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GENERALIZED RAY–KNIGHT THEORY AND LIMIT THEOREMS FOR SELF-INTERACTING RANDOM WALKS ON $\mathbb{Z}^1$

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We consider non-Markovian, self-interacting random walks (SIRW) on the one-dimensional integer lattice . The walk starts from the origin and at each step jumps to a neighboring site. The probability of jumping along a bond is proportional to w (number of previous jumps along that lattice bond), where $w: \mathbb{N} \to \mathbb{R}_+$ is a monotone weight function. Exponential and subexponential weight functions were considered in earlier papers. In the present paper we consider weight functions w with polynomial asymptotics. These weight functions define variants of the "reinforced random walk." We prove functional limit theorems for the local time processes of these random walks and local limit theorems for the position of the random walker at late times. A generalization of the Ray–Knight theory of local time arises.

1. Introduction. In the present paper we consider self-interacting random walks (SIRW) X_i on the one-dimensional integer lattice $\mathbb{Z}$, defined as follows: the walk starts from the origin of the lattice and at time $i+1$ it jumps to one of the two neighboring sites of X_i, so that the probability of jumping along a bond of the lattice is proportional to

$$w(\text{number of previous jumps along that bond}),$$

where

$$w: \mathbb{N} \to \mathbb{R}_+$$

is a weight function to be specified later. More formally, for a nearest neighbor walk $\underline{x}_0^i = (x_0, x_1, \dots, x_i)$ we define

$$r(\underline{x}_0^i) = \left\{ 0 \le j < i : (x_j, x_{j+1}) = (x_i, x_i + 1) \text{ or } (x_i + 1, x_i) \right\}, \tag{1.1}$$

$$l(\underline{x}_0^i) = \left\{ 0 \le j < i : (x_j, x_{j+1}) = (x_i, x_i - 1) \text{ or } (x_i - 1, x_i) \right\}; \tag{1.2}$$

that is, the number $r(\underline{x}_0^i)$ [respectively $l(\underline{x}_0^i)$] shows how many times had the walk $\underline{x}_0^i$ visited the edge adjacent from the right (respectively from the left) to the terminal site x_i. The random walk X_i is governed by the law

$$\begin{aligned} \mathbf{P}(X_{i+1} = X_i + 1 \| \underline{X}_0^i = \underline{x}_0^i) &= \frac{w(r(\underline{x}_0^i))}{w(r(\underline{x}_0^i)) + w(l(\underline{x}_0^i))} \\ &= 1 - \mathbf{P}(X_{i+1} = X_i - 1 \| \underline{X}_0^i = \underline{x}_0^i). \end{aligned} \tag{1.3}$$

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Our aim is to prove limit theorems for these random walks and their local time processes, with proper scaling.

The "true" self-avoiding random walks defined by the weight functions $w(n) = \exp\{-g \cdot n\}$, $g > 0$ [respectively, generalizations with weight functions $w(n) = \exp\{-g \cdot n^{\kappa}\}$, $g > 0$, $\kappa \in (0, 1)$], have been considered in Tóth (1995) [respectively, Tóth (1994)]. In the present paper we shall consider weight functions satisfying the following two conditions:

1. *Monotonicity*. Nonincreasing w defines a self-repelling random walk, while nondecreasing w defines a self-attracting one.
2. *Regular polynomial asymptotic behavior*. The most convenient way to formulate this condition is

$$w(n)^{-1} = 2^{-\alpha}(\alpha + 1)n^{\alpha} + 2^{1-\alpha}Bn^{\alpha-1} + \mathscr{O}(n^{\alpha-2}), \tag{1.4}$$

where $\alpha \in \mathbb{R}$ and $B \in \mathbb{R}$ are two constant parameters. Since in the definition (1.3) of jump probabilities only ratios of w's play any role, the constant factor in front of the leading term is chosen for convenience only. Note, that the next-to-leading term is assumed asymptotically "smooth." The monotonicity condition (1) implies that $\alpha < 0$ defines self-attracting walks, $\alpha > 0$ defines self-repelling walks, while for $\alpha = 0$, lower order terms determine the character of the self-interaction.

Due to Davis (1990) the recurrence properties of these walks, with $w(n) \sim n^{-\alpha}$, $\alpha \in \mathbb{R}$ are well understood: for $\alpha \in [-1, \infty)$, with probability 1, the walk visits infinitely often every site of the lattice, whereas for $\alpha \in (-\infty, -1)$ the walk eventually sticks to one edge of the lattice jumping back and forth on it, indefinitely. The case $\alpha = -1$ is somewhat special: $w(n) = w(0) + n$, $w(0) > 0$, defines the so-called (linearly) reinforced random walk. From the results of Pemantle (1988) it follows that in this case the random walk X_i has an asymptotic (random) distribution on $\mathbb{Z}$, *without any scaling*. These remarks suggest that only the cases $\alpha \in (-1, \infty)$ will show interesting, nontrivial scaling behavior. There are three essentially different regimes according to the value of the parameter α:

CASE A. $\alpha = 0$. We shall call this the *asymptotically free* case.

CASE B. $\alpha \in (0, \infty)$. We shall refer to this as the *polynomially self-repelling* case.

CASE C. $\alpha \in (-1, 0)$. This will be called the *weakly reinforced* case.

Cases A and B show similar scaling behavior and will be treated in parallel in the main body of the paper. Case C differs essentially from the first two, the results referring to this case will be presented in a separate note [Tóth (1996)].

A very special case of A, the "random walk with partial reflection at extrema" or "once reinforced random walk" has been recently considered in Davis (1994, 1995) and Nester (1994). These walks are defined by the weight function

$$w(n) = \begin{cases} \dfrac{2}{\delta}, & \text{for } n = 0, \\ 1, & \text{for } n \neq 0 \end{cases} \tag{1.5}$$

with the parameter $\delta > 2$. Davis and Nester prove pathwise convergence for these very special cases. See also the remark at the end of Section 3.

The outline of the paper is as follows: In Section 2 we define the random processes and variables which will appear as weak limits in later sections. In Section 3 we formulate the limit theorems referred to in the title. Section 4 contains the convenient representation of local time processes of the self-interacting random walks considered. In Section 5 we prove the limit theorems referring to the local times of SIRW—this section is the core of the paper. In Section 6 the limit theorem for the position of SIRW is proved. In the Appendix we analyse the "generalized Ray–Knight processes" defined in Section 2. The results presented in the Appendix are technically needed in the proof of Theorems 2A and 2B, but they are self-contained and might be interesting from a purely diffusion-theoretic point of view.

2. Generalized Ray–Knight processes I. The random processes and variables defined in this section will appear as weak limits in the limit theorems stated in the next section.

The squared Bessel process (BESQ$^{\delta}$) of generalized dimension $\delta \in \mathbb{R}_+$ is well understood. For exhaustive description of these processes and their properties we refer the reader to Revuz and Yor (1991). For our present purposes it is more convenient to consider $\frac{1}{2}$ times the conventional BESQ$^{\delta}$: the stochastic process $\mathbb{R}_+ \ni t \mapsto Z^{(\delta)}(t) \in \mathbb{R}_+$ which solves the SDE

$$dZ^{(\delta)}(t) = \frac{\delta}{2}\, dt + \sqrt{2Z^{(\delta)}(t)}\, dW(t), \qquad Z^{(\delta)}(0) \in \mathbb{R}_+, \tag{2.1}$$

where $W(t)$ is a standard Brownian motion. These processes are well defined for any $\delta \geq 0$. The infinitesimal generator of the Feller semigroup acting on the Banach space

$$C_0[0,\infty) = \left\{ f \in C[0,\infty) \colon \lim_{x\to\infty} f(x) = 0 \right\} \tag{2.2}$$

corresponding to the process $Z^{(\delta)}(\cdot)$ is

$$G^{(\delta)} = x\frac{\partial^2}{\partial x^2} + \frac{\delta}{2}\frac{\partial}{\partial x} \tag{2.3}$$

defined on the domain

$$\mathscr{D}^{(\delta)} = \Big\{ f \in C_0[0,\infty) \cap C^2(0,\infty) \colon \big[G^{(\delta)} f \in C_0[0,\infty)\big] \wedge \Big[\lim_{x \searrow 0} x f''(x) = 0\Big]\Big\}. \tag{2.4}$$

Let σ_0 be the first hitting of zero by $Z^{(\delta)}$:

$$\sigma_0 = \sigma_0(Z^{(\delta)}(\cdot)) = \inf\{ t > 0 \mid Z^{(\delta)}(t) = 0 \}. \tag{2.5}$$

It is well known that for $\delta \geq 2$, $\sigma_0 = \infty$ a.s. and for $0 \leq \delta < 2$, $\sigma_0 < \infty$ a.s. Furthermore, if $\delta = 0$, then zero is an absorbing point, that is, for $t \geq \sigma_0$, $Z^{(0)}(t) \equiv 0$.

For $0 \leq \delta < 2$, we denote by $\tilde{Z}^{(\delta)}(\cdot)$ the process stopped at σ_0:

$$\tilde{Z}^{(\delta)}(t) = \begin{cases} Z^{(\delta)}(t), & \text{for } t \in [0, \sigma_0), \\ 0, & \text{for } t \in [\sigma_0, \infty). \end{cases} \tag{2.6}$$

The process $\tilde{Z}^{(\delta)}(\cdot)$ is naturally defined for $\delta < 0$, too, as the solution of the SDE (2.1) until the first hitting of 0, and identically zero afterward. For $\delta \in (-\infty, 2)$, the generator of the process $\tilde{Z}^{(\delta)}(\cdot)$ is the operator $\tilde{G}^{(\delta)}$ formally given in (2.3), but now defined on the domain

$$\widetilde{\mathscr{D}}^{(\delta)} = \Big\{ f \in C_0[0,\infty) \cap C^2(0,\infty) \colon \big[G^{(\delta)} f \in C_0[0,\infty)\big] \wedge \Big[\lim_{x \searrow 0} G^{(\delta)} f(x) = 0\Big]\Big\}. \tag{2.7}$$

Let $\delta > 0$ and two other parameters, $a \geq 0$ and $h \geq 0$, be fixed. We define the process $S^{(\delta)}_{a,h}(\cdot)$ by patching together three different, independent BESQ's: $\tilde{Z}^{(2-\delta)}_l(\cdot)$, $Z^{(\delta)}(\cdot)$ and $\tilde{Z}^{(2-\delta)}_r(\cdot)$, in the following way:

$$S^{(\delta)}_{a,h}(y) = \begin{cases} \big(\tilde{Z}^{(2-\delta)}_l(-y) \big\| \tilde{Z}^{(2-\delta)}_l(0) = h\big), & \text{for } y \in (-\infty, 0], \\ \big(Z^{(\delta)}(y) \big\| Z^{(\delta)}(0) = h\big), & \text{for } y \in [0, a], \\ \big(\tilde{Z}^{(2-\delta)}_r(y-a) \big\| \tilde{Z}^{(2-\delta)}_r(0) = Z^{(\delta)}(a)\big), & \text{for } y \in [a, \infty); \end{cases} \tag{2.8}$$

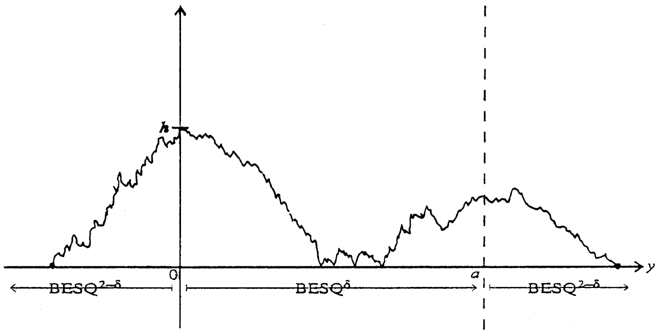
δ, a and h are fixed parameters of the process and $y \in \mathbb{R}$ is the "time" variable. The process $S^{(\delta)}_{a,h}(\cdot)$ is graphically represented on Fig. 1. For reasons which will soon become obvious we call the process $S^{(\delta)}_{a,h}(\cdot)$ a *generalized Ray–Knight process*.

We denote

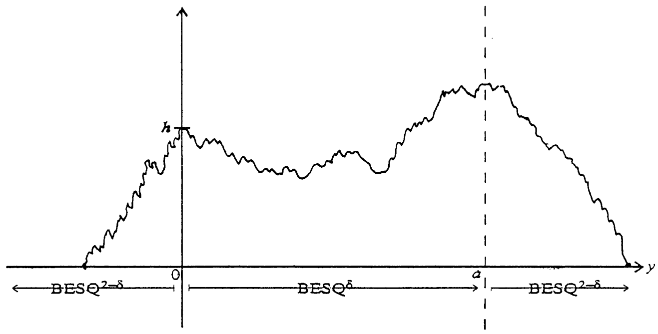
$$\omega^{(\delta)-}_{a,h} = \omega^-\left(S^{(\delta)}_{a,h}\right) = \inf\{y \leq 0 | S^{(\delta)}_{a,h}(y) > 0\} \tag{2.9}$$

$$\omega^{(\delta)+}_{a,h} = \omega^+\left(S^{(\delta)}_{a,h}\right) = \sup\{y \geq a | S^{(\delta)}_{a,h}(y) > 0\}. \tag{2.10}$$

For any $\delta > 0$, that is, $2 - \delta < 2$, $|\omega^{\pm}|$ are finite a.s.



(a)



(b)

FIG. 1. *The generalized Ray–Knight process* $S^{(\delta)}_{|x|,h}(y)$. (a) $\delta \in (0, 2)$; (b) $\delta \in [2, \infty)$.

Since the process $\tilde{Z}^{(2-\delta)}(\cdot)$ almost surely hits 0 in finite time and it is stopped at this hitting time, the process $S^{(\delta)}_{a,h}(\cdot)$ almost surely has compact support and the total area under $S^{(\delta)}_{a,h}(\cdot)$,

$$T^{(\delta)}_{a,h} = \int_{\omega^{(\delta)-}_{a,h}}^{\omega^{(\delta)+}_{a,h}} S^{(\delta)}_{a,h}(y)\,dy = \int_{-\infty}^{\infty} S^{(\delta)}_{a,h}(y)\,dy, \tag{2.11}$$

is almost surely finite. For any δ, a and h, the random variable $T^{(\delta)}_{a,h}$ defined in (2.11) has an absolutely continuous distribution. Let

$$\varrho^{(\delta)}(t,\ a,\ h) = \frac{\partial}{\partial t}\mathbf{P}(T^{(\delta)}_{a,h} < t) \tag{2.12}$$

be the density of the distribution of $T^{(\delta)}_{a,h}$. From scaling the BESQ processes we easily get

$$\lambda^2 \varrho^{(\delta)}(\lambda^2 t, \lambda a, \lambda h) = \varrho^{(\delta)}(t, a, h) \tag{2.13}$$

for any $\lambda > 0$. Define $\mathbb{R}_+ \times \mathbb{R} \ni (t, x) \mapsto \pi^{(\delta)}(t, x) \in \mathbb{R}_+$ as

$$\pi^{(\delta)}(t, x) = \int_0^\infty \varrho^{(\delta)}\left(\frac{t}{2}, |x|, h\right) dh. \tag{2.14}$$

The scaling property (2.13) of $\varrho^{(\delta)}$ implies

$$\lambda \pi^{(\delta)}(\lambda^2 t, \lambda x) = \pi^{(\delta)}(t, x). \tag{2.15}$$

We denote by $\hat{\varrho}^{(\delta)}$ and $\hat{\pi}^{(\delta)}$ the Laplace transforms of $\varrho^{(\delta)}$ (respectively, $\pi^{(\delta)}$):

$$\hat{\varrho}^{(\delta)}(s, a, h) = s \int_0^\infty \exp(-st)\varrho^{(\delta)}(t, a, h)\, dt = s\mathbf{E}\big(\exp\{-sT^{(\delta)}_{a,h}\}\big), \tag{2.16}$$

$$\hat{\pi}^{(\delta)}(s, x) = s \int_0^\infty \exp(-st)\pi^{(\delta)}(t, x)\, dt = \int_0^\infty \hat{\varrho}^{(\delta)}(2s, |x|, h)\, dh. \tag{2.17}$$

These functions scale as

$$\lambda^2 \hat{\varrho}^{(\delta)}(\lambda^{-2}s, \lambda a, \lambda h) = \hat{\varrho}^{(\delta)}(s, a, h), \tag{2.18}$$

$$\lambda \hat{\pi}^{(\delta)}(\lambda^{-2}s, \lambda x) = \hat{\pi}^{(\delta)}(s, x). \tag{2.19}$$

In the *particular case*, $\delta = 2$, $S^{(2)}_{a,h}(\cdot)$ is well known: according to the by now classical Ray–Knight theorems it is identical to the *local time process* of standard one-dimensional Brownian motion stopped at appropriately chosen sampling times. More precisely, let $\mathscr{B}(t)$ be a standard Brownian motion on $\mathbb{R}$ and $\mathscr{L}(x, t)$ be its local time process. The Ray–Knight theorems [see Chapter XI of Revuz and Yor (1991)] state that given $x \in \mathbb{R}$, $h \geq 0$ fixed, if we stop the Brownian motion $\mathscr{B}(\cdot)$ at the stopping time

$$\mathscr{T}_{x,h} = \inf\{ t \geq 0: \mathscr{L}(x, t) > h\} \tag{2.20}$$

and consider the (shifted) local time process

$$\mathscr{S}_{x,h}(y) = \mathscr{L}(x - \operatorname{sgn}(x)y, \mathscr{T}_{x,h}), \qquad y \in (-\infty, \infty), \tag{2.21}$$

then

$$\mathscr{S}_{x,h}(\cdot) \quad =_{d.} S^{(2)}_{|x|,h}(\cdot), \tag{2.22}$$

where $=_d$ stands for equality in distribution. Since

$$\mathscr{T}_{x,h} = \int_{-\infty}^\infty \mathscr{S}_{x,h}(y)\, dy \tag{2.23}$$

clearly holds, we actually have

$$\big(\mathscr{S}_{x,h}(\cdot), \mathscr{T}_{x,h}\big) =_{d.} \big(S^{(2)}_{|x|,h}(\cdot), T^{(2)}_{|x|,h}\big). \tag{2.24}$$

This is exactly the content of the Ray–Knight theorems on Brownian local time. [See XI.2.2. and XI.2.3. of Revuz and Yor (1991).]

