frac{tY}{y\xi_n}\right)\right)I(|Y| \le y\xi_n)\\
&\qquad = \int_{-y\xi_n}^{y\xi_n}\left(1-\exp\left(\frac{tx}{y\xi_n}\right)\right)dF(x) = \int_{-1}^{1}(1-e^{tx})\,dF(y\xi_n x).
\end{aligned}\tag{2.15}$$

First we assume that $1 < \alpha < 2$. Since $EX = 0$, we obtain that

$$\begin{aligned}
&\int_{-1}^{1}(1-e^{tx})\,dF(y\xi_n x)\\
&\qquad = \int_{-1}^{1}(1+tx-e^{tx})\,dF(y\xi_n x) - t\int_{-1}^{1}x\,dF(y\xi_n x)\\
&\qquad = -\int_{0}^{1}(1+tx-e^{tx})\,dP\{X > y\xi_n x\}\\
&\qquad\quad + \int_{-1}^{0}(1+tx-e^{tx})\,dP\{X \le y\xi_n x\} + \frac{t}{y\xi_n}EXI\{|X| > y\xi_n\}.
\end{aligned}\tag{2.16}$$

Using (1.2), (H1), (H5) and (2.6), we get

$$
\begin{aligned}
\frac{t}{y\xi_n} EXI(X \geq y\xi_n) &= \frac{t}{y\xi_n} \int_{y\xi_n}^{\infty} x\,dF(x) \\
&= t(1 - F(y\xi_n)) + \frac{t}{y\xi_n} \int_{y\xi_n}^{\infty} (1 - F(x))\,dx \\
&= \frac{t(c_1 + r_{n,1}(y))l(y\xi_n)}{(y\xi_n)^{\alpha}} \\
&\quad + \frac{t}{y\xi_n(\alpha - 1)} (y\xi_n)^{1-\alpha} l(y\xi_n)(c_1 + r_{n,2}(y)) \\
&= \frac{tl(y\xi_n)}{(y\xi_n)^{\alpha}(\alpha - 1)} c_1\alpha(1 + r_{n,3}(y)) \\
&= \frac{c_1 \alpha t\, x_n}{n(\alpha - 1)y^{\alpha}}(1 + r_{n,4}(y)).
\end{aligned} \tag{2.17}
$$

Similar arguments give

$$
\sup_{d \leq y \leq D} \left| \frac{1}{y\xi_n} EXI\{X < -y\xi_n\} \Big/ \left(\frac{\alpha c_2 x_n}{(\alpha - 1)y^{\alpha} n} \right) + 1 \right| \to 0
$$

as $n \to \infty$ and, therefore,

$$
\frac{t}{y\xi_n} EXI\{|X| > y\xi_n\} = \frac{(c_1 - c_2)\alpha x_n}{n(\alpha - 1)y^{\alpha}}(1 + r_{n,5}(t, y)). \tag{2.18}
$$

Integration by parts yields

$$
\begin{aligned}
&-\int_0^1 (1 + tx - e^{tx})\,dP\{X \geq y\xi_n x\} \\
&\qquad = -(1 + t - e^t)(1 - F(y\xi_n x)) + \int_0^1 (t - te^{tx})(1 - F(y\xi_n x))\,dx.
\end{aligned}
$$

As in the steps leading to (2.18), we have

$$
(1 + t - e^t)(1 - F(y\xi_n x)) = \frac{x_n}{n} \left\{ (1 + te^{-t}) \frac{c_1}{y^{\alpha}} + r_{n,6}(t, y) \right\}. \tag{2.19}
$$

Using (1.2) and (H1)–(H3) again, we obtain

$$
t \int_0^1 (1 - e^{tx})(1 - F(y\xi_n x))\,dx = t\frac{x_n}{n} \left\{ \frac{c_1}{y^{\alpha}} \int_0^1 \frac{1 - e^{tx}}{x^{\alpha}}\,dx + r_{n,7}(t, y) \right\}. \tag{2.20}
$$

It is easy to see that

$$
t \int_0^1 \frac{1 - e^{tx}}{x^{\alpha}}\,dx - (1 + t - e^t) = \alpha \int_0^1 \frac{1 + tx - e^{tx}}{x^{\alpha+1}}\,dx. \tag{2.21}
$$

Similarly,

$$
\int_{-1}^0 (1 + tx - e^{tx})\,dF(y\xi_n x) = \frac{x_n}{n} \left\{ \frac{c_2\alpha}{y^{\alpha}} \int_0^1 \frac{1 - tx - e^{-tx}}{x^{\alpha+1}}\,dx + r_{n,8}(t, y) \right\}. \tag{2.22}
$$

Putting (2.16)–(2.22) together, we obtain Lemma 2.1, when $1 < \alpha < 2$.

