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RANDOM FOURIER SERIES AND CONTINUOUS ADDITIVE FUNCTIONALS OF LÉVY PROCESSES ON THE TORUS

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Let X be an exponentially killed Lévy process on T^n, the n-dimensional torus, that satisfies a sector condition. (This includes symmetric Lévy processes.) Let $\mathcal{F}_e$ denote the extended Dirichlet space of X. Let $h \in \mathcal{F}_e$ and let $\{h_y, y \in T^n\}$ denote the set of translates of h. That is, $h_y(\cdot) = h(\cdot - y)$. We consider the family of zero-energy continuous additive functions $\{N_t^{[h_y]}, (y, t) \in T^n \times R^+\}$ as defined by Fukushima. For a very large class of random functions h we show that

$$J_\rho(T^n) = \int (\log N_\rho(T^n, \varepsilon))^{1/2} \, d\varepsilon < \infty$$

is a necessary and sufficient condition for the family $\{N_t^{[h_y]}, (y, t) \in T^n \times R^+\}$ to have a continuous version almost surely. Here $N_\rho(T^n, \varepsilon)$ is the minimum number of balls of radius ε in the metric ρ that covers T^n, where the metric ρ is the energy metric. We argue that this condition is the natural extension of the necessary and sufficient condition for continuity of local times of Lévy processes of Barlow and Hawkes.

Results on the bounded variation and p-variation (in t) of $N_t^{[h_y]}$, for y fixed, are also obtained for a large class of random functions h.

1. Introduction. Let $X = \{X_t, t \in R^+\}$ be a Markov process with state space $\mathcal{S}$, a locally compact metric space. We are interested in the joint continuity of families of continuous (in t) additive functionals of X. Perhaps the simplest example of such a continuous additive functional of X is

$$L_t^f = \int_0^t f(X_s) \, ds, \tag{1.1}$$

where f is a bounded real-valued continuous function on $\mathcal{S}$. This definition (1.1) of L_t^f can often be extended to generalized functions f, such as measures or distributions. In the broadest context we are interested in the dependence of L_t^f on f, as f varies in some space of generalized functions. In particular, we ask: when is $(f, t) \mapsto L_t^f$ continuous? In this paper we examine this question in considerable detail when X is a Lévy process in T^n, the n-dimensional

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torus, and the "family" of distributions consists of the set of translates of a fixed distribution f on T^n.

In [11], we use the traditional approach of [4], for extending the definition (1.1) of L_t^f to a class of measures f. In [11] we assume that $X = \{X_t, t \in R^+\}$ is a strongly symmetric Markov process with respect to some σ-finite reference measure m on the state space $\mathscr{S}$. To be more explicit, let $u^1(x, y)$ denote the 1-potential density with respect to m and let $f = \mu$ be a finite positive measure on $\mathscr{S}$ such that $U^1\mu(x) = \int u^1(x, y)\, d\mu(y)$, $x \in \mathscr{S}$, is bounded. Then, as is well known, we can define a continuous additive functional L_t^μ of X which extends (1.1). Here L_t^μ is characterized by the relationship

$$E^x\left(\int_0^\infty e^{-t}\, dL_t^\mu\right) = U^1\mu(x), \qquad \forall\, x \in \mathscr{S},$$

where μ is referred to as the Revuz measure of L_t^μ. The basic result of [11], Theorem 1.1, holds in a very general setting. However, it is easier to explain and more relevent to this paper if we assume that $\mathscr{S}$ has a group structure. Then, as a generalization of the problem of joint continuity of local times, we considered, in [11], the question of the continuity of the family of continuous additive functionals $\{L_t^{\mu_y}, (y, t) \in \mathscr{S} \times R^+\}$, where μ_y is the translate of μ by y, that is, $\mu_y(A) = \mu(A + y)$ for all Borel sets $A \subset \mathscr{S}$. As is well known, if $u^1(x, y) < \infty$ for all x, $y \in \mathscr{S}$, then X has local times at all $x \in \mathscr{S}$. In the above notation we write the local time process as $\{L_t^{\delta_y}, (y, t) \in \mathscr{S} \times R^+\}$, where $\delta_0 = \delta$ is the point mass at $0 \in \mathscr{S}$. Necessary and sufficient conditions for the continuity of $\{L_t^{\delta_y}, (y, t) \in \mathscr{S} \times R^+\}$ were obtained in [10], solely in terms of the metric

$$\tau(x, y) = \left(u^1(x, x) + u^1(y, y) - 2u^1(x, y)\right)^{1/2}. \tag{1.2}$$

When X is a Lévy process on R^1 or T^1, this result is due to Barlow and Hawkes for sufficiency (see [1] and [3]) and to Barlow for necessity (see [2]). They do not require that X is symmetric. They show that $\{L_t^{\delta_y}, (y, t) \in \mathscr{S} \times R^+\}$ has a continuous version almost surely if and only if

$$J_\tau([0, 1]) = \int_0^\infty (\log N_\tau([0, 1], \varepsilon))^{1/2}\, d\varepsilon < \infty, \tag{1.3}$$

where $N_\tau([0, 1], \varepsilon)$ is the minimum number of balls of radius ε in the metric τ that covers $[0, 1]$.

For symmetric Lévy processes we set $u^1(x, y) = u^1(y - x) = u^1(x - y)$ and $\tau(x, y) = \tau(y - x)$. Thus, in this case,

$$\tau(x - y) = \sqrt{2}\left(u^1(0) - u^1(y - x)\right)^{1/2}. \tag{1.4}$$

To generalize this result to Markov processes without local times, it is natural to consider the "energy" metric

$$\tau(x, y) = \left(\iint u^1(r, s)\, d(\mu_x(r) - \mu_y(r))\, d(\mu_x(s) - \mu_y(s))\right)^{1/2} \tag{1.5}$$

and ask whether there are classes of measures μ for which $\{L_t^{\mu_y}, (y,t) \in \mathscr{S} \times R^+\}$ has a continuous version almost surely if and only if (1.3) holds. We show in [11] that, at least for many smooth measures, the answer to this question is no. (This will be discussed further below.) However, we show in this paper that if we replace μ by a random distribution, then the answer to this question, roughly speaking, is almost always yes.

From this point on let X be an exponentially killed Lévy process in T^n, with Lévy exponent ψ, that satisfies a sector condition. That is,

$$E(e^{ikX_t}) = e^{-t(1+\psi(k))} \qquad \forall\, k \in Z^n, \tag{1.6}$$

and

$$|\psi(k)| \le C \operatorname{Re} \psi(k) \qquad \forall\, k \in Z^n, \tag{1.7}$$

for some constant $0 < C < \infty$. Note that it follows from (1.6) that $\operatorname{Re}\psi(k) \ge 0$ so that all symmetric Lévy processes trivially satisfy this sector condition, since, in this case, $\operatorname{Im}\psi(k) = 0$. In addition, we assume throughout this paper that ψ satisfies the following mild regularity condition:

$$\operatorname{Re}\psi(k) \le C \operatorname{Re}\psi(jk) \qquad \forall\, k \in Z^n \text{ and integers } j \ge 1. \tag{1.8}$$

If $\psi(k)$ is never purely imaginary and $\lim_{k\to\infty}|\psi(k)| = \infty$, then (1.7) is equivalent to

$$|\psi(k)| \le C(1 + \operatorname{Re}\psi(k)) \qquad \forall\, k \in Z^n. \tag{1.9}$$

In what follows, for $k \in Z^n$ and $x \in T^n$, we write kx for $(k \cdot x)$. The notation $|k| \le K$ is an abbreviation for the set $\{k \in Z^n \colon |k| \le K\}$ for some positive number K.

When f is the trigonometric polynomial

$$f(x) = \sum_{|k|\le K} a_k e^{ikx}, \qquad a_k \in C^1, \tag{1.10}$$

the initial formulation of a continuous additive functional of X, (1.1), can be written as

$$L_t^f = \sum_{|k|\le K} a_k \hat{\nu}_t(k), \tag{1.11}$$

where

$$\hat{\nu}_t(k) = \int_0^t e^{ikX_s}\, ds = \int_{T^n} e^{iky}\, d\nu_t(y), \qquad k \in Z^n, \tag{1.12}$$

are the Fourier coefficients of the occupation measure ν_t of X. We set

$$L_t^{f_y} = \int_0^t f(X_s - y)\, ds, \qquad y \in T^n, \tag{1.13}$$

where f is as in (1.10) and $f_y(x) = f(x-y)$. This gives us a simple example of a family of continuous additive functionals of X. Analogous to (1.11), we have

$$L_t^{f_y} = \sum_{|k| \le K} a_k \hat{\nu}_t(k) e^{-iky}. \tag{1.14}$$

We only consider f real and hence assume that $\bar{a}_k = a_{-k}$ both here and in all the extensions that follow. Thus, clearly, $L_t^{f_y}$ (and all its extensions) are real.

We employ two methods for extending L_t^f to generalized functions. One approach is that of Fukushima using the theory of Dirichlet spaces. In the other, we use results and techniques from the theory of random Fourier series. Both approaches give rise to the same extension. Indeed, the interplay between these two approaches is one of the interesting aspects of this paper. It is because our proofs ultimately employ techniques from the theory of random Fourier series, that we work with the state space T^n.

We begin with the Fourier series approach which is easier to explain. As long as $\{a_k \hat{\nu}_t(k)\}_{k \in Z^n} \in l^2$, we can extend (1.14) and consider

$$L(y, t) = \sum_{k \in Z^n} a_k \hat{\nu}_t(k) e^{-iky}, \tag{1.15}$$

where the equality sign denotes the standard identification of a function in $L^2(T^n)$ with its Fourier series. Define

$$\|f\|_2 = \left(\int_{T^n} |f(y)|^2 \, dy \right)^{1/2}. \tag{1.16}$$

By Plancherel's theorem

$$\|L(\,\cdot\,, t)\|_2 = (2\pi)^n \left(\sum_{k \in Z^n} |a_k|^2 |\hat{\nu}_t(k)|^2 \right)^{1/2}. \tag{1.17}$$

Consequently, by Corollary 3.1,

$$E\|L(\,\cdot\,, t)\|_2 \le C t^{1/2} \left(\sum_{k \in Z^n} \frac{|a_k|^2}{1 + \operatorname{Re} \psi(k)} \right)^{1/2}. \tag{1.18}$$

Thus we can extend (1.13) to distributions

$$f(x) = \sum_{k \in Z^n} a_k e^{ikx} \tag{1.19}$$

as long as

$$\sum_{k \in Z^n} \frac{|a_k|^2}{1 + \operatorname{Re} \psi(k)} < \infty. \tag{1.20}$$

Under condition (1.20) we want to determine when the family $\{L(y, \cdot), y \in T^n\}$, defined in (1.15), constitutes a continuous family of continuous additive functionals. By this we mean that not only is $L(y, t)$ a continuous additive functional in t for each fixed y but it is also almost surely jointly continuous in (y, t). (Whenever we write almost surely without qualification, we mean almost surely with respect to P^x for each $x \in T^n$.)

Since $\hat{\nu}_t(k)$ is a continuous additive functional for each k, it is clear that $\{L(y,\cdot),\, y \in T^n\}$ will be a continuous family of continuous additive functionals whenever the Fourier series (1.15) converges locally uniformly almost surely in (y,t). That is, whenever (1.15) converges uniformly almost surely on $T^n \times [0,t^*]$ for all $t^* > 0$.

Let (P^0, Ω') be the probability space of X. For each fixed $t \in R^+$, $L(\cdot,t)$ is a random Fourier series with respect to (P^0,Ω'). In order to show that $L(y,t)$ is almost surely jointly continuous in (y,t), we must be able to show that $L(\cdot,t)$ is almost surely continuous in y. However, continuity properties of random Fourier series are well understood only when the coefficients are sign invariant. With respect to (1.15), this means only when $\{a_k\hat{\nu}_t(k)\}_{k\in Z^n} =_{\mathscr{D}} \{a_k\varepsilon_k\hat{\nu}_t(k)\}_{k\in Z^n}$, where $\{\varepsilon_k\}$ is an independent identically distributed sequence of random variables, independent of X, defined on the probability space (P,Ω), with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2$. We cannot expect this condition to hold in general but it will if we replace the original sequence $\{a_k\}_{k\in Z^n}$ by the random sequence $\{a_k\varepsilon_k\}_{k\in Z^n}$. Clearly, if $\{a_k\}_{k\in Z^n}$ satisfies (1.20), then so does $\{a_k\varepsilon_k(\omega);\, k \in Z^n\}$ for each $\omega \in \Omega$. Thus we can define the family of continuous additive functionals $\{L(y,t,\omega),\, (y,t,\omega) \in T^n \times R^+ \times \Omega\}$, given by

$$L(y,t,\omega) = \sum_{k\in Z^n} a_k\hat{\nu}_t(k)\varepsilon_k(\omega)e^{-iky}. \tag{1.21}$$

We now ask: when will $\{L(y,t,\omega),\, (y,t) \in T^n \times R^+\}$ converge locally uniformly almost surely in (y,t) for almost all $\omega \in \Omega$? (Whenever $\{L(y,t,\omega),\, (y,t) \in T^n \times R^+\}$ converges locally uniformly for any $\omega \in \Omega$, it constitutes a continuous [in (y,t)] family of continuous additive functionals of X.) A definitive answer to this question is given in Theorem 1.1. [Note that this question is actually a special case of the one posed in the paragraph following (1.20). Rather than asking whether $\{L(y,\cdot),\, y \in T^n\}$ constitutes a continuous family of continuous additive functionals for a specific sequence $\{a_k\}$ for which (1.20) holds, we ask if it constitutes a continuous family of continuous additive functionals for almost all sequences $\{a_k\varepsilon_k(\omega)\}$ for which (1.20) holds.]

Before stating Theorem 1.1 let us recall Fukushima's method, using Dirichlet spaces, for extending (1.1) [and hence (1.13)] to generalized functions. Let

$$\mathscr{F}_e = \left\{ h \in L^2(T^n, dx) \,\middle|\, \mathscr{E}(h,h) =_{\text{def}} \sum_{k\in Z^n} (1 + \operatorname{Re}\psi(k))|\hat{h}|^2(k) < \infty \right\}$$

be the (extended) Dirichlet space of X. Let f be the distribution

$$f(x) = \sum_{k\in Z^n} a_k e^{ikx} \tag{1.22}$$

in the dual space

$$\mathscr{F}_e^* = \left\{ v \in \mathscr{S}'(T^n) \,\middle|\, \mathscr{E}^*(v,v) =_{\text{def}} \sum_{k\in Z^n} \frac{|\hat{v}|^2(k)}{1 + \operatorname{Re}\psi(k)} < \infty \right\}. \tag{1.23}$$

We call the metric induced by the norm $\mathscr{E}^*$ the energy metric.

We define L_t^f to be the zero-energy continuous additive functional, $N_t^{[h]}$, which arises in Fukushima's decomposition of $h(X_t)$, where

$$h(x) = \sum_{k \in Z^n} \frac{a_k}{1+\psi(k)} e^{ikx}. \tag{1.24}$$

[Note that by the sector condition, (1.7), $h \in \mathscr{F}_e$.] Details will be given in Section 2.

The condition that $f \in \mathscr{F}_e^*$ is precisely the condition that $\{a_k, k \in Z^n\}$ satisfies (1.20). Of course, if f is in $\mathscr{F}_e^*$, the same will be true of

$$f_{y,\omega}(x) = \sum_{k \in Z^n} a_k \varepsilon_k(\omega) e^{ik(x-y)} \tag{1.25}$$

for each (y, ω). Let

$$\begin{aligned} \rho(x, y) &= (\mathscr{E}^*(f_x - f_y, f_x - f_y))^{1/2} \\ &= \left(\sum_{k \in Z^n} \frac{|a_k|^2}{1 + \operatorname{Re}\psi(k)} \sin^2 \frac{(x-y)k}{2} \right)^{1/2}. \end{aligned} \tag{1.26}$$

We now give the main result of this paper.

