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GAUSSIAN CHAOS AND SAMPLE PATH PROPERTIES OF ADDITIVE FUNCTIONALS OF SYMMETRIC MARKOV PROCESSES

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Let X be a strongly symmetric Hunt process with α-potential density $u^{\alpha}(x, y)$. Let

$$\mathscr{G}_{\alpha}^{2} = \left\{ \mu | \iint (u^{\alpha}(x, y))^{2}\, d\mu(x)\, d\mu(y) < \infty \right\}$$

and let L_t^{μ} denote the continuous additive functional with Revuz measure μ. For a set of positive measures $\mathscr{M} \subset \mathscr{G}_{\alpha}^{2}$, subject to some additional regularity conditions, we consider families of continuous (in time) additive functionals $L = \{L_t^{\mu}, (t, \mu) \in R^{+} \times \mathscr{M}\}$ of X and a second-order Gaussian chaos $H_{\alpha} = \{H_{\alpha}(\mu), \mu \in \mathscr{M}\}$ which is associated with L by an isomorphism theorem of Dynkin.

A general theorem is obtained which shows that, with some additional regularity conditions depending on X and $\mathscr{M}$, if H_{α} has a continuous version on $\mathscr{M}$ almost surely, then so does L and, furthermore, that moduli of continuity for H_{α} are also moduli of continuity for L.

Special attention is given to Lévy processes in R^n and T^n, the n-dimensional torus, with $\mathscr{M}$ taken to be the set of translates of a fixed measure. Many concrete examples are given, especially when X is Brownian motion in R^n and T^n for $n = 2$ and 3. For certain measures μ on T^n and processes, including Brownian motion in T^3, necessary and sufficient conditions are given for the continuity of $\{L_t^{\mu}, (t, \mu) \in R^{+} \times \mathscr{M}\}$, where $\mathscr{M}$ is the set of all translates of μ.

1. Introduction. In this paper we study the continuity of families of additive functionals of symmetric Markov processes. Let us briefly consider this question heuristically. Let $\{X_t, t \in R^{+}\}$ be a symmetric Markov process with locally compact state space S. One may think of the local time of X_s, up to time t, at a point $x \in S$, as

$$L_t^x = \lim_{\varepsilon \to 0} \int_0^t \delta_{x,\varepsilon}(X_s)\, ds, \tag{1.1}$$

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whenever the limit exists in some sense, where $\{\delta_{x,\varepsilon}\}$ is a family of approximate delta functions at x. Considerable effort has been spent over the last 30 years to find reasonable conditions for the continuity of the stochastic process $\mathscr{T} = \{L_t^x, (t, x) \in R^+ \times S\}$. Some historical background is given in [14], in which we obtain necessary and sufficient conditions for the continuity of $\mathscr{T}$ for Markov processes with symmetric transition probability density functions.

Local times exist only for a relatively narrow class of Markov processes. A Lévy process in R^n has a local time only when $n = 1$. When local times do not exist, and even when they do, one can consider continuous additive functionals of a Markov process determined by measures on the state space of the process. We may think of these as

$$L_t^\mu = \lim_{\varepsilon \to 0} \int_S \int_0^t \delta_{x,\varepsilon}(X_s)\, ds\, d\mu(x), \tag{1.2}$$

where μ is a positive measure on S. In this case, depending on the measure, such limits exist for all Lévy processes in R^n, for all $n \geq 1$. For some family of measures $\mathscr{M}$ for which (1.2) exists, endowed with some topology, we consider the question of the continuity of $L = \{L_t^\mu, (t, \mu) \in R^+ \times \mathscr{M}\}$. The papers of Bass [2] and Bass and Khoshnevisan [3] are the only prior works we know that pursue this question.