Next we assume that $\alpha = 1$ and X is symmetric. By (1.2) and (H1)–(H3), we have

$$\begin{aligned}
&\int_{-1}^{1}(1 - e^{tx})\,dP\{X \le y\xi_n x\} \\
&\qquad = \int_{-1}^{1}(1 + tx - e^{tx})\,dF(y\xi_n x) \qquad (2.23)\\
&\qquad = \frac{x_n}{n}\left\{\frac{c_1}{y}\int_0^1 \frac{2 - e^{tx} - e^{-tx}}{x^2}\,dx + r_{n,9}(t, y)\right\},
\end{aligned}$$

which gives Lemma 2.1.

By an analogous argument, for $0 < \alpha < 1$, we obtain

$$\int_{-1}^{1}(1 - e^{tx})\,dF(y\xi_n x) = \frac{x_n}{n}\left\{-\frac{f(t)}{y^\alpha} + r_{n,10}(t, y)\right\}. \qquad (2.24)$$

Now the proof of Lemma 2.1 is complete. □

Having Lemma 2.1, we can easily get an upper bound for I_3 in (2.13).

LEMMA 2.2. *If the conditions of Theorem* 1.1 *are satisfied, then*

$$\begin{aligned}
&\sup_{a_\varepsilon \le y \le A}\left(\inf_{0 \le t < \infty} \exp\left(-\frac{t(1-2\varepsilon)x_n}{n-1}\right) E\exp\left(\frac{tY}{y\xi_n}\right) I\{|Y| \le y\xi_n\}\right)^{n-1} \\
&\qquad \le \exp\big(-x_n((1-2\varepsilon)\tau - (c_1 + c_2)A^{-\alpha} + o(1))\big).
\end{aligned}$$

PROOF. By the definition of a_ε and (2.2), we have

$$a_\varepsilon^\alpha > \max\left(\lim_{t \downarrow 0} f'(t), 0\right).$$

So for any $y \in [a_\varepsilon, A]$, there is a unique t_y such that

$$f'(t_y) = (1 - 2\varepsilon)y^\alpha. \qquad (2.25)$$

From Lemma 2.1 it follows that

$$\begin{aligned}
&\sup_{a_\varepsilon \le y \le A}\left(\inf_{0 \le t < \infty} \exp\left(-\frac{t(1-2\varepsilon)x_n}{n-1}\right) E\exp\left(\frac{tY}{y\xi_n}\right) I\{|Y| \le y\xi_n\}\right)^{n-1} \\
&\quad \le \sup_{a_\varepsilon \le y \le A}\left(\exp\left(-\frac{t_y(1-2\varepsilon)x_n}{n-1}\right) E\exp\left(\frac{t_y Y}{y\xi_n}\right) I\{|Y| \le y\xi_n\}\right)^{n-1} \\
&\quad \le \sup_{a_\varepsilon \le y \le A} \exp\left(-t_y(1-2\varepsilon)x_n - x_n\left(\frac{c_1+c_2}{y^\alpha} - \frac{c_1+c_2}{A^\alpha} - \frac{f(t_y)}{y^\alpha} + o(1)\right)\right) \qquad (2.26)\\
&\quad = \exp\left(x_n\left(\frac{c_1+c_2}{A^\alpha} + o(1)\right)\right) \\
&\qquad \times \sup_{a_\varepsilon \le y \le A} \exp\left(-x_n\left((1-2\varepsilon)t_y + \frac{c_1+c_2}{y^\alpha} - \frac{f(t_y)}{y^\alpha}\right)\right).
\end{aligned}$$

Let

$$g(y) = -(1-2\varepsilon)t_y - \frac{c_1+c_2}{y^\alpha} + \frac{f(t_y)}{y^\alpha}$$

and y_0 be defined by $t_{y_0} = \tau$. It follows from (2.25) that

$$\begin{aligned} g'(y) &= -(1-2\varepsilon)t'_y + \frac{\alpha(c_1+c_2)}{y^{\alpha+1}} + \frac{f'(t_y)t'_y}{y^\alpha} - \frac{\alpha f(t_y)}{y^{\alpha+1}} \\ &= \frac{\alpha}{y^{\alpha+1}}(c_1+c_2-f(t_y)). \end{aligned}$$

It follows from (2.2)–(2.4) that t_y is an increasing function of y. Thus $g(y)$ reaches its maximum at $y = y_0$ and

$$g(y_0) = -(1-2\varepsilon)\tau - \frac{c_1+c_2}{y_0^\alpha} + \frac{f(\tau)}{y_0^\alpha} = -(1-2\varepsilon)\tau.$$