THEOREM 1.1. *For almost every $\omega \in \Omega$, the family of continuous additive functionals $\{L_t^{f_{y,\omega}}, (y, t) \in T^n \times R^+\}$ has a version with continuous sample paths almost surely if and only if*

$$J_\rho(T^n) = \int_0^\infty \left(\log N_\rho(T^n, \varepsilon)\right)^{1/2} d\varepsilon < \infty, \tag{1.27}$$

where $N_\rho(T^n, \varepsilon)$ is the minimum number of balls of radius ε in the metric ρ that covers T^n.

Furthermore, if (1.27) *holds, then, for almost every $\omega \in \Omega$,*

$$L(y, t, \omega) = \sum_{k \in Z^n} a_k \hat{\nu}_t(k) \varepsilon_k(\omega) e^{-iky} \tag{1.28}$$

converges almost surely locally uniformly in $(y, t) \in T^n \times R^+$ and is a continuous version of $\{L_t^{f_{y,\omega}}, (y, t) \in T^n \times R^+\}$.

As mentioned above, by "almost surely" we mean almost surely with respect to P^x for each $x \in T^n$. We point out in Section 2 that $N_t^{[h]}$ is defined up to equivalence, that is, P^x almost surely for q.e. x. When we say that $L =_{\text{def}} L(y, t, \omega)$ is a continuous version of $L^f =_{\text{def}} L_t^{f_{y,\omega}}$, we mean that L is continuous P^x almost surely for each $x \in T^n$ and $L = L^f$, P^x almost surely for q.e. x.

Before commenting further on Theorem 1.1, it is useful to state the next theorem which describes an important property of the continuous additive functionals which appear in Theorem 1.1.

THEOREM 1.2. *For each $y \in T^n$, for almost all $\omega \in \Omega$, the stochastic process $\{L_t^{f_{y,\omega}}, t \in [0,1]\}$ is of bounded variation in t almost surely, if and only if $\{a_k\} \in l_2$.*

1.2.1. The continuous additive functionals that we are considering are different from those usually studied. It is traditional to study continuous additive functionals, say A_t, which are positive and hence increasing. This corresponds to taking $A_t = L_t^f$ with f a smooth measure, the Revuz measure of A_t. More generally, one studies continuous additive functionals A_t which are the difference of two positive continuous additive functionals and hence of bounded variation. This corresponds, to taking f to be the difference of smooth measures. (See [5], Chapter 5. The class of smooth measures is quite large. It contains all bounded Borel measures not charging polar sets. Since smooth measures may be infinite, the difference of two smooth measures must be defined with some care.) Theorem 1.2 shows that even if we start with f a bounded smooth measure, in general, for almost all $\omega \in \Omega$, $L_t^{f_\omega}$ will not be of bounded variation in t, where $f_\omega =_{\mathrm{def}} f_{0,\omega}$. Therefore, f_ω will not be the difference of smooth measures. Thus we see that the natural framework for Theorem 1.1 is Fukushima's approach to the study of continuous additive functionals via Dirichlet spaces since this approach encompasses distributions which are not necessarily differences of smooth measures.

1.2.2. We began this presentation by posing the problem of the joint continuity of the family of continuous additive functionals $\{L_t^{f_y}, y \in T^n\}$ for a fixed f of the form (1.22) for which the coefficients $\{a_k\}$ satisfy (1.20), and proceeded to solve a natural modification of this problem to that of the joint continuity of the family of continuous additive functionals $\{L_t^{f_{y,\omega}}, y \in T^n\}$ for almost all $\omega \in \Omega$. As we remarked in Section 1.2.1, this allows us to show how the Dirichlet space approach to the study of continuous additive functionals, based on distributions, enters naturally, even when the original problem concerned the more traditional case in which f is a bounded smooth measure. However, from the perspective of Dirichlet spaces, we can look upon Theorem 1.1 differently. In a certain heuristic sense, it gives necessary and sufficient conditions for the continuity of the family of continuous additive functionals $\{L_t^{f_y}, y \in T^n\}$ for "almost every" f in the (dual) Dirichlet space $\mathscr{F}_e^*$.

1.2.3. Theorem 1.1 shows that under (1.27) both $\{L(y,t,\omega), (y,t) \in T^n \times R^+\}$ and $\{L_t^{f_{y,\omega}}, (y,t) \in T^n \times R^+\}$ constitute a continuous family of continuous additive functionals, and, furthermore, they are equivalent as stochastic processes. We show, in the proof of Theorem 1.1, that even when (1.27) does not hold, we can still identify $L(y,t,\omega)$ with $L_t^{f_{y,\omega}}$ for each fixed $y \in T^n$. Thus we can consider these two processes as interchangeable.

1.2.4. It is interesting to note, as pointed out in [5], that the metric ρ, which arises so naturally in the study of $L(y,t)$ and $L_t^{f_y}$, is related to the "energy integral" of classical potential theory. Suppose that f is actually a signed measure, that is, that (1.19) is the Fourier series of a signed measure, say μ on T^n. Then f_y is the Fourier series of a signed measure, μ_y on T^n, with $\mu_y(A) = \mu(A+y)$ for all measurable sets $A \subseteq T^n$. In this case ρ is equivalent to the metric τ given in (1.5). [The sum in (1.26) is the Fourier series of the integral in (1.5).] Thus Theorem 1.1 does address the question raised in the paragraph containing (1.5) and extends it. Recall that when $\{a_k\} \in l^2$, $f_{y,\omega}$ is the difference of two positive finite measures. Furthermore, $f_{y,\omega}$ can be a positive finite measure. This is discussed in Section 6.

For an important class of Lévy processes and distributions, condition (1.27) can be made much more explicit. First,

$$\sum_{j=2}^{\infty} \frac{\left(\sum_{|l|\geq j}[|a_l|^2/(1+\operatorname{Re}\psi(l))]\right)^{1/2}}{j(\log j)^{1/2}} < \infty \tag{1.29}$$

always implies (1.27). Also, if $\psi(k)$ and $|a_k|$ depend only on $|k|$, for all $k \in Z^n$ and are regularly varying in $|k|$, then (1.29) and (1.27) are equivalent. (See Remark 6.1.) This condition on ψ is satisfied by symmetric stable processes. Radially symmetric Fourier coefficients can be obtained as the Fourier coefficients of the 1-potential density of any radially symmetric Lévy process.

Let us return now to the problem posed initially: what can be said about the continuity of $L_t^{f_y}$ on $(T^n \times R^+)$ for a fixed sequence $\{a_k\}$? As we remarked above, this seems to be a very difficult question. However, a reinterpretation of the necessary and sufficient condition of Barlow and Hawkes for the joint continuity of local times gives the following result.

THEOREM 1.3. *Let $X = \{X_t, t \in R^+\}$ be a Lévy process in T^n. Assume that $\{\hat{\nu}_t(k)\}_{k\in Z^n}$, the Fourier coefficients of the occupation measure ν_t of X_s up to time t, is contained in l^2. This is a necessary and sufficient condition for the local time $L_t^x =_{\mathrm{def}} L_t^{\delta_x}$ of X to exist, and we can write*

$$L_t^x = \sum_{k\in Z} \hat{\nu}_t(k)e^{ikx} \tag{1.30}$$

in the sense of convergence in $L^2(T^n)$. Furthermore, the following are equivalent:

(i) *$\{L_t^x, (x,t) \in T^n \times R^+\}$ has a continuous version almost surely*;

(ii) $\sum_{k\in Z} \varepsilon_k \hat{\nu}_t(k)e^{ikx}, \qquad x \in T^n,$
converges uniformly almost surely, where $\{\varepsilon_k\}$ is independent of X;

(iii) (1.27) *holds* (*with $a_k = 1$ in* (1.26), *for all $k \in Z^n$*).

1.3.1. Fix $t = t_0 > 0$. Theorem 1.3 implies that, almost surely with respect to Ω', $\{L_{t_0}^x, x \in T^n\}$ is in the Pisier algebra if and only if $\{L_t^x, (x,t) \in T^n \times R^+\}$ has a continuous version. See [7], page 213.

Theorem 1.3 shows that when the local time exists the continuity of the randomized and nonrandomized versions of (1.15) are essentially equivalent. However, this is not the case in general. To see this, we describe some results from [11] which show that (1.27) is neither a necessary nor a sufficient condition for the continuity almost surely of $\{L_t^{f_y}, (y, t) \in T^n \times R^+\}$ for all f of the form (1.22).

Let f be a finite positive measure μ; that is, suppose that (1.19) is the Fourier series of a finite positive measure μ on T^n. We show in [11], under the assumption that X is symmetric, that $\{L_t^{f_y}, (y, t) \in T^n \times R^+\}$ has a version with continuous sample paths if

$$\int \log N_d(T^n, \varepsilon)\, d\varepsilon < \infty, \tag{1.31}$$

where

$$d(x, y) = \Big(\sum_{l \in Z^n} \beta(l) |a_l|^2 \sin^2 \frac{l(x-y)}{2} \Big)^{1/2} \tag{1.32}$$

and

$$\beta(l) = \sum_{k \in Z^n} \frac{1}{(1 + \psi(k-l))(1 + \psi(k))}. \tag{1.33}$$

It is easier to see the relationship between d and τ if we note that, in analogy with (1.5), we have

$$d(x, y) = \Big(\iint (u^1(r, s))^2 \, d(\mu_x(r) - \mu_y(r))\, d(\mu_x(s) - \mu_y(s)) \Big)^{1/2}. \tag{1.34}$$

Clearly, (1.31) is a more restrictive condition than (1.27) (recall that when f is a finite positive measure τ and ρ are equivalent). In fact, d is only defined for dimension $n \le 3$, since otherwise $\beta(0) = \infty$. Furthermore, under certain smoothness conditions on $\{a_k\}_{k \in Z^n}$ and $\{\psi(k)\}_{k \in Z^n}$, which include stable processes in T^3 of index greater than 3/2, Theorem 1.5 of [11] states that $\{L_t^{f_y}, (y, t) \in T^n \times R^+\}$ is continuous almost surely if and only if

$$\int (\log N_d(T^n, \varepsilon))^{1/2}\, d\varepsilon < \infty. \tag{1.35}$$

Let us now consider the different results obtained applied to Brownian motion on T^3. By Theorem 1.1, $\{L_t^{f_{y,\omega}}, (y, t) \in T^n \times R^+\}$ has a version with continuous sample paths, for almost all $\omega \in \Omega$, if

$$|a_k| = O\Big(\frac{1}{|k|^{1/2} (\log |k|)^{1+\varepsilon}} \Big) \quad \text{for some } \varepsilon > 0. \tag{1.36}$$

However, this assertion is false when $\varepsilon = 0$. By Theorem 1.5 of [11], if f is a finite positive measure with smooth Fourier coefficients $\{L_t^{f_y}, (y, t) \in T^n \times R^+\}$ has a version with continuous sample paths if

$$|a_k| = O\Big(\frac{1}{|k| (\log |k|)^{1+\varepsilon}} \Big) \quad \text{for some } \varepsilon > 0 \tag{1.37}$$

and in this case this assertion is also false if $\varepsilon = 0$. Thus we have two different sets of necessary and sufficient conditions for the continuity of continuous additive functionals of Lévy processes according to whether the Fourier coefficients of the distribution f are smooth or highly oscillatory. Finally, note that neither of these results is trivial, since $\{a_k \hat{\nu}_t(k)\} \in l^1$ if

$$|a_k| = O\left(\frac{1}{|k|^2(\log|k|)^{1+\varepsilon}}\right) \quad \text{for some } \varepsilon > 0 \tag{1.38}$$

and not necessarily when $\varepsilon = 0$.

As mentioned above, $L_t^{f_{y,\omega}}$ has zero energy and hence, in a certain sense, zero quadratic variation (see Theorem 5.2). On the other hand, we saw in Theorem 1.2 that, in general, $L_t^{f_{y,\omega}}$ is not of bounded variation in t. It is therefore of interest to study the p-variation in t of $L_t^{f_{y,\omega}}$. For any function N_t, we define the dyadic p-variation by

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} |N_{i/2^n} - N_{(i-1)/2^n}|^p. \tag{1.39}$$

In order to say something about this we impose some conditions on X and $f_{y,\omega}$.

We assume that $\{\psi(k)\}$ and the sequence $\{a_k\}$, in the definition of $f_{y,\omega}$, satisfy the conditions that for every $\varepsilon > 0$ there exists a constant C_ε, independent of k, such that

$$C_\varepsilon^{-1}|k|^{\beta-\varepsilon} \le \operatorname{Re}\psi(k) \le C_\varepsilon |k|^{\beta+\varepsilon} \tag{1.40}$$

and

$$C_\varepsilon^{-1}|k|^{-\alpha-\varepsilon} \le |a_k| \le C_\varepsilon |k|^{-\alpha+\varepsilon} \tag{1.41}$$

for all $k \in T^n$, $k \neq 0$. We also require that $2\alpha < n$, and $2\alpha + \beta > n$, which implies that $\{a_k\}_{k\in Z^n} \notin l_2$ and that (1.20) is satisfied. Therefore, by Theorem 3.1, for fixed $y \in T^n$, $\{L_t^{f_{y,\omega}}, t \in [0,1]\}$ is continuous almost surely for almost all $\omega \in \Omega$. [Actually under these conditions $\{L_t^{f_{y,\omega}}, (y,t) \in T^n \times R^+\}$ is continuous almost surely for almost all $\omega \in \Omega$.]

THEOREM 1.4. *Let $f_{y,\omega}$ be a distribution for which $\{\psi(k)\}$ and $\{a_k\}$ satisfy the conditions given in* (1.40) *and* (1.41). *Let*

$$p_0 = \frac{2\beta}{2\beta + 2\alpha - n}. \tag{1.42}$$

(*So that* $1 < p_0 < 2$.)

(i) *If $p < p_0$, then, for almost every $\omega \in \Omega$,*

$$\limsup_{m\to\infty} \sum_{i=1}^{2^m} \left|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\right|^p = \infty \quad a.s. \tag{1.43}$$

(ii) *If* $p > p_0$, *then, for almost every* $\omega \in \Omega$,

$$\lim_{m\to\infty} \sum_{i=1}^{2^m} \left|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\right|^p = 0 \quad a.s. \tag{1.44}$$

Functions satisfying (1.40) and (1.41) include sequences $\{\psi(k)\}$ and $\{a_k\}$ which are regularly varying functions of $|k|$ at ∞ with index β and $-\alpha$, respectively.

For a more concrete example, let X be the projection onto T^n of the exponentially killed symmetric stable process of order β in R^n and let f be the probability measure $v(x)\,dx$, where $v(x)$ is the 1-potential density of the Lévy process which is the projection onto T^n of the symmetric stable process of order $n-\alpha$ in R^n. In this case we can take $\psi(k) = |k|^\beta$ and $|a_k| = |k|^{-\alpha}$.

Certainly Lévy processes on R^n seem more natural than Lévy processes on T^n. Theorems 1.1–1.3 have versions in R^n and probably Theorem 1.4 does as well. However, on R^n the randomized distributions $f_{y,\omega}$ are replaced by generalized stationary Gaussian processes. Although the mathematics remains the same, the continuous additive functionals that we get for Lévy processes on R^n seem so specialized that we do not pursue this line at this time except to point out in Remark 4.1 what the continuous additive functionals on R^n look like.