Now, using the straightforward identity

$$\int_0^\infty \mathbb{1}\big(\mathscr{T}_{x,h} < t\big)\, dh = \mathscr{L}(x, t) \tag{2.25}$$

we readily get

$$\begin{aligned}\pi^{(2)}(t,x) &= \frac{\partial}{\partial t}\int_0^\infty \mathbf{P}\big(\mathscr{T}_{x,h} < t\big)\,dh = \frac{\partial}{\partial t}\mathbf{E}\big(\mathscr{L}(x,t)\big)\\ &= \frac{1}{\sqrt{2\pi t}}\exp\left(\frac{-x^2}{2t}\right)\end{aligned}\tag{2.26}$$

or, equivalently,

$$\hat{\pi}^{(2)}(s,x) = \frac{\sqrt{s}}{2}\exp(-\sqrt{s}|x|);\tag{2.27}$$

that is, $\mathbb{R} \ni x \mapsto \pi^{(2)}(t,x) \in (0,\infty)$ [respectively, $\mathbb{R} \ni x \mapsto \hat{\pi}^{(2)}(s,x) \in (0,\infty)$] are the *densities of the distribution of the Brownian motion* stopped at time t (respectively, stopped at an independent random time of exponential distribution with expectation s^{-1}). For the special values $\delta = 2$ and $\delta = 0$ of the dimension parameter, formulas relating to functionals of the squared Bessel process become more explicit and, consequently, (2.26) and (2.27) could have been established by direct computation, *without any reference to Ray–Knight theorems.* However, this is *not* the case for other values of the parameter δ. Nevertheless, Theorem 4 formulated and proved in the Appendix generalizes the statement made above: from Theorem 4 it follows that indeed given $t \in (0,\infty)$ [respectively $s \in (0,\infty)$] fixed, the function $x \mapsto \pi^{(\delta)}(t,x)$ [respectively $x \mapsto \hat{\pi}^{(\delta)}(s,x)$] is a probability density, that is, for any $t \in (0,\infty)$ [respectively $s \in (0,\infty)$],

$$\int_{-\infty}^{\infty}\pi^{(\delta)}(t,x)\,dx = 1 = \int_{-\infty}^{\infty}\hat{\pi}^{(\delta)}(s,x)\,dx.\tag{2.28}$$

The two assertions of (2.28) are, of course, equivalent: $\hat{\pi}^{(\delta)}(s,\cdot)$ is the distribution $\pi^{(\delta)}(t,\cdot)$ observed at a "random time" of exponential distribution with mean value s^{-1}. Furthermore, using the scaling relations (2.15) [respectively (2.19)], one can eliminate the t (respectively s) parameters from these integrals; that is, one has to prove, say, the right-hand side equality with $s = 1$. However, for $\delta \neq 2$ the integrals in (2.28) seem to evade any attempt at explicit computation. The integrals cannot be computed even for $\delta = 1$, which is another very special case. We shall prove a further generalization of (2.28) in the Appendix.

There are no explicit formulas for the distributions $\pi^{(\delta)}$; however, according to recent results of Carmona, Petit and Yor (1995) and Davis (1995) they coincide with the one-dimensional marginal distributions of the "Brownian motion perturbed at extrema." See also the remark at the end of Section 3.

In Cases A and B (that is, for the asymptotically free and polynomially self-repelling walks) the generalized Ray–Knight processes $S^{(\delta)}_{a,h}(\cdot)$ will arise as scaling limits of properly defined local time processes, and later the distributions $\hat{\pi}^{(\delta)}(s,x)\,dx$ will arise as weak limits for the properly scaled position of the SIRW's at late times.

REMARK. BESQ^{δ} and $\widetilde{\mathrm{BESQ}}^{2-\delta}$ processes with $\delta > 0$ also have been recently found as local time processes of reflecting Brownian motion perturbed by its local time at zero, that is, $X_t = |\mathscr{B}_t| - \mu\mathscr{L}(0, t)$ with $\mu = 2/\delta$. For details of different approaches to this problem, see, for example, Le Gall and Yor (1986), Carmona, Petit and Yor (1994), Perman (1995) or Werner (1995); the most general approach is that of Carmona, Petit and Yor (1995).

3. Limit theorems. The present section is divided into two subsections: in Section 3.1. we formulate limit theorems for the local time processes and hitting times of the SIRW's. In Section 3.2. we formulate the limit theorems for the position of the SIRW at late times.

3.1. *The local time process and hitting times.* Our first results are limit theorems for the local time processes of the random walks X_i, stopped at appropriately defined stopping times. We define the (bond) local time process

$$L(l, i) = \#\{0 \le j < i: X_j = l,\ X_{j+1} = l - 1\}, \qquad l \in \mathbb{Z},\ i \in \mathbb{N}, \tag{3.1.1}$$

and stopping times

$$\begin{aligned} &T^{>}_{k,-1} = 0,\\ &T^{>}_{k,m} = \inf\{i > T^{>}_{k,m-1}: X_{i-1} = k-1, X_i = k\}, \qquad k > 0, m \ge 0. \end{aligned} \tag{3.1.2}$$

$$\begin{aligned} &T^{<}_{k,0} = 0,\\ &T^{<}_{k,m} = \inf\{i > T^{<}_{k,m-1}: X_{i-1} = k+1, X_i = k\} \qquad k \ge 0, m \ge 1. \end{aligned} \tag{3.1.3}$$

In plain words, $L(l, i)$ is the number of leftward jumps on the bond $l \to l-1$ performed by the random walk up to time i, $T^{>}_{k,m}$ is the time of the $m+1$th arrival to the lattice site k coming from the left and $T^{<}_{k,m}$ is the time of the mth arrival to the lattice site k coming from the right.

In formula (3.1.4) and thereafter the superscript asterisk (*) stands for either $<$ or $>$. We consider the following shifted (bond) local time processes of the walk stopped at $T^{*}_{k,m}$:

$$S^{*}_{k,m}(l) = L(k - l, T^{*}_{k,m}) \tag{3.1.4}$$

$S^{*}_{k,m}(l)$ is roughly half of the total number of jumps across the bond $\{k - l - 1, k - l\}$:

$$\begin{aligned} &\#\{0 \le j < T^{*}_{k,m}: \{X_j, X_{j+1}\} = \{k - l - 1, k - l\}\}\\ &\qquad = 2S^{*}_{k,m}(l) + \mathbb{1}_{[0,k)}(l). \end{aligned} \tag{3.1.5}$$

Denote

$$\omega^{*-}_{k,m} = \omega^{-}(S^{*}_{k,m}) = \inf\{l \le 0: S^{*}_{k,m}(l) > 0\}, \tag{3.1.6}$$

$$\omega^{*+}_{k,m} = \omega^{+}(S^{*}_{k,m}) = \sup\{l \ge k: S^{*}_{k,m}(l) > 0\}. \tag{3.1.7}$$

In plain words, $k-\omega^{*+}_{k,m}$ (respectively $k-\omega^{*-}_{k,m}-1$) is the leftmost (respectively rightmost) site visited by the stopped walk $\underline{X}_0^{T^*_{k,m}}$.

From (3.1.5) it clearly follows that

$$T^*_{k,m} = 2\sum_{l=\omega^{*-}_{k,m}}^{\omega^{*+}_{k,m}} S^*_{k,m}(l) + k = 2\sum_{l=-\infty}^{\infty} S^*_{k,m}(l) + k. \tag{3.1.8}$$

Looking at the formal definitions only, in principle, these local times or hitting times might be infinite, that is, it could happen that the site $k \in \mathbb{Z}$ is never hit. From the results of Davis (1990) it follows that in the cases considered in the present paper, with probability 1, this is *not* the case: all the random variables defined above are finite almost surely.

The following two theorems and their corollaries describe the precise asymptotics of the *local time processes* $S^*_{k,m}(\cdot)$ and *hitting times* $T^*_{k,m}$ in the asymptotically free and polynomially self-repelling cases.

THEOREM 1A (Asymptotically free case: $\alpha = 0$). *The limit*

$$\delta = 2w(0)^{-1} + 2\sum_{j=1}^{\infty}\left(w(2j)^{-1} - w(2j-1)^{-1}\right) \tag{3.1.9}$$

exists; $\delta \in (0, 2]$ *for self-repelling walks* (*i.e.*, w *nonincreasing*) *and* $\delta \in [2, \infty)$ *for self-attracting walks* (*i.e.*, w *nondecreasing*).

Let $x \in [0,\infty)$, $h \geq 0$ *and* $* = <$ *or* $>$ *be fixed. In the* $A \to \infty$ *limit the following weak convergence holds in the space* $\mathbb{R}_- \times \mathbb{R}_+ \times D(-\infty,\infty)$:

$$\begin{aligned}&\left(\frac{\omega^{*-}_{[Ax],[Ah]}}{A}, \frac{\omega^{*+}_{[Ax],[Ah]}}{A}, \frac{S^*_{[Ax],[Ah]}([Ay])}{A}\right)\\ &\qquad \Rightarrow \left(\omega^{(\delta)-}_{x,h}, \omega^{(\delta)+}_{x,h}, S^{(\delta)}_{x,h}(y)\right).\end{aligned} \tag{3.1.10}$$

THEOREM 1B [Polynomially self-repelling case: $\alpha \in (0,\infty)$]. *Denote*

$$\beta = \frac{1}{2\alpha+1} \in (0,1). \tag{3.1.11}$$

Let $x \in [0,\infty)$, $h \geq 0$ *and* $* = <$ *or* $>$ *be fixed. In the* $A \to \infty$ *limit the following weak convergence holds in the space* $\mathbb{R}_- \times \mathbb{R}_+ \times D(-\infty,\infty)$:

$$\begin{aligned}&\left(\frac{\omega^{*-}_{[Ax],[A\beta h]}}{A}, \frac{\omega^{*+}_{[Ax],[A\beta h]}}{A}, \frac{S^*_{[Ax],[A\beta h]}([Ay])}{A\beta}\right)\\ &\qquad \Rightarrow \left(\omega^{(1)-}_{x,h}, \omega^{(1)+}_{x,h}, S^{(1)}_{x,h}(y)\right).\end{aligned} \tag{3.1.12}$$

The generalized Ray–Knight process $S^{(\delta)}_{x,h}(\cdot)$ and random variables $\omega^{(\delta)\pm}_{x,h}$ appearing on the right hand side of (3.1.10) and (3.1.12) are defined in (2.8)–(2.10).

REMARK. It seems rather surprising (at least for the author) that in the polynomially self-repelling case (Case B), the exponent α is reflected only in the constant scaling factor β and the limit process is unaffected; it is always a squared Wiener process.

Immediate corollaries of the previous theorems are the following limit laws for the hitting times defined in (3.1.2) and (3.1.3):

COROLLARY 1A (Asymptotically free case: $\alpha = 0$). *Let x, h, $*$ and δ be as in Theorem 1A. Then*

$$\frac{T^*_{[Ax],[Ah]}}{2A^2} \Rightarrow T^{(\delta)}_{x,h} \tag{3.1.13}$$

as $A \to \infty$.

COROLLARY 1B [Polynomially self-repelling case: $\alpha \in (0, \infty)$]. *Let x, h, $*$ and β be as in Theorem 1B. Then*

$$\frac{T^*_{[Ax],[A\beta h]}}{2A^2\beta} \Rightarrow T^{(1)}_{x,h}. \tag{3.1.14}$$

as $A \to \infty$.

The random variables $T^{(\delta)}_{x,h}$ appearing on the right-hand side of (3.1.13) and (3.1.14) are defined in (2.11). These corollaries will be used in the proof of Theorems 2A and 2B.

3.2. *Local limit theorem for the position of the random walk.* The second group of results concerns the limiting distribution of the SIRW X_n for late times. We denote by $P(n,k)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, the distribution of our self-interacting random walk at time n,

$$P(n,k) = \mathbf{P}(X_n = k), \tag{3.2.1}$$

and by $R(s,k)$, $s \in \mathbb{R}_+$, $k \in \mathbb{Z}$, the distribution of the walk observed at an independent random time θ_s, of geometric distribution

$$\mathbf{P}(\theta_s = n) = (1 - e^{-s})e^{-sn}, \tag{3.2.2}$$

$$R(s,k) = \mathbf{P}(X_{\theta_s} = k) = (1 - e^{-s}) \sum_{n=0}^{\infty} e^{-sn} P(n,k). \tag{3.2.3}$$

We define the following rescaled "densities" of the above distributions

$$\pi_A(s,x) = A^{1/2} P([At], [A^{1/2}x]), \tag{3.2.4}$$

$$\hat{\pi}_A(s,x) = A^{1/2} R(A^{-1}s, [A^{1/2}x]) \tag{3.2.5}$$

$t, s \in \mathbb{R}_+$, $x \in \mathbb{R}$. It is straightforward that $\hat{\pi}_A$ is exactly the Laplace transform of π_A.

THEOREM 2A (Asymptotically free case: $\alpha = 0$). *For any $s \in \mathbb{R}_+$ and almost all $x \in \mathbb{R}$,*

$$\hat{\pi}_A(s, x) \to \hat{\pi}^{(\delta)}(s, x) \tag{3.2.6}$$

as $A \to \infty$, where the probability density $\mathbb{R} \ni x \mapsto \hat{\pi}^{(\delta)}(s, x)$ is that defined in (2.17) *with δ given in* (3.1.9).

THEOREM 2B [Polynomially self-repelling case: $\alpha \in (0, \infty)$]. *For any $s \in \mathbb{R}_+$ and almost all $x \in \mathbb{R}$,*

$$\hat{\pi}_A(s, x) \to \beta^{1/2}\hat{\pi}^{(1)}(s, \beta^{1/2}x) \tag{3.2.7}$$

as $A \to \infty$, where the probability density $\mathbb{R} \ni x \mapsto \hat{\pi}^{(1)}(s, x)$ is that defined in (2.17) *with $\delta = 1$ and β is given in* (3.1.11).

These are of course *local limit theorems* for the self-interacting random walks, observed at an independent random time $\theta_{s/A}$ of geometric distribution with mean $e^{-s/A}(1 - e^{-s/A})^{-1} \sim A/s$. In particular, the (integral) limit laws follow:

$$\text{Case A, for } \alpha = 0, \qquad \mathbf{P}(A^{-1/2}X_{\theta_{s/A}} < x) \to \int_{-\infty}^{x} \hat{\pi}^{(\delta)}(s, y)\, dy; \tag{3.2.8}$$

$$\text{Case B, for } \alpha \in (0, \infty), \qquad \mathbf{P}(A^{-1/2}X_{\theta_{s/A}} < x) \to \int_{-\infty}^{\beta^{1/2}x} \hat{\pi}^{(1)}(s, y)\, dy. \tag{3.2.9}$$

These are a little bit short of stating the limit theorems for *deterministic time*:

$$\text{Case A, for } \alpha = 0, \qquad \mathbf{P}(A^{-1/2}X_{[At]} < x) \to \int_{-\infty}^{x} \pi^{(\delta)}(t, y)\, dy; \tag{3.2.10}$$

$$\text{Case B, for } \alpha \in (0, \infty), \qquad \mathbf{P}(A^{-1/2}X_{[At]} < x) \to \int_{-\infty}^{\beta^{1/2}x} \pi^{(1)}(t, y)\, dy \tag{3.2.11}$$

Of course, we can conclude that the sequence $A^{-1/2}X_{[At]}$, with $t \in \mathbb{R}_+$ fixed and $A \to \infty$, is tight and if it converges in distribution, then (3.2.10)/(3.2.11) also hold.