In [14], in studying the continuity of the local time process $\mathscr{T}$, we used an isomorphism theorem of Dynkin, which enabled us to show that the continuity of $\mathscr{T}$ was equivalent to the continuity of an associated Gaussian process on S. Since the sample path properties of Gaussian processes are very well understood, we were able to use them to obtain many new results about local times of Markov processes. A different version of Dynkin's isomorphism theorem associates a second-order Gaussian chaos on $\mathscr{M}$ with L. In this paper we first prove a general theorem which shows that the continuity of this Gaussian chaos implies the continuity of L, subject to various additional conditions. These conditions are removed when we specialize to the case of Lévy processes in R^n and T^n, the n-dimensional torus, with $\mathscr{M}$ taken to be the set of translates of a fixed measure. Furthermore, using known results about sample path properties of Gaussian chaoses, concrete sufficient conditions for the continuity and modulus of continuity of L are obtained.

Even when $\mathscr{M}$ is restricted to the set of translates of a fixed finite measure, the diversity of the processes L is vastly greater than its subset $\mathscr{T}$. In some cases descriptions of L in terms of the associated Gaussian chaoses lead to weak results. In other cases the estimates obtained are quite sharp and for certain important Lévy processes, including Brownian motion and other stable processes in T^3, taken together with certain finite measures, we show that L is continuous if and only if the associated Gaussian chaos is continuous.

Let $X = (\Omega, \mathscr{F}_t, X_t, P^x)$, $t \in R^+$, denote a strongly symmetric Hunt process with lifetime ζ and locally compact separable state space S with reference measure m. The full definition of these terms is given in [14]. For the purposes of this paper, it is enough to just say that X has a symmetric transition probability density $p_t(x, y)$. Let $\alpha > 0$. As usual, we define the α-potential density

$$u^{\alpha}(x, y) = \int_0^{\infty} e^{-\alpha t} p_t(x, y)\, dt \tag{1.3}$$

and, setting

$$u_{\delta}^{\alpha}(x, y) = \int_{\delta}^{\infty} e^{-\alpha t} p_t(x, y)\, dt, \tag{1.4}$$

assume that, for all $\delta > 0$,

$$u_{\delta}^{\alpha}(x, y) < \infty \qquad \forall\, x, y \in S. \tag{1.5}$$

We also consider $u^0(x, y)$ for $\alpha = 0$ and define $u_{\delta}^0(x, y)$ as in (1.4). When dealing with $u^0(x, y)$ we assume that (1.5) holds. As usual, we sometimes drop the superscript 0 when dealing with u^0 or u_{δ}^0. We are primarily concerned with Markov processes for which $u^{\alpha}(x, x) = \infty$ for some, or all, $x \in S$. This is the fundamental difference between the processes considered in this paper and those considered in [14].

We assume that $\int u^{\alpha}(x, y) f(y)\, dm(y)$ is a bounded continuous function on S for some, equivalently all, $\alpha > 0$, for all bounded measurable functions f on S which vanish outside of a compact subset of S. When we consider $u(x, y)$ we assume that this is also the case for $\int u(x, y) f(y)\, dm(y)$. These are the smoothness hypotheses on the potential given in Chapter 6, 4.1 and 4.2, of [4]. Theorem 1.1 is expressed in terms of an auxiliary function $h_{(\alpha)}$. By the smoothness hypotheses on the potential, we can always find strictly positive bounded functions f in $L^1(dm)$ such that

$$h_{(\alpha)}(x) = \int u^{\alpha}(x, y) f(y)\, dm(y) \tag{1.6}$$

is continuous and bounded. We define $U^{\alpha} h_{(\alpha)} \mu(\cdot) = \int u^{\alpha}(\cdot, y) h_{(\alpha)}(y)\, d\mu(y)$.