Now the proof of Lemma 2.2 is complete. □

We now return to obtain upper bound of I_3 in (2.13). Lemma 2.2 implies that

$$I_3 \le \frac{2(c_1+c_2+1)}{a_\varepsilon^\alpha} x_n \exp\left(-x_n\left((1-2\varepsilon)\tau - \frac{c_1+c_2}{A^\alpha} + o(1)\right)\right). \tag{2.27}$$

Putting (2.8), (2.9), (2.12) and (2.27) together, we obtain

$$\limsup_{n\to\infty} \frac{1}{x_n} \log P\{T_n \ge x_n\} \le -\tau. \tag{2.28}$$

Next we show that

$$\liminf_{n\to\infty} \frac{1}{x_n} \log P\{T_n \ge x_n\} \ge -\tau. \tag{2.29}$$

Let

$$a = \left(\frac{f'(\tau)}{1+2\varepsilon}\right)^{1/\alpha} \tag{2.30}$$

and $\{Z, Z_i, 1 \le i < \infty\}$ be a sequence of i.i.d. r.v.'s with distribution function G, where

$$dG(x) = \frac{I\{-a\xi_n \le x \le a\xi_n\}}{P\{|X| \le a\xi_n\}}\, dF(x). \tag{2.31}$$

Elementary arguments give

$$
\begin{aligned}
P\{T_n \geq x_n + 1\} &\geq nP\{S_n \geq (1+x_n)|X_1|,\ |X_i| < |X_1|,\ 2 \leq i \leq n\} \\
&\geq nP\bigg\{\sum_{2\leq i\leq n} X_i \geq x_n|X_1|,\ |X_i| < |X_1|, 2 \leq i \leq n, \\
&\qquad\qquad a\xi_n < |X_1| \leq (1+\varepsilon)a\xi_n\bigg\} \\
&\geq nP\bigg\{\sum_{2\leq i\leq n} X_i \geq x_n(1+\varepsilon)a\xi_n,\ |X_i| \leq a\xi_n,\ 2\leq i\leq n\bigg\} \\
&\quad \times P\{a\xi_n < |X_1| \leq (1+\varepsilon)a\xi_n\} \\
&= nP\{a\xi_n < |X| \leq (1+\varepsilon)a\xi_n\}P\{|X_i| \leq a\xi_n,\ 2\leq i\leq n\} \\
&\quad \times P\bigg\{\sum_{2\leq i\leq n} X_i \geq x_n(1+\varepsilon)a\xi_n \,\bigg|\, |X_i| \leq a\xi_n,\ 2\leq i\leq n\bigg\} \\
&= nP\{a\xi_n < |X| \leq (1+\varepsilon)a\xi_n\}(P\{|X| \leq a\xi_n\})^{n-1} \\
&\quad \times P\bigg\{\sum_{1\leq i\leq n-1} Z_i \geq x_n(1+\varepsilon)a\xi_n\bigg\}.
\end{aligned}
\tag{2.32}
$$

In terms of (1.2), we have

$$
\begin{aligned}
&nP\{a\xi_n < |X| \leq (1+\varepsilon)a\xi_n\} \\
&\quad = n\bigg\{\frac{c_1+c_2+o(1)}{(a\xi_n)^\alpha}l(a\xi_n) - \frac{c_1+c_2+o(1)}{((1+\varepsilon)a\xi_n)^\alpha}l((1+\varepsilon)a\xi_n)\bigg\} \\
&\quad = n(c_1+c_2)\frac{l(a\xi_n)}{(a\xi_n)^\alpha}\bigg(1 - \frac{1}{(1+\varepsilon)^\alpha} + o(1)\bigg) \\
&\quad = \frac{n(c_1+c_2)l(\xi_n)}{a^\alpha\xi_n^\alpha}\bigg(1 - \frac{1}{(1+\varepsilon)^\alpha} + o(1)\bigg) \\
&\quad = \frac{(c_1+c_2)x_n}{a^\alpha}\bigg(1 - \frac{1}{(1+\varepsilon)^\alpha} + o(1)\bigg).
\end{aligned}
$$

Similar calculations yield

$$
(P\{|X| \leq a\xi_n\})^{n-1} = \exp\bigg(-x_n\bigg(\frac{c_1+c_2}{a^\alpha} + o(1)\bigg)\bigg). \tag{2.33}
$$

The lower bound for $P\{\sum_{1\leq i\leq n-1} Z_i \geq x_n(1+\varepsilon)a\xi_n\}$ is based on the following lemmas:

