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STABLE WINDINGS

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We derive the asymptotic laws of winding numbers for planar isotropic stable Lévy processes and walks of index $\alpha \in (0, 2)$.

1. Introduction. This paper deals with the asymptotic behaviour of winding numbers of planar isotropic stable processes. The asymptotic study of the winding numbers of a planar Brownian motion B was initiated by Spitzer (1958), who proved the following celebrated result: If $(\theta_t,\ t \geq 0)$ denotes the continuous determination of the argument of B started away from the origin, then $2\theta_t/\log t$ converges in distribution toward a standard Cauchy law as $t \to \infty$. We refer to Yor [(1992), Chapter 5] and the references therein for much more on this topic. Our main purpose is to present an analogue of Spitzer's theorem, when the Brownian motion is replaced by an isotropic stable Lévy process of index $\alpha \in (0, 2)$ (the winding number θ is then defined by "filling in the jumps with straight lines").

Because the latter is transient (which contrasts with the Brownian case), one expects that it will wind more slowly, and actually our main result implies that $\theta_t/\sqrt{\log t}$ converges in distribution to some centered Gaussian law as $t \to \infty$. In Section 4, we show that a similar result holds for the isotropic stable random walk.

Our approach is closely related to that we used in Bertoin and Werner (1994) to investigate Brownian windings. Typically, we shall not work directly with the stable process, but rather with an analogue of the Ornstein–Uhlenbeck process obtained from Z by an exponential change of scale and speed. The point is that this new process is positive recurrent, so that ergodic theory applies. The second key ingredient, which follows from a result due to Graversen and Vuolle-Apiala (1986), is that θ suitably time-changed is in fact a symmetric Lévy process.

The paper is structured as follows: We first recall and derive relevant facts concerning isotropic planar stable processes. Then we state and prove our main result on the asymptotic behaviour of the stable winding number. Finally, in Section 4, we consider the random walk analogue.

2. Preliminaries. Let $z = (Z_t,\ t \geq 0)$ be a standard isotropic stable process of index $\alpha \in (0, 2)$ valued in the complex plane. In other words, Z has

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stationary independent increments, its sample path is right-continuous and has left limits (i.e. cadlag), and

$$E_0(\exp\{i\langle \lambda, Z_t\rangle\}) = \exp\{-t|\lambda|^\alpha\}$$

for all $t \geq 0$ and $\lambda \in \mathbf{C}$, where P_z refers to the law of the process Z started from $z \in \mathbf{C}$, and $\langle \cdot, \cdot \rangle$ refers to the Euclidean inner product. We will implicitly work from now on under the law $P = P_1$, unless otherwise stated.

Recall that Z is transient, that is,

$$\lim_{t\to\infty} |Z_t| = \infty \quad \text{almost surely,}$$

and that single points are polar,

$$P(Z_t = z \text{ for some } t > 0) = 0$$

for all $z \in \mathbb{C}$.

In the sequel, it will be useful to invoke the expression of Z as a subordinated planar Brownian motion. Specifically, let $B = (B_t,\ t \geq 0)$ denote a complex-valued Brownian motion started from 1 and let $S = (S(t),\ t \geq 0)$ be an independent stable subordinator with index $\alpha/2$ started from 0, that is,

$$E(\exp\{-\mu S(t)\}) = \exp\{-t\mu^{\alpha/2}\}$$

for all $t \geq 0$, $\mu \geq 0$. Then the process $(B_{2S(t)},\ t \geq 0)$ is a standard isotropic stable process of index α. Recall that the Lévy measure of S, that is, the intensity of its jumps, is

$$\frac{\alpha}{2\Gamma(1-\alpha/2)} s^{-1-\alpha/2} 1_{\{s>0\}}\, ds.$$

It follows that the Lévy measure ν of Z is

$$\begin{aligned}\nu(dx) &= \frac{\alpha}{2\Gamma(1-\alpha/2)} \int_0^\infty s^{-1-\alpha/2} P(B_{2s} - 1 \in dx)\, ds \\ &= \frac{\alpha}{8\pi\Gamma(1-\alpha/2)} \left(\int_0^\infty s^{-2-\alpha/2} \exp(-|x|^2/(4s))\, ds \right) dx,\end{aligned}$$

and finally

$$\nu(dx) = \frac{\alpha 2^{-1+\alpha/2}\Gamma(1+\alpha/2)}{\pi\Gamma(1-\alpha/2)} |x|^{-2-\alpha}\, dx. \tag{1}$$