To any continuous additive functional A_t of X we can associate a positive σ-finite measure ν_A called the Revuz measure of A_t. The measure ν_A is defined by the formula

$$\nu_A(g) = \lim_{t \to 0} \frac{1}{t} E^m \left(\int_0^t g(X_s)\, dA_s \right) \tag{1.7}$$

for all bounded continuous functions g on S, and A_t is uniquely determined by ν_A. We will use the notation L_t^{μ} for the continuous additive functional with Revuz measure μ, and often refer to L_t^{μ} as the continuous additive functional determined by μ. Not every σ-finite measure is the Revuz measure of a continuous additive functional. The set of all Revuz measures of continuous additive functionals of X will be denoted by $\mathrm{Rev}(X)$. A complete characterization of $\mathrm{Rev}(X)$ is known; see, for example, [6] and [16]. For our purposes it will be enough to note that a sufficient condition for $\mu \in \mathrm{Rev}(X)$ is that $U^{\alpha}\mu(x)$ is bounded, or, more generally, that $U^{\alpha} h_{(\alpha)} \mu(x)$ is bounded for some $\alpha \geq 0$.

For positive measures μ on S, define

$$\mathscr{G}_{\alpha}^{1} = \left\{ \mu \middle| \iint u^{\alpha}(x, y)\, d\mu(x)\, d\mu(y) < \infty \right\} \tag{1.8}$$

and

$$\mathscr{G}_{\alpha}^{2} = \left\{ \mu \middle| \iint \left(u^{\alpha}(x, y)\right)^{2} d\mu(x)\, d\mu(y) < \infty \right\}. \tag{1.9}$$

Let $\mathscr{G}_{\alpha, F}^{1}, \mathscr{G}_{\alpha, F}^{2}$ denote the set of finite measures in $\mathscr{G}_{\alpha}^{1}, \mathscr{G}_{\alpha}^{2}$, respectively. The Cauchy–Schwarz inequality shows that

$$\mathscr{G}_{\alpha, F}^{2} \subseteq \mathscr{G}_{\alpha, F}^{1}. \tag{1.10}$$

Throughout this paper α is a fixed number greater than or equal to 0. As usual, we denote $\mathscr{G}_{0}^{1}$ and $\mathscr{G}_{0}^{2}$ by $\mathscr{G}^{1}$ and $\mathscr{G}^{2}$.

We are concerned with the sample path properties of the stochastic process

$$L = \left\{ L_{t}^{\mu}, (t, \mu) \in R^{+} \times \mathscr{G}_{\alpha, \mathscr{R}}^{2} \right\}, \tag{1.11}$$

where $\mathscr{G}_{\alpha, \mathscr{R}}^{2} = \mathscr{G}_{\alpha}^{2} \cap \operatorname{Rev}(X)$. An isomorphism theorem of Dynkin given in [5] associates L with a second-order Gaussian chaos $H_{\alpha} = \{H_{\alpha}(\mu),\ \mu \in \mathscr{G}_{\alpha}^{2}\}$. A Gaussian chaos is a family of second-order terms in the Hermite polynomial expansion of random variables in $L^{2}(\gamma)$, where γ is the canonical Gaussian product measure on R^{N}. We describe H_{α} in Section 2 in which we also give, in Theorems 2.1 and 2.2, proofs of versions of Dynkin's theorem which we use.

We begin with a general theorem which states that a family of continuous additive functionals of a Markov process is jointly continuous if the associated Gaussian chaos is continuous, subject to various additional conditions. In several subsequent theorems we impose regularity conditions which enable us to eliminate or simplify these additional conditions.

For any set $\mathscr{C}$ we denote by $\mathscr{B}(\mathscr{C})$ the set of bounded functions on $\mathscr{C}$ with the topology induced by the sup-norm. Occasionally, we will simply say that a stochastic process is continuous to mean that the process has a version which is continuous almost surely.