The outline of this paper is as follows. In Section 2 we describe more carefully the family of continuous additive functionals $L_t^{f_{y,\omega}}$ and its relation to the random Fourier series $L(y,t,\omega)$ and, assuming Theorems 2.1 and 2.2, which are results on the uniform convergence of random Fourier series, prove Theorem 1.1. In Section 3 we obtain many interesting properties of the Fourier coefficients of the occupation measure of X. These are used in Section 4 to prove Theorems 2.1 and 2.2. Section 5 deals with the p-variation in t of the zero-energy continuous additive functionals $L_t^{f_{y,\omega}}$. Finally, in Section 6 we briefly consider the case when $f_{y,\omega}$ is a measure or signed measure.

2. Continuous additive functionals of zero energy. In this section we give a more complete description of the zero-energy continuous additive functionals defined by Fukushima which we denoted by L_t^f in the Introduction, and go on to prove Theorem 1.1. Fukushima's monograph, [5], only considers symmetric Dirichlet spaces. Consequently, the results in [5] can only be applied to symmetric Markov processes. In [12] many of the results from [5] are extended to more general Dirichlet spaces. It is these results that we actually use in this paper since we only require that X satisfy the sector condition. Nevertheless, we only give references to [5] and leave it to the reader to check the details in [12].

By Theorem 5.2.2 of [5] to each $h \in \mathscr{F}_e$ there is naturally associated, up to equivalence, a continuous additive functional $N_t^{[h]} \in \mathscr{N}_c$. Here $\mathscr{N}_c$ denotes the class of continuous additive functionals (in the sense of [5], page 124) N_t such that $E^x(|N_t|) < \infty$ q.e. for each $t > 0$ and such that $e(N) = 0$. (Here q.e. for "quasi-everywhere" means for all $x \in T^n$ except for a set of capacity zero; see [5]. The energy $e(A)$ of an additive functional A_t is defined by

$$e(A) = \lim_{t\to 0} \frac{1}{2t} E^m(A_t^2), \tag{2.1}$$

whenever the limit exists. Here $E^m =_{\text{def}} \int E^x\, m(dx)$, where m is Lebesgue measure on T^n. Two continuous additive functionals A_t and B_t are said to be equivalent if $A_t = B_t$, P^x almost surely for q.e. x.)

The association of $N_t^{[h]}$ with h in [5] is made as follows. Choose a version $\tilde{h}$ of h which is quasi-continuous. There is a unique way (up to equivalence) to decompose $\tilde{h}(X_t) - \tilde{h}(X_0)$ as a sum

$$\tilde{h}(X_t) - \tilde{h}(X_0) = M_t^{[h]} - N_t^{[h]}, \tag{2.2}$$

where $N_t^{[h]} \in \mathscr{N}_c$ and $M_t^{[h]} \in \mathscr{M}^0$. Here $\mathscr{M}^0$ denotes the class of additive functionals M_t of finite energy such that, for each $t > 0$, $E^x(M_t^2) < \infty$ and $E^x(M_t) = 0$ q.e. (In this paper $N_t^{[h]}$ in this paper is the negative of that used in [5].) Recall that for any $f \in \mathscr{F}_e^*$ we defined L_t^f to be $N_t^{[h]}$. [See the paragraph containing (1.22)–(1.24).]

The proof of Theorem 1.1 depends very strongly on the fact that the random Fourier series $L(y, t, \omega)$, defined in (1.21), converges locally uniformly almost surely on $(P \times P^0, \Omega \times \Omega')$. Let us elaborate on this. Consider

$$\tilde{L}(y, t) =_{\text{def}} \sum_{k \in Z^n} a_k \hat{\nu}_t(k, \omega') \varepsilon_k(\omega) e^{-iky}. \tag{2.3}$$

Of course, $\tilde{L}(y, t)$ is also equal to $L(y, t, \omega)$. We write $L(y, t, \omega)$ to emphasize that for each $\omega \in \Omega$ we have a stochastic process on (P^0, Ω'). When we write $\tilde{L}(y, t)$ we emphasize that we have a stochastic process on $(P \times P^0, \Omega \times \Omega')$. We shall also use the following notation:

$$\begin{aligned} \tilde{L}_N(y, t) &= \sum_{|k| \le N} a_k \hat{\nu}_t(k) \varepsilon_k e^{-iky}, \\ \tilde{L}_N^c(y, t) &= \sum_{|k| > N} a_k \hat{\nu}_t(k) \varepsilon_k e^{-iky}. \end{aligned} \tag{2.4}$$

The next two theorems give the continuity properties of $\tilde{L}(y, t)$. We state them here and use them in the proof of Theorem 1.1. They are proved in Section 4 using several lemmas proved in Section 3.

THEOREM 2.1. *If* (1.27) *holds the stochastic process* $\{\tilde{L}(y, t), (y, t) \in T^n \times R^+\}$ *has a version with continuous sample paths almost surely. If* (1.27) *does not hold, then, for almost all* $\omega \in \Omega$,

$$\sup_N \sup_{y \in T^n} \left| \sum_{|k| \le N} a_k \hat{\nu}_t(k, \omega') \varepsilon_k e^{-iky} \right| = \infty \tag{2.5}$$

on a set of measure greater than 0 *in* Ω'.

The following theorem is the key to identifying $L(y, t, \omega)$ with the family of continuous additive functionals $L_t^{f_{y,\omega}}$.

THEOREM 2.2. *Let* $\|\cdot\|_\infty$ *denote* $\sup_{y\in T^n,\, t\in[0,t^*]} |\cdot|$ *for some* $t^* > 0$, *and*

$$\tilde{L}_{N^c} = L_{N^c}(y, t) = \sum_{|k|>N} a_k \hat{\nu}_t(k)\varepsilon_k e^{-iky} \tag{2.6}$$

and assume that (1.27) *holds. Then*

$$\lim_{N\to\infty} E(\|\tilde{L}_{N^c}\|_\infty) = 0, \tag{2.7}$$

which implies, in particular, that, for almost all $\omega \in \Omega$, $L_N(y, t, \omega)$ *converges uniformly in the sup-norm on* $\mathscr{C}(T^n \times [0, t^*])$, *the space of continuous functions on* $T^n \times [0, t^*]$.

PROOF OF THEOREM 1.1. We will first show that (1.27) implies that, for almost all $\omega \in \Omega$, $\{L_t^{f_{y,\omega}}, (y, t) \in (T^n \times R^+)\}$ has a version with continuous sample paths. Set

$$h_{y,\omega}(x) = \sum_{k\in Z^n} \frac{a_k}{1+\psi(k)} \varepsilon_k(\omega) e^{ik(x-y)}. \tag{2.8}$$

It is easy to see that $h_{y,\omega} \in \mathscr{F}_e$. [When (1.27) holds we have that, for almost all $\omega \in \Omega$, $h_{y,\omega}(x)$ is actually continuous almost surely with respect to (P^0, Ω').] Furthermore, defining

$$h_{y,\omega,N}(x) = \sum_{|k|\le N} \frac{a_k}{1+\psi(k)} \varepsilon_k(\omega) e^{ik(x-y)}, \tag{2.9}$$

we see that $\lim_{N\to\infty} h_{y,\omega,N} = h_{y,\omega}$ in $\mathscr{F}_e$ with respect to the Dirichlet norm $\mathscr{E}$. [Convergence also holds in $C(T^n)$ with sup-norm for almost all $\omega \in \Omega$.]

It is clear that $h_{y,\omega,N} = U^0 f_{y,\omega,N}$, where

$$f_{y,\omega,N}(x) = \sum_{|k|\le N} a_k \varepsilon_k(\omega) e^{ik(x-y)}. \tag{2.10}$$

(Note that U^0 denotes the zero-resolvent operator with respect to the killed process, which corresponds to the one-resolvent operator with respect to the unkilled process.) Therefore, by (5.2.22) of [5] (more precisely, by the analog for transient processes) or, alternatively, by (5.3.10) of [5] [see, in particular, the last equation on page 144), we have that

$$N_t^{[h_{y,\omega,N}]} = \int_0^t f_{y,\omega,N}(X_s)\, ds. \tag{2.11}$$

Obviously, $\int_0^t f_{y,\omega,N}(X_s)\, ds = \tilde{L}_N(y, t) =_{\text{def}} L_N(y, t, \omega)$. Therefore, for all $\omega \in \Omega$, $N_t^{[h_{y,\omega,N}]} = L_N(y, t, \omega)$. We see from Theorems 2.1 and 2.2 that, for almost all $\omega \in \Omega$, $N_t^{h_{y,\omega,N}} \to L(y, t, \omega)$ uniformly almost surely on every finite interval of t as $N \to \infty$. By Corollary 1(ii) to Theorem 5.2.2 of [5], we also see that, for some subsequence $N_n \to \infty$, $N_t^{[h_{y,\omega,N_n}]} \to N_t^{[h_{y,\omega}]}$ uniformly on every finite interval of t as $N_n \to \infty$, P^x almost surely for q.e. $x \in T^n$. (In fact, this last convergence holds for any choice of the ε_k's.) These two observations show that when (1.27) holds, for almost all $\omega \in \Omega$, $\{L(y,t,\omega), (y,t) \in T^n \times R^+\}$ is a continuous version of $N_t^{[h_{y,\omega}]}$ or, equivalently, of $L_t^{f_{y,\omega}}$.

We now show that when (1.27) does not hold there exists a countable dense set $D \subset T^n$ so that, for almost all $\omega \in \Omega$, $\sup_{y\in D} N_t^{[h_{y,\omega}]}$ or, equivalently, $\sup_{y\in D} L_t^{f_{y,\omega}}$ is unbounded with probability greater than 0. Fix y and t. Since

$$L(y,t,\omega) = \sum_{k\in Z^n} a_k \hat{\nu}_t(k)\varepsilon_k(\omega)e^{-iky} \tag{2.12}$$

converges in $L^2(T^n)$, there exists a subsequence N_n such that $L_{N_n}(y,t)$ converges to $L(y,t)$ almost surely. As in the first part of this proof, this shows that, for almost all $\omega \in \Omega$, $L(y,t,\omega)$ is a version of $N_t^{[h_{y,\omega}]}$ for fixed y and t. Let $D \subset T^n$ be a countable dense set. Then, for almost all $\omega \in \Omega$,

$$L(y,t,\omega) = N_t^{[h_{y,\omega}]} \qquad \forall\, y \in D\ P^x \text{ a.s. q.e. } x. \tag{2.13}$$

It follows from (2.13) that, for almost all $\omega \in \Omega$,

$$\sup_{y\in D} N_t^{[h_{y,\omega}]} = \sup_{y\in D} L(y,t,\omega), \qquad P^x \text{ a.s. q.e. } x. \tag{2.14}$$

Let $x \in T^n$ be a point at which (2.14) holds P^x almost surely and fix $\omega' \in \Omega'$. Consider $\tilde{L}(y,t)$ defined in (2.3) which we now write as $\tilde{L}(y,t,\omega')$ to emphasize the fact that ω' is held fixed. Since $\tilde{L}(y,t,\omega')$ is a separable process with respect to Ω, $\sup_{y\in D} \tilde{L}(y,t,\omega') = \sup_{y\in T^n} \tilde{L}(y,t,\omega')$ almost surely with respect to Ω. Assume that $\sup_{y\in T^n} \tilde{L}(y,t,\omega') < \infty$ on a set of positive measure in Ω. Then, since this is a tail event, $\sup_{y\in T^n} \tilde{L}(y,t,\omega') < \infty$ almost surely with respect to Ω. It follows from Billard's theorem (see [5], Chapter 5, Theorem 3) that the series $\tilde{L}(y,t,\omega')$ converges uniformly almost surely with respect to Ω. Therefore,

$$\sup_N \sup_{y\in T^n} \left| \sum_{|k|\le N} a_k \hat{\nu}_t(k,\omega')\varepsilon_k e^{-iky} \right| < \infty \tag{2.15}$$

almost surely with respect to Ω. However, (2.15) cannot occur for ω' in a subset of Ω' of measure 1 because that would contradict (2.5). Therefore, for almost all $\omega \in \Omega$, $\sup_{y\in D} L(y,t,\omega) = \infty$ on a set of P^x measure greater than 0. [In fact, it is the set for which (2.5) holds.] Note that if (2.5) holds on a subset of Ω' with P^x measure c greater that 0, then it also holds on a subset of Ω' with P^y measure c for all $y \in T^n$. This is because the effect on $\tilde{L}(y,t,\omega')$ of shifting a path by some $z \in T^n$ is simply to multiply the $\hat{\nu}_t(k)$ in (2.5) by e^{ikz} and, clearly, this does not change the value of the left-hand side of (2.5). Thus we see that, for almost all $\omega \in \Omega$, $\sup_{y\in D} N_t^{[h_{y,\omega}]} = \infty$ on a set of P^x measure greater than 0, q.e. x. □

3. Fourier coefficients of occupation measures. The proofs of Theorems 2.1 and 2.2 will follow from fairly standard techniques in the theory of random Fourier series once we have a good understanding of $\{\hat{\nu}_t(k)\}_{k\in Z^n}$, the Fourier coefficients of the occupation measure of X, defined in (1.12). Let Y be a real- or complex-valued random variable. As usual, we denote $(E|Y|^q)^{1/q}$ by $\|Y\|_q$.

LEMMA 3.1. *For any fixed $T < \infty$, there exists a constant $0 < C_T < \infty$ independent of $t \in [0, T]$, $k \in Z^n$ and integers $m \geq 1$ such that*

$$\|\hat{\nu}_t(k)\|_{2m} \leq C_T \sqrt{m} \|\hat{\nu}_t(k)\|_2 \qquad \forall\, k \in Z^n. \tag{3.1}$$

Furthermore, for any $p > 0$, there exist finite positive constants $A_{p,T}, B_{p,T}$ independent of $t \in [0, T]$ and $k \in Z^n$ such that

$$A_{p,T}\|\hat{\nu}_t(k)\|_2 \leq \|\hat{\nu}_t(k)\|_p \leq B_{p,T}\|\hat{\nu}_t(k)\|_2 \qquad \forall\, k \in Z^n. \tag{3.2}$$

A result of this nature is obtained for Brownian motion in [7], Chapter 17, Section 3.