Because Z is discontinuous, one cannot define its winding number θ just as for Brownian motion (via the continuous determination of its argument), but it is easy to circumvent this difficulty. Consider a path on a finite time interval $[0, t]$ and "fill in" the gaps due to the jumps with line segments. We obtain the curve of a continuous function $f\colon [0,1] \to \mathbf{C}$, with $f(0) = 1$. Since 0 is polar and Z has no jumps across 0, almost surely, we have $f(u) \neq 0$ for every $u \in [0,1]$. The final value of the continuous determination of the argument of f, which takes the value 0 at $f(0)$, does not depend on the actual choice for f; we denote it by θ_t and call $\theta = (\theta_t,\ t \geq 0)$ the process of the winding number of Z around 0. It is clear that θ has cadlag paths, no jumps of absolute length greater than π and that, for all $t \geq 0$,

$$\exp(i\theta_t) = \frac{Z_t}{|Z_t|}.$$

Conversely, the three preceding properties characterize the winding number process.

We now recall the following result due to Graversen and Vuolle-Apiala (1986). Introduce the clock

$$H(t) = \int_0^t |Z_s|^{-\alpha} \, ds, \qquad t \geq 0,$$

and its inverse

$$T(u) = \inf\{t \geq 0, H(t) > u\}.$$

Then the scaling property of Z yields that the process $Z_{T(u)}/|Z_{T(u)}|$, $u \geq 0$, is a Lévy process valued in the unit circle. Let us mention at this stage that this property is definitely correct, but that the independence between $|Z|$ and $Z_{T(\cdot)}/|Z_{T(\cdot)}|$ stated in the same paper is incorrect. Indeed, the processes $|Z_{T(\cdot)}|$ and $Z_{T(\cdot)}/|Z_{T(\cdot)}|$ jump at the same times, which implies that they cannot be independent. As the time change $T(\cdot)$ depends only upon $|Z|$, $|Z|$ and $Z_{T(\cdot)}/|Z_{T(\cdot)}|$ are not independent either.

We then deduce the following lemma.

LEMMA 1. *The time-changed process* $(\theta_{T(u)}, u \geq 0)$ *is a real-valued symmetric Lévy process. It has no Gaussian component and its Lévy measure has support in* $[-\pi, \pi]$.

PROOF. It is clear from the foregoing text that $\theta_{T(\cdot)}$ is a real-valued Lévy process and that its Lévy measure has support in $[-\pi, \pi]$. The symmetry assertion is plain from the isotropy of Z. The feature that $\theta_{T(\cdot)}$ has no Gaussian component is intuitively clear, because Z has no Gaussian component either. To be rigorous, we can use a result of Blumenthal and Getoor (1961) on the Hölder continuity of stable processes. For every $r \in [1/2, 1/\alpha)$,

$$\lim_{t \to 0+} |Z_t - 1| t^{-r} = 0 \quad \text{a.s.}$$

It is easy to deduce that

$$\lim_{t \to 0+} |\theta_{T(t)}| t^{-r} = 0 \quad \text{a.s.},$$

which implies [using, e.g., Theorem IV.6 in Gihman and Skorohod (1975), page 332] that $\theta_{T(\cdot)}$ has no Gaussian component. $\square$

The next lemma describes the Lévy measure of $\theta_{T(\cdot)}$. Denote by dz the Lebesgue measure on $\mathbf{C}$ and, for every complex number $z \neq 0$, let $\phi(z)$ denote the determination of its argument valued in $(-\pi, \pi]$.