REMARK. Carmona, Petit and Yor (1995) and Davis (1995) considered the "Brownian motion perturbed at extrema," that is, the stochastic process formally defined by

$$Y_t = B_t + \alpha \sup_{s \le t} Y_s + \beta \inf_{s \le t} Y_s. \tag{3.2.12}$$

It turns out that for $\alpha = \beta = 2/\delta$ these processes have the same family of local time processes as those appearing in our Theorems 1A and 1B. Furthermore, Davis (1995) proves that the properly scaled *once reinforced random walk*, defined by (1.5), converges to $Y_.$ *as a process*. In Theorems 2A and 2B we prove only convergence of one-dimensional distributions, but for a much wider family of self-interacting random walks. Actually, our limiting one-dimensional distributions coincide with the one-dimensional marginal distributions of the process Y_t. The problem of higher-dimensional distributions of our general SIRW's remains open.

4. Representation of the local time process in terms of Pólya urns.

4.1. *Generalized Pólya urn schemes.* Given two weight functions

$$r\colon \mathbb{N} \to \mathbb{R}_+ \tag{4.1.1}$$

$$b\colon \mathbb{N} \to \mathbb{R}_+, \tag{4.1.2}$$

a generalized Pólya urn scheme is a Markov chain (ρ_i, β_i) on $\mathbb{N} \times \mathbb{N}$ with transition probabilities

$$\mathbf{P}((\rho_{i+1}, \beta_{i+1}) = (k+1, l) \| (\rho_i, \beta_i) = (k, l)) = \frac{r(k)}{r(k) + b(l)}, \tag{4.1.3}$$

$$\mathbf{P}((\rho_{i+1}, \beta_{i+1}) = (k, l+1) \| (\rho_i, \beta_i) = (k, l)) = \frac{b(l)}{r(k) + b(l)} \tag{4.1.4}$$

and no other transitions allowed. Usually the initial values $(\rho_0, \beta_0) = (0, 0)$ are assumed and β_i and ρ_i are interpreted as the number of blue and red marbles, respectively, drawn from the urn up to time i. Denote by τ_m the time when the mth red marble is drawn and by $\mu(m)$ the number of blue marbles drawn before the mth red one:

$$\tau_m = \min\{ i \mid \rho_i = m\}, \tag{4.1.5}$$

$$\mu(m) = \beta_{\tau_m}. \tag{4.1.6}$$

The following functions are essential in the study of the Pólya urn scheme defined above:

$$R_p(n) = \sum_{j=0}^{n-1} (r(j))^{-p}, \qquad p \in \mathbb{N}, \tag{4.1.7}$$

$$B_p(n) = \sum_{j=0}^{n-1} \big(b(j)\big)^{-p}, \qquad p \in \mathbb{N}. \tag{4.1.8}$$

We shall be particularly interested in $p = 1, 2, 3, 4$.

LEMMA 1. *For any $m \in \mathbb{N}$ and $\lambda < \min\{r(j)\colon 0 \le j \le m-1\}$ the following identity holds*:

$$\mathbf{E}\left(\prod_{j=0}^{\mu(m)-1} \left(1 + \frac{\lambda}{b(j)}\right)\right) = \prod_{j=0}^{m-1} \left(1 - \frac{\lambda}{r(j)}\right)^{-1}. \tag{4.1.9}$$

In particular,

$$\mathbf{E}(B_1(\mu(m))) = R_1(m), \tag{4.1.10}$$

$$\begin{aligned}\mathbf{E}([B_1(\mu(m)) - \mathbf{E}\,B_1(\mu(m))]^2)&\\ = R_2(m) + \mathbf{E}(B_2(\mu(m))),&\end{aligned} \tag{4.1.11}$$

$$\begin{aligned}\mathbf{E}([B_1(\mu(m)) - \mathbf{E}\,B_1(\mu(m))]^4)&\\ = 6R_4(m) + 9R_2^2(m) + 6R_1^2(m)R_2(m)&\\ - 12R_1(m)\,\mathbf{E}(B_1(\mu(m))B_2(\mu(m)))&\\ + 8R_1(m)\,\mathbf{E}(B_3(\mu(m)))&\\ + 12(R_1^2(m) + R_2(m))\,\mathbf{E}(B_2(\mu(m)))&\\ + 6\,\mathbf{E}(B_1^2(\mu(m))B_2(\mu(m)))&\\ - 8\,\mathbf{E}(B_1(\mu(m))B_3(\mu(m))) + 6\,\mathbf{E}(B_4(\mu(m)))&\\ + 6\,\mathbf{E}(B_2(\mu(m)))^2.&\end{aligned} \tag{4.1.12}$$

PROOF. The proof of (4.1.9) follows from standard martingale considerations. One possibility is using Rubin's representation of the generalized Pólya urn scheme, given in the Appendix of Davis (1990). Expanding (4.1.9) to fourth order in λ yields (4.1.10)–(4.1.12). We leave the standard details of this proof as an exercise for the reader. □

REMARK. The explicit form of the expressions on the right-hand sides of (4.1.10) and (4.1.11) will be used later. The right-hand side of (4.1.12) looks rather discouraging, but in our concrete application, we shall need only estimates on its order of magnitude.

4.2. *The local time process.* For the sake of definiteness we consider the case of superscript $>$, that is, we stop the SIRW at the hitting time $T^{>}_{k,m}$. The case of superscript $<$ is done in a very similar way, with straightforward slight changes.

Let $(\rho_i^{(l)}, \beta_i^{(l)})$, $l \in \mathbb{Z}$, be *independent* Pólya urn schemes with weight functions

$$r^{(l)}(j) = w(2j+1), \qquad b^{(l)}(j) = w(2j), \quad \text{for } l \in (-\infty, 0] \cup [k+1, \infty), \tag{4.2.1}$$

$$r^{(l)}(j) = w(2j), \qquad b^{(l)}(j) = w(2j+1), \quad \text{for } l \in [1, k-1], \tag{4.2.2}$$

$$r^{(l)}(j) = w(2j) + 1, \qquad b^{(l)}(j) = w(2j), \quad \text{for } l = k. \tag{4.2.3}$$

Denote by $\mu^{(l)}(m)$ the random variables defined in (4.1.6), the superscript l shows to which of the urn schemes it belongs.

The extension to *self-interacting walks* of Knight's (1963) description of the local time process $S^{>}_{k,m}(l)$, $l \in \mathbb{Z}$, *as a Markov chain* is formally exhaustively explained in several papers; see, for example, Davis (1990) or Tóth (1995). According to these arguments $S^{>}_{k,m}(l)$, $l \in \mathbb{Z}$, is obtained by patching together three homogeneous Markov chains in the following way:

1. In the interval $l \in (0, k-2)$, that is, steps $0 \to 1, 1 \to 2, \ldots, (k-2) \to (k-1)$,

$$S^{>}_{k,m}(0) = m, \qquad S^{>}_{k,m}(l+1) = \mu^{(l+1)}(S^{>}_{k,m}(l)+1), \qquad l = 0, 1, \ldots, k-2. \tag{4.2.4}$$

   This process will be the object of a "first Ray–Knight theorem."

2. The single step $(k-1) \to k$ is exceptional:

$$\begin{aligned} S^{>}_{k,m}(k-1) &= \text{given by (4.2.4)}, \\ S^{>}_{k,m}(k) &= \mu^{(k)}(S^{>}_{k,m}(k-1)+1). \end{aligned} \tag{4.2.5}$$

3. In the intervals $l \in (-\infty, 0)$ [respectively, $l \in (k+1, \infty)$], that is, steps $0 \to -1, -1 \to -2, -2 \to -3, \ldots$ [respectively, $k \to (k+1), (k+1) \to (k+2), (k+2) \to (k+3), (k+3) \to (k+4), \ldots$],

$$S^{>}_{k,m}(0) = m, \qquad S^{>}_{k,m}(l-1) = \mu^{(l)}(S^{>}_{k,m}(l)), \qquad l = 0, -1, -2, \ldots \tag{4.2.6}$$

   respectively

$$\begin{aligned} S^{>}_{k,m}(k) &= \text{ given by (4.2.5)}, \\ S^{>}_{k,m}(l+1) &= \mu^{(l+1)}(S^{>}_{k,m}(l)), \qquad l = k, k+1, k+2, \ldots]. \end{aligned} \tag{4.2.7}$$

Due to (4.2.1) these last two Markov chains have the same transition laws. These will be the object of a "second Ray–Knight theorem."

The proof of Theorems 1A and 1B will consist of proving limit theorems for the Markov chains given in (4.2.4) [respectively, (4.2.6)–(4.2.7)] and proving that the single exceptional step given in (4.2.5) does not have any effect on the limit (i.e., it does not spoil continuity of the limit process).

5. Proof of Theorems 1A and 1B.

5.1. *Preparations.* As suggested by the representation of the local times given in the previous section, we consider two homogeneous Markov chains $\mathscr{Z}(l)$ and $\widetilde{\mathscr{Z}}(l)$, $l = 0, 1, 2, \ldots$, on the state space $\mathbb{N}$, defined as follows:

$$\mathscr{Z}(l+1) = \mu^{(l+1)}(\mathscr{Z}(l)+1), \qquad \widetilde{\mathscr{Z}}(l+1) = \tilde{\mu}^{(l+1)}(\widetilde{\mathscr{Z}}(l)), \tag{5.1.1}$$

where the processes $\{\mu^{(l)}(\cdot)\}_{l\in\mathbb{N}}$ are those defined in (4.1.5) and (4.1.6), belonging to i.i.d. Pólya urn schemes $\{(\rho_i^{(l)}, \beta_i^{(l)})\}_{l\in\mathbb{N}}$, with weight functions

$$r(j) = w(2j), \qquad b(j) = w(2j+1). \tag{5.1.2}$$

Similarly, the processes $\{\tilde{\mu}^{(l)}(\cdot)\}_{l\in\mathbb{N}}$ belong to i.i.d. Pólya urn schemes $\{(\tilde{\rho}_i^{(l)}, \tilde{\beta}_i^{(l)})\}_{l\in\mathbb{N}}$, with weight functions

$$\tilde{r}(j) = w(2j+1), \qquad \tilde{b}(j) = w(2j). \tag{5.1.3}$$

We shall also need the hitting time

$$\tilde{\sigma}_0 = \tilde{\sigma}_0(\widetilde{\mathscr{Z}}(\cdot)) = \inf\{l: \widetilde{\mathscr{Z}}(l) = 0\}. \tag{5.1.4}$$

From (5.1.1), (4.1.5) and (4.1.6) we see that $\tilde{\sigma}_0$ is actually the extinction time of $\widetilde{\mathscr{Z}}(\cdot)$:

$$\widetilde{\mathscr{Z}}(l) \equiv 0 \quad \text{for } l \geq \tilde{\sigma}_0. \tag{5.1.5}$$

Lemma 1 suggests the introduction of the functions

$$U_p(n) = \sum_{j=0}^{n-1} (w(2j))^{-p}, \qquad p = 1, 2, 3, 4, \tag{5.1.6}$$

$$V_p(n) = \sum_{j=0}^{n-1} \big(w(2j+1)\big)^{-p}, \qquad p = 1, 2, 3, 4. \tag{5.1.7}$$

Using formulas (4.1.10) and (4.1.11) of Lemma 1 and the functions introduced above, we get the identities

$$\mathbf{E}(V_1(\mathscr{Z}(l+1)) \| \mathscr{Z}(l) = n) = U_1(n+1), \tag{5.1.8}$$

$$\mathbf{D}^2(V_1(\mathscr{Z}(l+1)) \| \mathscr{Z}(l) = n) = U_2(n+1) + \mathbf{E}(V_2(\mathscr{Z}(l+1)) \| \mathscr{Z}(l) = n), \tag{5.1.9}$$

$$\mathbf{E}(U_1(\widetilde{\mathscr{Z}}(l+1)) \| \widetilde{\mathscr{Z}}(l) = n) = V_1(n), \tag{5.1.10}$$

$$\mathbf{D}^2(U_1(\widetilde{\mathscr{Z}}(l+1)) \| \widetilde{\mathscr{Z}}(l) = n) = V_2(n) + \mathbf{E}(U_2(\widetilde{\mathscr{Z}}(l+1)) \| \widetilde{\mathscr{Z}}(l) = n). \tag{5.1.11}$$

As both functions $n \mapsto U_1(n)$ and $n \mapsto V_1(n)$ are *bijections* between $\mathbb{N}$ and their ranges it is more convenient to consider the Markov chains

$$\mathscr{Y}(l) = V_1(\mathscr{Z}(l)), \qquad \widetilde{\mathscr{Y}}(l) = U_1(\widetilde{\mathscr{Z}}(l)), \qquad l = 0, 1, 2, \ldots, \tag{5.1.12}$$

instead of $\mathscr{Z}(l)$ [respectively $\widetilde{\mathscr{Z}}(l)$]. With this change of variable, formulas (5.1.8)–(5.1.11) transform as

$$\mathbf{E}(\mathscr{Y}(l+1) \| \mathscr{Y}(l) = x) = U_1(V_1^{-1}(x) + 1), \tag{5.1.13}$$

$$\begin{aligned}\mathbf{D}^2(\mathscr{Y}(l+1)\|\mathscr{Y}(l)=x) = U_2(V_1^{-1}(x)+1)&\\ +\mathbf{E}(V_2\circ V_1^{-1}(\mathscr{Y}(l+1))\|\mathscr{Y}(l)=x),&\end{aligned} \tag{5.1.14}$$

$$\mathbf{E}(\widetilde{\mathscr{Y}}(l+1)\|\widetilde{\mathscr{Y}}(l)=x) = V_1\circ U_1^{-1}(x), \tag{5.1.15}$$

$$\begin{aligned}\mathbf{D}^2(\widetilde{\mathscr{Y}}(l+1)\|\widetilde{\mathscr{Y}}(l)=x) = V_2\circ U_1^{-1}(x),&\\ +\mathbf{E}(U_2\circ U_1^{-1}(\widetilde{\mathscr{Y}}(l+1))\|\widetilde{\mathscr{Y}}(l)=x).&\end{aligned} \tag{5.1.16}$$

We introduce the functions F, G: $\mathrm{Ran}(V_1)\to\mathbb{R}$ and $\tilde{F}$, $\tilde{G}$: $\mathrm{Ran}(U_1)\to\mathbb{R}$:

$$\begin{aligned}F(x) &= \mathbf{E}(\mathscr{Y}(l+1)\|\mathscr{Y}(l)=x)-x\\ &= U_1(V_1^{-1}(x)+1)-x,\end{aligned} \tag{5.1.17}$$

$$\begin{aligned}G(x) &= \mathbf{E}([\mathscr{Y}(l+1)-\mathbf{E}(\mathscr{Y}(l+1)\|\mathscr{Y}(l)=x)]^2\|\mathscr{Y}(l)=x)\\ &= U_2(V_1^{-1}(x)+1)+\mathbf{E}(V_2\circ V_1^{-1}(\mathscr{Y}(1))\|\mathscr{Y}(0)=x),\end{aligned} \tag{5.1.18}$$

$$\begin{aligned}\tilde{F}(x) &= \mathbf{E}(\widetilde{\mathscr{Y}}(l+1)\|\widetilde{\mathscr{Y}}(l)=x)-x\\ &= V_1\circ U_1^{-1}(x)-x,\end{aligned} \tag{5.1.19}$$

$$\begin{aligned}\tilde{G}(x) &= \mathbf{E}([\widetilde{\mathscr{Y}}(l+1)-\mathbf{E}(\widetilde{\mathscr{Y}}(l+1)\|\widetilde{\mathscr{Y}}(l)=x)]^2\|\widetilde{\mathscr{Y}}(l)=x)\\ &= V_2\circ U_1^{-1}(x)+\mathbf{E}(U_2\circ U_1^{-1}(\widetilde{\mathscr{Y}}(1))\|\widetilde{\mathscr{Y}}(0)=x).\end{aligned} \tag{5.1.20}$$

Since $\mathscr{Y}(\cdot)$ and $\widetilde{\mathscr{Y}}(\cdot)$ are *Markov chains*, from (5.1.13)–(5.1.20) it follows that the processes

$$\begin{aligned}\mathscr{M}(l) &= \mathscr{Y}(l)-\mathscr{Y}(0)-\textstyle\sum_{j=0}^{l-1}F(\mathscr{Y}(j)),\\ \widetilde{\mathscr{M}}(l) &= \widetilde{\mathscr{Y}}(l)-\widetilde{\mathscr{Y}}(0)-\textstyle\sum_{j=0}^{l-1}\tilde{F}(\widetilde{\mathscr{Y}}(j))\end{aligned} \tag{5.1.21}$$

are *martingales* with quadratic variation processes

$$\begin{aligned}\langle\mathscr{M},\mathscr{M}\rangle(l) &= \sum_{j=0}^{l-1}G(\mathscr{Y}(j)),\\ \langle\widetilde{\mathscr{M}},\widetilde{\mathscr{M}}\rangle(l) &= \sum_{j=0}^{l-1}\tilde{G}(\widetilde{\mathscr{Y}}(j)).\end{aligned} \tag{5.1.22}$$

Later, when proving tightness, we shall also need the functions $H\colon\mathrm{Ran}(V_1)\to\mathbb{R}$ and $\tilde{H}\colon\mathrm{Ran}(U_1)\to\mathbb{R}$:

$$H(x) = \mathbf{E}([\mathscr{Y}(l+1)-\mathbf{E}(\mathscr{Y}(l+1)\|\mathscr{Y}(l)=x)]^4\|\mathscr{Y}(l)=x), \tag{5.1.23}$$

$$\tilde{H}(x) = \mathbf{E}([\widetilde{\mathscr{Y}}(l+1)-\mathbf{E}(\widetilde{\mathscr{Y}}(l+1)\|\widetilde{\mathscr{Y}}(l)=x)]^4\|\widetilde{\mathscr{Y}}(l)=x). \tag{5.1.24}$$

5.2. *Asymptotics of the relevant functions.* In the present subsection we give the asymptotics of the relevant functions F, G, H, $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ to be used in the proof of Theorems 1A and 1B. All formulas are valid for *large values* of the variable and are obtained from (1.4) in a rather straightforward way. The three cases listed at the end of the Introduction show essentially different asymptotic behavior. These essentially different asymptotics explain why exactly these are the different regimes.