PROOF.

$$\begin{aligned}
&E(|\hat{\nu}_t(k)|^{2m})\\
&\quad = E\biggl(\int_0^t \int_0^t \exp(ik(X_s - X_r))\, dr\, ds\biggr)^m\\
&\quad = \int_{[0,t]^{2m}} E\biggl(\exp\biggl(ik\biggl(\sum_{j=1}^m X_{s_j}\biggr)\biggr)\exp\biggl(-ik\biggl(\sum_{j=1}^m X_{r_j}\biggr)\biggr)\biggr)\prod_{j=1}^m dr_j\, ds_j\\
&\quad = \sum_\pi \int_{0\leq t_{\pi_1}\leq\cdots\leq t_{\pi_{2m}}\leq t} E\biggl(\exp\biggl(ik\biggl(\sum_{j=1}^{2m}\varepsilon_{\pi_j}X_{t_{\pi_j}}\biggr)\biggr)\biggr)\prod_{j=1}^{2m} dt_j\\
&\quad = \sum_\pi \int_{0\leq t_{\pi_1}\leq\cdots\leq t_{\pi_{2m}}\leq t}\\
&\qquad\quad \times E\biggl(\exp\biggl(ik\sum_{v=1}^{2m}\biggl(\sum_{j=v}^{2m}\varepsilon_{\pi_j}\biggr)(X_{t_{\pi_v}} - X_{t_{\pi_{v-1}}})\biggr)\biggr)\prod_{j=1}^{2m} dt_j\\
&\quad = \sum_\pi \int_{0\leq t_{\pi_1}\leq\cdots\leq t_{\pi_{2m}}\leq t} \prod_{v=1}^{2m}\exp\biggl(-(t_{\pi_v} - t_{\pi_{v-1}})\\
&\qquad\qquad\qquad \times \biggl(1 + \psi\biggl(\biggl(\sum_{j=v}^{2m}\varepsilon_{\pi_j}\biggr)k\biggr)\biggr)\biggr)\prod_{j=1}^{2m} dt_j,
\end{aligned} \tag{3.3}$$

where π runs over all permutations of $\{1, 2, \ldots, 2m\}$, $\varepsilon_{\pi_j} = \pm 1$ depending on whether $t_{\pi_j} \in \{s_1, \ldots, s_m\}$ or $t_{\pi_j} \in \{r_1, \ldots, r_m\}$ and we set $t_{\pi_0} = 0$. In particular, if v is even, then $\sum_{j=v}^{2m}\varepsilon_{\pi_j} \neq 0$, so that using (1.7) and (1.8), we see that for v even

$$\left|\exp\left(-(t_{\pi_v}-t_{\pi_{v-1}})\left(1+\psi\left(\left(\sum_{j=v}^{2m}\varepsilon_{\pi_j}\right)k\right)\right)\right)\right| \tag{3.4}$$
$$\leq \exp(-C(t_{\pi_v}-t_{\pi_{v-1}})(1+|\psi(k)|)).$$

For v odd we simply use the fact that $\operatorname{Re}\psi(k)\geq 0$ so that

$$\begin{aligned} E(|\hat{\nu}_t(k)|^{2m}) &\leq (2m)!\int_{0\leq t_1\leq\cdots\leq t_{2m}\leq t}\prod_{j=1}^{m}\exp(-C(t_{\pi_{2j}}-t_{\pi_{2j-1}}) \\ &\qquad\times(1+|\psi(k)|))\prod_{j=1}^{2m}dt_j \\ &\leq (2m)!\left(\int_{0\leq t_1\leq t_3\leq\cdots\leq t_{2m-1}\leq t}\prod_{j=1}^{m}dt_{2j-1}\right) \\ &\qquad\times\left(\int_0^t\exp(-Cs(1+|\psi(k)|))\,ds\right)^m \\ &\leq (2m)!\frac{t^m}{m!}\left(\int_0^t\exp(-Cs(1+|\psi(k)|))\,ds\right)^m. \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned} \|\hat{\nu}_t(k)\|_{2m}^2 &\leq C_1 mt\left(\int_0^t\exp(-C_2 s(1+|\psi(k)|))\,ds\right) \\ &\leq Cmt^2\left(1\wedge\frac{1}{t(1+|\psi(k)|)}\right). \end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned} E(|\hat{\nu}_t(k)|^2) &= \int_0^t\int_0^t E(e^{ik(X_s-X_r)})\,dr\,ds \\ &= 2\operatorname{Re}\int_0^t\left(\int_r^t E(e^{ik(X_s-X_r)})\,ds\right)dr \\ &= 2\operatorname{Re}\int_0^t e^{-r}\left(\int_r^t e^{-(s-r)(1+\psi(k))}\,ds\right)dr \\ &= 2\operatorname{Re}\int_0^t e^{-r}\left(\frac{1-e^{-(t-r)(1+\psi(k))}}{1+\psi(k)}\right)dr \\ &= 2\operatorname{Re}\int_0^t\left(\frac{e^{-r}-e^{-t(1+\psi(k))}e^{r\psi(k)}}{1+\psi(k)}\right)dr \\ &= 2\operatorname{Re}\left(\frac{1-e^{-t}}{1+\psi(k)}-\frac{e^{-t}-e^{-t(1+\psi(k))}}{\psi(k)(1+\psi(k))}\right) \\ &= 2\operatorname{Re}\left(\frac{e^{-t(1+\psi(k))}-1+(1-e^{-t})(1+\psi(k))}{\psi(k)(1+\psi(k))}\right). \end{aligned} \tag{3.7}$$

Let S_δ denote the region in the complex plane defined by

$$S_\delta = \{z = x + iy \mid x \geq 0, \delta|y| \leq x\}$$

and let $h_t(z)$ be the analytic function defined on $\{\operatorname{Re} z \geq 0\}$ by

$$h_t(z) = \frac{e^{-t(1+z)} - 1 + (1 - e^{-t})(1+z)}{z(1+z)}$$

for $z \neq 0$ and $h(0) = 0$. In view of the above, (3.1) will follow from the next lemma. Then (3.2) follows easily from (3.1) via Hölder's inequality (see, e.g., [8], the proof of Lemma 4.1). □

LEMMA 3.2. *For any $\delta > 0$ we can find some $C_\delta > 0$ such that*

$$\operatorname{Re} h_t(z) \geq C_\delta t^2 e^{-t}\left(1 \wedge \frac{1}{t|1+z|}\right) \tag{3.8}$$

for all $z \in S_\delta$.

PROOF. Let D denote the region in the complex plane defined by

$$D = \{z = x + iy \mid x \geq 0 \text{ and } |y| \leq 1 + x\}.$$

We prove this lemma by first establishing (3.8) for $z \in D$ and then for $z \in S_\delta \cap D^c$.

Note that on D, for all $a \geq 0$,

$$\operatorname{Re}\left(\int_0^a e^{-u(1+z)}\,du\right) \geq 0. \tag{3.9}$$

Therefore, comparing the third line of (3.7) with the last, which is $2\operatorname{Re} h_t(z)$, we see that

$$\begin{aligned}\operatorname{Re}\, h_t(z) &\geq e^{-t}\operatorname{Re}\int_0^t\left(\int_r^t e^{-(s-r)(1+z)}\,ds\right)dr \\ &= e^{-t}t^2\operatorname{Re}\left(\frac{e^{-t(1+z)} - 1 + t(1+z)}{(t(1+z))^2}\right).\end{aligned} \tag{3.10}$$

Let $\tilde{D}$ denote the sector in the complex plane defined by

$$\tilde{D} = \{\eta = u + iv \mid |v| \leq u\}$$

and let $g(\eta)$ be the entire analytic function defined by

$$g(\eta) = \frac{e^{-\eta} - 1 + \eta}{\eta^2}$$

for $\eta \neq 0$ and $g(0) = 0$. In order to obtain (3.8) for $z \in D$, we need only show that there exists a constant $C > 0$ such that

$$\operatorname{Re} g(\eta) \geq C\left(1 \wedge \frac{1}{|\eta|}\right), \qquad \forall\, \eta \in \tilde{D}. \tag{3.11}$$

To obtain (3.11), we begin by noting that for $|\eta| \le 1$ we have

$$\begin{aligned} \left|g(\eta) - \frac{1}{2}\right| &= \frac{1}{|\eta|^2}\left|e^{-\eta} - 1 + \eta - \frac{\eta^2}{2}\right| \\ &= \frac{1}{|\eta|^2}\left|\sum_{n=3}^{\infty} \frac{(-\eta)^n}{n!}\right| \\ &\le \sum_{n=3}^{\infty} \frac{1}{n!} \le 0.3. \end{aligned} \tag{3.12}$$

Thus we see that

$$\operatorname{Re} g(\eta) \ge 0.2 \qquad \forall\, |\eta| \le 1. \tag{3.13}$$

Next we note that, for $\eta \in \tilde{D}$, $\operatorname{Re} \eta \ge 0$, which implies that

$$\left|\frac{e^{-\eta} - 1}{\eta^2}\right| \le \frac{2}{|\eta|^2}. \tag{3.14}$$

Also $\eta \in \tilde{D}$ implies that

$$|\eta|^2 = u^2 + v^2 \le 2u^2, \tag{3.15}$$

so that

$$\operatorname{Re} \frac{1}{\eta} = \frac{u}{|\eta|^2} \ge \frac{1}{\sqrt{2}|\eta|}. \tag{3.16}$$

Therefore, since

$$\operatorname{Re}\ g(\eta) \ge \operatorname{Re} \frac{1}{\eta} - \left|\frac{e^{-\eta} - 1}{\eta^2}\right|, \tag{3.17}$$

we see that (3.11) holds for all $\eta \in \tilde{D}$ such that $|\eta| \ge 4\sqrt{2}$.

Consider

$$\tilde{D}' = \{\eta \in \tilde{D} \mid 1 \le |\eta| \le 4\sqrt{2}\}.$$

To complete the proof of (3.11), we need only show that

$$\operatorname{Re} g(\eta) > 0 \qquad \forall\, \eta \in \tilde{D}'. \tag{3.18}$$

Since g is an entire analytic function on $\tilde{D}'$, $\operatorname{Re} g(\eta)$ is harmonic. Therefore, the restriction of $\operatorname{Re} g(\eta)$ to $\tilde{D}'$ takes its minimum value on the boundary of $\tilde{D}'$. Since we already know that $\operatorname{Re} g(\eta) > 0$ on

$$\tilde{D}' \cap \left(\{|\eta| = 1\} \cup \{|\eta| = 4\sqrt{2}\}\right), \tag{3.19}$$

in order to obtain (3.18) it suffices to show that $\operatorname{Re} g(\eta) > 0$ on $\tilde{D}' \cap \{\eta \mid |v| = u\}$, that is, for $\{\eta \mid |v| = u,\ 1/\sqrt{2} \le |v| \le 4\}$. We have

$$\begin{aligned} \operatorname{Re} g(|v| + iv) &= \operatorname{Re}\left(\frac{e^{-|v|}(\cos v - i \sin v) - 1 + |v| + iv}{2i|v|v}\right) \\ &= (v - e^{-|v|} \sin v)/(2|v|v). \end{aligned} \tag{3.20}$$

This last term is an even function of v and is clearly strictly positive for $1/\sqrt{2} \le v \le 4$. Thus we obtain (3.11) and consequently (3.8) for $z \in D$.

Let us now consider the case $z \in S_\delta \cap D^c$. We write out explicitly

$$\begin{aligned}
&\operatorname{Re}\, h_t(x+iy)\\
&\quad = \operatorname{Re}\left(\frac{e^{-t(1+x)}(\cos ty + i\sin(-ty) - 1 + (1-e^{-t})(1+x+iy)}{(x^2+x-y^2+i(2x+1)y)}\right)\\
&\quad = \frac{1}{|z(1+z)|^2}\big((e^{-t(1+x)}\cos ty - 1 + (1-e^{-t})(1+x))(x^2+x-y^2)\\
&\qquad + (e^{-t(1+x)}\sin(-ty) + (1-e^{-t})y)(2x+1)y\big)
\end{aligned} \tag{3.21}$$

and note that

$$\begin{aligned}
&(e^{-t(1+x)}\cos ty - 1 + (1-e^{-t})(1+x))(x^2+x-y^2)\\
&\quad + (e^{-t(1+x)}\sin(-ty) + (1-e^{-t})y)(2x+1)y\\
&\qquad = (e^{-t} - e^{-t(1+x)}\cos ty)(y^2-x^2-x)\\
&\qquad\quad + (1-e^{-t})(xy^2+y^2+x^2+x^3) + e^{-t(1+x)}\sin(-ty)(2x+1)y\\
&\qquad = e^{-t(1+x)}(1-\cos ty)(y^2-x^2-x) + e^{-t}(1-e^{-tx})(y^2-x^2-x)\\
&\qquad\quad + (1-e^{-t})(xy^2+y^2+x^2+x^3) + e^{-t(1+x)}\sin(-ty)(2xy+y)\\
&\qquad = e^{-t(1+x)}(1-\cos ty)(y^2-x^2-x)\\
&\qquad\quad + \left\{1-e^{-t}-e^{-t}(1-e^{-tx})/x\right\}(x^2+x^3) + (1-e^{-t})(xy^2+y^2)\\
&\qquad\quad + e^{-t}(1-e^{-tx})y^2 + e^{-t(1+x)}\sin(-ty)(2xy+y)\\
&\qquad \ge e^{-t(1+x)}(1-\cos ty)(y^2-x^2-x)\\
&\qquad\quad + (1-e^{-t}-te^{-t})(x^2+x^3) + (1-e^{-t}-te^{-t(1+x)})(xy^2+y^2)\\
&\qquad \ge e^{-t(1+x)}(1-\cos ty)(y^2-x^2-x) + \frac{t^2}{2}e^{-t}(x^2+x^3)\\
&\qquad\quad + te^{-t}(1-e^{-tx})(xy^2+y^2).
\end{aligned}$$

Hence, since $y^2 \ge x^2 + x$ on D^c, we have that

$$\begin{aligned}
\operatorname{Re}\, h_t(z) &\ge \frac{te^{-t}(1-e^{-tx})xy^2}{|z(1+z)|^2}\\
&\ge C\frac{te^{-t}(1-e^{-tx})}{|1+z|},
\end{aligned} \tag{3.22}$$

where we also use the fact that $x+iy \in S_\delta$. This gives the lower bound

$$C\frac{te^{-t}}{|1+z|}$$

whenever $e^{-tx} \leq 1/2$, while if $e^{-tx} \geq 1/2$ we get the lower bound

$$Ct^2 e^{-t} e^{-tx} \frac{x}{1+x} \geq Ct^2 e^{-t}$$

since, on $D^c \cap S_\delta$, $(1/\delta)x \geq y \geq 1+x$. This completes the proof of the lemma. $\square$

It is useful to state the following simple corollary of the proof of Lemma 3.1.

COROLLARY 3.1. *For any fixed $T < \infty$, there exists a constant $0 < C_T$ independent of $t \in [0, T]$, $k \in Z^n$ and all integers $m \geq 1$ and an absolute constant C such that, for all $k \in Z^n$,*

$$\begin{aligned} C_T t^2 \left(1 \wedge \frac{1}{t(1 + \operatorname{Re} \psi(k))}\right) &\leq \|\hat{\nu}_t(k)\|_2^2 \\ &\leq Ct^2 \left(1 \wedge \frac{1}{t(1 + \operatorname{Re} \psi(k))}\right). \end{aligned}$$

PROOF. The upper bound follows from (3.6). For the lower bound we use (3.7), Lemma 3.2 and the fact that, by (1.7),

$$1 + |\psi(k)| \leq C(1 + \operatorname{Re} \psi(k)), \tag{3.23}$$

where C is a constant independent of k. $\square$

LEMMA 3.3.

$$E \exp\left(\lambda \left| \frac{(1 + \operatorname{Re} \psi(k))^{1/2} (\hat{\nu}_t(k) - \hat{\nu}_s(k))}{|t-s|^{1/2}} \right|^2 \right) < \infty \tag{3.24}$$

for some $\lambda > 0$ sufficiently small, which can be chosen independently of k, t and s.