LEMMA 2.3. *Let $t_n = \tau/(a\xi_n)$. If the conditions of Theorem* 1.1 *are satisfied, then as $n \to \infty$, we have*

$$1 - Ee^{t_n Z} = -\frac{x_n f(\tau)}{(n-1)a^\alpha} + o\left(\frac{x_n}{n}\right), \tag{2.34}$$

$$EZe^{t_n Z} = \frac{x_n \xi_n}{n a^\alpha} f'(\tau)) + o\left(\frac{x_n \xi_n}{n}\right), \tag{2.35}$$

$$EZ^2 e^{t_n Z} = \frac{x_n (a\xi_n)^2}{n a^\alpha} f''(\tau) + o\left(\frac{x_n \xi_n^2}{n}\right). \tag{2.36}$$

PROOF. Using (2.31), we get

$$\begin{aligned}
1 - Ee^{t_n Z} &= E(1 - e^{t_n Z}) \\
&= \int_{-a\xi_n}^{a\xi_n} \frac{1 - e^{t_n x}}{P\{|X| \le a\xi_n\}} \, dF(x) \\
&= \frac{1}{P\{|X| \le a\xi_n\}} \int_{-1}^{1} (1 - e^{\tau x}) \, dP\{X \le a\xi_n x\}.
\end{aligned}$$

We showed in the proof of Lemma 2.1 [cf. (2.16)–(2.24)] that

$$\int_{-1}^{1} (1 - e^{\tau x}) \, dF(a\xi_n x) = \frac{x_n}{n-1}\left(-\frac{f(\tau)}{a^\alpha} + o(1)\right).$$

Observing that

$$P\{|X| \le a\xi_n\} = 1 + o(1),$$

the proof of (2.34) is complete.

If $1 < \alpha < 2$, then we have

$$\begin{aligned}
EZe^{t_n Z} &= \frac{1}{P\{|X| \le a\xi_n\}} \int_{-a\xi_n}^{a\xi_n} x e^{t_n x} \, dF(x) \\
&= \frac{1}{P\{|X| \le a\xi_n\}} \left\{ EXI(|X| \le a\xi_n) + \int_{-a\xi_n}^{a\xi_n} x(e^{t_n x} - 1) \, dF(x) \right\} \\
&= \frac{1}{P\{|X| \le a\xi_n\}} \left\{ EXI(|X| \le a\xi_n) - a\xi_n \int_0^1 x(e^{\tau x} - 1) \, dP\{X > a\xi_n x\} \right. \\
&\qquad \left. + a\xi_n \int_{-1}^{0} x(e^{\tau x} - 1) \, dF(a\xi_n x) \right\}.
\end{aligned} \tag{2.37}$$

It follows from (2.18) that

$$EXI\{|X| \le a\xi_n\} = \frac{a\xi_n\, x_n}{n}\left\{\frac{\alpha(c_2 - c_1)}{(\alpha-1)a^\alpha} + o(1)\right\}.$$

Following the arguments in (2.19)–(2.21), we obtain

$$\begin{aligned}
-&\int_0^1 x(e^{\tau x}-1)\,dP\{X > a\xi_n x\}\\
&= -(e^\tau - 1)P\{X > a\xi_n\} + \int_0^1 (1 - F(a\xi_n x))d(x(e^{\tau x}-1))\\
&= -(e^\tau - 1)(1 - F(a\xi_n)) + \int_0^1 (1 - F(a\xi_n x))(e^{\tau x} - 1 + \tau x e^{\tau x})\,dx \qquad (2.38)\\
&= \frac{x_n}{n}\left(-(e^\tau - 1)\frac{c_1}{a^\alpha} + o(1)\right) + \frac{x_n}{n}\left(\frac{c_1}{a^\alpha}\int_0^1 \frac{e^{\tau x} - 1 + \tau x e^{\tau x}}{x^\alpha}\,dx + o(1)\right)\\
&= \frac{x_n\,\alpha\, c_1}{n\, a^\alpha}\left\{\int_0^1 \frac{e^{\tau x}-1}{x^\alpha}\,dx + o(1)\right\}.
\end{aligned}$$

As in (2.38), we have

$$\int_{-1}^0 x(e^{\tau x}-1)\,dF(a\xi_n x) = \frac{x_n\,\alpha c_2}{n\,a^\alpha}\left\{\int_0^1 \frac{1-e^{-\tau x}}{x^\alpha}\,dx + o(1)\right\}. \qquad (2.39)$$

Putting (2.37)–(2.39) together, we obtain

$$\begin{aligned}
EZe^{t_n Z} &= \frac{a\xi_n x_n}{na^\alpha}\left\{\frac{\alpha(c_2-c_1)}{\alpha-1} + \alpha c_1\int_0^1 \frac{e^{\tau x}-1}{x^\alpha} + \alpha c_2 \int_0^1 \frac{1-e^{-\tau x}}{x^\alpha}\,dx + o(1)\right\}\\
&= \frac{x_n\, a\xi_n}{na^\alpha}\{f'(\tau) + o(1)\},
\end{aligned}$$

which is exactly (2.35).