LEMMA 2. *The Lévy measure of $\theta_{T(\cdot)}$ is the image of the Lévy measure ν of Z by the mapping $z \to \phi(1+z)$. As a consequence, $E((\theta_{T(u)})^2) = uk(\alpha)$, where*

$$k(\alpha) = \frac{\alpha 2^{-1+\alpha/2}\Gamma(1+\alpha/2)}{\pi\Gamma(1-\alpha/2)} \int_{\mathbf{C}} |z|^{-2-\alpha}|\phi(1+z)|^2\, dz. \tag{2}$$

PROOF. The jumps of the winding number are induced by those of the stable process. The precise relation is in the obvious notation

$$\Delta\theta_t = \theta_t - \theta_{t-} = \phi(Z_t/Z_{t-}) = \phi(1 + (\Delta Z_t)/Z_{t-}).$$

So, if for every $a \in \mathbf{C} \setminus \{0\}$, we denote by μ_a the image of the Lévy measure ν of Z under the mapping $z \to \phi(1 + z/a)$, an application of the compensation formula shows that for every measurable function $f\colon (-\pi, \pi] \to [0, \infty)$,

$$\begin{aligned} E\biggl(\sum_{0\le u\le 1} f(\Delta\theta_{T(u)})\biggr) &= E\biggl(\sum_{t\ge 0} 1_{\{H(t)\le 1\}} f(\Delta\theta_t)\biggr) \\ &= E\biggl(\int_0^\infty 1_{\{H(t)\le 1\}}\mu_{Z_t}(f)\, dt\biggr) \\ &= E\biggl(\int_0^1 |Z_{T(u)}|^\alpha \mu_{Z_{T(u)}}(f)\, du\biggr). \end{aligned}$$

On the other hand, the scaling property shows that the image of ν by a contraction $z \to z/a$ is $|a|^{-\alpha}\nu$, so that

$$\mu_a = |a|^{-\alpha}\mu_1.$$

In conclusion,

$$E\biggl(\sum_{0\le u\le 1} f(\Delta\theta_{T(u)})\biggr) = \mu_1(f(\phi)),$$

which proves our first assertion. The second follows immediately, using (1) and the absence of Gaussian component for $\theta_{T(\cdot)}$. □

We now focus on some special values of α: Note that $k(\alpha) < \infty$ for every $\alpha \in (0, 2)$ and

$$\lim_{\alpha\to 2-} k(\alpha) = \infty, \tag{3}$$

which is hardly surprising. More intriguing is the fact that

$$\lim_{\alpha\to 0+} k(\alpha) = \frac{\pi^2}{3}. \tag{4}$$

One can rewrite (2) as

$$k(\alpha) = \frac{\alpha 2^{\alpha/2}\Gamma(1+\alpha/2)}{\pi\Gamma(1-\alpha/2)} \int_0^\infty \int_0^\pi \frac{r\theta^2\, dr\, d\theta}{(1 + r^2 - 2r\cos\theta)^{1+\alpha/2}}. \tag{5}$$

For $\alpha = 1$ (i.e., for the Cauchy process), this expression can be explicitly computed. First use the fact that

$$\int_0^\infty \frac{r\,dr}{(1+r^2-2ru)^{3/2}} = \frac{1}{1-u};$$

see, for example, formula 2.264(6) in Gradshteyn and Ryzhik (1980). This yields that

$$k(1) = \frac{1}{\pi\sqrt{2}} \int_0^\pi \frac{u^2\,du}{1-\cos u}.$$

Integrating by parts twice, one finds

$$k(1) = \frac{-4\sqrt{2}}{\pi} \int_0^{\pi/2} \log(\sin u)\,du.$$

Using the trick

$$\begin{aligned}\int_0^{\pi/2} \log(\sin u)\,du = \int_0^{\pi/2} \log(\cos u)\,du &= \frac{1}{2}\int_0^{\pi/2} \log\left(\frac{\sin(2u)}{2}\right)du\\ &= \frac{1}{2}\int_0^{\pi/2} \log(\sin u)\,du - \frac{\pi\log 2}{4},\end{aligned}$$

one gets

$$k(1) = 2\sqrt{2}\log 2, \tag{6}$$

which is numerically approximatively 1.96 and smaller than the limit for $k(0+)$. Hence the function $k(\alpha)$ is not monotonous on $(0,2)$.