Asymptotically free case: $\alpha = 0$ (Case A). From (1.4) we get

$$U_1(n) = n - B\log n + u + \mathscr{O}(n^{-1}), \tag{5.2.1}$$

$$V_1(n) = n - B\log n + v + \mathscr{O}(n^{-1}), \tag{5.2.2}$$

$$V_2(n) = n + \mathscr{O}(\log n) = U_2(n). \tag{5.2.3}$$

In (5.2.1) [respectively, (5.2.2)], u and v are two real constants. We define

$$\delta = 2\lim_{n\to\infty}(U(n+1) - V(n)) = 2 + 2(u - v). \tag{5.2.4}$$

In the self-repelling case, that is, $w(k+1) \le w(k)$, we write

$$\begin{aligned}\frac{\delta}{2} &= w(0)^{-1} + \sum_{j=1}^{\infty}[w(2j)^{-1} - w(2j-1)^{-1}]\\ &= 1 - \sum_{j=0}^{\infty}[w(2j+1)^{-1} - w(2j)^{-1}]\end{aligned} \tag{5.2.5}$$

and hence we conclude

$$0 < 2w(0)^{-1} \le \delta \le 2. \tag{5.2.6}$$

On the other hand, in the case of self-attraction, that is, $w(k+1) \ge w(k)$, we write

$$\begin{aligned}\frac{\delta}{2} &= w(0)^{-1} - \sum_{j=1}^{\infty}[w(2j-1)^{-1} - w(2j)^{-1}]\\ &= 1 + \sum_{j=0}^{\infty}[w(2j)^{-1} - w(2j+1)^{-1}],\end{aligned} \tag{5.2.7}$$

which implies

$$2 \le \delta \le 2w(0)^{-1} < \infty. \tag{5.2.8}$$

The asymptotics of the functions F, $\tilde{F}$, G, $\tilde{G}$, H and $\tilde{H}$ is given in the next lemma.

LEMMA 2A (Asymptotically free case: $\alpha = 0$). *The following asymptotics hold for* $x \gg 1$:

$$F(x) = \frac{\delta}{2} + \mathscr{O}(x^{-1}), \tag{5.2.9}$$

$$\tilde{F}(x) = \frac{2-\delta}{2} + \mathscr{O}(x^{-1}), \tag{5.2.10}$$

$$G(x) = 2x + \mathscr{O}(\log x) = \tilde{G}(x), \tag{5.2.11}$$

$$H(x) = \mathscr{O}(x^2) = \tilde{H}(x). \tag{5.2.12}$$

PROOF. Clearly

$$F(x) = U_1(n+1) - V_1(n) \quad \text{with } n = V_1^{-1}(x), \tag{5.2.13}$$

$$\tilde{F}(x) = V_1(n) - U_1(n) \quad \text{with } n = U_1^{-1}(x). \tag{5.2.14}$$

From (5.2.1) it follows that

$$U_1^{-1}(x) = x + \mathscr{O}(\log x) = V_1^{-1}(x). \tag{5.2.15}$$

Using (5.2.1), (5.2.2), (5.2.15) and (5.2.13) [respectively, (5.2.14)], we easily get (5.2.9) [respectively, (5.2.10)].

To prove (5.2.11) note first that

$$U_2 \circ U_1^{-1}(x) = x + \mathscr{O}(\log x) = V_2 \circ V_1^{-1}(x). \tag{5.2.16}$$

Hence, using (5.1.17)–(5.1.19), (5.2.9)/(5.2.10) and Jensen's inequality,

$$\begin{aligned} &\mathbf{E}(U_2 \circ U_1^{-1}(\mathscr{Y}(1)) \| \mathscr{Y}(0) = x) \\ &\qquad = x + \mathscr{O}(\log x) = \mathbf{E}(V_2 \circ V_1^{-1}(\mathscr{Y}(1)) \| \mathscr{Y}(0) = x). \end{aligned} \tag{5.2.17}$$

Using this in the expressions (5.1.18)–(5.1.20), we get (5.2.11).

The details of the proof of (5.2.12) are lengthy and not very illuminating. The first three terms on the right-hand side of (4.1.12) are estimated directly. For the remaining seven terms, one applies repeatedly the method of the proof of (5.2.23). We omit these details. □

Polynomially self-repelling case: $\alpha \in (0, \infty)$ (Case B). In this case, (1.4) implies

$$U_1(n) = n^{\alpha+1} + \left(\frac{B}{\alpha} + \frac{\alpha+1}{2}\right) n^{\alpha} + \mathscr{O}(n^{\alpha-1} \vee 1), \tag{5.2.18}$$

$$V_1(n) = n^{\alpha+1} + \left(\frac{B}{\alpha} - \frac{\alpha+1}{2}\right) n^{\alpha} + \mathscr{O}(n^{\alpha-1} \vee 1), \tag{5.2.19}$$

$$U_2(n) = \frac{(\alpha+1)^2}{2\alpha+1} n^{2\alpha+1} + \mathscr{O}(n^{2\alpha}) = V_2(n). \tag{5.2.20}$$

We shall denote

$$\beta = \frac{1}{2\alpha + 1}. \tag{5.2.21}$$

Using (5.2.18)–(5.2.20), we get the following asymptotic expressions for the functions F, $\tilde{F}$, G, $\tilde{G}$, H and $\tilde{H}$.

LEMMA 2B [Polynomially self-repelling case: $\alpha \in (0, \infty)$]. *The following asymptotics hold for* $x \gg 1$:

$$F(x) = \frac{\alpha + 1}{2} x^{\alpha/(\alpha+1)} + \mathscr{O}(x^{(\alpha-1)/(\alpha+1)} \vee 1) = \tilde{F}(x), \tag{5.2.22}$$

$$G(x) = 2\frac{(\alpha + 1)^2}{2\alpha + 1} x^{(2\alpha+1)/(\alpha+1)} + \mathscr{O}(x^{2\alpha/(\alpha+1)}) = \tilde{G}(x), \tag{5.2.23}$$

$$H(x) = \mathscr{O}(x^{(4\alpha+2)/(\alpha+1)}) = \tilde{H}(x). \tag{5.2.24}$$

PROOF. From (5.2.18) and (5.2.19) we get

$$U_1^{-1}(x) = x^{1/(\alpha+1)} + \mathscr{O}(1) = V_1^{-1}(x). \tag{5.2.25}$$

Now, from (5.2.18), (5.2.19), (5.2.25) and (5.2.13) [respectively, (5.2.14)], the asymptotic formulas (5.2.22) follow directly.

The derivation of (5.2.23) is slightly more complicated: first note that from (5.2.19) and (5.2.25) it follows that there are two finite constants, say C_1 and C_2, so that for any $x > 0$ and $z \geq 0$,

$$\begin{aligned} &C_1 x^{\alpha/(\alpha+1)}(z - x) \\ &\quad \leq V_2 \circ V_1^{-1}(z) - V_2 \circ V_1^{-1}(x) \\ &\quad \leq C_1 x^{\alpha/(\alpha+1)}(z - x) + C_2 x^{-1/(\alpha+1)}(z - x)^2. \end{aligned} \tag{5.2.26}$$

We insert this in the definition (5.1.18) of the function G and get

$$\begin{aligned} C_1 x^{\alpha/(\alpha+1)} F(x) &\leq G(x) - [(U_2(V_1^{-1}(x) + 1) + V_2 \circ V_1^{-1}(x))] \\ &\leq C_1 x^{\alpha/(\alpha+1)} F(x) + C_2 x^{-1/(\alpha+1)} G(x). \end{aligned} \tag{5.2.27}$$

From these bounds and the explicitly known asymptotics of the functions involved, the asymptotics (5.2.23) of the function G now follows in a straightforward way. An identical derivation holds also for the function $\tilde{G}$.

The proof of (5.2.24) goes through very similar steps, but it is considerably longer. Again, the first three terms on the right-hand side of (4.1.12) are estimated directly and the remaining seven terms are estimated by considerations similar to (5.2.26) and (5.2.27). As the details are lengthy and of no particular interest, we do not present them here.

5.3. *Scaling.* The proper scaling of the processes $\mathscr{Y}(\cdot)$ and $\widetilde{\mathscr{Y}}(\cdot)$ is determined by the dominant terms in the asymptotics of the functions F, G (respectively $\tilde{F}$, $\tilde{G}$). The scaling of the processes $\mathscr{Z}(\cdot)$ and $\widetilde{\mathscr{Z}}(\cdot)$ will be later determined by the functional relations (5.1.12).

Asymptotically free case: $\alpha = 0$ (Case A). Equations (5.2.9)–(5.2.11) suggest the scaling

$$Y_A(t) = A^{-1}\mathscr{Y}([At]), \qquad \tilde{Y}_A(t) = A^{-1}\widetilde{\mathscr{Y}}([At]). \tag{5.3.1}$$

The rescaled martingales $M_A(\cdot)$ and $\tilde{M}_A(\cdot)$ and their quadratic variation processes will be

$$\begin{aligned} M_A(t) &= A^{-1}\mathscr{M}([At]) \\ &= Y_A(t) - Y_A(0) - \int_0^{A^{-1}[At]} F(AY_A(s))\,ds, \end{aligned} \tag{5.3.2}$$

$$\begin{aligned} \tilde{M}_A(t) &= A^{-1}\widetilde{\mathscr{M}}([At]) \\ &= \tilde{Y}_A(t) - \tilde{Y}_A(0) - \int_0^{A^{-1}[At]} \tilde{F}(A\tilde{Y}_A(s))\,ds, \end{aligned} \tag{5.3.3}$$

$$\begin{aligned} \langle M_A, M_A\rangle(t) &= A^{-2}\langle \mathscr{M}, \mathscr{M}\rangle([At]) \\ &= \int_0^{A^{-1}[At]} A^{-1}G(AY_A(s))\,ds, \end{aligned} \tag{5.3.4}$$

$$\begin{aligned} \langle \tilde{M}_A, \tilde{M}_A\rangle(t) &= A^{-2}\langle \widetilde{\mathscr{M}}, \widetilde{\mathscr{M}}\rangle([At]) \\ &= \int_0^{A^{-1}[At]} A^{-1}\tilde{G}(A\tilde{Y}_A(s))\,ds. \end{aligned} \tag{5.3.5}$$

Polynomially self-repelling case: $\alpha \in (0, \infty)$ (Case B). Now, (5.2.22) and (5.2.23) suggest

$$Y_A(t) = (\beta A)^{-(\alpha+1)}\mathscr{Y}([At]), \qquad \tilde{Y}_A(t) = (\beta A)^{-(\alpha+1)}\widetilde{\mathscr{Y}}([At]). \tag{5.3.6}$$

The rescaled martingales $M_A(\cdot)$ and $\tilde{M}_A(\cdot)$ and their quadratic variation processes will be now

$$\begin{aligned} M_A(t) &= (\beta A)^{-(\alpha+1)}\mathscr{M}([At]) \\ &= Y_A(t) - Y_A(0) - \int_0^{A^{-1}[At]} \beta^{-1}(\beta A)^{-\alpha} F((\beta A)^{\alpha+1}Y_A(s))\,ds, \end{aligned} \tag{5.3.7}$$

$$\begin{aligned} \tilde{M}_A(t) &= (\beta A)^{-(\alpha+1)}\widetilde{\mathscr{M}}([At]) \\ &= \tilde{Y}_A(t) - \tilde{Y}_A(0) - \int_0^{A^{-1}[At]} \beta^{-1}(\beta A)^{-\alpha} \tilde{F}((\beta A)^{\alpha+1}\tilde{Y}_A(s))\,ds, \end{aligned} \tag{5.3.8}$$

$$\begin{aligned} \langle M_A, M_A\rangle(t) &= (\beta A)^{-2(\alpha+1)}\langle \mathscr{M}, \mathscr{M}\rangle\,([At]) \\ &= \int_0^{A^{-1}[At]} \beta^{-1}(\beta A)^{-(2\alpha+1)} G((\beta A)^{\alpha+1}Y_A(s))\,ds \end{aligned} \tag{5.3.9}$$

$$\begin{aligned}\langle \tilde{M}_A, \tilde{M}_A\rangle(t) &= (\beta A)^{-2(\alpha+1)}\langle \widetilde{\mathscr{M}}, \widetilde{\mathscr{M}}\rangle([At]) \\ &= \int_0^{A^{-1}[At]} \beta^{-1}(\beta A)^{-(2\alpha+1)}\tilde{G}((\beta A)^{\alpha+1}\tilde{Y}_A(s))\, ds.\end{aligned} \tag{5.3.10}$$

The functional relations (5.1.12), the asymptotics (5.2.15) [respectively, (5.2.25)] and the scaling (5.3.1) [respectively, (5.3.6)] determine the proper scaling of the processes $\mathscr{Z}(\cdot)$ and $\widetilde{\mathscr{Z}}(\cdot)$:

Asymptotically free case: $\alpha = 0$ (Case A):

$$Z_A(t) = A^{-1}\mathscr{Z}([At]), \qquad \tilde{Z}_A(t) = A^{-1}\widetilde{\mathscr{Z}}([At]). \tag{5.3.11}$$

Polynomially self-repelling case: $\alpha \in (0, \infty)$ (Case B):

$$Z_A(t) = (\beta A)^{-1}\mathscr{Z}([At]), \qquad \tilde{Z}_A(t) = (\beta A)^{-1}\widetilde{\mathscr{Z}}([At]). \tag{5.3.12}$$

5.4. *Tightness.* Given the asymptotic estimates (5.2.9)–(5.2.12) [respectively, (5.2.22)–(5.2.24)], the proof of tightness is rather standard: we have to check Kolmogorov's criterion; that is, the conditions of Theorem 12.3 from Billingsley (1968). We give the details of the proof for the processes $Y_A(\cdot)$ in the asymptotically free case ($\alpha = 0$); the proof for the other cases is completely identical.

Let M_l, $l = 0, 1, 2, \ldots$, be an arbitrary discrete parameter martringale and write

$$\xi_l = M_l - M_{l-1}, \qquad l = 1, 2, 3, \ldots. \tag{5.4.1}$$

The following identity holds:

$$\begin{aligned}\mathbf{E}((M_l - M_k)^4) &= 6 \sum_{j=k+1}^{l} \mathbf{E}(\xi_j^2(M_{j-1} - M_k)^2) \\ &\quad + 4 \sum_{j=k+1}^{l} \mathbf{E}(\xi_j^3(M_{j-1} - M_k)) + \sum_{j=k+1}^{l} \mathbf{E}(\xi_j^4),\end{aligned} \tag{5.4.2}$$

which, via Jensen's inequality, yields

$$\begin{aligned}\mathbf{E}((M_l - M_k)^4) &\le 6 \sum_{j=k+1}^{l} \mathbf{E}(\mathbf{E}(\xi_j^2 \,|\, \mathscr{F}_{j-1})(M_{j-1} - M_k)^2) \\ &\quad +4 \sum_{j=k+1}^{l} \sqrt{\mathbf{E}(\mathbf{E}(\xi_j^4|\mathscr{F}_{j-1})^{3/2}(M_{j-1} - M_k)^2)} \\ &\quad + \sum_{j=k+1}^{l} \mathbf{E}(\mathbf{E}(\xi_j^4 \,|\, \mathscr{F}_{j-1})).\end{aligned} \tag{5.4.3}$$

Applied to the martingale $M_A(\cdot)$ defined in (5.3.2) this gives

$$
\begin{aligned}
&\mathbf{E}((M_A(t) - M_A(s))^4) \\
&\quad \le 6\int_{[As]/A}^{[At]/A} \mathbf{E}(A^{-1}G(AY_A(r))(M_A(r) - M_A(s))^2)\,dr \\
&\qquad + 4A^{-1/2}\int_{[As]/A}^{[At]/A} \sqrt{\mathbf{E}((A^{-2}H(AY_A(r)))^{3/2}(M_A(r) - M_A(s))^2)}\,dr \\
&\qquad + A^{-1}\int_{[As]/A}^{[At]/A} \mathbf{E}(A^{-2}H(AY_A(r)))\,dr.
\end{aligned} \tag{5.4.4}
$$

From the asymptotics (5.2.11) and (5.2.12) it follows that there exists a finite constant C, such that for any $y > 0$,

$$A^{-1}G(Ay) < C(y+1), \qquad A^{-2}H(Ay) < C(y^2+1). \tag{5.4.5}$$

Consider the stopping times

$$\tau_{y,A} = \inf\{t \ge 0: Y_A(t) \ge y\}. \tag{5.4.6}$$

From (5.4.4) we easily get for $t, s \le \tau_{y,A}$,

$$
\begin{aligned}
&\mathbf{E}((M_A(t \wedge \tau_{y,A}) - M_A(s \wedge \tau_{y,A}))^4) \\
&\qquad \le K(y)(|t-s| \vee A^{-1})^2,
\end{aligned} \tag{5.4.7}
$$

where $K(y)$ is a constant depending on y only. Hence, applying Theorem 12.3 of Billingsley (1968), we get the tightness of the martingales $M_A(t \wedge \tau_{y,A})$ for any $y < \infty$. This also implies tightness of the martingales $M_A(t)$. From (5.3.2) it is straightforward to see that tightness of the processes $M_A(\cdot)$ and $Y_A(\cdot)$ is equivalent.