We still assume that $1 < \alpha < 2$. Following the proof of (2.35), we get

$$\begin{aligned}
EZ^2 e^{t_n Z} &= \frac{1}{P\{|X| \le a\xi_n\}}\int_{-a\xi_n}^{a\xi_n} x^2 e^{t_n x}\,dF(x)\\
&= \frac{(a\xi_n)^2}{P\{|X| \le a\xi_n\}}\int_{-1}^1 x^2 e^{\tau x}\,dF(a\xi_n x)\\
&= \frac{(a\xi_n)^2}{P\{|X| \le a\xi_n\}}\left\{-\int_0^1 x^2 e^{\tau x}\,dP\{X > a\xi_n x\} + \int_{-1}^0 x^2 e^{\tau x}\,dF(a\xi_n x)\right\}\\
&= \frac{(a\xi_n)^2}{P\{|X| \le a\xi_n\}}\left\{-e^\tau(1 - F(a\xi_n))\right.\\
&\qquad + \int_0^1 (1 - F(a\xi_n x))(2xe^{\tau x} + x^2\tau e^{\tau x})\,dx\\
&\qquad \left. - e^\tau F(-a\xi_n) - \int_{-1}^0 F(a\xi_n x)(2xe^{\tau x} + x^2 \tau e^{\tau x})\,dx\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x_n(a\xi_n)^2}{n\,a^\alpha}\bigg\{-c_1e^\tau + c_1\int_0^1 \frac{2xe^{\tau x}+x^2\tau e^{\tau x}}{x^\alpha}\,dx \\
&\qquad\qquad - c_2e^\tau - c_2\int_{-1}^0 \frac{2xe^{\tau x}+x^2\tau e^{\tau x}}{|x|^\alpha}\,dx + o(1)\bigg\} \\
&= \frac{x_n(a\xi_n)^2}{n\,a^\alpha}\bigg\{-c_1e^\tau + c_1\int_0^1 \frac{1}{x^\alpha} d(x^2e^{\tau x}) \\
&\qquad\qquad - c_2e^\tau + c_2\int_0^1 \frac{1}{x^\alpha} d(x^2e^{-\tau x}) + o(1)\bigg\} \\
&= \frac{x_n(a\xi_n)^2\,\alpha}{n\,a^\alpha}\bigg\{c_1\int_0^1 \frac{e^{\tau x}}{x^{\alpha-1}}\,dx + c_2\int_0^1 \frac{e^{-\tau x}}{x^{\alpha-1}}\,dx + o(1)\bigg\} \\
&= \frac{x_n(a\xi_n)^2}{na^\alpha}\{f''(\tau)+o(1)\}.
\end{aligned}
$$

The proofs for (2.35) and (2.36) in the case $0 < \alpha \le 1$ are similar and will be left to the reader. □

LEMMA 2.4. *Let $\{\eta, \eta_i,\ 1 \le i < \infty\}$ be a sequence of i.i.d. r.v.'s and assume*

$$H = \sup\{h\colon\ Ee^{h\eta} < \infty\} > 0.$$

For $0 < h < H$ we define

$$m(h) = E\eta e^{\eta h}/Ee^{\eta h} \quad \text{and} \quad \sigma^2(h) = E\eta^2/Ee^{\eta h} - m^2(h).$$

If $m(h) \ge x + 2\sigma(h)/n^{1/2}$, then we have

$$P\bigg\{\sum_{1\le i\le n} \eta_i \ge nx\bigg\} \ge \tfrac{3}{4}(Ee^{h\eta})^n \exp(-nhm(h) - 2h\sigma(h)n^{1/2}).$$

PROOF. The proof is a standard application of the well-known conjugate method [cf. Petrov (1965)]. The details can be found in Shao (1994). □