3. Asymptotic windings for stable processes. We now state the main result of this paper:

THEOREM 1. *The family of processes*

$$\left(r^{-1/2}\theta_{\exp(rt)},\ t \geq 0\right)$$

converges in distribution on $D([0,\infty), R)$ *endowed with the Skorohod topology, as* $r \to \infty$, *to* $(\beta_{c(\alpha)t},\ t \geq 0)$, *where* $\beta = (\beta_t,\ t \geq 0)$ *is a standard one-dimensional Brownian motion started from* 0 *and*

$$c(\alpha) = \frac{\alpha 2^{-1-\alpha/2}}{\pi} \int_{\mathbf{C}} |z|^{-2-\alpha} |\phi(1+z)|^2\,dz.$$

Let us first make some remarks:

1. It is interesting to recall that in the Brownian case, one can extend Spitzer's theorem and get convergence in the sense of finite-dimensional distributions but not in the sense of Skorohod [see Durrett (1984)].
2. Because Z is transient, the difference between θ and the winding number around an arbitrary fixed point $z \neq 1$ is bounded and converges as time goes to infinity. As a consequence, the extension of Theorem 1 to the winding numbers about several points is obvious. This contrasts again with the Brownian case; see Pitman and Yor (1986).
3. It is possible to reinforce Theorem 1 and state a strong limit theorem, using, for example, the strong approximation results of Komlós, Major and

Tusnády (1976). This can be used to derive sample path properties for θ such as the law of the iterated logarithm. We leave the precise statements to the interested readers and refer to Bertoin and Werner (1994) and the references therein for the analogous results in the Brownian case.

4. Numerically, as for $k(\alpha)$ [(3), (4) and (6)], one gets

$$\lim_{\alpha \to 0+} c(\alpha) = \frac{\pi^2}{3}, \qquad \lim_{\alpha \to 2-} c(\alpha) = \infty \quad \text{and} \quad c(1) = 2\sqrt{2}\log 2, \tag{7}$$

which implies that the function $c(\alpha)$ is not monotonous on $(0, 2)$.

We now proceed to the proof of Theorem 1, which is divided into several steps. Introduce the process

$$\tilde{Z}_u = \exp(-u/\alpha) Z_{\exp u}, \qquad u \geq 0,$$

which bears the same relationship to Z as the Ornstein–Uhlenbeck process does to Brownian motion. Breiman (1968) pointed out that $\tilde{Z}$ is a stationary Markov process under P_0. If we denote by $p_t(\cdot)$ the semigroup of Z, that is,

$$p_t(z) = P_0(Z_t \in dz)/dz \quad \text{for } z \in \mathbf{C},$$

then the semigroup $q_u(\cdot)$ of $\tilde{Z}$ is given by

$$\begin{aligned} q_u(x, y) &= p_{\exp(u)-1}(e^{u/\alpha}y - x)e^{2u/\alpha} \\ &= (e^u - 1)^{-2/\alpha} e^{2u/\alpha} p_1\big((e^u - 1)^{-1/\alpha}(e^{u/\alpha}y - x)\big), \end{aligned} \tag{8}$$

where the ultimate identity stems from the scaling property.

The key point is the following lemma.

LEMMA 3. *$\tilde{Z}$ is ergodic.*

PROOF. Let $\tilde{P}_z$ be the law of $\tilde{Z}$ started at z and let $(\tilde{\mathscr{F}}_u)_{u \geq 0}$ be its natural filtration. Consider an invariant event Λ and let

$$f(z) = \tilde{P}_z(\Lambda).$$

The theorem of convergence of martingales gives that for every z,

$$1_{\{\Lambda\}} = \lim_{u \to \infty} \tilde{P}_z\big(\Lambda | \tilde{\mathscr{F}}_u\big) \quad \text{a.s.}$$

Because Λ is invariant, the Markov property yields

$$\tilde{P}_z\big(\Lambda | \tilde{\mathscr{F}}_u\big) = f\big(\tilde{Z}_u\big) \quad \text{a.s.}$$

On the other hand, we have $f(z) = \tilde{E}_z(f(\tilde{Z}_u))$, that is,

$$f(z) = \int_{\mathbf{C}} q_u(z, a) f(a)\, da. \tag{9}$$

For every fixed z, we deduce from (8) that

$$\lim_{u \to \infty} q_u(z, a) = p_1(a) \quad \text{pointwise} \tag{10}$$

and since

$$\int_{\mathbf{C}} q_u(z,a)\,da = 1 = \int_{\mathbf{C}} p_1(a)\,da,$$

Sheffé's lemma implies that the convergence in (10) holds in $L^1(da)$. Letting $u \to \infty$ in (9), we now find that f is constant and thus $\tilde{P}_z(\Lambda) = 0$ or 1. □