PROOF. We recall two properties of the Fourier coefficients $\{\hat{\nu}_t(k)\}$ which follow immediately from their definition (1.12): the additivity property

$$\hat{\nu}_t(k) - \hat{\nu}_s(k) = \hat{\nu}_{t-s}(k) \circ \theta_s \quad \text{for } t > s \tag{3.25}$$

and the transformation property

$$\hat{\nu}_t(k)(x + \omega) = e^{ikx} \hat{\nu}_t(k)(\omega) \qquad \forall\, x \in T^d, \tag{3.26}$$

where $x + \omega$ denotes the uniform shift of the path ω by $x \in T^d$, that is, $X_u(x + \omega) = x + X_u(\omega)$ for all $u \in R^+$. Using these and the Markov property, we see that

$$
\begin{aligned}
E^0(|\hat{\nu}_t(k) - \hat{\nu}_s(k)|^{2m}) &= E^0(|\hat{\nu}_{t-s}(k)|^{2m} \circ \theta_s) \\
&= E^0\big(E^{X_s}(|\hat{\nu}_{t-s}(k)|^{2m})\big) \\
&= \int_{T^d} p_s(x) E^x(|\hat{\nu}_{t-s}(k)(\omega)|^{2m})\, dx \\
&= \int_{T^d} p_s(x) E^0(|\hat{\nu}_{t-s}(k)(x+\omega)|^{2m})\, dx \\
&= e^{-s} E^0(|\hat{\nu}_{t-s}(k)|^{2m}),
\end{aligned}
\tag{3.27}
$$

where $p_s(x)$ denotes the density of the exponentially killed Lévy process X. Therefore, by (3.6), we have that

$$
\begin{aligned}
I_m &=_{\text{def}} E\left|\frac{(1+\operatorname{Re}\psi(k))^{1/2}(\hat{\nu}_t(k)-\hat{\nu}_s(k))}{|t-s|^{1/2}}\right|^{2m} \\
&= e^{-s}\frac{(1+\operatorname{Re}\psi(k))^m}{|t-s|^m}\|\hat{\nu}_{t-s}(k)\|_{2m}^{2m} \\
&\le C^m m^m.
\end{aligned}
\tag{3.28}
$$

It is now easy to see that for λ sufficiently small $\lambda^q I_q/q! \le \delta^q$ for some $\delta < 1$, which gives (3.24). □

LEMMA 3.4. *Let $t^* > 0$. Then*

$$
E \sup_{0\le t\le t^*} (1+\operatorname{Re}\psi(k))^{1/2}|\hat{\nu}_t(k)|^2 \le C t^*\left(1 \vee \log\frac{1}{t^*}\right),
\tag{3.29}
$$

where C is a constant independent of k, and

$$
E \sup_{\substack{|s-t|\le\delta \\ 0\le s,\, t\le t^*}} (1+\operatorname{Re}\psi(k))|\hat{\nu}_s(k)-\hat{\nu}_t(k)|^2 \le C_{t^*}\delta\left(1\vee\log\frac{1}{\delta}\right),
\tag{3.30}
$$

where $C_{t^*} > 0$ is a constant independent of k but, in general, dependent on t^*.

PROOF. Set $Y(t) = (1+\operatorname{Re}\psi(k))^{1/2}\hat{\nu}_t(k)$. Since $\psi(0) = 0$ and $\hat{\nu}_t(0) = t$, both (3.29) and (3.30) are trivially true when $k = 0$. Otherwise, by Lemma 3.3,

$$
P\left(\frac{\sqrt{2\lambda}\,|Y(t)-Y(s)|}{|t-s|^{1/2}} > v\right) \le Ce^{-v^2/2}.
\tag{3.31}
$$

It follows from [9], Chapter 2, Theorem 3.1, applied separately to the real and imaginary parts of $Y(t)$, that

$$
\left(E\sup_{t\le t^*}|Y(t)|^2\right)^{1/2} \le C\Big(\sup_{t\le t^*}(E|Y(t)|^2)^{1/2} + (t^*)^{1/2} + J_q(T^n)\Big),
\tag{3.32}
$$

where $q(s,t) = |t-s|^{1/2}/\sqrt{2\lambda}$. Note that, given (3.31), we get (3.14) of Theorem 3.1, Chapter 2 of [9]. Now, since $EY(t)$ is not constant here, we get (3.32) rather than (3.5) in [9]. [Of course, by (3.6), $\sup_{t\le t^*} E|Y(t)|^2 \le t^*$.] Equation (3.29) now follows from a simple estimate of $J_q(T^n)$. Note that although Theorem 3.1, Chapter 2 of [9] refers to a version of $Y(t)$, since $Y(t)$ is continuous in t, we can take the version to be $Y(t)$ itself.

In an even more direct fashion (3.31) implies (3.4) of Theorem 3.1, Chapter 2 of [9] which implies (3.30) by a simple calculation. □

Recall that we defined (P^0, Ω') as the probability space of X and (P, Ω) as the probability space of $\{\varepsilon_k\}$, where the two probability spaces are independent. In addition to $\{\varepsilon_k\}$, we also consider $\{\varepsilon'_k\}_{k\in Z^n}$, independent Rademacher random variables, and $\{g_k\}_{k\in Z^n}$ and $\{g'_k\}_{k\in Z^n}$, independent normal random variables with mean 0 and variance 1, all defined on (P, Ω), with all four sequences being independent of each other and of X. We define $\{\tilde\varepsilon_k\}_{k\in Z^n}$, where $\tilde\varepsilon_k = \varepsilon_k + i\varepsilon'_k$, and $\{\tilde g_k\}$, where $\tilde g_k = g_k + ig'_k$. Let E^0 denote expectation with respect to the probability space (P^0, Ω') and let E_G denote expectation with respect to the probability space (P, Ω). Let E denote expectation with respect to the product space $\Omega \times \Omega'$.

LEMMA 3.5. *Let $\{\alpha_k\}$ be complex numbers and assume that*

$$\kappa(s,t) =_{\mathrm{def}} \left(|t-s| \sum_{k\in Z^n} \frac{|\alpha_k|^2}{1+\operatorname{Re}\psi(k)}\right)^{1/2} < \infty. \tag{3.33}$$

Then

$$E\,\exp\left(\lambda\left|\frac{\operatorname{Re}(\sum_{k\in Z^n} \alpha_k(\hat\nu_t(k) - \hat\nu_s(k))\tilde g_k)}{\kappa(s,t)}\right|\right) < \infty \tag{3.34}$$

and

$$E\,\exp\left(\lambda\left|\frac{\operatorname{Re}(\sum_{k\in Z^n} \alpha_k(\hat\nu_t(k) - \hat\nu_s(k))\tilde\varepsilon_k)}{\kappa(s,t)}\right|\right) < \infty \tag{3.35}$$

for some $\lambda > 0$ sufficiently small.

PROOF. We first prove (3.34). Note that for ω' fixed, $\operatorname{Re}(\sum_{k\in Z^n} \alpha_k(\hat\nu_t(k,\omega') - \hat\nu_s(k,\omega'))\tilde g_k)$ is a normal random variable with mean 0 and variance $\sum_{k\in Z^n} |\alpha_k|^2 |\hat\nu_t(k,\omega') - \hat\nu_s(k,\omega')|^2$. Hence

$$\begin{aligned}\operatorname{Re}&\left(\sum_{k\in Z^n} \alpha_k(\hat\nu_t(k,\omega') - \hat\nu_s(k,\omega'))\tilde g_k\right)\\ &=_{\mathscr{D}} \left(\sum_{k\in Z^n} |\alpha_k|^2 |\hat\nu_t(k,\omega') - \hat\nu_s(k,\omega')|^2\right)^{1/2} g_1\end{aligned}$$

from which we get

$$
\begin{aligned}
&E^0 E_G \exp\biggl(u\biggl|\mathrm{Re}\biggl(\sum_{k\in Z^n}\alpha_k(\hat\nu_t(k)-\hat\nu_s(k))\tilde g_k\biggr)\biggr|\biggr)\\
&\quad = E^0\exp\biggl(\frac{u^2}{2}\sum_{k\in Z^n}|\alpha_k|^2|\hat\nu_t(k)-\hat\nu_s(k)|^2\biggr)\\
&\quad \le E^0\exp\biggl(\frac{u^2}{2}\sum_{k\in Z^n}|\alpha_k|^2|\hat\nu_{t-s}(k)|^2\biggr)
\end{aligned}
\tag{3.36}
$$

for some $u>0$ sufficiently small. In the last step we used the additivity and transformation properties, (3.25) and (3.26), as well as the Markov property, as in the proof of Lemma 3.3.

It now follows from the multiple Hölder inequality and (3.6) that

$$
\begin{aligned}
&E^0\biggl(\frac{u^2}{2}\sum_{k\in Z^n}|\alpha_k|^2|\hat\nu_{t-s}(k)|^2\biggr)^j\\
&\quad=\biggl(\frac{u^2}{2}\biggr)^j\sum_{k_1,\dots,k_j\in Z^n}\biggl(\prod_{i=1}^j|\alpha_{k_i}|^2\biggr)E\biggl(\prod_{i=1}^j|\hat\nu_{t-s}(k_i)|^2\biggr)\\
&\quad\le\biggl(\frac{u^2}{2}\biggr)^j\sum_{k_1,\dots,k_j\in Z^n}\prod_{i=1}^j|\alpha_{k_i}|^2\|(|\hat\nu_{t-s}(k_i)|^2)\|_j\\
&\quad\le\biggl(\frac{u^2}{2}\biggr)^j\sum_{k_1,\dots,k_j\in Z^n}\prod_{i=1}^j|\alpha_{k_i}|^2\|\hat\nu_{t-s}(k_i)\|_{2j}^2\\
&\quad\le\biggl(\frac{u^2}{2}\biggr)^j\sum_{k_1,\dots,k_j\in Z^n}\prod_{i=1}^j\frac{C|\alpha_{k_i}|^2 j|t-s|}{1+\mathrm{Re}\,\psi(k_i)}\\
&\quad\le\biggl(\frac{u^2}{2}\biggr)^j\biggl(Cj\sum_{k\in Z^n}\frac{|\alpha_k|^2|t-s|}{1+\mathrm{Re}\,\psi(k)}\biggr)^j.
\end{aligned}
\tag{3.37}
$$

It is easy to see that, for $u=\lambda^{1/2}/\kappa(s,t)$ for some $\lambda>0$ sufficiently small, we have (3.34). (The argument is similar to the one in the last sentence of the proof of Lemma 3.3.)

Since we can take $\tilde g_k=\varepsilon_k|g_k|+i\varepsilon'_k|g'_k|$ for all $k\in Z^n$ in (3.34) without changing its value, we get (3.35) by Jensen's inequality. □

THEOREM 3.1. *If* (1.20) *holds, then, for each fixed* $y\in T^n$, *the stochastic process* $\{\tilde L(y,t), t\in R^+\}$ [*see* (2.3)] *has a continuous version almost surely.*

PROOF. It follows immediately from Lemma 3.5 and Theorem 11.6 of [8] (see the remark on the bottom of page 300 of [8]) that, for each fixed $y\in T^n$, the stochastic process

$$\mathrm{Re}\Big(\sum_{k\in Z^n} a_k\hat{\nu}_t(k)\tilde{\varepsilon}_k e^{-iky}\Big), \qquad t\in T^n, \tag{3.38}$$

has a version with continuous sample paths almost surely. This clearly implies that $\{\tilde{L}(y,t), t\in R^+\}$ does also. □

4. Random Fourier series. We are concerned with the sample path properties of the random Fourier series $\tilde{L}(y,t)$ defined in (2.3). Recall that the sequences $\{\hat{\nu}_t(k)\}_{k\in Z^n}$ and $\{\varepsilon_k\}_{k\in Z^n}$ are independent of each other. However, the $\{\hat{\nu}_t(k)\}$ are not independent of each other and so $\tilde{L}(y,t)$ is not a sum of independent Banach space–valued random variables. This makes the proofs of Theorems 2.1 and 2.2 a little more delicate.

Rather than studying $\tilde{L}(y,t)$ it is often easier to analyze

$$\tilde{H}(y,t) = \mathrm{Re}\Big(\sum_{k\in Z^n} a_k\hat{\nu}_t(k)\tilde{g}_k e^{-iky}\Big), \qquad (y,t)\in T^n\times R^+, \tag{4.1}$$

since, for fixed $\omega'\in\Omega'$,

$$\tilde{H}(y,t,\omega') = \mathrm{Re}\Big(\sum_{k\in Z^n} a_k\hat{\nu}_t(k,\omega')\tilde{g}_k e^{-iky}\Big), \qquad (y,t)\in T^n\times R^+, \tag{4.2}$$

is a stationary Gaussian process. [Recall that (P^0,Ω') is the probability space of X.] It is not difficult to pass from results about $\tilde{H}(y,t,\omega')$ to results about $\tilde{L}(y,t)$.

PROOF OF THEOREM 2.1. We say that a stochastic process is continuous almost surely if it has a version with continuous sample paths almost surely. Assume that (1.27) is satisfied. We first show that, for P^0 almost all $\omega'\in\Omega'$ and any $t^*>0$, $\{\tilde{H}(y,t,\omega'), (y,t)\in(T^d\times[0,t^*])\}$ is continuous almost surely with respect to (P,Ω). (See the paragraph preceding Lemma 3.5 for notation.) We have

$$\begin{aligned}
&E_G(\tilde{H}(y,s,\omega')-\tilde{H}(x,t,\omega'))^2\\
&\quad\le 2(E_G(\tilde{H}(y,s,\omega')-\tilde{H}(y,t,\omega'))^2+E_G(\tilde{H}(y,t,\omega')-\tilde{H}(x,t,\omega'))^2)\\
&\quad= 2\sum_{k\in Z^n}|a_k|^2|\hat{\nu}_s(k)-\hat{\nu}_t(k)|^2+8\sum_{k\in Z^n}|a_k|^2|\hat{\nu}_t(k)|^2\sin^2\frac{(y-x)k}{2}\\
&\quad\le 2\sum_{k\in Z^n}|a_k|^2|\hat{\nu}_s(k)-\hat{\nu}_t(k)|^2+8\sum_{k\in Z^n}|a_k|^2\sup_{t\le t*}|\hat{\nu}_t(k)|^2\sin^2\frac{(x-y)k}{2}.
\end{aligned} \tag{4.3}$$

It follows from [7], Chapter 15, Section 3, Theorem 2, that $\{H(y,t,\omega'), (y,t)\in(T^n\times[0,t^*])\}$ is continuous almost surely for ω' in a set of measure 1, if the two Gaussian processes

$$U(t,\omega') = \mathrm{Re}\Big(\sum_{k\in Z^n} a_k\hat{\nu}_t(k,\omega')\tilde{g}_k\Big), \qquad t\in[0,t^*], \tag{4.4}$$

and

$$V(y,\omega') = \mathrm{Re}\bigg(\sum_{k\in Z^n} a_k \sup_{t\le t*} |\hat{\nu}_t(k,\omega')|\tilde{g}_k e^{-iky}\bigg), \qquad y\in T^n, \tag{4.5}$$

are continuous almost surely for ω' in a set of measure 1. We shall do this by showing that both $U=\{U(t), t\in[0,t^*]\}$ and $V=\{V(y), y\in T^d\}$ are continuous almost surely on $(\Omega\times\Omega')$. For U this follows from Lemma 3.5 using precisely the same proof that showed that for fixed $y\in T^n$ the process in (3.38) is continuous almost surely. For V it follows from (3.29) and [9], Chapter 1, Theorem 1.1, along with [9], Chapter 2, Lemma 3.6, since we can write

$$V(y) = \mathrm{Re}\bigg(\sum_{k\in Z^n} \frac{a_k}{(1+\mathrm{Re}\,\psi(k))^{1/2}} \sup_{t\le t^*}(1+\mathrm{Re}\,\psi(k))^{1/2}|\hat{\nu}_t(k)|\tilde{g}_k e^{-iky}\bigg). \tag{4.6}$$

This shows that, for almost all $\omega'\in\Omega'$, $\{\tilde{H}(y,t,\omega'), (y,t)\in(T^d\times[0,t^*])\}$ is continuous almost surely and hence so is

$$\sum_{k\in Z^n} a_k\hat{\nu}_t(k,\omega')g_k e^{-iky}, \qquad (y,t)\in T^n\times[0,t^*]. \tag{4.7}$$

Using a comparison theorem (Theorem 5.1 of [6]), we see that, for P^0 almost all $\omega'\in\Omega'$,

$$\sum_{k\in Z^n} a_k\hat{\nu}_t(k,\omega')\varepsilon_k e^{-iky}, \qquad (y,t)\in T^n\times[0,t^*], \tag{4.8}$$

is continuous almost surely and so, by Fubini's theorem, we see that $\{\tilde{L}(y,t), (y,t)\in T^n\times[0,t^*]\}$ is continuous almost surely. Since this is true for all $t^*>0$, it is true for $t\in R^+$. Furthermore, since $\hat{\nu}_t(k,\omega'+x)=\hat{\nu}_t(k,\omega')e^{ikx}$, we also have that this holds for P^x almost all $\omega'\in\Omega'$ for each $x\in T^n$.