5.5. *Identification of the limiting processes.* Assume, with some abuse of notation, that $Y_A(\cdot)$ and $\tilde{Y}_A(\cdot)$ are weakly convergent *subsequences*:

$$Y_A(\cdot) \Rightarrow Y(\cdot), \qquad \tilde{Y}_A(\cdot) \Rightarrow \tilde{Y}(\cdot). \tag{5.5.1}$$

From this it follows that the martingales $M_A(\cdot)$ and $\tilde{M}_A(\cdot)$ converge weakly, too:

$$M_A(\cdot) \Rightarrow M(\cdot), \qquad \tilde{M}_A(\cdot) \Rightarrow \tilde{M}(\cdot). \tag{5.5.2}$$

Asymptotically free case: $\alpha = 0$ (Case A). We use (5.3.2)–(5.3.5) and the asymptotics (5.2.9)–(5.2.11) and get

$$M(t) = Y(t) - Y(0) - \frac{\delta}{2}t, \qquad \langle M, M\rangle(t) = 2\int_0^t Y(s)\,ds, \tag{5.5.3}$$

$$\tilde{M}(t) = \tilde{Y}(t) - \tilde{Y}(0) - \frac{2-\delta}{2}t, \qquad \langle \tilde{M}, \tilde{M}\rangle(t) = 2\int_0^t \tilde{Y}(s)\,ds. \tag{5.5.4}$$

These relations yield the following SDE's for $Y(\cdot)$ [respectively, $\tilde{Y}(\cdot)$]:

$$
\begin{aligned}
dY(t) &= \frac{\delta}{2}\,dt + \sqrt{2Y(t)}\,dW(t), \\
d\tilde{Y}(t) &= \frac{2-\delta}{2}\,dt + \sqrt{2\tilde{Y}(t)}d\tilde{W}(t),
\end{aligned}
\tag{5.5.5}
$$

which are valid as long as $Y(t) > 0$ [respectively, $\tilde{Y}(t) > 0$]. These are precisely the SDE's of the BESQ^{δ} (respectively, $\widetilde{\mathrm{BESQ}}^{2-\delta}$) processes described in Section 2.

Polynomially self-repelling case: $\alpha \in (0, \infty)$ (Case B). Now, (5.3.7)–(5.3.10), (5.2.22), (5.2.23) and the explicit value of β given in (5.2.21) lead to

$$
M(t) = Y(t) - Y(0) - \frac{(\alpha+1)(2\alpha+1)}{2}\int_0^t Y(s)^{\alpha/(\alpha+1)}\,ds, \tag{5.5.6}
$$

$$
\langle M, M\rangle(t) = 2(\alpha+1)^2\int_0^t Y(s)^{(2\alpha+1)/(\alpha+1)}\,ds, \tag{5.5.7}
$$

$$
\tilde{M}(t) = \tilde{Y}(t) - \tilde{Y}(0) - \frac{(\alpha+1)(2\alpha+1)}{2}\int_0^t \tilde{Y}(s)^{\alpha/(\alpha+1)}\,ds, \tag{5.5.8}
$$

$$
\langle \tilde{M}, \tilde{M}\rangle(t) = 2(\alpha+1)^2\int_0^t \tilde{Y}(s)^{(2\alpha+1)/(\alpha+1)}\,ds. \tag{5.5.9}
$$

We write these relations again as SDE's:

$$
\begin{aligned}
dY(t) &= \frac{(\alpha+1)(2\alpha+1)}{2} Y(t)^{\alpha/(\alpha+1)}\,dt \\
&\quad + \sqrt{2}(\alpha+1)Y(t)^{(2\alpha+1)/(2\alpha+2)}\,dW(t), \\
d\tilde{Y}(t) &= \frac{(\alpha+1)(2\alpha+1)}{2} \tilde{Y}(t)^{\alpha/(\alpha+1)}\,dt \\
&\quad + \sqrt{2}(\alpha+1)\tilde{Y}(t)^{(2\alpha+1)/(2\alpha+2)}\,d\tilde{W}(t).
\end{aligned}
\tag{5.5.10}
$$

These SDE's are again valid as long as $Y(t) > 0$ [respectively, $\tilde{Y}(t) > 0$]. The SDE's in (5.5.10) are easily identified as the SDE's of the $(\alpha+1)$th power of BESQ^1.

The SDE's (5.5.5) and (5.5.10) *determine uniquely the limit processes* $Y(\cdot)$ *and* $\tilde{Y}(\cdot)$ *as long as these processes do not hit the boundary* $0 = \partial\mathbb{R}_+$. In order to identify completely the limit processes we have to describe precisely their behavior at the boundary.

5.6. *Reflection at the boundary.* In this subsection we prove that the limit processes $Y(\cdot)$ are *reflected instantaneously at* $0 = \partial\mathbb{R}_+$. For $y \in \mathbb{R}_+$ let us denote

$$
\tau_y = \inf\{l \geq 0 : \mathscr{Y}(l) \geq y\}, \tag{5.6.1}
$$

$$
\tau_{y,A} = \inf\{t \geq 0 : Y_A(t) \geq y\}. \tag{5.6.2}
$$

We shall prove that for any $\eta > 0$,

$$\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(\tau_{y,A} > \eta \| Y_A(0) = 0) = 0, \tag{5.6.3}$$

from which the assertion follows.

Asymptotically free case: $\alpha = 0$ (Case A). In this case,

$$\tau_{y,A} = A^{-1}\tau_{Ay}. \tag{5.6.4}$$

We apply directly the optional sampling theorem [see Breiman (1968)] to the martingale $\mathscr{M}(l)$ defined in (5.1.21) with the stopping time τ_y:

$$\begin{aligned} 0 &= \mathbf{E}(\mathscr{M}(\tau_y \wedge l) \| \mathscr{Y}(0) = 0) \\ &= \mathbf{E}(\mathscr{Y}(\tau_y \wedge l) \| \mathscr{Y}(0) = 0) - \mathbf{E}\Bigg(\sum_{j=0}^{\tau_y \wedge l - 1} F(\mathscr{Y}(j)) \| \mathscr{Y}(0) = 0 \Bigg). \end{aligned} \tag{5.6.5}$$

However, from (5.2.5) and (5.2.7) we see that

$$\inf_{x\geq 0} F(x) = \min\left\{ \frac{2}{w(0)}, 2 \right\} > 0. \tag{5.6.6}$$

From (5.6.5) and (5.6.6) we conclude

$$\mathbf{E}(\tau_y \wedge l \| \mathscr{Y}(0) = 0) \leq \max\left\{ \frac{w(0)}{2}, \frac{1}{2} \right\} \mathbf{E}(\mathscr{Y}(\tau_y \wedge l) \| \mathscr{Y}(0) = 0). \tag{5.6.7}$$

Using this inequality it follows that for any $t < \infty$,

$$\begin{aligned} &\limsup_{A\to\infty} \mathbf{E}(\tau_{y,A} \wedge t \| Y_A(0) = 0) \\ &\quad = \limsup_{A\to\infty} \mathbf{E}(A^{-1}(\tau_{Ay} \wedge [At]) \| \mathscr{Y}(0) = 0) && (5.6.8) \\ &\quad \leq \max\left\{ \frac{w(0)}{2}, \frac{1}{2} \right\} \limsup_{A\to\infty} \mathbf{E}(A^{-1}\mathscr{Y}(\tau_{Ay} \wedge [At]) \| \mathscr{Y}(0) = 0). && (5.6.9) \end{aligned}$$

In order to estimate the right-hand side of (5.6.8) and (5.6.9), note first that the following simple bound holds for the biggest jump of $\mathscr{Y}(\cdot)$ before $\tau_y \wedge l$:

$$\mathbf{E}\Big(\max_{1\leq k\leq \tau_y \wedge l} |\mathscr{Y}(k) - \mathscr{Y}(k-1)| \Big) \leq \Big(l \max_{0\leq z\leq y} H(z) \Big)^{1/4}. \tag{5.6.10}$$

Hence, using the asymptotics (5.2.12) of the function H, we get the "overshoot bound"

$$\limsup_{A\to\infty} \mathbf{E}\left(A^{-1}\mathscr{Y}(\tau_{Ay} \wedge [At]) \middle\| \mathscr{Y}(0) = 0 \right) \leq y, \tag{5.6.11}$$

which combined with (5.6.9) leads us to

$$\limsup_{A\to\infty} \mathbf{E}(\tau_{y,A} \wedge t \| Y_A(0) = 0) \leq \max\left\{ \frac{w(0)}{2}, \frac{1}{2} \right\} y, \tag{5.6.12}$$

which is valid for any $t < \infty$. Hence, via Markov's inequality, (5.6.3) follows for the asymptotically free case.

Polynomially self-repelling case: $\alpha \in (0, \infty)$ (Case B). This is slightly more complicated. In this case,

$$\tau_{y,A} = A^{-1}\tau_{(\beta A)^{\alpha+1}y}. \tag{5.6.13}$$

For any $x > 0$, $z \geq 0$, the following inequality holds:

$$\begin{aligned} z^{1/(\alpha+1)} - x^{1/(\alpha+1)} \geq{}& \frac{1}{\alpha+1}x^{-\alpha/(\alpha+1)}(z-x) \\ & - \frac{1}{2}\frac{\alpha}{(\alpha+1)^2}x^{-(2\alpha+1)/(\alpha+1)}(z-x)^2. \end{aligned} \tag{5.6.14}$$

From this it follows that

$$\begin{aligned} &\mathbf{E}(\mathscr{Y}(l+1)^{1/(\alpha+1)} \| \mathscr{Y}(l) = x) - x^{1/(\alpha+1)} \\ &\quad \geq \frac{1}{\alpha+1}x^{-\alpha/(\alpha+1)}F(x) - \frac{1}{2}\frac{\alpha}{(\alpha+1)^2}x^{-(2\alpha+1)/(\alpha+1)}G(x) \\ &\quad = \frac{1}{2(2\alpha+1)} + \mathscr{O}(x^{-1/(\alpha+1)} \vee x^{-1/(\alpha+1)}). \end{aligned} \tag{5.6.15}$$

Fix x_0 so that for $x \geq x_0$,

$$\mathbf{E}(\mathscr{Y}(l+1)^{1/(\alpha+1)} \| \mathscr{Y}(l) = x) - x^{1/(\alpha+1)} > \frac{1}{4(2\alpha+1)} \tag{5.6.16}$$

and denote $x_1 = (x_0^{1/(\alpha+1)} + 1)^{\alpha+1}$. We consider the sequence of sampling times

$$k_0 = 0, \qquad k_l = \min\{i > k_{l-1} : \mathscr{Y}(i) > x_0\} \tag{5.6.17}$$

and the time-changed process

$$\widehat{\mathscr{Y}}(l) = \mathscr{Y}(k_l), \qquad l \geq 1. \tag{5.6.18}$$

Due to (5.6.16) the process

$$\widehat{\mathscr{N}}(l) = \widehat{\mathscr{Y}}(l)^{1/(\alpha+1)} - \frac{1}{4(2\alpha+1)}l \tag{5.6.19}$$

is *submartingale*. We define the following sequence of stopping times:

$$s_0 = 0, \tag{5.6.20}$$

$$t_i = \min\{l > s_{i-1} : \mathscr{Y}(l) \geq x_1\}, \tag{5.6.21}$$

$$s_i = \min\{l > t_i : \mathscr{Y}(l) \leq x_0\}. \tag{5.6.22}$$

For $y \geq x_1$ fixed, let

$$\hat{\tau}_y = \inf\{l > 0 : \widehat{\mathscr{Y}}(l) \geq y\}, \tag{5.6.23}$$

$$r_y = \max\{i : s_i \leq \tau_y\}, \tag{5.6.24}$$

$$\hat{\hat{\tau}}_y = \sum_{i=1}^{r_y+1} (t_i - s_{i-1}). \tag{5.6.25}$$

In plain words, $\hat{\tau}_y$ denotes the time spent in the interval (x_0, ∞), r_y denotes the number of downcrossings of the interval $[x_0, x_1]$ and $\hat{\hat{\tau}}_y$ denotes the time spent in the interval $[0, x_1)$ by the process $\mathscr{Y}(l)$, before τ_y. Clearly

$$\tau_y \leq \hat{\tau}_y + \hat{\hat{\tau}}_y \tag{5.6.26}$$

and

$$r_y + 1 \leq \hat{\tau}_y. \tag{5.6.27}$$

Applying the optional sampling theorem for the submartingale $\widehat{\mathscr{N}}(l)$ defined in (5.6.19), we get

$$\begin{aligned} \mathbf{E}(\hat{\tau}_y \wedge l \| \mathscr{Y}(0) = 0) &\leq 4(2\alpha + 1)\, \mathbf{E}(\widehat{\mathscr{Y}}(\hat{\tau}_y \wedge l)^{1/(\alpha+1)} \| \mathscr{Y}(0) = 0) \\ &\leq 4(2\alpha + 1)\, \mathbf{E}(\widehat{\mathscr{Y}}(\hat{\tau}_y \wedge l) \| \mathscr{Y}(0) = 0)^{1/(\alpha+1)}. \end{aligned} \tag{5.6.28}$$

Hence

$$\begin{aligned} &\limsup_{A\to\infty} \mathbf{E}(A^{-1}(\hat{\tau}_{(\beta A)^{\alpha+1} y} \wedge [At]) \| \mathscr{Y}(0) = 0) \\ &\qquad \leq 4(2\alpha + 1) \limsup_{A\to\infty} \mathbf{E}(A^{-1}\widehat{\mathscr{Y}}(\hat{\tau}_{(\beta A)^{\alpha+1} y} \wedge [At]) \| \mathscr{Y}(0) = 0)^{1/(\alpha+1)}. \end{aligned} \tag{5.6.29}$$

An estimate identical to (5.6.10) on the biggest jump of the process $\widehat{\mathscr{Y}}(\cdot)$ and the asymptotics (5.2.24) of the function H yields now the "overshoot bound"

$$\limsup_{A\to\infty} \mathbf{E}((\beta A)^{-(\alpha+1)} \widehat{\mathscr{Y}}(\hat{\tau}_{(\beta A)^{\alpha+1} y} \wedge [At]) \| \mathscr{Y}(0) = 0) \leq y, \tag{5.6.30}$$

which combined with (5.6.29) leads to

$$\limsup_{A\to\infty} \mathbf{E}(A^{-1}(\hat{\tau}_{(\beta A)^{\alpha+1} y} \wedge [At]) \| \mathscr{Y}(0) = 0) \leq 4(2\alpha + 1) y^{1/(\alpha+1)}. \tag{5.6.31}$$

Hence, by Markov's inequality,

$$\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(A^{-1}\hat{\tau}_{(\beta A)^{\alpha+1} y} > \eta \| \mathscr{Y}(0) = 0) = 0. \tag{5.6.32}$$

To estimate $\hat{\hat{\tau}}_y$, note first that the random variables

$$t_i - s_{i-1}, \qquad i = 1, 2, 3, \ldots, \tag{5.6.33}$$

are *uniformly stochastically bounded*:

$$m = \max_{x \le x_0} \mathbf{E}(\tau_{x_1} \| \mathscr{Y}(0) = x) \tag{5.6.34}$$

is finite since the Markov chain $\mathscr{Y}(\cdot)$ is not trapped in any finite interval. Using in turn this fact, Markov's inequality and (5.6.32), we find

$$\begin{aligned}
&\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(A^{-1}\hat{\hat{\tau}}_{(\beta A)^{\alpha+1}y} > \eta \| \mathscr{Y}(0) = 0) \\
&\quad \le \lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}\Big([A^{-1}\hat{\hat{\tau}}_{(\beta A)^{\alpha+1}y} > \eta] \\
&\qquad\qquad \wedge [A^{-1}(r_{(\beta A)^{\alpha+1}y} + 1) \le \varepsilon] \big\| \mathscr{Y}(0) = 0\Big) \qquad (5.6.35)\\
&\qquad + \lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(A^{-1}(r_{(\beta A)^{\alpha+1}y} + 1) > \varepsilon \| \mathscr{Y}(0) = 0) \\
&\quad \le \frac{m\varepsilon}{\eta} + \lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(A^{-1}\hat{\tau}_{(\beta A)^{\alpha+1}y} > \varepsilon \| \mathscr{Y}(0) = 0) = \frac{m\varepsilon}{\eta}.
\end{aligned}$$

Letting $\varepsilon \to 0$ on the right-hand side of (5.6.35), we get

$$\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}\left(A^{-1}\hat{\hat{\tau}}_{(\beta A)^{\alpha+1}y} > \eta \big\| \mathscr{Y}(0) = 0\right) = 0. \tag{5.6.36}$$

Finally, (5.6.26), (5.6.32) and (5.6.36) imply (5.6.3) for the polynomially self-repelling case, $\alpha \in (0, \infty)$.