We now deduce the following limit theorem for the clock:

COROLLARY 1. *Almost surely,*

$$\lim_{u\to\infty} \frac{H(e^u)}{u} = 2^{-\alpha}\frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)}. \tag{11}$$

PROOF. A change of variables yields

$$H(e^u) - H(1) = \int_1^{\exp u} |Z_s|^{-\alpha}\,ds = \int_0^u |\tilde{Z}_v|^{-\alpha}\,dv$$

and the ergodic theorem ensures that almost surely,

$$\lim_{u\to\infty}\frac{H(e^u)}{u} = E_0(|Z_1|^{-\alpha}).$$

It remains to calculate $E_0(|Z_1|^{-\alpha})$. The representation of Z as a subordinated Brownian motion gives

$$E_0(|Z_1|^{-\alpha}) = E_0\big((2S(1))^{-\alpha/2}\big)E_0(|B_1|^{-\alpha}).$$

Using the identity

$$q^{-c} = \frac{1}{\Gamma(c)}\int_0^\infty e^{-qt}t^{c-1}\,dt \quad \text{for } c \in (0,1),$$

we get

$$\begin{aligned} E_0\big(S(1)^{-\alpha/2}\big) &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty E\big(\exp(-tS(1))\big)t^{-1+\alpha/2}\,dt \\ &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty \exp(-t^{\alpha/2})t^{-1+\alpha/2}\,dt \\ &= \frac{2}{\alpha\Gamma(\alpha/2)}\int_0^\infty s\exp(-s)\,ds = \frac{1}{\Gamma(1+\alpha/2)}. \end{aligned}$$

On the other hand,

$$E_0(|B_1|^{-\alpha}) = 2^{-\alpha/2}\Gamma(1-\alpha/2)$$

and the corollary is proven. □

PROOF OF THEOREM 1. Because θ can be thought of as a symmetric centered Lévy process $\theta_{T(\cdot)}$ with finite variance, time-changed by an increasing clock H, θ is a square-integrable martingale and according to Lemma 2,

$$\langle \theta, \theta \rangle_t = k(\alpha) H(t).$$

Thus the process $(r^{-1/2}\theta_{\exp(ru)},\ u \geq 0)$ is also a square-integrable martingale with bracket

$$r^{-1}\langle \theta, \theta \rangle_{\exp(ru)} = k(\alpha) H(e^{ru})/r.$$

According to Corollary 1, the right-hand side converges almost surely as $r \to \infty$ to

$$uk(\alpha)2^{-\alpha}\Gamma(1-\alpha/2)/\Gamma(1+\alpha/2).$$

Because the jumps of θ are bounded in absolute value by π, the theorem now follows from Theorem VIII.3.11 in Jacod and Shiryaev (1987). □

4. Asymptotic windings of stable random walks. In this section, we are going to compare the windings around 0 of the stable process $(Z_t,\ t \geq 0)$ and of the stable random walk $(Z_n,\ n \in \mathbf{N})$. Windings of planar random walks with finite second moments have been investigated by Bélisle (1989), who showed that they behave asymptotically like large windings of a Brownian motion. In contrast, windings of stable random walks behave eventually exactly like windings of stable processes:

Let Z and θ be defined as in the previous section. We denote by $(\varphi_n,\ n \geq 0)$ the windings number around 0 of the stable random walk $(Z_n,\ n \geq 0)$ (again defined by filling in the gaps by straight lines).

Note that for all $n \geq 0$,

$$\varphi_{n+1} - \varphi_n = \theta_{n+1} - \theta_n \quad \text{provided that} \quad |\theta_{n+1} - \theta_n| < \pi. \tag{12}$$

We are going to establish the following proposition, which in particular [combined with (12) and Theorem 1] implies that $\varphi_n / \sqrt{c(\alpha)\log n}$ converges in law toward a normal Gaussian random variable.

PROPOSITION 1. *Almost surely*, $|\theta_{n+1} - \theta_n| < \pi$ *for all large enough* n.