Suppose that (1.27) does not hold. Write

$$\tilde{L}(y,t) = \sum_{k\in Z^n} \frac{a_k}{(1+\mathrm{Re}\,\psi(k))^{1/2}}(1+\mathrm{Re}\,\psi(k))^{1/2}|\hat{\nu}_t(k)|\varepsilon_k e^{-iky} \tag{4.9}$$

and note that, by (3.2) and Corollary 3.1,

$$E((1+\mathrm{Re}\,\psi(k))^{1/2}|\hat{\nu}_t(k)|) \ge C_1(t\wedge t^{1/2}) \tag{4.10}$$

and

$$E((1+\mathrm{Re}\,\psi(k))|\hat{\nu}_t(k)|^2) \le C_2 t \tag{4.11}$$

for constants $0<C_1\le C_2<\infty$, which are independent of $t\in[0,t^*]$ and k. Thus, if we consider $\tilde{L}(y,t)$ for fixed $t>0$, we see from [9], Chapter 1, Theorem 1.1, along with [9], Chapter 2, Lemma 3.6, that, for $t>0$,

$$\sup_N \sup_{y\in T^n}\bigg|\sum_{|k|\le N} a_k\hat{\nu}_t(k)\varepsilon_k e^{-iky}\bigg| = \infty \tag{4.12}$$

on a set of measure greater than 0 in $(\Omega \times \Omega')$. However, the event in (4.12) is a tail event with respect to $\{\varepsilon_k\}$. Thus we get (2.5). This completes the proof of Theorem 2.1. □

PROOF OF THEOREM 2.2. Let

$$\tilde{H}_{N^c}(y,t) = \mathrm{Re}\Bigg(\sum_{|k|>N} a_k \hat{\nu}_t(k)\tilde{g}_k e^{-iky}\Bigg), \qquad (y,t) \in T^n \times [0,t^*], \tag{4.13}$$

and, for fixed $\omega' \in \Omega'$,

$$\tilde{H}_{N^c}(y,t,\omega') = \mathrm{Re}\Bigg(\sum_{|k|\geq N} a_k \hat{\nu}_t(k,\omega')\tilde{g}_k e^{-iky}\Bigg), \qquad (y,t) \in T^n \times [0,t^*]. \tag{4.14}$$

By the contraction principle (see, e.g., [9], Chapter 2, Theorem 4.9) and simple manipulations between real- and complex-valued processes, it is sufficient to obtain (2.7) with $\tilde{L}_{N^c}$ replaced by $\tilde{H}_{N^c}(y,t)$.

Consider the stationary Gaussian process $\tilde{H}_{N^c}(y,t,\omega')$, given in (4.14), and the metric

$$\begin{aligned} d((y,s),(x,t)) &= d_{N^c}((y,s),(x,t);\omega') \\ &= (E_G(\tilde{H}_{N^c}(y,s,\omega') - \tilde{H}_{N^c}(x,t,\omega'))^2)^{1/2}. \end{aligned} \tag{4.15}$$

Then, similar to (4.3), we have

$$d((y,s),(x,t)) \leq d_1(s,t) + d_2(y,x), \tag{4.16}$$

where

$$d_1(s,t) = \Bigg(2\sum_{|k|>N} |a_k|^2 |\hat{\nu}_s(k) - \hat{\nu}_t(k)|^2\Bigg)^{1/2} \tag{4.17}$$

and

$$d_2(y,x) = \Bigg(8\sum_{|k|>N} |a_k|^2 \sup_{t\leq t*} |\hat{\nu}_t(k)|^2 \sin^2 \frac{(y-x)k}{2}\Bigg)^{1/2}. \tag{4.18}$$

We have $d \leq d_1 + d_2$, where d, d_1 and d_2 are metrics on $T =_{\mathrm{def}} T^n \times [0,t^*]$, $[0,t^*]$ and T^n, respectively. As usual, for any metric space (S,χ), we set $B_\chi(x,\varepsilon) = \{y \in S \mid \chi(x,y) < \varepsilon\}$. Let λ be the normalized Lebesgue measure on $[0,t^*]$ (i.e., $\lambda([0,t^*]) = 1$), and let θ be the Haar measure on T^n and let $m \equiv \lambda \times \theta$ be the product measure on T. It is easy to see that

$$B_d((y,s),\varepsilon) \supset B_{d_1}(s,\varepsilon/2) \times B_{d_2}(y,\varepsilon/2) \tag{4.19}$$

and, consequently, that

$$m(B_d((y,s),\varepsilon)) \geq \lambda(B_{d_1}(s,\varepsilon/2))\theta(B_{d_2}(y,\varepsilon/2)). \tag{4.20}$$

We then have by Theorem 11.18 of [8] (applied to $\tilde{H}_{N^c}$ and $-\tilde{H}_{N^c}$) that

$$
\begin{aligned}
E_G(\|\tilde{H}_{N^c}(y,t,\omega')\|_\infty) \le C\biggl(&\sup_{s\in[0,t^*]}\int\biggl(\log\frac{1}{\lambda(B_{d_1}(s,\varepsilon))}\biggr)^{1/2}d\varepsilon \\
&+\sup_{y\in T^n}\int\biggl(\log\frac{1}{\theta(B_{d_2}(y,\varepsilon))}\biggr)^{1/2}d\varepsilon\biggr) \\
= {} & \mathrm{I}+\mathrm{II}.
\end{aligned}
\tag{4.21}
$$

We define a new homogeneous metric β on $[0,t^*]$ by

$$
\begin{aligned}
\beta(u) &= \beta(t+u,t) \\
&= \biggl(2\sum_{|k|>N}|a_k|^2\sup_{|s-t|\le|u|}|\hat{\nu}_s(k)-\hat{\nu}_t(k)|^2\biggr)^{1/2} \\
&= \biggl(2\sum_{|k|>N}\frac{|a_k|^2}{(1+\operatorname{Re}\psi(k))}\sup_{|s-t|\le|u|}(1+\operatorname{Re}\psi(k))|\hat{\nu}_s(k)-\hat{\nu}_t(k)|^2\biggr)^{1/2}.
\end{aligned}
\tag{4.22}
$$

It is clear that $d_1(s,t)\le\beta(|s-t|)=\beta(0,|s-t|)$, which implies that $B_\beta(0,\varepsilon)\subseteq B_{d_1}(s,\varepsilon)$. Therefore,

$$
\mathrm{I}\le C\int\biggl(\log\frac{1}{\lambda(B_\beta(0,\varepsilon))}\biggr)^{1/2}d\varepsilon. \tag{4.23}
$$

Recall that β is a random variable on Ω'. Using [9], Chapter 2, Lemma 2.4, we see that

$$
E^0\mathrm{I}\le C\int_0^\infty\biggl(\log\frac{1}{\lambda(\{x\mid E^0\beta(x)<\varepsilon\})}\biggr)^{1/2}d\varepsilon. \tag{4.24}
$$

By Lemma 3.4,

$$
E^0\beta(x)\le\biggl(C_{t^*}\sum_{|k|>N}\frac{|a_k|^2}{(1+\operatorname{Re}\psi(k))}x\biggl(1\vee\log\frac{1}{x}\biggr)\biggr)^{1/2}. \tag{4.25}
$$

Note that the upper limit in (4.24) can be taken to be $E^0\beta(t^*)$. Doing this and substituting the right-hand side of (4.25) for $E^0\beta(x)$ in (4.24), we see that

$$
\begin{aligned}
E^0\mathrm{I} &\le C_{t^*}\biggl(\sum_{|k|>N}\frac{|a_k|^2}{(1+\operatorname{Re}\psi(k))}\biggr)^{1/2}\int_0^{(t^*(1\vee\log 1/t^*))^{1/2}}\biggl(\log\frac{1}{\varepsilon^4}\biggr)^{1/2}d\varepsilon \\
&\le C_{t^*}\biggl(\sum_{|k|>N}\frac{|a_k|^2}{(1+\operatorname{Re}\psi(k))}\biggr)^{1/2}.
\end{aligned}
\tag{4.26}
$$

We treat II in a similar fashion. The metric d_2 is homogeneous on T^n. We set $\gamma(u)=d_2(t,t+u)$. Similarly to how we obtained (4.24), we have

$$
E^0\mathrm{II}\le C\int_0^\infty\biggl(\log\frac{1}{\theta(\{u\mid E^0\gamma(u)<\varepsilon\})}\biggr)^{1/2}d\varepsilon. \tag{4.27}
$$

Let

$$\delta_N(u) = \left(\sum_{|k|>N} (|a_k|^2/(1+\operatorname{Re}\psi(k))) \sin^2(|u|/2) \right)^{1/2}.$$

By Lemma 3.4,

$$E^0\gamma(u) \le C\left(t^*\left(1 \vee \log\frac{1}{t^*}\right)\right)^{1/2} \delta_N(u). \tag{4.28}$$

Thus, similarly to how we obtained (4.26), we have

$$\begin{aligned} E^0\mathrm{II} &\le C\left(t^*\left(1 \vee \log\frac{1}{t^*}\right)\right)^{1/2} \\ &\quad \times \int_0^{(\sum_{|k|>N}(|a_k|^2/(1+\operatorname{Re}\psi(k))))^{1/2}} \left(\log\frac{1}{\theta(\{u \mid \delta_N(u) < \varepsilon\})}\right)^{1/2} d\varepsilon. \end{aligned} \tag{4.29}$$

By (1.26), $\delta_N(u) \le \rho(a+u, a) =_{\mathrm{def}} \rho(u)$. Therefore,

$$\begin{aligned} N_\rho(T^n, \varepsilon) &\ge \frac{1}{\theta(\{u \mid \rho(u) < \varepsilon\})} \\ &\ge \frac{1}{\theta(\{u \mid \delta_N(u) < \varepsilon\})}. \end{aligned} \tag{4.30}$$

[For the first inequality in (4.30) we simply note that the Haar measure of $N_\rho(T^n, \varepsilon)$ balls must be greater than the Haar measure of T^n, which is 1.] Substituting (4.30) in (4.29), we get

$$\begin{aligned} E^0\mathrm{II} &\le C\left(t^*\left(1 \vee \log\frac{1}{t^*}\right)\right)^{1/2} \\ &\quad \times \int_0^{(\sum_{|k|>N}(|a_k|^2/(1+\operatorname{Re}\psi(k))))^{1/2}} (\log N_\rho(T^n, \varepsilon))^{1/2}\, d\varepsilon. \end{aligned} \tag{4.31}$$

Since $J_\rho(T^n, \varepsilon) < \infty$, the integral in (4.31) goes to 0 as N goes to ∞. Combining (4.26) and (4.31), we get (2.7).

By Lévy's inequality (see, e.g., [9], Chapter 2, Lemma 4.1), we actually have

$$\lim_{N_0\to\infty} E\left(\sup_{N\ge N_0} \|\tilde{H}_{N^c}\|_\infty\right) = 0, \tag{4.32}$$

which verifies the statement about uniform convergence. □

REMARK 4.1. The continuity properties of the random Fourier series in (1.21) remain the same if the $\{\varepsilon_k\}$ are replaced by an independent identically distributed sequence of normal random variables. The resulting Gaussian random Fourier series can readily be extended to R^n; the series in (1.21) cannot. This is one way in which the results of this section can be extended to R^n-valued Lévy processes $X = \{X_t, t \in R^+\}$ for which

$$Ee^{i\xi X_t} = e^{-t(1+\psi(\xi))}. \tag{4.33}$$

Let F be a σ-finite positive measure on R^n and let $\tilde{B}$ be rescaled complex-valued white noise on R^n; that is, for measurable sets A, $D \subset R^n$,

$$E \exp \operatorname{Re}(\bar{z}\tilde{B}(A)) = \exp -F(A)|z|^2, \qquad z \in C, \tag{4.34}$$

and if $A \cap D = \phi$, $\tilde{B}(A)$ and $\tilde{B}(D)$ are independent. Let Ω be the probability space of $\tilde{B}$. We define the generalized Gaussian process

$$f_{y,\omega}(x) = \operatorname{Re} \int e^{i\xi(x-y)} \tilde{B}(d\xi, \omega), \qquad \omega \in \Omega. \tag{4.35}$$

As long as

$$\int \frac{1}{1 + \operatorname{Re} \psi(\xi)} F(d\xi) < \infty, \tag{4.36}$$

we can define the continuous additive functional (in t)

$$L_t^{f_{y,\omega}} = \operatorname{Re} \int \hat{\nu}_t(\xi) e^{-i\xi y} \tilde{B}(d\xi, \omega), \tag{4.37}$$

where $\hat{\nu}_t(\xi)$ is the Fourier transform of the occupation measure of X up to time t. We can then show that, for almost all $\omega \in \Omega$, $\{L_t^{f_{y,\omega}}, (y,t) \in R^n \times R^+\}$ is continuous almost surely with respect to the probability space of X, under conditions that are completely analogous to those obtained for processes in T^n.

5. *p*-variation. In this section we prove Theorems 1.2 and 1.4 and also briefly consider the quadratic variation of $\{L_t^{f_{y,\omega}}, (y,t) \in T^n \times [0,1]\}$ in Theorem 5.2. We precede the proofs of these theorems with a series of auxiliary results. Recall that the probability space of $\{\varepsilon_k\}$ is denoted by (P, Ω) with expectation operator E_ε and the probability space of X is denoted by (P_X^y, Ω') with expectation operator E^y. Here y denotes the starting point of the Lévy process. Unless otherwise stated, we assume that the process starts at 0, so that the expectation operator on the product space $\Omega \times \Omega'$ is $E = E^0 E_\varepsilon$. (In what follows $\|\cdot\|_p$ is with respect to E.)

In this section we will take $t \in [0,1]$. Actually the inequalities obtained are valid for t in any bounded interval, but, in general, the constants depend on the size of the interval.

LEMMA 5.1. *For all p, $q > 0$, there exist constants $A_{p,q}$, and $B_{p,q}$ such that*

$$\Big\| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \Big\|_p \le A_{p,q} \Big\| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \Big\|_q \tag{5.1}$$

and

$$\|L_t^{f_{y,\omega}}\|_p \le B_{p,q} \|L_t^{f_{y,\omega}}\|_q. \tag{5.2}$$

PROOF. It follows from the first four lines of (3.37) followed by (3.1) that, for any $p > 0$, there exist finite positive constants A'_p, B'_p independent of $t \in [0, 1]$ such that

$$A'_p \left\| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right\|_1 \le \left\| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right\|_p \le B'_p \left\| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right\|_1 . \tag{5.3}$$

[For the left-hand side of (5.3), when $p < 1$, one also needs Hölder's inequality.] Clearly, (5.1) follows from (5.3).