5.7. *Absorption at the boundary.* We prove that the limit processes $\tilde{Y}(\cdot)$ are *absorbed at* $0 = \partial\mathbb{R}_+$. When proving this latter assertion we also get the *weak convergence of the extinction times* $\tilde{\sigma}_{0,A}$.

For $x \in \mathbb{R}_+$ we denote

$$\tilde{\sigma}_x = \inf\{l \ge 0 : \tilde{\mathscr{Y}}(l) \le x\}, \tag{5.7.1}$$

$$\tilde{\sigma}_{x,A} = \inf\{t \ge 0 : \tilde{Y}_A(t) \le x\} \tag{5.7.2}$$

We prove now that for any $\eta > 0$,

$$\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(\tilde{\sigma}_{0,A} > \eta \| \tilde{Y}_A(0) = y) = 0. \tag{5.7.3}$$

Asymptotically free case: $\alpha = 0$ (Case A). In this case (5.7.3) is equivalent to

$$\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(\tilde{\sigma}_0 > A\eta \| \tilde{\mathscr{Y}}(0) = Ay) = 0. \tag{5.7.4}$$

$\delta > 2$ and $0 < \delta \le 2$ are treated separately.

Case A1. First we consider the self-attracting cases with $\delta > 2$. From (5.2.10) it follows that there exists an $x_0 < \infty$ such that for $x \ge x_0$,

$$\tilde{F}(x) \le \frac{2-\delta}{4} < 0 \tag{5.7.5}$$

and thus

$$\mathscr{N}(l) = \widetilde{\mathscr{Y}}(l) + \frac{\delta - 2}{4} l \tag{5.7.6}$$

is *supermartingale* as long as $\widetilde{\mathscr{Y}}(l) \geq x_0$. Applying the optional sampling theorem to the supermartingale $\mathscr{N}(l)$ we get for $y > x_0$,

$$\mathbf{E}(\tilde{\sigma}_{x_0} \| \widetilde{\mathscr{Y}}(0) = y) \leq \frac{4}{\delta - 2} y. \tag{5.7.7}$$

Now we prove (5.7.4):

$$\begin{aligned}
&\lim_{y \to 0} \limsup_{A \to \infty} \mathbf{P}(\tilde{\sigma}_0 > A\eta \| \widetilde{\mathscr{Y}}(0) = Ay) \\
&\quad \leq \lim_{y \to 0} \limsup_{A \to \infty} \mathbf{P}(\tilde{\sigma}_{x_0} > A\eta/2 \| \widetilde{\mathscr{Y}}(0) = Ay) \\
&\qquad + \limsup_{A \to \infty} \sup_{0 \leq x \leq x_0} \mathbf{P}(\tilde{\sigma}_0 > A\eta/2 \| \widetilde{\mathscr{Y}}(0) = x).
\end{aligned} \tag{5.7.8}$$

Applying Markov's inequality and (5.7.7) we get

$$\lim_{y \to 0} \limsup_{A \to \infty} \mathbf{P}\left(\tilde{\sigma}_{x_0} > \frac{A\eta}{2} \middle\| \widetilde{\mathscr{Y}}(0) = Ay \right) \leq \lim_{y \to 0} \frac{8y}{(\delta - 2)\eta} = 0. \tag{5.7.9}$$

On the other hand, since x_0 is constant independent of A, the second limit on the right hand side of (5.7.8) clearly vanishes. Hence (5.7.4) for this case.

Case A2. Next we deal with the cases when $\delta \leq 2$. Choose

$$\gamma < \frac{\delta}{2} \leq 1. \tag{5.7.10}$$

For any $x > 0$, $z \geq 0$, the following inequality holds:

$$\begin{aligned}
z^\gamma - x^\gamma \leq \gamma x^{\gamma - 1}(z - x) - \frac{\gamma(1 - \gamma)}{2} x^{\gamma - 2}(z - x)^2 \\
+ \frac{\gamma(\gamma - 1)(\gamma - 2)}{6} x^{\gamma - 3}(z - x)^3
\end{aligned} \tag{5.7.11}$$

From this it follows that

$$\begin{aligned}
&\mathbf{E}(\widetilde{\mathscr{Y}}(l+1)^\gamma \| \widetilde{\mathscr{Y}}(l) = x) - x^\gamma \\
&\quad \leq \gamma x^{\gamma - 1} \tilde{F}(x) - \frac{\gamma(1 - \gamma)}{2} x^{\gamma - 2} \tilde{G}(x) \\
&\qquad + \frac{\gamma(\gamma - 1)(\gamma - 2)}{6} x^{\gamma - 3} \tilde{H}(x)^{3/4} \\
&\quad = -\frac{\gamma(\delta - 2\gamma)}{2} x^{\gamma - 1} + \mathscr{O}(x^{\gamma - 3/2}),
\end{aligned} \tag{5.7.12}$$

where we have used the asymptotics (5.2.10)–(5.2.12). Fix $x_0 < \infty$ such that for $x \geq x_0$,

$$\mathbf{E}(\widetilde{\mathscr{Y}}(l+1)^\gamma \| \widetilde{\mathscr{Y}}(l) = x) - x^\gamma \leq -\frac{\gamma(\delta - 2\gamma)}{4} x^{\gamma - 1}. \tag{5.7.13}$$

Thus

$$\mathscr{N}(l) = \widetilde{\mathscr{Y}}(l)^{\gamma} + \frac{\gamma(\delta - 2\gamma)}{4} \sum_{k=0}^{l-1} \widetilde{\mathscr{Y}}(k)^{\gamma-1} \tag{5.7.14}$$

is *supermartingale* as long as $\mathscr{Y}(l) > x_0$, Hence, for $y \geq x_0$,

$$\begin{aligned} &\mathbf{E}\biggl(\tilde{\sigma}_{x_0}\biggl[\sup_{0\leq k\leq \tilde{\sigma}_{x_0}-1} \widetilde{\mathscr{Y}}(k)\biggr]^{\gamma-1} \biggl\| \widetilde{\mathscr{Y}}(0) = y\biggr) \\ &\quad \leq \mathbf{E}\biggl(\sum_{k=0}^{\tilde{\sigma}_{x_0}-1} \widetilde{\mathscr{Y}}(k)^{\gamma-1} \biggl\| \widetilde{\mathscr{Y}}(0) = y\biggr) \\ &\quad \leq \frac{4}{\gamma(\delta - 2\gamma)} y^{\gamma}. \end{aligned} \tag{5.7.15}$$

Since $\widetilde{\mathscr{Y}}(k)^{\gamma}$, $0 \leq k \leq \tilde{\sigma}_{x_0}$, itself is a supermartingale, we also have

$$\mathbf{P}\biggl(\sup_{0\leq k\leq \tilde{\sigma}_{x_0}-1} \widetilde{\mathscr{Y}}(k) > \lambda \biggl\| \widetilde{\mathscr{Y}}(0) = y\biggr) \leq \frac{y^{\gamma}}{\lambda^{\gamma}}. \tag{5.7.16}$$

Finally, using (5.7.15) and (5.7.16) we derive

$$\begin{aligned} &\mathbf{P}\biggl(\tilde{\sigma}_{x_0} > \frac{A\eta}{2} \biggl\| \widetilde{\mathscr{Y}}(0) = Ay\biggr) \\ &\quad \leq \mathbf{P}\biggl(\biggl[\tilde{\sigma}_{x_0} > \frac{A\eta}{2}\biggr] \wedge \biggl[\sup_{0\leq k\leq \tilde{\sigma}_{x_0}-1} \widetilde{\mathscr{Y}}(k) \leq A\lambda\biggr] \biggl\| \widetilde{\mathscr{Y}}(0) = Ay\biggr) \\ &\qquad + \mathbf{P}\biggl(\sup_{0\leq k\leq \tilde{\sigma}_{x_0}-1} \widetilde{\mathscr{Y}}(k) > A\lambda \biggl\| \widetilde{\mathscr{Y}}(0) = Ay\biggr) \\ &\quad \leq \mathbf{P}\biggl(\tilde{\sigma}_{x_0}\biggl[\sup_{0\leq k\leq \tilde{\sigma}_{x_0}-1} \widetilde{\mathscr{Y}}(k)\biggr]^{\gamma-1} > \frac{A^{\gamma}\lambda^{\gamma-1}\eta}{2} \biggl\| \widetilde{\mathscr{Y}}(0) = Ay\biggr) \\ &\qquad + \mathbf{P}\biggl(\sup_{0\leq k\leq \tilde{\sigma}_{x_0}-1} \widetilde{\mathscr{Y}}(k) > A\lambda \biggl\| \widetilde{\mathscr{Y}}(0) = Ay\biggr) \\ &\quad \leq \frac{8}{\gamma(\delta - 2\gamma)} \frac{y^{\gamma}}{\eta\lambda^{\gamma-1}} + \frac{y^{\gamma}}{\lambda^{\gamma}}. \end{aligned} \tag{5.7.17}$$

Hence

$$\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(\tilde{\sigma}_{x_0} > A\eta/2 \| \widetilde{\mathscr{Y}}(0) = Ay) = 0 \tag{5.7.18}$$

and from an argument identical to (5.7.8) we get (5.7.4).

Polynomially self-repelling case: $\alpha \in (0, \infty)$ (Case B). In this case (5.7.3) is equivalent to

$$\lim_{y\to 0} \limsup_{A\to\infty} \mathbf{P}(\tilde{\sigma}_0 > A\eta \| \widetilde{\mathscr{Y}}(0) = A^{\alpha+1} y) = 0. \tag{5.7.19}$$

The proof is completely identical to the previous one, with the choice of

$$\gamma < \frac{1}{2(\alpha+1)} < \frac{1}{2}. \tag{5.7.20}$$

We omit the repetition of the details.

5.8. *End of proof.* Collecting the results of Sections 5.4, 5.5, 5.6 and 5.7 we conclude the following:

In the *asymptotically free case*, $\alpha = 0$ (Case A),

$$Z_A(\cdot) \Rightarrow Z^{(\delta)}(\cdot) \tag{5.8.1}$$

in the space $D([0,\infty))$ and

$$(\tilde{Z}_A(\cdot), \tilde{\sigma}_{0,A}) \Rightarrow (\tilde{Z}^{(2-\delta)}(\cdot), \tilde{\sigma}_0) \tag{5.8.2}$$

in the space $D([0,\infty)) \times [0,\infty)$, where $Z^{(\delta)}(\cdot)$ is the BESQ$^{\delta}$ process and $\tilde{Z}^{(2-\delta)}(\cdot)$ is the $\widetilde{\text{BESQ}}^{2-\delta}$ process defined in Section 2. The parameter δ is given in (5.2.4).

In the *polynomially self-repelling case*, $\alpha \in (0,\infty)$ (Case B),

$$Z_A(\cdot) \Rightarrow Z^{(1)}(\cdot) \tag{5.8.3}$$

in the space $D([0,\infty))$ and

$$(\tilde{Z}_A(\cdot), \tilde{\sigma}_{0,A}) \Rightarrow (\tilde{Z}^{(1)}(\cdot), \tilde{\sigma}_0) \tag{5.8.4}$$

in the space $D([0,\infty)) \times [0,\infty)$.

Given the representation of the local time process described in Section 4.2, Theorem 1A (respectively, Theorem 1B), follows directly from (5.8.1) and (5.8.2) [respectively (5.8.3) and (5.8.4)] after noting that due to (4.1.11) it is easily seen that the single exceptional step (4.2.5) does not spoil the continuity of the limit process $S^*_{x,h}(y)$ at $y = x$. $\square$

Corollaries 1A and 1B follow directly from Theorems 1A and 1B, respectively. Note that the *joint convergence* of the processes $S^*_{[Ax],[Ah]}([A\cdot])/A$ and extinction times $\omega^{*\pm}_{[Ax],[Ah]}/A$ is needed in this proof.

6. Proof of Theorems 2A and 2B. We prove a general abstract version of Theorems 2A and 2B:

Let $\mathbb{R} \ni y \mapsto S_{a,h}(y) \in \mathbb{R}_+$ be a *generalized Ray–Knight process*, as defined in Section A.5 of the Appendix and let $T_{a,h}$ be the total area under $S_{a,h}(\cdot)$. Define the functions $\varrho(t,a,h)$, $\pi(t,x)$, $\hat{\varrho}(s,a,h)$ and $\hat{\pi}(t,x)$ via formulas (2.12), (2.14), (2.16), and (2.17), respectively, with the superscript (δ) erased. According to Theorem 4 (proved in the Appendix) the functions $\mathbb{R} \ni x \mapsto \pi(t,x) \in \mathbb{R}_+$ and $\mathbb{R} \ni x \mapsto \hat{\pi}(s,x) \in \mathbb{R}_+$ are *probability densities*.

Consider a nearest neighbor, self-interacting random walk X_i on $\mathbb{Z}$ with law given by (1.1)–(1.3), with arbitrary weight function $w(\cdot)$ and denote by $T^*_{k,m}$ the hitting times defined in (3.1.2) and (3.1.3). Further, let $P(n,k)$ and $R(s,k)$

be the distributions of the random walk X_i at time n and at the geometrically distibuted time θ_s, respectively, as given in (3.2.1)–(3.2.3).

THEOREM 2′. *Assume there are two constants $\beta > 0$ and $\gamma > 0$ so that for any $x > 0$, $h > 0$ and $* = <$ or $>$,*

$$\frac{T^*_{[Ax],\,[A^\gamma \beta h]}}{2\beta A^{1+\gamma}} \Rightarrow T_{x,\,h} \tag{6.1}$$

as $A \to \infty$. Then for any $s > 0$ and almost all $x \in \mathbb{R}$,

$$\hat{\pi}_A(s, x) \to \beta^{1/(1+\gamma)} \hat{\pi}(s, \beta^{1/(1+\gamma)} x) \tag{6.2}$$

as $A \to \infty$, where $\hat{\pi}_A(s, x)$ is the properly rescaled distribution of the random walk:

$$\hat{\pi}_A(s, x) = A^{1/(1+\gamma)} R(A^{-1}s, [A^{1/(1+\gamma)}x]). \tag{6.3}$$

PROOF. We note first that

$$P(n, k) = \mathbf{P}(X_n = k) = \sum_{m=0}^{\infty} [\mathbf{P}(T^>_{k,\,m} = n) + \mathbf{P}(T^<_{k,\,m} = n)]. \tag{6.4}$$

On the other hand, from the definition (6.3) of $\hat{\pi}_A$,

$$\hat{\pi}_A(s, x) = \frac{1 - e^{-s/A}}{s/A} s A^{-\gamma/(1+\gamma)} \sum_{n=0}^{\infty} e^{-ns/A} P(n, [A^{1/(1+\gamma)}x]). \tag{6.5}$$

Combining (6.4) and (6.5) we are led to

$$\begin{aligned} \hat{\pi}_A(s, x) = \frac{1 - \exp(-s/A)}{s/A} s A^{-\gamma/(1+\gamma)} \sum_{m=0}^{\infty} \Bigg[&\mathbf{E}\bigg(\exp\bigg\{ -\frac{s}{A} T^>_{[A^{1/(1+\gamma)}x],\, m} \bigg\} \bigg) \\ &+ \mathbf{E}\bigg(\exp\bigg\{ -\frac{s}{A} T^<_{[A^{1/(1+\gamma)}x],\, m} \bigg\} \bigg) \Bigg]. \end{aligned} \tag{6.6}$$

Defining

$$\hat{\varrho}^*_A(s; x, h) = s\, \mathbf{E}\left(\exp\left\{ -\frac{s}{2\beta A} T^*_{[A^{1/(1+\gamma)}x],[A^{\gamma/(1+\gamma)}\beta h]} \right\} \right) \tag{6.7}$$

(6.6) reads

$$\hat{\pi}_A(s, x) = \frac{1 - e^{-s/A}}{s/A} \frac{1}{2} \int_0^\infty \big(\hat{\varrho}^>_A(2\beta s, x, h) + \hat{\varrho}^<_A(2\beta s, x, h) \big)\, dh. \tag{6.8}$$

From (6.1) it follows that for any $s > 0$, $x \in \mathbb{R}$ and $h > 0$,

$$\hat{\varrho}^*_A(s, x, h) \to \hat{\varrho}(s, x, h) \tag{6.9}$$

as $A \to \infty$ and the functions $\hat{\varrho}$ and $\hat{\pi}$ scale as follows: $\forall\, \lambda > 0$,

$$\lambda \hat{\varrho}(\lambda^{-1}s, \lambda^{1/(1+\gamma)}x, \lambda^{\gamma/(1+\gamma)}h) = \hat{\varrho}(s, x, h) \tag{6.10}$$

$$\lambda^{1/(1+\gamma)} \hat{\pi}(\lambda^{-1}s, \lambda^{1/(1+\gamma)}x) = \hat{\pi}(s, x). \tag{6.11}$$

by Faton's lemma, equations (6.8), (6.9) and (6.11) imply for any $x \in \mathbb{R}$,

$$\begin{aligned} \liminf_{A\to\infty} \hat{\pi}_A(s, x) &\geq \int_0^\infty \hat{\varrho}(2\beta s, |x|, h)\, dh \\ &= \hat{\pi}(\beta s, x) = \beta^{1/(1+\gamma)} \hat{\pi}(s, \beta^{1/(1+\gamma)}x). \end{aligned} \tag{6.12}$$

On the other hand, by Theorem 4,

$$\int_{-\infty}^{\infty} \hat{\pi}_A(s, x)\, dx = 1 = \int_{-\infty}^{\infty} b^{1/(1+\gamma)} \hat{\pi}(s, b^{1/(1+\gamma)}x)\, dx. \tag{6.13}$$

From (6.12) and (6.13) follows the statement of Theorem 2′. □

Clearly, the statements of Theorems 2A and 2B are just particular cases of Theorem 2′.