The proof is based on a Borel–Cantelli-type argument. For every $x_0 \in (0, \infty)$ and $n \geq 1$, one has

$$P\big(|\theta_{n+1} - \theta_n| \geq \pi \,\big|\, |Z_n| = x_0\big) = P_{x_0}\big(|\theta_1| \geq \pi\big).$$

In order to estimate this probability, we first put down some notation and recall a couple of relevant results. We define the stopping time

$$\sigma = \inf\{t \geq 0,\ |\theta_t| \geq \pi\}.$$

Suppose that $x_0 > 1$ and let us consider the points A and B in the complex plane defined by

$$\{A, B\} = \left\{z \in \mathbf{C},\ |z - x_0| = x_0/2,\ \Re(z) < x_0,\ |\Im(z)| = \sqrt{x_0}/2\right\},$$

where $\Re(z)$ and $\Im(z)$ stand for the real and imaginary part of z, respectively. Let Ω denote the convex cone with vertex at x_0, whose boundary contains A and B. Let Π denote the strip $\{z, |\Im(z)| < \sqrt{x_0}/2\}$ and D the disk of radius $x_0/2$ centered at x_0. We then define the sets

$$\begin{aligned}
\Pi_0 &= D \cap \Pi,\\
\Pi_1 &= D \setminus \Pi,\\
\Pi_2 &= \Omega \setminus D,\\
\Pi_3 &= \mathbf{C} \setminus (\Pi_0 \cup \Pi_1 \cup \Pi_2) = \mathbf{C} \setminus (D \cup \Omega).
\end{aligned}$$

Note that $x_0 \in \Pi_0$. For $i \in \{1, 2, 3\}$, the corresponding hitting times T_i are defined by

$$T_i = \inf\{t > 0, Z_t \in \Pi_i\}.$$

Also, let $T = \min(T_1, T_2, T_3)$. Geometric considerations show that $T \leq \sigma$.

We are going to estimate the probability of the three events $\{T_1 = T \leq \sigma < 1\}$, $\{T_2 = T \leq \sigma < 1\}$ and $\{T_3 = T \leq \sigma < 1\}$ separately. We will essentially use the fact that if $(X_t, t \geq 0)$ denotes a standard symmetric one-dimensional stable process of index α started from 0 (for instance the imaginary part of Z), then for some fixed constant c_0 and for all $x > 1$, one has [see, e.g., Zolotarev (1986)]

$$P\Big(\sup_{s<1}|X_s| > x/2\Big) \leq 2P(|X_1| > x/2) \leq c_0 x^{-\alpha}. \tag{13}$$

In particular, this implies that

$$P_0\Big(\sup_{s<1}|Z_s| > x\Big) \leq 2P_0(|Z_1| > x) \leq 4P_0(|X_1| > x/2) \leq 2c_0 x^{-\alpha}. \tag{14}$$

Suppose $T_1 = T \leq \sigma < 1$. Then, for some $t \in [T_1, T_1 + 1]$,

$$\Re(Z_t) \leq 0 \leq \Re(Z_{T_1}) - x_0/2$$

(otherwise, $\sigma \geq T_1 + 1 > 1$). Hence, the strong Markov property at T_1, together with (13), yields readily that

$$P_{x_0}(T_1 = T \leq \sigma < 1) \leq c_0(x_0)^{-\alpha/2} c_0(x_0)^{-\alpha} = (c_0)^2 (x_0)^{-3\alpha/2}. \tag{15}$$

Similarly, if $T_3 = T \leq \sigma < 1$, then for some $t \in [T_3, T_3 + 1]$,

$$|\Im(Z_t) - \Im(Z_{T_3})| > \sqrt{x_0}/2.$$

Hence,

$$P_{x_0}(T_3 = T \leq \sigma < 1) \leq (c_0)^2 (x_0)^{-3\alpha/2}. \tag{16}$$

Let us put $x_0' = 2\arcsin(1/\sqrt{x_0})$ and define $U = \inf\{t > 0, |Z_t - Z_0| > x_0/2\}$. Note that Ω is a wedge of angle x_0' with vertex at x_0. The isotropy of Z shows that

$$\begin{aligned}
P_{x_0}(T = T_2 \leq \sigma < 1) &\leq P_{x_0}(U < 1 \text{ and } Z_U \in \Pi_2)\\
&\leq 2\frac{x_0'}{2\pi} P_0\Big(|Z_1| > \frac{x_0}{2}\Big)\\
&\leq \frac{c_0}{\pi} x_0'(x_0)^{-\alpha}.
\end{aligned} \tag{17}$$