To obtain (5.2), we recall that, by definition,

$$L_t^{f_{y,\omega}} = \sum_k a_k \varepsilon_k \hat{\nu}_t(k) e^{-iky}. \tag{5.4}$$

Furthermore, by the Khintchine inequalities (see, e.g., Lemma 4.1 of [8]) for any $p > 0$, there exist finite positive constants $A'_{2,p}$ and $B'_{2,p}$ such that

$$\begin{aligned} A'_{2,p} \left| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right|^{p/2} &\le E_\varepsilon \left| \sum_k a_k \varepsilon_k \hat{\nu}_t(k) \right|^p \\ &\le B'_{2,p} \left| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right|^{p/2} . \end{aligned} \tag{5.5}$$

It follows from (5.4) and (5.5) that

$$\begin{aligned} A'_{2,p} \left| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right|^{p/2} &\le E_\varepsilon |L_t^{f_{y,\omega}}|^p \\ &\le B'_{2,p} \left| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right|^{p/2} . \end{aligned} \tag{5.6}$$

Consequently, taking expectation with respect to E^0 and using first (5.1) and then (5.6), we get

$$\begin{aligned} \| L_t^{f_{y,\omega}} \|_p^p &\le B'_{2,p} \left\| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right\|_{p/2}^{p/2} \\ &\le B'_{2,p} A_{p/2,q/2} \left\| \sum_k |a_k|^2 |\hat{\nu}_t(k)|^2 \right\|_{q/2}^{p/2} \\ &\le B'_{2,p} A_{p/2,q/2} (A'_{2,q})^{-1} \| L_t^{f_{y,\omega}} \|_q^p . \end{aligned} \tag{5.7}$$

This gives us (5.2). □

LEMMA 5.2. *There exist finite positive constants* C_p, *independent of* r, $t \in [0, 1]$, *such that*

$$C_p^{-1} E \left| L_t^{f_{y,\omega}} \right|^p \le E \left| L_{r+t}^{f_{y,\omega}} - L_r^{f_{y,\omega}} \right|^p \le C_p E \left| L_t^{f_{y,\omega}} \right|^p . \tag{5.8}$$

PROOF. We repeat the argument from (5.4)–(5.6) of the previous proof but with (5.4) replaced by

$$L_{r+t}^{f_{y,\omega}} - L_r^{f_{y,\omega}} = \sum_k a_k \varepsilon_k (\hat{\nu}_{r+t}(k) - \hat{\nu}_r(k)) e^{-iky} \tag{5.9}$$

to obtain

$$\begin{aligned} A'_{2,p}\left|\sum_k |a_k|^2 |\hat{\nu}_{r+t}(k) - \hat{\nu}_r(k)|^2\right|^{p/2} &\le E_\varepsilon \left|L_{r+t}^{f_{y,\omega}} - L_r^{f_{y,\omega}}\right|^p \\ &\le B'_{2,p}\left|\sum_k |a_k|^2 |\hat{\nu}_{r+t}(k) - \hat{\nu}_r(k)|^2\right|^{p/2}. \end{aligned} \tag{5.10}$$

Consequently, using the additivity and transformation properties, (3.25) and (3.26), as well as the Markov property, as in the proof of Lemma 3.3, we see that

$$\begin{aligned} E\left|L_{r+t}^{f_{y,\omega}} - L_r^{f_{y,\omega}}\right|^p &\le B'_{2,p} E^0 \left|\sum_k |a_k|^2 |\hat{\nu}_t(k)|^2\right|^{p/2} \\ &\le \frac{B'_{2,p}}{A'_{2,p}} E\left|L_t^{f_{y,\omega}}\right|^p. \end{aligned} \tag{5.11}$$

This gives us the upper bound in (5.8). A similar analysis gives the lower bound. □

LEMMA 5.3. *Let*

$$V_{p,m}(L_1^{f_{y,\omega}}) = \sum_{i=1}^{2^m} \left|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\right|^p \tag{5.12}$$

and

$$D_m = \sum_{i=1}^{2^m} E^0\left(\left|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\right|^p \mid \mathscr{F}_{(i-1)/2^m}\right). \tag{5.13}$$

There exist constants $0 < C'_1, C'_2 < \infty$, *independent of* m, *such that*

$$E\left(V_{p,m}(L_1^{f_{y,\omega}}) - D_m\right)^2 \le C'_1 2^m \left\|L_{1/2^m}^{f_{y,\omega}}\right\|_2^{2p} \tag{5.14}$$

and

$$E_\varepsilon D_m^2 \le C'_2 (E_\varepsilon D_m)^2. \tag{5.15}$$

PROOF. We write

$$\begin{aligned} &V_{p,m}(L_1^{f_{y,\omega}}) - D_m \\ &\quad = \sum_{i=1}^{2^m} \left(\left|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\right|^p - E^0\left(\left|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\right|^p \mid \mathscr{F}_{(i-1)/2^m}\right)\right) \end{aligned} \tag{5.16}$$

and note that the summands are orthogonal. Therefore, by Lemma 5.2, we have that

$$\begin{aligned}
&E\big((V_{p,m}(L_1^{f_{y,\omega}}) - D_m)^2\big) \\
&\quad \le \sum_{i=1}^{2^m} E\Big(\big(\big|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\big|^p - E^0\big(\big|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\big|^p \,\big|\, \mathscr{F}_{(i-1)/2^m}\big)\big)^2\Big) \\
&\quad \le \sum_{i=1}^{2^m} E\big(\big|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\big|^{2p}\big) \qquad (5.17)\\
&\quad \le C_{2p} 2^m \big\|L_{1/2^m}^{f_{y,\omega}}\big\|_{2p}^{2p} \\
&\quad \le C_{2p} B_{2p,2}^{2p} 2^m \big\|L_{1/2^m}^{f_{y,\omega}}\big\|_{2}^{2p}.
\end{aligned}$$

This gives us (5.14).

We now obtain (5.15). Note that

$$D_m = \sum_{i=1}^{2^m} E^{X_{(i-1)/2^m}} \big|L_{1/2^m}^{f_{y,\omega}}\big|^p. \tag{5.18}$$

We use $X^{(1)}, X^{(2)}$ to denote two independent copies of X. We use $\nu^{(1)}$, $\nu^{(2)}$ to denote the corresponding versions of ν and ${}^{(1)}L_{1/2^m}^{f_{y,\omega}}$ and ${}^{(2)}L_{1/2^m}^{f_{y,\omega}}$ to denote the corresponding versions of $L_{1/2^m}^{f_{y,\omega}}$. We use $E_{(1)}^y$, $E_{(2)}^y$ to distinguish between expectations with respect to $X^{(1)}$ and $X^{(2)}$. We can thus write

$$D_m^2 = \sum_{i,j=1}^{2^m} E_{(1)}^{X_{(i-1)/2^m}} E_{(2)}^{X_{(j-1)/2^m}} \big|{}^{(1)}L_{1/2^m}^{f_{y,\omega}}\big|^p \big|{}^{(2)}L_{1/2^m}^{f_{y,\omega}}\big|^p. \tag{5.19}$$

Thus

$$\begin{aligned}
E_\varepsilon(D_m^2) &= \sum_{i,j=1}^{2^m} E_{(1)}^{X_{(i-1)/2^m}} E_{(2)}^{X_{(j-1)/2^m}} E_\varepsilon\big(\big|{}^{(1)}L_{1/2^m}^{f_{y,\omega}}\big|^p \big|{}^{(2)}L_{1/2^m}^{f_{y,\omega}}\big|^p\big) \\
&\le \sum_{i,j=1}^{2^m} E_{(1)}^{X_{(i-1)/2^m}} E_{(2)}^{X_{(j-1)/2^m}} E_\varepsilon\big(\big|{}^{(1)}L_{1/2^m}^{f_{y,\omega}}\big|^{2p}\big)^{1/2} \qquad (5.20)\\
&\qquad \times E_\varepsilon\big(\big|{}^{(2)}L_{1/2^m}^{f_{y,\omega}}\big|^{2p}\big)^{1/2} \\
&\le \left(\sum_{i=1}^{2^m} E^{X_{(i-1)/2^m}} \big(E_\varepsilon\big|(L_{1/2^m}^{f_{y,\omega}})\big|^{2p}\big)^{1/2}\right)^2.
\end{aligned}$$

It follows from (5.6) that

$$\big(E_\varepsilon\big|L_{1/2^m}^{f_{y,\omega}}\big|^{2p}\big)^{1/2} \le (B'_{2,p})^{1/2}(A'_{2,p})^{-1/2} E_\varepsilon\big|L_{1/2^m}^{f_{y,\omega}}\big|^p. \tag{5.21}$$

Thus

$$E_\varepsilon(D_m^2) \le \frac{B'_{2,p}}{A'_{2,p}} \left(\sum_{i=1}^{2^m} E^{X_{(i-1)/2^m}} \left(E_\varepsilon \left| \left(L^{f_{y,\omega}}_{1/2^m} \right) \right|^p \right) \right)^2 \tag{5.22}$$

$$= \frac{B'_{2,p}}{A'_{2,p}} (E_\varepsilon(D_m))^2, \tag{5.23}$$

which is (5.15). This completes the proof of Lemma 5.3. □

Let ζ be an exponential random variable with mean 1 that denotes the lifetime of X.

LEMMA 5.4. *There exist constants* $0 < C_1'', C_2'' < \infty$, *independent of* m, *such that*

$$P_\varepsilon(D_m > C_1'' 2^m \| L^{f_{y,\omega}}_{1/2^m} \|_2^p (1 \wedge \zeta)) \ge C_2'' > 0 \tag{5.24}$$

uniformly in X.

PROOF. We use the Paley–Zygmund inequality

$$P_\varepsilon(D_m > \lambda E_\varepsilon(D_m)) \ge (1-\lambda)^2 \frac{(E_\varepsilon(D_m))^2}{E_\varepsilon(D_m^2)}. \tag{5.25}$$

(See, e.g., [7], Inequality 2, page 8.)

Note that, by (5.6),

$$\begin{aligned} E_\varepsilon(D_m) &= \sum_{i=1}^{2^m} E^{X_{(i-1)/2^m}} E_\varepsilon\left(\left| L^{f_{y,\omega}}_{1/2^m} \right|^p \right) \\ &\ge A'_{2,p} \sum_{i=1}^{2^m} E^{X_{(i-1)/2^m}} \left| \sum_k a_k^2 |\hat{\nu}_{1/2^m}(k)|^2 \right|^{p/2}. \end{aligned} \tag{5.26}$$

Using (3.26), we see that the distribution of $\{|\hat{\nu}_t(k)|^2\}_{k=0}^\infty$ is the same for all starting points of the Lévy process. Hence, by (5.6),

$$\begin{aligned} E_\varepsilon(D_m) &\ge A'_{2,p} \sum_{i=1}^{2^m} 1_{\{\zeta > (i-1)/2^m\}} E^0 \left| \sum_k |a|_k^2 |\hat{\nu}_{1/2^m}(k)|^2 \right|^{p/2} \\ &\ge A'_{2,p} (B'_{2,p})^{-1} 2^m \| L^{f_{y,\omega}}_{1/2^m} \|_p^p (1 \wedge \zeta) \\ &\ge A'_{2,p} (B'_{2,p} B_{2,p})^{-1} 2^m \| L^{f_{y,\omega}}_{1/2^m} \|_2^p (1 \wedge \zeta) \end{aligned} \tag{5.27}$$

uniformly in X. This and (5.15) give us the lemma. □

THEOREM 5.1. *Let* $p \ge 1$ *and assume that*

$$\lim_{m \to \infty} 2^{m/2p} \| L^{f_{y,\omega}}_{1/2^m} \|_2 = 0 \tag{5.28}$$

and

$$\lim_{m\to\infty} 2^{m/p} \| L^{f_{y,\omega}}_{1/2^m} \|_2 = \infty. \tag{5.29}$$

Then

$$\limsup_{m\to\infty} \sum_{i=1}^{2^m} \left| L^{f_{y,\omega}}_{i/2^m} - L^{f_{y,\omega}}_{(i-1)/2^m} \right|^p = \infty \quad a.s. \tag{5.30}$$

PROOF. By (5.14) and (5.28), $V_{p,m}(L^{f_{y,\omega}}_1) - D_m$ converges to 0 in probability as $m \to \infty$. Hence there exists a subsequence $\{m_j\}$, such that

$$\lim_{m_j\to\infty} (V_{p,m_j}(L^{f_{y,\omega}}) - D_{m_j}) = 0 \quad \text{a.s.} \tag{5.31}$$

Thus, in order to prove this theorem, it suffices to show that

$$\limsup_{m_j\to\infty} D_{m_j} = \infty \quad \text{a.s.} \tag{5.32}$$

Let

$$A_{m_j} = \left\{ D_{m_j} > C_1'' 2^{m_j} \| L^{f_{y,\omega}}_{1/2^{m_j}} \|_2^p (1 \wedge \zeta) \right\}. \tag{5.33}$$

It follows from Lemma 5.4 and (5.29) that

$$\limsup_{m_j\to\infty} D_{m_j} = \infty \tag{5.34}$$

on $\limsup_{m_j\to\infty} A_{m_j}$ and

$$P_\varepsilon\Big(\limsup_{m_j\to\infty} A_{m_j} \Big) \ge C_2'' \tag{5.35}$$

uniformly in X.

It is useful to write out the statements contained in (5.34) and (5.35) in greater detail. They say that, for almost all paths of the Lévy process X,

$$\limsup_{m_j\to\infty} \sum_{i=1}^{2^{m_j}} E^{X_{(i-1)/2^{m_j}}} \left(\left| \sum_k a_k \varepsilon_k \hat{\nu}_{1/2^{m_j}}(k) \right|^p \right) = \infty \tag{5.36}$$

on a set $Q \subset \Omega$ with $P_\varepsilon(Q) \ge C_2'' > 0$. However, it is easy to see that (5.36) is a tail event in (Ω, P_ε), since, by (1.12),

$$\sum_{i=1}^{2^{m_j}} E^{X_{(i-1)/2^{m_j}}} \left(\left| \sum_{|k|\le k_0} a_k \varepsilon_k \hat{\nu}_{1/2^{m_j}}(k) \right|^p \right) \le 2^{m_j(1-p)} \left| \sum_{|k|\le k_0} |a_k| \right|^p \le C \tag{5.37}$$

for some finite constant C independent of m_j. Thus (5.34) holds almost surely with respect to P_ε uniformly in X, which gives us (5.30). □

Let $\Delta_m = \{0 = t_0 < t_1 < \cdots < t_m = 1\}$ be a partition of $[0, 1]$ and let $|\Delta_m| = \max_{1\le i\le m}(t_i - t_{i-1})$. Define

$$V_{p,\Delta_m}(L_1^{f_{y,\omega}}) = \sum_{i=1}^{m} \left|L_{t_i}^{f_{y,\omega}} - L_{t_{i-1}}^{f_{y,\omega}}\right|^p. \tag{5.38}$$

Proof of Theorem 1.2. Assume that $\{a_k\} \notin l^2$. We will show that in this case both (5.28) and (5.29) are satisfied when $p = 1$. We first consider (5.28). Let $M > 0$. We have, by Corollary 3.1, that

$$\begin{aligned} t^{-1}\left\|L_t^{f_{y,\omega}}\right\|_2^2 &= t^{-1}\sum_k |a_k|^2 E^0|\hat{\nu}_t(k)|^2 \\ &\le Ct^{-1}\left(t^2\sum_{|k|\le M}|a_k|^2 + 2t\sum_{|k|\ge M}\frac{|a_k|^2}{1+\operatorname{Re}\psi(k)}\right) \\ &= C\left(t\sum_{|k|\le M}|a_k|^2 + 2\sum_{|k|\ge M}\frac{|a_k|^2}{1+\operatorname{Re}\psi(k)}\right), \end{aligned} \tag{5.39}$$

where C is a constant independent of $t \in [0, 1]$ and M. This shows that, for all $t \le t_0(M)$ for some $t_0(M)$ sufficiently small,

$$t^{-1}\left\|L_t^{f_{y,\omega}}\right\|_2^2 \le C\sum_{|k|\ge M}\frac{|a_k|^2}{1+\operatorname{Re}\psi(k)}. \tag{5.40}$$

Since this holds for all M, we get (5.28). Also, by Corollary 3.1, we have

$$t^{-2}\left\|L_t^{f_{y,\omega}}\right\|_2^2 \ge C\sum_{t\operatorname{Re}\psi(k)\le 1}|a_k|^2, \tag{5.41}$$

which gives us (5.29). Therefore, by Theorem 5.1, $\{L_t^{f_{y,\omega}}, t \in R^+\}$ is not of bounded variation.