Now we return to the lower bound of $P\big\{\sum_{1\le i\le n-1} Z_i \ge x_n(1+\varepsilon)a\xi_n\big\}$. Let $m(t_n) = EZe^{t_nZ}/Ee^{t_nZ}$ and $\sigma^2(t_n) = EZ^2e^{t_nZ}/Ee^{t_nZ} - m^2(t_n)$. It follows from Lemma 2.3 and (2.30) that

$$m(t_n) \ge \frac{x_n(1+\varepsilon)a\xi_n}{n-1} + 2\frac{\sigma(t_n)}{n^{1/2}} \tag{2.40}$$

and

$$\frac{\sigma(t_n)}{n^{1/2}} = o\bigg(\frac{x_n\xi_n}{n}\bigg). \tag{2.41}$$

So we can apply Lemma 2.4 and obtain

$$P\left\{\sum_{1\le i\le n-1} Z_i \ge x_n(1+\varepsilon)a\xi_n\right\}$$
$$\ge \tfrac{3}{4}(Ee^{t_n Z})^{n-1}\exp(-(n-1)t_n m(t_n) - 2t_n\sigma(t_n)(n-1)^{1/2}).$$

By (2.30), (2.34), (2.40) and (2.41) and noting that $f(\tau) = c_1 + c_2$, we have

$$\begin{aligned} &P\left\{\sum_{1\le i\le n-1} Z_i \ge x_n(1+\varepsilon)a\xi_n\right\} \\ &\quad\ge \frac{3}{4}\exp\left(x_n\left(\frac{f(\tau)}{a^\alpha}+o(1)\right) - (n-1)\frac{\tau}{a\xi_n}\left(\frac{x_n}{n}\frac{a\xi_n}{a^\alpha}f'(\tau)+o\left(\frac{x_n\xi_n}{n}\right)\right)\right) \\ &\quad= \frac{3}{4}\exp\left(\frac{x_n}{a^\alpha}(f(\tau)-\tau f'(\tau)+o(1))\right) \\ &\quad= \frac{3}{4}\exp\left(x_n\left(\frac{c_1+c_2}{a^\alpha} - \tau(1+2\varepsilon)+o(1)\right)\right). \end{aligned} \tag{2.42}$$

Putting (2.32), (2.33) and (2.42) together, one can easily obtain

$$\begin{aligned} P\{T_n \ge 1+x_n\} \ge{}& \frac{(c_1+c_2)x_n}{a^\alpha}\left(1-\frac{1}{(1+\varepsilon)^\alpha}+o(1)\right) \\ &\times \exp\left(-x_n\left(\frac{c_1+c_2}{a^\alpha}+o(1)\right)\right) \\ &\times \frac{3}{4}\exp\left(x_n\left(\frac{c_1+c_2}{a^\alpha}-\tau(1+2\varepsilon)+o(1)\right)\right), \end{aligned}$$

which yields (2.29). $\square$

3. Proof of Theorem 3.1. The proof of the upper bound in Theorem 1.3 requires a stronger version of Theorem 1.1.

THEOREM 3.1. *Assume that* $\{x_n,\ 1 \le n < \infty\}$ *is a sequence of positive numbers satisfying* (1.6) *and the conditions of Theorem* 1.1 *are satisfied. Then for any* $0 < \varepsilon < 1/2$ *there exists* $\theta > 1$ *such that*

$$P\left\{\max_{n\le k\le\theta n} T_k \ge x_n\right\} \le \exp(-(1-\varepsilon)x_n\tau) \tag{3.1}$$

for all large enough n.

PROOF. Let $\gamma = (1-(1-\varepsilon/2)^{1/4})/3$. It is clear that

$$P\left\{\max_{n\le k\le\theta n} T_k \ge x_n\right\} \le P\{T_n \ge (1-3\gamma)x_n\} + P\left\{\max_{n<k\le\theta n}\frac{S_k-S_n}{M_k} \ge 3\gamma x_n\right\}. \tag{3.2}$$

By Theorem 1.1, we have

$$P\{T_n \ge (1-3\gamma)x_n\} \le \exp\big(-(1-\varepsilon/2)\tau x_n\big) \tag{3.3}$$

if n is sufficiently large. Let ξ_n be defined as in (2.6) and $\delta > 0$. Truncation arguments yield

$$
\begin{aligned}
&P\left\{\max_{n<k\le\theta n}\frac{S_k-S_n}{M_k}\ge 3\gamma x_n\right\}\\
&\quad\le P\left\{\max_{n<k\le\theta n}\frac{1}{M_k}\sum_{n<i\le k}X_iI(|X_i|\le\xi_n)\ge 2\gamma x_n\right\}\\
&\qquad+P\left\{\max_{n<k\le\theta n}\frac{1}{M_k}\sum_{n<i\le k}|X_i|I(|X_i|>\xi_n)\ge\gamma x_n\right\}\\
&\quad\le P\{M_n\le\delta\xi_n\}+P\left\{\max_{n<k\le n\theta}\sum_{n<i\le k}X_iI(|X_i|\le\xi_n)\ge 2\gamma\delta x_n\xi_n\right\}\\
&\qquad+P\left\{\sum_{n<i\le n\theta}I(|X_i|\ge\xi_n)\ge\gamma x_n\right\}.
\end{aligned}
\tag{3.4}
$$