Combining (15), (16) and (17) shows eventually that for some constant $c_1 > 0$, for all $x_0 > 1$,

$$P_{x_0}(\sigma < 1) \le c_1(x_0)^{-\min(3\alpha/2,\, \alpha+1/2)}.$$

In particular, we choose $\varepsilon > 0$ small enough such that $\varepsilon < 1/2$, $\varepsilon\alpha < 1/2$ and $(1+\varepsilon)\alpha < 2$. Then

$$P_{x_0}(\sigma < 1) \le c_1(x_0)^{-\alpha(1+\varepsilon)}.$$

Thus, for all $n \ge 1$,

$$(18) \qquad P(|\theta_{n+1} - \theta_n| \ge \pi) \le P(|Z_n| \le 1) + c_1 E\big(|Z_n|^{-\alpha(1+\varepsilon)}\big).$$

Let us now consider the stable process $Y = Z - 1$, which is started from 0 and therefore has the usual scaling property. Symmetry considerations and the scaling property for Y show that

$$(19) \qquad P(|Z_n| \le 1) \le P(|Y_n| \le 1) = P\big(|Y_1| \le n^{-1/\alpha}\big).$$

However, as the density of Y_1 in the plane is bounded (this can be seen, for instance, by inverting the Fourier transform of Y_1, or alternatively using explicit computations as in Corollary 1),

$$(20) \qquad P\big(|Y_1| \le n^{-1/\alpha}\big) \le c_2 n^{-2/\alpha}$$

for some fixed c_2. Similarly, symmetry considerations and the scaling property of Y show that

$$(21) \qquad E\big(|Z_n|^{-\alpha(1+\varepsilon)}\big) \le E\big(|Y_n|^{-\alpha(1+\varepsilon)}\big) \le n^{-(1+\varepsilon)} E\big(|Y_1|^{-\alpha(1+\varepsilon)}\big).$$

This last expectation is finite since the density of Y_1 in the plane is bounded and $\alpha(1+\varepsilon) < 2$. Eventually, for some constant c_3, for all $n \ge 1$, (18), (19), (20) and (21) imply that

$$P(|\theta_{n+1} - \theta_n| \ge \pi) \le c_3\big(n^{-2/\alpha} + n^{-(1+\varepsilon)}\big).$$

Note that $2/\alpha > 1$. Thus, Borel–Cantelli's lemma implies Proposition 1.

5. Remarks. We now conclude with some remarks.

1. It seems likely that our results still hold in the more general case, where the isotropy hypothesis is relaxed. For example, if Z is a α-stable planar process, such that no projection of Z is a subordinator, one expects Theorem 1 will be true. However, our proofs do not generalize to this case, partially because the time-changed argument is not a symmetric Lévy process anymore.
2. Suppose $X = (X^1, X^2, X^3)$ is a Brownian motion in $\mathbf{R}^3$ started from $(1, 0, 0)$. Consider the local time of X^3 at 0 and its right-continuous inverse $(\tau_l,\ l \ge 0)$. It is well known that $\log(\tau(t))/\log t$ converges almost surely toward 2 as $t \to \infty$. On the other hand, the process $Z = (X^1_{\tau(\cdot)}, X^2_{\tau(\cdot)})$ is then a Cauchy process, for which Theorem 1 applies. Hence, if ψ denotes the winding number of the process $(Y_t,\ t \ge 0)$ defined as

$$Y_t = Z_{\sigma(t)},$$

with $\sigma = \sup\{s \le t,\ X_s^3 = 0\}$ (i.e., Y_t is the last-visited point on the plane $\{x_3 = 0\}$ by X before time t), then $\psi_t/\sqrt{\log t}$ is asymptotically Gaussian. Hence, the trace of a three-dimensional Brownian motion on a plane winds differently than Brownian motion itself.

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