Assume now that $\{a_k\} \in l^2$. We provide two different proofs in this case. The first one is a direct calculation. Using (3.25), (3.26) and Corollary 3.1, we see that

$$\begin{aligned} E^0 V_{1,\Delta_m}(L_1^{f_{y,\omega}}) &\le C\sum_{i=1}^{m}\left(\sum_k |a_k|^2 E^0|\hat{\nu}_{t_i}(k) - \hat{\nu}_{t_{i-1}}(k)|^2\right)^{1/2} \\ &\le C\left(\sum_{i=1}^{m}(t_i - t_{i-1})\right)\left(\sum_k |a_k|^2\right)^{1/2} \\ &\le C' \end{aligned} \tag{5.42}$$

for some constant C' independent of m. Let $\{\Delta_m\}$ be a sequence of partitions of $[0, 1]$, such that $\lim_{m\to\infty}|\Delta_m| = 0$. We will show that for this sequence

$$\sup_m V_{1,\Delta_m}(L_1^{f_{y,\omega}}) < \infty. \tag{5.43}$$

It follows from the triangle inequality that adding terms to Δ_m increases (5.38). Therefore, we can assume that $\Delta_m \subset \Delta_{m+1}$. In this case $V_{1,\Delta_m}(L_1^{f_{y,\omega}})$ is increasing in m. By the monotone convergence theorem and (5.42),

$$E \sup_m V_{1,\Delta_m}(L_1^{f_{y,\omega}}) = \sup_m E V_{1,\Delta_m}(L_1^{f_{y,\omega}}) \le C'. \tag{5.44}$$

Thus

$$\sup_m V_{1,\Delta_m}(L_1^{f_{y,\omega}}) < \infty \quad \text{a.s.} \tag{5.45}$$

Finally, it follows from Theorem 3.1 that, for each $y \in T^n$ for almost all ω, the stochastic process $\{L_t^{f_{y,\omega}}, t \in R^+\}$ is a continuous function of t. This and (5.45) show that it is also of bounded variation.

The second proof of bounded variation when $\{a_k\} \in l^2$ is more abstract. Consider the distribution

$$f_{y,\omega} = \sum_k a_k \varepsilon_k e^{-iky}. \tag{5.46}$$

By the Schwarz inequality,

$$E_\varepsilon \int_{[0,2\pi]^n} |f_{y,\omega}|\, dy \le (2\pi)^n \biggl(\sum_k |a_k|^2\biggr)^{1/2}. \tag{5.47}$$

This shows that $f_{y,\omega} \in L_1(T^n)$ almost surely. Thus, almost surely, $f_{y,\omega} = f^+(\omega) - f^-(\omega)$, where $f^+(\omega)$ and $f^-(\omega)$ are positive functions on T^n. Let $\mu_1(\omega)$ be the measure on T^n with density $f^+(\omega)$ and let $\mu_2(\omega)$ be the measure on T^n with density $f^-(\omega)$. One can check by looking at the potentials of the corresponding continuous additive functionals that

$$L_t^{f_{y,\omega}} = Ł_t^{\mu_1(\omega)} - Ł_t^{\mu_2(\omega)}, \tag{5.48}$$

where we use Ł to denote the classical continuous additive functionals with respect to a positive measure. Since both continuous additive functionals on the right-hand side of (5.48) are increasing, it follows that $L_t^{f_{y,\omega}}$ is of bounded variation in t. This completes the proof of Theorem 1.2. □

The following estimates are used in the proof of Theorem 1.4.

LEMMA 5.5. *Let X be a symmetric Lévy process and let $f_{y,\omega}$ be a distribution for which $\{\psi(k)\}$ and $\{a_k\}$ satisfy the conditions given in* (1.40) *and* (1.41). *Then, for each $\varepsilon > 0$, there exist positive constants C_ε and t_ε such that, for all $t \le t_\varepsilon$,*

$$C_\varepsilon^{-1} t^{\gamma_1} \le t^{-2/p} \| L_t^{f_{y,\omega}} \|_2^2 \le C_\varepsilon t^{\gamma_2} \tag{5.49}$$

and

$$t^{-1/p} \| L_t^{f_{y,\omega}} \|_2^2 \le C_\varepsilon t^{\gamma_3}, \tag{5.50}$$

where

$$\gamma_1 = (2\beta(p/p_0 - 1) + 10\varepsilon)/(p(\beta + \varepsilon)),$$
$$\gamma_2 = (2\beta(p/p_0 - 1) - 10\varepsilon)/(p(\beta - \varepsilon)),$$
$$\gamma_3 = (\beta(2p/p_0 - 1) - 10\varepsilon)/(p(\beta - \varepsilon)).$$

PROOF. To get (5.49), we note that

$$t^{-2/p}\|L_t^{f_{y,\omega}}\|_2^2 = t^{-2/p}\sum_k |a_k|^2 E^0|\hat{\nu}_t(k)|^2. \tag{5.51}$$

By Corollary 3.1 there exist constants C and C' such that

$$\begin{aligned} Ct^2 \sum_{t\,\mathrm{Re}\,\psi(k)\le 1} |a_k|^2 &\le \sum_{k\in Z^n} |a_k|^2 E^0|\hat{\nu}_t(k)|^2 \\ &\le C'\left(t^2 \sum_{t\,\mathrm{Re}\,\psi(k)\le 1} |a_k|^2 + 2t \sum_{t\,\mathrm{Re}\,\psi(k)\ge 1} \frac{|a_k|^2}{1+\mathrm{Re}\,\psi(k)}\right). \end{aligned} \tag{5.52}$$

The rest follows by simple estimates using (1.40) and (1.41). [The two terms in the last expression in (5.52) are comparable.] The proof of (5.50) is similar. □

PROOF OF THEOREM 1.4. We first prove (1.44). By Chebyshev's inequality, Lemma 5.2 and Lemma 5.1, in that order, we have

$$\begin{aligned} P\big(V_{p,m}(L_1^{f_{y,\omega}}) > \delta_m\big) &\le \frac{E\big(V_{p,m}(L_1^{f_{y,\omega}})\big)}{\delta_m} \\ &= \frac{1}{\delta_m}\sum_{i=1}^{2^m} E\big|L_{i/2^m}^{f_{y,\omega}} - L_{(i-1)/2^m}^{f_{y,\omega}}\big|^p \\ &\le C_p \frac{2^m}{\delta_m}\big\|L_{1/2^m}^{f_{y,\omega}}\big\|_p^p \\ &\le \frac{C_p B_{p,2}}{\delta_m}\big(2^{2m/p}\big\|L_{1/2^m}^{f_{y,\omega}}\big\|_2^2\big)^{p/2}. \end{aligned} \tag{5.53}$$

We see from Lemma 5.5, since $p > p_0$, that, for any admissible α, β, p and p_0, we can find an $\varepsilon > 0$ such that

$$2^{2m/p}\big\|L_{1/2^m}^{f_{y,\omega}}\big\|_2^2 \le C_\varepsilon 2^{-m\eta} \tag{5.54}$$

for all m sufficiently large, where $\eta > 0$. Let $\delta_m = 2^{-mp\eta/4}$. It follows from (5.53) and (5.54) that

$$P\big(V_{p,m}(L_1^{f_{y,\omega}}) > 2^{-mp\eta/4}\big) \le C_\varepsilon 2^{-mp\eta/4}. \tag{5.55}$$

We get (1.44) by the Borel–Cantelli lemma.

The proof of (1.43) follows easily from Theorem 5.1 and Lemma 5.5. In this case, $p < p_0$. Therefore, by the left-hand side of (5.49), we can find an $\varepsilon > 0$ such that

$$2^{2m/p}\|L^{f_{y,\omega}}_{1/2^m}\|_2^2 \ge (C_\varepsilon)^{-1}2^{m\eta} \tag{5.56}$$

for all m sufficiently large, where $\eta > 0$. This gives (5.29). On the other hand, since $1 < p_0 < 2$, $2p/p_0 - 1 > 0$ for all $p \ge 1$ and hence for all $1 \le p < p_0$. This shows that (5.28) is satisfied and we get (5.30) for all $1 \le p < p_0$. However, since, for almost all ω, $\{L_t^{f_{y,\omega}}, t \in R^+\}$ is continuous, it is easy to see that it is also true for all $p < p_0$. Thus we get (1.43). This completes the proof of Theorem 1.4. □

Under (1.20) we know by Theorem 5.2.2 of [5] that, for all $\omega \in \Omega$ and any $y \in T^n$, $L_t^{f_{y,\omega}}$ is a continuous additive functional of zero energy almost surely and hence has zero quadratic variation in the sense of (5.2.20) of [5]. It is rather simple to show this, and a bit more, directly.

THEOREM 5.2. *If* (1.20) *holds, then, for almost all* $\omega \in \Omega$,

$$\lim_{|\Delta_m| \to 0} E\left|V_{2,\Delta_m}(L_1^{f_{y,\omega}})\right|^q = 0 \tag{5.57}$$

for all $q > 0$. *That is, the quadratic variation of* $L_1^{f_{y,\omega}}$ *is* 0, *in the sense of convergence in* $L^q(T^n)$ *for all* $q > 0$.

PROOF. We have

$$E\left|V_{2,\Delta_m}(L_1^{f_{y,\omega}})\right|^q = E\left|\sum_{i=1}^m \left|L_{t_i}^{f_{y,\omega}} - L_{t_{i-1}}^{f_{y,\omega}}\right|^2\right|^q. \tag{5.58}$$

By Lemmas 5.1 and 5.2 all the moments of $|L_{t_i}^{f_{y,\omega}} - L_{t_{i-1}}^{f_{y,\omega}}|$ are equivalent. It then follows from the proofs of Lemmas 3.5 and 3.1 [see, in particular, (3.37) and (3.1)] that this property can be extended to linear combinations of these variables. Thus the last line in (5.58) is less than or equal to

$$C_q\left(E\sum_{i=1}^m \left|L_{t_i}^{f_{y,\omega}} - L_{t_{i-1}}^{f_{y,\omega}}\right|^2\right)^q. \tag{5.59}$$

The rest of the proof is straightforward. We have

$$\begin{aligned}\sum_{i=1}^m E\left|L_{t_i}^{f_{y,\omega}} - L_{t_{i-1}}^{f_{y,\omega}}\right|^2 &\le \sum_{i=1}^m E\left|L_{t_i - t_{i-1}}^{f_{y,\omega}}\right|^2 \\ &= \sum_{i=1}^m \sum_{k \in Z^n} |a_k|^2 E^0 |\hat{\nu}_{t_i - t_{i-1}}(k)|^2.\end{aligned} \tag{5.60}$$

Choose a sequence $\delta_j \downarrow 0$ and then choose m_j such that

$$\sum_{|\Delta_{m_j}|(1+\operatorname{Re}\psi(k))\geq\delta_j} \frac{|a_k|^2}{1+\operatorname{Re}\psi(k)} \leq \delta_j. \tag{5.61}$$

We can assume that $|\Delta_{m_j}|$ is decreasing. Thus, with $m = m_j$, the last term in (5.60)

$$\begin{aligned} &\leq C\sum_{i=1}^{m_j}\bigg((t_i-t_{i-1})^2 \sum_{(t_i-t_{i-1})(1+\operatorname{Re}\psi(k))\leq\delta_j} |a_k|^2 \\ &\qquad\qquad + (t_i-t_{i-1}) \sum_{(t_i-t_{i-1})(1+\operatorname{Re}\psi(k))\geq\delta_j} \frac{|a_k|^2}{1+\operatorname{Re}\psi(k)}\bigg) \\ &\leq C\sum_{i=1}^{m_j}\bigg((t_i-t_{i-1})\delta_j \sum_{k\in Z^n} \frac{|a_k|^2}{1+\operatorname{Re}\psi(k)} + (t_i-t_{i-1})\delta_j\bigg) \\ &\leq C\delta_j\bigg(\sum_{k\in Z^n} \frac{|a_k|^2}{1+\operatorname{Re}\psi(k)} + 1\bigg). \end{aligned} \tag{5.62}$$

Since this inequality holds for all $|\Delta_m| \leq |\Delta_{m_j}|$, the proof follows. □

6. Continuous additive functionals of signed measures. When $\{a_k\}_{k\in Z^n} \in l_2$, it follows from Theorem 1.2 that, for each $y \in T^n$ and $\omega \in \Omega$, $\{L_t^{f_{y,\omega}}, t \in [0, 1]\}$ is a function of bounded variation. The results in this paper also apply in this case and give some insight into the behavior of these more classical continuous additive functionals. Let $f_{y,\omega}$ be as defined in (1.25) and consider

$$\phi(x, y) = \bigg(\sum_{k\in Z^n} |a_k|^2 \sin^2\frac{(x-y)k}{2}\bigg)^{1/2}. \tag{6.1}$$

Assume that $J_\phi(T^n) < \infty$ [see (1.27)]. This is a necessary and sufficient condition for the continuity almost surely of $\{f_{y,\omega}(x), x \in T^n\}$ for each $y \in T^n$. (Although since the $f_{y,\omega}$, as y varies, are all translates of each other, it is enough to know this for $y = 0$.) Let $M(\omega) = \sup_{x\in T^n} |f_{0,\omega}(x)|$ and consider

$$q(x, \omega) = 2M(\omega) + f_{0,\omega}(x) \tag{6.2}$$

so that, obviously, $q(x, \omega) \geq 0$ for ω in a subset $\mathscr{O} \subseteq \Omega$ of measure 1. Let $\omega \in \mathscr{O}$ and consider

$$L_t^{q_{y,\omega}} = \int_0^t q(X_s - y, \omega)\, ds. \tag{6.3}$$

Since $J_\phi(T^n) < \infty$ implies that $J_\tau(T^n) < \infty$, Theorem 1.1 shows that $\{L_t^{q_{y,\omega}}, (y, t) \in T^n \times R^+\}$ is continuous almost surely for all Lévy processes satisfying (1.7) and (1.8). Of course, we do not need Theorem 1.1 to tell us this since $J_\phi(T^d) < \infty$ implies that $\{q(x, \omega), x \in T^n\}$ is continuous almost surely. Still it is interesting to see how (1.27) operates for these positive continuous additive functionals which are quite different from local times.

Now let us consider that $\{a_k\}_{k\in Z^n} \in l_2$ but that $J_\phi(T^n)$ is not necessarily finite. In this case $f_{y,\omega}$ need not have continuous sample paths, nevertheless, it is in $L^1(T^n)$. Thus it is the distribution of a signed measure. By Theorems 3.1 and 1.1 the family of continuous (in t) additive functionals

$$L_t^{f_{y,\omega}} = \int_0^t f(X_s - y, \omega)\,ds, \qquad (y,t) \in T^n \times R^+, \tag{6.4}$$

is continuous in (y,t) almost surely for almost all $\omega \in \Omega$ if and only if (1.27) holds. This shows that Theorem 1.1 is applicable and nontrivial for these more classical types of continuous additive functionals.

REMARK 6.1. That (1.29) implies (1.27), without any conditions on $\{a_k\}$ or $\{\psi(k)\}$, is well known and follows from [9], Chapter 7, Lemma 1.1. The reverse implication, when $\operatorname{Re}\psi(k)$ and $|a_k|$ depend only on $|k|$ for all $k \in Z^n$ and are regularly varying in $|k|$, is proved by an argument similar to the one used in the second half of Lemma 6.3 in [11].

REMARK 6.2. Clearly, if (1.15) has a continuous version almost surely, then $Y = \sum_{k\in Z^n} a_k \hat{\nu}_t(k)$ must exist as a random variable. What we must show is that if $Y < \infty$ almost surely, then $EY < \infty$. We can do this, using the Paley–Zygmund lemma (see, e.g., [7], Inequality 2, page 8), by showing that $(E|Y|)^2 \geq E(|Y|)^2$, but only with some regularity conditions on $\{a_k\}$ and $\{\psi(k)\}$, such as the ones in the previous remark.

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