As in (2.12), we have

$$
\begin{aligned}
P&\left\{\sum_{n<i\le n\theta}I(|X_i|\ge\xi_n)\ge\gamma x_n\right\}\\
&\le\left(\frac{3(\theta-1)P\{|X|\ge\xi_n\}}{\gamma x_n}\right)^{\gamma x_n}\\
&\le\left(\frac{6(\theta-1)(c_1+c_2)l(\xi_n)}{\gamma x_n\xi_n^{\alpha}}\right)^{\gamma x_n}\\
&\le\left(\frac{12(\theta-1)(c_1+c_2)}{\gamma}\right)^{\gamma x_n}\le\exp(-2\tau x_n),
\end{aligned}
\tag{3.5}
$$

provided that θ is close enough to 1. It follows from (1.2) that

$$
\begin{aligned}
P\{M_n\le\delta\xi_n\}&=(P\{|X|\le\delta\xi_n\})^n\\
&=\left(1-\frac{c_1+c_2+o(1)}{(\delta\xi_n)^{\alpha}}\,l(\delta\xi_n)\right)^n\\
&\le\exp\left(-\frac{nl(\delta\xi_n)(c_1+c_2)}{2(\delta\xi_n)^{\alpha}}\right)\\
&\le\exp\left(-x_n\frac{c_1+c_2}{4\delta^{\alpha}}\right)\le\exp(-2x_n\tau)
\end{aligned}
\tag{3.6}
$$

if $\delta^{\alpha}<(c_1+c_2)/(8\tau)$.

Arguing as in the proof of Lemma 2.3, we have

$$\lim_{x\to\infty}EX^2I\{|X|\le x\}\Big/\left(\frac{\alpha(c_1+c_2)}{2-\alpha}x^{2-\alpha}l(x)\right)=1,$$

$$\lim_{x\to\infty}E|X|I\{|X|\ge x\}\Big/\left(\frac{\alpha(c_1+c_2)}{\alpha-1}x^{1-\alpha}l(x)\right)=1\quad\text{if}\quad 1<\alpha<2$$

and

$$\lim_{x\to\infty} E|X|I\{|X|\le x\}\Big/\left(\frac{\alpha(c_1+c_2)}{1-\alpha}x^{1-\alpha}l(x)\right)=1 \quad \text{if} \quad 0<\alpha<1.$$

Hence by (2.6) we get

$$\sum_{n<i\le\theta n}|EX_iI(|X_i|\le\xi_n)|\le\tfrac{1}{2}\gamma\delta x_n\xi_n,$$

$$\sum_{n<i\le\theta n}\operatorname{Var}(X_iI(|X_i|\le\xi_n))\le\tfrac{1}{8}(\gamma\delta x_n\xi_n)^2,$$

provided $\theta-1$ is small enough. Therefore, by the Ottawiani inequality [cf. Chow and Teicher (1988), page 74]

$$\begin{aligned}
&P\left\{\max_{n<k\le\theta n}\sum_{n<i\le k}X_iI(|X_i|\le\xi_n)\ge 2\gamma\delta x_n\xi_n\right\}\\
&\quad\le 2P\left\{\sum_{n<i\le n\theta}(X_iI(|X_i|\le\xi_n)-EX_iI(|X_i|\le\xi_n))\ge\gamma\delta x_n\xi_n\right\}\\
&\quad\le 2\exp(-4\tau x_n)\left(E\exp\left(\frac{4\tau}{\gamma\delta x_n}(XI(|X|\le\xi_n)-EXI(|X|\le\xi_n))\right)\right)^{n(\theta-1)}\\
&\quad\le 2\exp(-4\tau x_n)(1+K_1\xi_n^{-2}EX^2I(|X|\le\xi_n))^{n(\theta-1)}\\
&\quad\le 2\exp(-4\tau x_n)(1+K_2\xi_n^{-\alpha}l(\xi_n))^{n(\theta-1)}\\
&\quad\le 2\exp(-4\tau x_n)\left(1+K_3\frac{x_n}{n}\right)^{n(\theta-1)}\\
&\quad\le \exp(-2\tau x_n)
\end{aligned}\tag{3.7}$$

if $\theta-1$ is small enough, where K_1, K_2 and K_3 are constants, depending only on α, c_1, c_2 and δ. From (3.2)–(3.6) and (3.7) we obtain

$$P\left\{\max_{n\le k\le\theta n}T_k\ge x_n\right\}\le\exp\left(-\left(1-\frac{\varepsilon}{2}\right)\tau x_n\right)+\exp\left(-\frac{c_1+c_2}{4\delta^\alpha}x_n\right)+2\exp(-2\tau x_n),$$

as desired. □

Now we are ready to prove Theorem 1.3. Using the well-known subsequence method, Theorem 3.1 implies that

$$\limsup_{n\to\infty}\frac{T_n}{\log\log n}\le\frac{1}{\tau}\quad\text{a.s.}\tag{3.8}$$