APPENDIX

Generalized Ray–Knight processes II. This appendix is devoted to the proof of (2.28). Actually, we define a more general notion of Ray–Knight process and we prove (2.28) in a much more general context. This Appendix is completely self-contained and we think that it might be interesting on its own, from a purely diffusion-theoretic point of view.

A.1. *Conjugate diffusions.* Let

$$a\colon \overset{\circ}{\mathbb{R}}_+ \to (0, \infty), \qquad b\colon \overset{\circ}{\mathbb{R}}_+ \to (-\infty, \infty) \tag{A.1.1}$$

be twice continuously differentiable functions and define the second order differential operators

$$[Gf](x) = \tfrac{1}{2} a(x) f''(x) + (\tfrac{1}{4} a'(x) + b(x)) f'(x), \tag{A.1.2}$$

$$[Hf](x) = \tfrac{1}{2} a(x) f''(x) + (\tfrac{1}{4} a'(x) - b(x)) f'(x). \tag{A.1.3}$$

We call the operators G and H a *conjugate pair* of diffusion generators on $\mathbb{R}_+$. The *analytic content* of this conjugacy is the (equivalent) pair of commutation relations

$$\frac{d}{dx} G = H^* \frac{d}{dx}, \qquad \frac{d}{dx} H = G^* \frac{d}{dx}, \tag{A.1.4}$$

where G^* and H^* are the formal Lebesgue adjoints of G and H, respectively:

$$\begin{aligned}[G^*f](x) &= \tfrac{1}{2}a(x)f''(x) + (\tfrac{3}{4}a'(x) - b(x))f'(x)\\ &\quad + (\tfrac{1}{4}a''(x) - b'(x))f(x)\end{aligned} \tag{A.1.5}$$

$$\begin{aligned}[H^*f](x) &= \tfrac{1}{2}a(x)f''(x) + (\tfrac{3}{4}a'(x) + b(x))f'(x)\\ &\quad + (\tfrac{1}{4}a''(x) + b'(x))f(x).\end{aligned} \tag{A.1.6}$$

Integration by parts yields the identities

$$\begin{aligned}\int_{x_0}^{x_1}[G^*f](y)g(y)\,dy &= \int_{x_0}^{x_1} f(y)[Gg](y)\,dy\\ &\quad + \{[H{\textstyle\int} f](y)g(y) - \tfrac{1}{2}a(y)f(y)g'(y)\}_{x_0}^{x_1}\end{aligned} \tag{A.1.7}$$

$$\begin{aligned}\int_{x_0}^{x_1}[H^*f](y)g(y)\,dy &= \int_{x_0}^{x_1} f(y)[Hg](y)\,dy\\ &\quad + \{[G{\textstyle\int} f](y)g(y) - \tfrac{1}{2}a(y)f(y)g'(y)\}_{x_0}^{x_1}\end{aligned} \tag{A.1.8}$$

where by $\int f$ we denoted the function $[\int f](y) = \int_0^y f(z)\,dz$ and we adopted the notation $\{h(y)\}_{x_0}^{x_1} = h(x_1) - h(x_0)$.

Define the functions

$$u(x) = \sqrt{\frac{2}{a(x)}}\exp\left\{-\int_1^x \frac{2b(y)}{a(y)}dy\right\} \tag{A.1.9}$$

$$v(x) = \sqrt{\frac{2}{a(x)}}\exp\left\{\int_1^x \frac{2b(y)}{a(y)}\,dy\right\}. \tag{A.1.10}$$

With the help of these functions we can express the differential operators G, H, G^* and H^* as

$$G = \frac{1}{v}\frac{d}{dx}\frac{1}{u}\frac{d}{dx}, \qquad G^* = \frac{d}{dx}\frac{1}{u}\frac{d}{dx}\frac{1}{v}, \tag{A.1.11}$$

$$H = \frac{1}{u}\frac{d}{dx}\frac{1}{v}\frac{d}{dx}, \qquad H^* = \frac{d}{dx}\frac{1}{v}\frac{d}{dx}\frac{1}{u}. \tag{A.1.12}$$

We consider two diffusion processes on $\mathbb{R}_+$: X_t and Y_t with generators G (respectively, H). More precisely, the generators of X_t (respectively, Y_t) restricted to smooth functions with compact support in $\overset{\circ}{\mathbb{R}}_+$ act as G (respectively, H). The diffusions X_t and Y_t are uniquely determined by these generators as long as they do not hit the boundary $\{0\} = \partial\mathbb{R}_+$. The ambiguity in the behavior of the processes X_t and Y_t at 0 is eliminated in the following way: X_t is *reflected instantaneously at* 0 [see Definition VII.3.11. in Revuz and Yor (1991)] and Y_t is *stopped at*

$$\tau_0 = \inf\{t : Y_t = 0\}. \tag{A.1.13}$$

We call the processes X_t and Y_t *conjugate diffusions*. The *scale functions* and *speed measures* of the processes X_t (respectively, Y_t) are $r(x)$ and $n(dx)$ [respectively, $s(x)$ and $m(dx)$]. According to standard results about one-dimensional diffusions [see Exercise VII.3.20. in Revuz and Yor (1991)], on $\overset{\circ}{\mathbb{R}}_+$ we have

$$r(x) = \int^x u(y)dy, \qquad n(dx) = v(x)\,dx, \tag{A.1.14}$$

$$s(x) = \int^x v(y)dy, \qquad m(dx) = u(x)\,dx. \tag{A.1.15}$$

The lower limits in the integrals defining r and s are chosen in an arbitrary way. Formulas (A.1.14) and (A.1.15) give the *probabilistic content* of conjugacy of the diffusions X_t and Y_t: the derivative of the scale function of one is the Radon–Nikodym derivative of the speed measure of the other, and vice versa. In accordance with the behavior at the boundary described in the previous paragraph, we define

$$n(\{0\}) = 0, \qquad m(\{0\}) = \infty. \tag{A.1.16}$$

The conjugacy of a pair of diffusions is invariant under diffeomorphisms of $\overset{\circ}{\mathbb{R}}_+$: let

$$\Lambda: \overset{\circ}{\mathbb{R}}_+ \to \overset{\circ}{\mathbb{R}}_+ \tag{A.1.17}$$

be a C^2 bijection which has a C^2 inverse Λ^{-1}, and preserves the orientation of the half-line $\mathbb{R}_+$:

$$\lim_{x \searrow 0} \Lambda(x) = 0, \qquad \lim_{x \nearrow \infty} \Lambda(x) = \infty. \tag{A.1.18}$$

Consider the diffusions

$$\tilde{X}_t = \Lambda(X_t), \qquad \tilde{Y}_t = \Lambda(Y_t). \tag{A.1.19}$$

It is easy to check that if X_t and Y_t are a conjugate pair of diffusions on $\mathbb{R}_+$, then so are $\tilde{X}_t$ and $\tilde{Y}_t$, with

$$\tilde{a} = [(\Lambda')^2 \cdot a] \circ \Lambda^{-1}, \tag{A.1.20}$$

$$\tilde{b} = [\Lambda' \cdot b] \circ \Lambda^{-1}. \tag{A.1.21}$$

Our notion of conjugacy of the pair of diffusions X_t, Y_t is closely related to the conjugacy notion introduced in Biane (1985). By time-reversing the process Y_t,

$$\hat{Y}_t = Y_{\tau_0 - t}, \qquad 0 \le t \le \tau_0, \tag{A.1.22}$$

we get a transient diffusion $\hat{Y}_t$ stopped at its *last hitting* of $y \in \mathbb{R}_+$. The diffusions $\hat{Y}_t$ and X_t are conjugate in Biane's sense. Biane introduced his notion of conjugacy in order to generalize the Cieselski–Taylor identities. This fact suggests that the identity proved in Section A.4 [and, consequently, (2.28) too] must have some connection to the Cieselski–Taylor identities that we have not been able to elucidate yet.

A.2. *Boundary conditions.* Throughout this Appendix we shall use the following stopping times: for $y \in [0, \infty)$,

$$\sigma_y = \inf\{t: X_t = y\}, \tag{A.2.1}$$

$$\tau_y = \inf\{t: Y_t = y\} \tag{A.2.2}$$

with the usual convention $\inf \varnothing = \infty$.

We impose some conditions on the behavior of the diffusions X_t and Y_t near 0 and ∞:

Condition at 0. We give two equivalent formulations—one referring to the diffusion X_t and the other to Y_t—of the same single condition

$$\int_0^1 \left(\int_y^1 u(z)dz \right) v(y)\, dy = \int_0^1 [r(1) - r(y)] n(dy) < \infty, \tag{A.2.3}$$

$$\int_0^1 \left(\int_0^z v(y)dy \right) u(z)\, dz = \int_0^1 \left[s(z) - s(0)\right] m(dz) < \infty. \tag{A.2.4}$$

[We could have chosen any positive number instead of 1 as upper limit of integration in (A.2.3)/(A.2.4).] The left-hand sides in the two formulas (A.2.3) and (A.2.4) are clearly the same, so we emphasize again that these are just two different formulations of the same condition. In probabilistic terms, these conditions are equivalent to

$$(\sigma_x | X_0 = 0) \longrightarrow_{\mathbf{P}} 0 \quad \text{as } x \to 0, \tag{A.2.5}$$

$$(\tau_0 | Y_0 = x) \longrightarrow_{\mathbf{P}} 0 \quad \text{as } x \to 0, \tag{A.2.6}$$

where $\to_{\mathbf{P}}$ stands for convergence in probability. In plain words, (A.2.5) [respectively, (A.2.6)] means that X_t does not stick to 0 (respectively, Y_t can hit 0) in finite time.

In particular, from (A.2.4) it also follows that

$$s(x) - s(0) = \int_0^x v(y)\, dy < \infty \tag{A.2.7}$$

and we can choose

$$s(x) = \int_0^x v(y)\, dy, \quad \text{that is,} \quad s(0) = 0. \tag{A.2.8}$$

Conditions at ∞. Again, we give two equivalent formulations of the boundary condition at infinity. The first formulation refers to the diffusion X_t, the second one to Y_t:

$$\int_1^\infty \left(\int_y^\infty u(z)\, dz \right) v(y)\, dy = \int_1^\infty [r(\infty) - r(y)] n(dy) = \infty, \tag{A.2.9}$$

$$\int_1^\infty \left(\int_1^z v(y)\, dy \right) u(z)\, dz = \int_1^\infty [s(z) - s(1)] m(dz) = \infty. \tag{A.2.10}$$

[Now, we could have chosen any positive number instead of 1 as lower limit of integration in (A.2.9)/(A.2.10).] Clearly, the left-hand sides of (A.2.9) and (A.2.10) are the same again. The probabilistic content of these conditions is the following: for any fixed $y \geq 0$,

$$(\sigma_x | X_0 = y) \longrightarrow_{\mathbf{P}} \infty \quad \text{as } x \to \infty, \tag{A.2.11}$$

$$(\tau_y \| Y_0 = x) \longrightarrow_{\mathbf{P}} \infty \quad \text{as } x \to \infty. \tag{A.2.12}$$

Condition (A.2.11) [respectively, (A.2.12)] means that X_t does not escape to infinity (respectively Y_t does not come in from infinity) in finite time.

In addition to condition (A.2.9)/(A.2.10) we also impose

$$\lim_{x \nearrow \infty} s(x) = s(\infty) = \infty. \tag{A.2.13}$$

We do not give the standard (but lengthy) details here of the derivation of the equivalence of the analytic versus probabilistic formulations (A.2.3)$\sim$(A.2.5), (A.2.4)$\sim$(A.2.6), (A.2.9)$\sim$(A.2.11) and (A.2.10)$\sim$(A.2.12).

It is straightforward to check that the boundary conditions (A.2.3)/(A.2.4), (A.2.9)/(A.2.10) and (A.2.13) are invariant under diffeomorphic images (A.1.17) and (A.1.18).

A.3. *Examples.*

EXAMPLE 1. Our main class of examples consists of *pairs of squared Bessel processes*. Fix $\delta > 0$ and let

$$a(x) = 2x, \qquad b(x) \equiv \frac{\delta - 1}{2}. \tag{A.3.1}$$

The corresponding generators G and H will be

$$G = x\frac{\partial^2}{\partial x^2} + \frac{\delta}{2}\frac{\partial}{\partial x}, \qquad H = x\frac{\partial^2}{\partial x^2} + \frac{2-\delta}{2}\frac{\partial}{\partial x}; \tag{A.3.2}$$

that is, X_t (respectively Y_t) will be the squared Bessel processes of $Z_t^{(\delta)}$ (respectively $\tilde{Z}_t^{(2-\delta)}$) given in Section 2.

The various functions arising in this case are

$$u(x) = x^{-\delta/2}, \qquad v(x) = x^{(\delta-2)/2}$$

$$r(x) = \begin{cases} \dfrac{2}{2-\delta} x^{(2-\delta)/2}, & \text{if } \delta \neq 2, \\ \ln x, & \text{if } \delta = 2, \end{cases} \qquad s(x) = \frac{2}{\delta} x^{\delta/2}. \tag{A.3.3}$$

Conditions (A.2.3)/(A.2.4), (A.2.9)/(A.2.10) and (A.2.13) are easily checked.

Using diffeomorphisms $\Lambda: \overset{\circ}{\mathbb{R}}_+ \to \overset{\circ}{\mathbb{R}}_+$ we get a wider class of examples. For example, $\Lambda(x) = \sqrt{x}$ transforms the pair of *squared* Bessel processes into a pair of Bessel processes of the same generalized dimensions δ (respectively $2-\delta$). With the particular choice $\delta = 1 = 2 - \delta$ and $\Lambda(x) = \sqrt{x}$ we get a pair of Brownian motions reflecting (respectively stopped at) 0: $X_t = |W_t|$, $Y_t = |W_{t\wedge\tau_0}|$.

EXAMPLE 2. second class of examples (not exploited in this paper) is provided by pairs of Brownian motions with drift:

$$a(x) \equiv 1, \qquad b(x) \equiv b > 0. \tag{A.3.4}$$

In this case the relevant functions will be

$$\begin{aligned} u(x) &= \sqrt{2}e^{-2bx}, & v(x) &= \sqrt{2}e^{2bx}, \\ r(x) &= \sqrt{2}\frac{1-e^{-2bx}}{2b}, & s(x) &= \sqrt{2}\frac{e^{2bx}-1}{2b}. \end{aligned} \tag{A.3.5}$$

The processes defined by these parameters will be X_t = Brownian motion with constant drift b reflected at 0 (respectively, Y_t = Brownian motion with constant drift $-b$ absorbed at 0). The boundary conditions (A.2.3)/(A.2.4), (A.2.9)/(A.2.10) and (A.2.13) are again easy to check.