To get the lower bound, let $q > 1$ and define $n_k = [\exp(k^q)]$. First we note that

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{T_n}{\log\log n} &\geq \limsup_{k\to\infty} \tfrac{T_{n_k}}{\log\log n_k} \\
&\geq \limsup_{k\to\infty} \frac{S_{n_k} - S_{n_{k-1}}}{M_{n_k}\log\log n_k} + \liminf_{k\to\infty} \frac{S_{n_{k-1}}}{M_{n_k}\log\log n_k} \\
&= \limsup_{k\to\infty} \frac{\max_{n_{k-1}<i\leq n_k}|X_i|}{M_{n_k}} \cdot \frac{S_{n_k} - S_{n_{k-1}}}{\max_{n_{k-1}<i\leq n_k}|X_i|\log\log n_k} \\
&\quad - \limsup_{k\to\infty} \frac{M_{n_{k-1}}}{M_{n_k}} \cdot \frac{-S_{n_{k-1}}}{M_{n_{k-1}}\log\log n_k}.
\end{aligned} \tag{3.9}
$$

Since $\{(S_{n_k} - S_{n_{k-1}})/\max_{n_{k-1}<i\leq n_k}|X_i|,\ 1 \leq k < \infty\}$ are independent random variables, it follows from Theorem 1.1 and the Borel–Cantelli lemma that

$$
\limsup_{k\to\infty} \frac{S_{n_k} - S_{n_{k-1}}}{\max_{n_{k-1}<i\leq n_k}|X_i|\log\log n_k} \geq \frac{1}{q\tau} \quad \text{a.s.} \tag{3.10}
$$

Next we show that

$$
\lim_{k\to\infty} M_{n_{k-1}}/M_{n_k} = 0 \quad \text{a.s.} \tag{3.11}
$$

Take y_k satisfying

$$
\lim_{k\to\infty} \frac{l(y_k)}{y_k^\alpha} \Bigg/ \left(\frac{1}{k^2 n_{k-1}}\right) = 1.
$$

For all $\varepsilon > 0$ we have

$$
\begin{aligned}
P\{M_{n_{k-1}} \geq \varepsilon M_{n_k}\} &\leq P\{M_{n_{k-1}} \geq y_k\} + P\left\{M_{n_k} \leq \frac{1}{\varepsilon} y_k\right\} \\
&\leq n_{k-1} P\{|X| \geq y_k\} + \left(P\left\{|X| \leq \frac{1}{\varepsilon} y_k\right\}\right)^{n_k} \\
&\leq n_{k-1} \frac{2(c_1+c_2)l(y_k)}{y_k^\alpha} + \left(1 - \frac{(c_1+c_2)l(y_k/\varepsilon)}{2((1/\varepsilon)y_k)^\alpha}\right)^{n_k} \\
&\leq \frac{4(c_1+c_2)}{k^2} + \exp\left(-\frac{\varepsilon(c_1+c_2)}{8k^2}\ \frac{n_k}{n_{k-1}}\right) \\
&\leq \frac{4(c_1+c_2)}{k^2} + \exp\left(-\frac{\varepsilon(c_1+c_2)}{8k^2} \exp((k-1)^{q-1})\right),
\end{aligned}
$$

and therefore the Borel–Cantelli lemma gives (3.11).

As in (3.8), there is a positive constant K such that

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{-S_{n_{k-1}}}{M_{n_{k-1}}\log\log n_k} &\leq \begin{cases} K \quad \text{a.s.}, & \text{for } 1 \leq \alpha < 2, \\ \limsup_{n\to\infty} \dfrac{\sum_{i=1}^{n_{k-1}}|X_i|}{M_{n_{k-1}}\log\log n_k} & \text{for } 0 < \alpha < 1 \end{cases} \\
&\leq K \quad \text{a.s.}
\end{aligned} \tag{3.12}
$$

Combining (3.9)–(3.12), we obtain

$$\limsup_{n\to\infty} T_n/\log\log n \geq \frac{1}{\tau} \quad \text{a.s.},$$

which completes the proof of Theorem 1.3. □

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