A.4. *Technical result.* We consider the function

$$\phi(y) = \mathbf{E}\left(\exp\left\{ -\int_0^{\tau_0} Y_s ds \right\} \middle\| Y_0 = y \right). \tag{A.4.1}$$

The function ϕ is the *unique bounded solution* on $\mathbb{R}_+$ of the ordinary differential equation

$$[H\phi](y) - y\phi(y) = 0 \tag{A.4.2}$$

with boundary condition

$$\phi(0) = 1, \tag{A.4.3}$$

which holds due to (A.2.4) or equivalently (A.2.6). Differentiating (A.4.2) and using the commutation relation (A.1.4), we get another very useful relation for ϕ:

$$\phi(y) = \big[G^*\phi'\big](y) - y\phi'(y). \tag{A.4.4}$$

The following simple lemma will be used in the proof of the forthcomming theorem:

LEMMA 3. *Assume that conditions* (A.2.10) *and* (A.2.13) *hold. The fundamental solution ϕ of the ODE* (A.4.2) *satisfies the following integral equation*

$$\phi(x) = \int_x^\infty \big(s(y) - s(x)\big)u(y)y\phi(y)\,dy. \tag{A.4.5}$$

PROOF. From the ODE (A.4.2), using the form (A.1.12) of the differential operator H, by one quadrature, we get

$$\frac{\phi'(y)}{v(y)} - \frac{\phi'(x)}{v(x)} = \int_x^y u(z) z \phi(z)\, dz. \tag{A.4.6}$$

As ϕ is decreasing (i.e., ϕ' is negative), from (A.4.6) it follows that the limit

$$A = -\lim_{y\to\infty} \frac{\phi'(y)}{v(y)} \tag{A.4.7}$$

exists and is finite. Taking the limit $y \to \infty$ in (A.4.6) we get

$$-\phi'(x) = Av(x) + v(x) \int_x^\infty u(y) y \phi(y)\, dy \tag{A.4.8}$$

and by a second quadrature

$$\phi(x) - \phi(y) = -\frac{\phi'(y)}{v(y)}(s(y) - s(x)) + \int_x^y \big(s(z) - s(x)\big) u(z) z \phi(z)\, dz. \tag{A.4.9}$$

We keep in mind that ϕ is positive and decreasing. From (A.2.10) and (A.2.13) it follows that

$$\lim_{x\to\infty} \phi(x) = \phi(\infty) = 0 \tag{A.4.10}$$

and

$$-\lim_{x\to\infty} \frac{\phi'(x)}{v(x)} = A = 0, \tag{A.4.11}$$

respectively. Otherwise the right-hand side of (A.4.9) would explode while the left-hand side would remain finite in the $y \to \infty$ limit. Thus, in the $y \to \infty$ limit, (A.4.9) transforms to

$$\phi(x) = -\lim_{y\to\infty}\left(\frac{\phi'(y)}{v(y)} s(y)\right) + \int_x^\infty \big(s(z) - s(x)\big) u(z) z \phi(z)\, dz. \tag{A.4.12}$$

Taking in (A.4.12) the $x \to \infty$ limit and using (A.4.10) it follows that not only (A.4.11), but actually

$$\lim_{y\to\infty} \frac{\phi'(y)}{v(y)} s(y) = 0 \tag{A.4.13}$$

holds and consequently the assertion (A.4.5) of the lemma is valid. $\square$

A second function considered will be

$$\begin{aligned}\psi(y) = \int_0^\infty \left\{ \int_0^\infty \mathbf{E}\left(\exp\left\{ -\int_0^t X_s\, ds \right\} \mathbb{1}\{X_t > y\} \,\middle\|\, X_0 = z \right) dt \right\} \\ \times \phi(z)\, dz.\end{aligned} \tag{A.4.14}$$

THEOREM 3. *Let X_t and Y_t be a pair of conjugate diffusions on $\mathbb{R}_+$ with the coefficients a and b satisfying conditions* (A.2.3)/(A.2.4), (A.2.9)/(A.2.10) *and* (A.2.13). *Then the functions ϕ and ψ defined in* (A.4.1) *and* (A.4.14), *respectively*, *are identical*:

$$\phi \equiv \psi. \tag{A.4.15}$$

PROOF. We first prove

$$\phi' \equiv \psi'. \tag{A.4.16}$$

Let $g(y)$ be a smooth positive test function with $\operatorname{supp}(g) \subset [\underline{y}, \overline{y}] \subset \mathring{\mathbb{R}}_+$ and

$$h(z) = \int_0^\infty \mathbf{E}\left(\exp\left\{-\int_0^t X_s\,ds\right\} g(X_t) \,\middle\|\, X_0 = z\right) dt. \tag{A.4.17}$$

The function h is bounded in $\mathbb{R}_+$ due to (A.2.3) or equivalently (A.2.5), and satisfies the following differential equation in $\mathring{\mathbb{R}}_+$:

$$[Gh](y) - yh(y) = -g(y). \tag{A.4.18}$$

Using in turn the definitions (A.4.14) and (A.4.17), the identity (A.4.4), the integration by parts formula (A.1.7) and the ODE (A.4.2), and finally the ODE (A.4.18), we get

$$\begin{aligned}
-\int_0^\infty g(y)\psi'(y)\,dy &= \int_0^\infty h(z)\phi(z)\,dz \\
&= \int_0^\infty h(z)([G^*\phi'](z) - z\phi'(z))\,dz \\
&= \int_0^\infty ([Gh](z) - zh(z))\phi'(z)\,dz \\
&\qquad + \{y\phi(y)h(y) - \tfrac{1}{2}a(y)\phi'(y)h'(y)\}_0^\infty \\
&= -\int_0^\infty g(z)\phi'(z)\,dz \\
&\qquad + \{y\phi(y)h(y) - \tfrac{1}{2}a(y)\phi'(y)h'(y)\}_0^\infty.
\end{aligned} \tag{A.4.19}$$

From this derivation it also follows that the limits $\lim_{y\to 0,\infty}\{y\phi(y)h(y) - \frac{1}{2}a(y)\phi'(y)h'(y)\}$ exist. We prove that these limits actually vanish:

$$\lim_{y\to\infty}\{y\phi(y)h(y) - \tfrac{1}{2}a(y)\phi'(y)h'(y)\} = 0, \tag{A.4.20}$$

$$\lim_{y\to 0}\{y\phi(y)h(y) - \tfrac{1}{2}a(y)\phi'(y)h'(y)\} = 0. \tag{A.4.21}$$

PROOF OF (A.4.20). Since $g(y) = 0$ for $y \geq \overline{y}$, using the strong Markov property of the diffusion X_t we can write

$$h(y) = \kappa(y)h(\overline{y}) \quad \text{if } y \geq \overline{y}, \tag{A.4.22}$$

where

$$\kappa(y) = \mathbf{E}\left(\exp\left\{-\int_0^{\sigma_{\overline{y}}} X_s\,ds\right\}\bigg\| X_0 = y\right), \qquad y \geq \overline{y}, \tag{A.4.23}$$

is the *unique bounded solution* in the interval $[\overline{y}, \infty)$ of the differential equation

$$[G\kappa](y) - y\kappa(y) = 0 \tag{A.4.24}$$

with

$$\kappa(\overline{y}) = 1. \tag{A.4.25}$$

By one quadrature (similar to the first step in the proof of Lemma 3) the differential equation (A.4.24) transforms to

$$\frac{\kappa'(y)}{u(y)} - \frac{\kappa'(x)}{u(x)} = \int_x^y v(z)z\kappa(z)\,dz. \tag{A.4.26}$$

As κ is decreasing (i.e., κ' is negative), from (A.4.26) it follows now that the limit

$$B = -\lim_{y\to\infty} \frac{\kappa'(y)}{u(y)} \tag{A.4.27}$$

exists and is finite.

Now, using (A.4.22), (A.4.11) and (A.4.27) we get

$$\begin{aligned}\lim_{y\to\infty} \frac{1}{2} a(y)h'(y)\phi'(y) &= h(\overline{y}) \lim_{y\to\infty} \frac{1}{2} a(y)\kappa'(y)\phi'(y) \\ &= h(\overline{y}) \lim_{y\to\infty} \frac{\kappa'(y)}{u(y)}\frac{\phi'(y)}{v(y)} = h(\overline{y})AB = 0.\end{aligned} \tag{A.4.28}$$

Next, from (A.2.10) and the finiteness of the integral on the left-hand side of (A.4.5) it follows that

$$\liminf_{y\to\infty} y\phi(y) = 0 \tag{A.4.29}$$

and hence

$$\liminf_{y\to\infty} y\phi(y)h(y) \leq h(\overline{y}) \liminf_{y\to\infty} y\phi(y) = 0 \tag{A.4.30}$$

However, the limit $\lim_{y\to\infty}\{y\phi(y)h(y) - \frac{1}{2}a(y)\phi'(y)h'(y)\}$ exists, so (A.4.28) and (A.4.30) imply (A.4.20).

PROOF OF (A.4.21). From boundedness of the functions ϕ and h it follows that

$$\lim_{y\to 0} y\phi(y)h(y) = 0 \tag{A.4.31}$$

so we have to prove only

$$\lim_{y\to 0} \tfrac{1}{2} a(y)h'(y)\phi'(y) = 0. \tag{A.4.32}$$

Using again the strong Markov property of the process X_t we write now

$$h(y) = \frac{h(0)}{\lambda(y)}, \qquad y \in [0, \underline{y}], \tag{A.4.33}$$

with

$$\lambda(y) = \mathbf{E}\biggl(\exp\biggl\{-\int_0^{\sigma_y} X_s\,ds\}\biggm\| X_0 = 0\biggr), \qquad y \in [0, \underline{y}]. \tag{A.4.34}$$

Let $0 \le y \le z \le \underline{y}$. Then

$$\begin{aligned} 1 \ge \frac{\lambda(z)}{\lambda(y)} &= \mathbf{E}\biggl(\exp\biggl\{-\int_0^{\sigma_z} X_s\,ds\biggr\}\biggm\| X_0 = y\biggr) \\ &\ge \exp\biggl\{-\mathbf{E}\biggl(\int_0^{\sigma_z} X_s\,ds \biggm\| X_0 = y\biggr)\biggr\}. \end{aligned} \tag{A.4.35}$$

By use of VII.3.8. of Revuz and Yor (1991) we find the expectation in the exponent on the right-hand side of (A.4.35):

$$\begin{aligned} \mathbf{E}\biggl(\int_0^{\sigma_z} X_s\,ds \biggm\| X_0 = y\biggr) &= \int_0^y [r(z) - r(y)]xv(x)\,dx \\ &\qquad + \int_y^z [r(z) - r(x)]xv(x)\,dx \\ &\le [r(z) - r(y)]zs(z). \end{aligned} \tag{A.4.36}$$

Hence

$$|\lambda'(y)| \le \lambda(y)yu(y)s(y) \tag{A.4.37}$$

and by (A.4.33)

$$|h'(y)| \le h(y)yu(y)s(y). \tag{A.4.38}$$

Using this and (A.4.5) we have

$$\begin{aligned} \frac{1}{2}|a(y)h'(y)\phi'(y)| = \left|\frac{h'(y)}{u(y)}\frac{\phi'(y)}{v(y)}\right| &\le h(y)ys(y)\int_y^\infty u(z)z\phi(z)\,dz \\ &\le h(y)y\int_0^\infty s(z)u(z)z\phi(z)\,dz = h(y)y\phi(0). \end{aligned} \tag{A.4.39}$$

The right-hand side clearly vanishes in the $y \to 0$ limit. Hence (A.4.32) and (A.4.21). □

Thus we proved (A.4.20) and (A.4.21) for arbitrary test functions g. From this and (A.4.19) the identity (A.4.16) follows. Equation (A.4.15) will follow from a straightforward monotone convergence argument: clearly the integrand in (A.4.14) is monotone decreasing in the parameter y, and by (A.2.9), or equivalently (A.2.11),

$$\lim_{y\to\infty} \mathbf{E}\biggl(\exp\biggl\{-\int_0^t X_s\,ds\biggr\}\mathbb{1}\{X_t > y\}\biggm\| X_0 = z\biggr) = 0. \tag{A.4.40}$$

Hence

$$\psi(\infty) = 0. \tag{A.4.41}$$

The assertion of the theorem follows from (A.4.16), (A.4.10) and (A.4.41). □

Let now Λ be a diffeomorphism, as given in (A.1.17) and (A.1.18) and define

$$\phi_{\Lambda}(y) = \mathbf{E}\left(\exp\left\{-\int_0^{\tau_0} \Lambda(Y_s)\,ds\right\}\middle\| Y_0 = y\right), \tag{A.4.42}$$

$$\begin{aligned}\psi_{\Lambda}(y) = \int_0^{\infty}\biggl\{\int_0^{\infty} \mathbf{E}\biggl(&\exp\left\{-\int_0^{t} \Lambda(X_s)\,ds\right\}\\ &\times \mathbb{1}\{X_t > y\}\biggm\| X_0 = z\biggr)\,dt\biggr\}\phi_{\Lambda}(z)\Lambda'(z)\,dz.\end{aligned} \tag{A.4.43}$$

Applying Theorem 3 to the conjugate pair of diffusions $\tilde{X}_t = \Lambda(X_t)$, $\tilde{Y}_t = \Lambda(Y_t)$ defined in (A.1.19) we easily get the following more general theorem.

THEOREM 3′. *Under the conditions of Theorem* 3 *the functions* ϕ_{Λ} *and* ψ_{Λ} *defined in* (A.4.42), *and* (A.4.43), *are identical*:

$$\phi_{\Lambda} \equiv \psi_{\Lambda}. \tag{A.4.44}$$

REMARK. As we mentioned already, in the light of Biane's results we expect that identity (A.4.15), or more generally (A.4.44), must be related to the generalized Ciesielski–Taylor identities. However, we could not clarify yet the intimate relation between the two. For more information on generalized Ciesielski–Taylor identities and more "mysterious" formulas, we refer the reader to Donati-Martin and Yor (1991), Yor (1991), Carmona, Petit and Yor (1994) and Donati-Martin, Song and Yor (1994).

A.5. *Generalized Ray–Knight processes and Proof of* (2.28). We define a wider notion of generalized Ray–Knight process. Let us fix two parameters $a \geq 0$ and $h \geq 0$ and define a process $(-\infty, \infty) \ni y \mapsto S_{a,h}(y) \in \mathbb{R}_+$ [with time variable $y \in (-\infty, +\infty)$] by patching together three independent diffusions, $X(\cdot)$, $Y_l(\cdot)$, and $Y_r(\cdot)$, in the following way:

$$S_{a,h}(y) = \begin{cases} (Y_l(-y) \| Y_l(0) = h), & \text{for } y \in (-\infty, 0],\\ (X(y) \| X(0) = h), & \text{for } y \in [0, a],\\ (Y_r(y-a) \| Y_r(0) = X(a)), & \text{for } y \in [a, \infty), \end{cases} \tag{A.5.1}$$

where the diffusions $Y_l(\cdot)$ and $Y_r(\cdot)$ are conjugate to the diffusion $X(\cdot)$ in the sense of the previous subsections. We define the random variable $T_{a,h}$ and the functions ϱ, π, $\hat{\varrho}$ and $\hat{\pi}$ by the formulas (2.11), (2.12), (2.14), (2.16) and (2.17), with the superscript (δ) erased.

The following assertion is a direct corollary of Theorem 3′, but due to its importance in our context, we prefer to formulate it as a separate theorem.

THEOREM 4. *For any fixed* $t > 0$ (*respectively* $s > 0$) *the functions* $(-\infty, \infty) \ni x \mapsto \pi(t, x) \in (0, \infty)$ [*respectively,* $(-\infty, \infty) \ni x \mapsto \hat{\pi}(s, x) \in (0, \infty)$] *are probability densities*; *that is*,

$$\int_{-\infty}^{\infty} \pi(t, x)\, dx = 1 = \int_{-\infty}^{\infty} \hat{\pi}(s, x)\, dx \tag{A.5.2}$$

holds for any generalized Ray–Knight process.

PROOF. Apply Theorem 3′ with $\Lambda(x) = sx$. It is straightforward to check that

$$\begin{aligned}\int_0^{\infty} \hat{\pi}(s, x)\, dx &= -\int_0^{\infty} \phi_{\Lambda}(h)\psi'_{\Lambda}(h)\, dh = -\int_0^{\infty} \phi_{\Lambda}(h)\phi'_{\Lambda}(h)\, dh \\ &= \tfrac{1}{2}(\phi_{\Lambda}^2(0) - \phi_{\Lambda}^2(\infty)) = \tfrac{1}{2}.\end{aligned} \tag{A.5.3}$$

Indeed, given the definition of $\hat{\pi}(s, x)$ via (2.8)-(2.11)-(2.16)-(2.17) on one hand and the definitions (A.4.42) and (A.4.43) of the functions ϕ_{Λ} and ψ_{Λ} on the other hand, the first equality follows from a careful application of the Feynman–Kac formula. (Actually this is the reason why we introduced and analyzed the functions ϕ and ψ.) The second equality follows from (A.4.44), and the last one from (A.4.3) and (A.4.10). □

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