THEOREM 1.1. *Let X be a Markov process satisfying all the conditions given above and let $\mathscr{M} \subseteq \mathscr{G}_{\alpha}^{2}$. Assume that we are given a topology $\mathscr{O}$ for $\mathscr{M}$ under which $\mathscr{M}$ is locally compact and has a countable base. Assume also that*:

(i) *$\mu \mapsto U^{\alpha}\mu$ and $\mu \mapsto U^{\alpha} h_{(\alpha)}\mu$ are continuous maps from $\mathscr{M}$ to $\mathscr{B}(S)$*;

(ii) *the associated second-order Gaussian chaos $H_{\alpha}(\mu)$ is continuous almost surely on $\mathscr{M}$.*

Then there exists a polar set $Q \subseteq S$, such that, if we restrict X and $\mathscr{M}$ to $S - Q$, we can find a continuous version of $\{L_{t}^{\mu}, (t, \mu) \in [0, \zeta) \times \mathscr{M}\}$.

REMARK 1.1. When X is a Lévy process on R^{n} or T^{n}, the n-dimensional torus, and the set of measures $\mathscr{M}$ is the set of translates of a fixed finite measure μ, the exclusion of a polar set is unnecessary and (i) can be eliminated. These results are given in Theorems 1.3 and 3.2.

REMARK 1.2. For many Markov processes, such as Lévy processes, the killing of time ζ is identically infinite. In this case the last term in Theorem 1.1 can be replaced by $\{L_t^\mu, (t,\mu) \in R^+ \times \mathscr{M}\}$. In general, when ζ is not infinite, in order to find a version of L_t^μ that is also continuous at ζ, we must restrict $\mathscr{M}$ to be a set of measures with common compact support. This is the content of Theorem 3.1.

REMARK 1.3. Since measures with bounded potentials do not charge polar sets, restricting $\mathscr{M}$ to $S - Q$ does not require us to alter the measures in $\mathscr{M}$.

Theorem 1.1 requires the continuity of an associated Gaussian chaos $H_\alpha(\mu)$. We now describe a well-known sufficient condition for the continuity of a Gaussian chaos. Define a metric on $\mathscr{G}_\alpha^2$:

$$\begin{aligned} d(\mu,\nu) &= d_\alpha(\mu,\nu) \\ &= \left(\iint (u^\alpha(x,y))^2 (d(\mu(x)-\nu(x)))(d(\mu(y)-\nu(y)))\right)^{1/2} \qquad (1.12) \\ &= \left(E(H_\alpha(\mu)-H_\alpha(\nu))^2\right)^{1/2}, \end{aligned}$$

where $H_\alpha(\mu)$ is the Gaussian chaos associated with the continuous additive functional with potential $U^\alpha\mu$. [The last equality is explained in (2.10) for $\alpha = 0$.]

THEOREM 1.2. *Let $H_\alpha = \{H_\alpha(\mu), \mu \in \mathscr{M}\}$ be a second-order Gaussian chaos and let $\mathscr{M} \subseteq \mathscr{G}_\alpha^2$ be a set of measures that is compact with respect to $(\mathscr{G}_\alpha^2, d)$, where d is given in* (1.12). *Assume that there exists a probability measure σ on $\mathscr{M}$ such that*

$$\lim_{\eta\to 0} \sup_{t\in\mathscr{M}} \int_0^\eta \log \frac{1}{\sigma(B_d(t,\varepsilon))}\, d\varepsilon = 0, \qquad (1.13)$$

where $B_d(t,\varepsilon)$ denotes the ball in the metric d, with center at t and radius $\varepsilon > 0$. Then H_α has a version which is continuous almost surely.

This is Theorem 11.22 of [9]. See Remark 2.2 for further explanation.

REMARK 1.4. Note that a sufficient condition for (1.13) is that

$$I(d,\mathscr{M}) =_{\text{def}} \int_0^\infty \log N_d(\mathscr{M},\varepsilon)\, d\varepsilon < \infty, \qquad (1.14)$$

where $N_d(\mathscr{M},\varepsilon)$ is the minimum number of balls of radius ε that covers $\mathscr{M}$ [$\log N_d(\mathscr{M},\cdot)$ is called the metric entropy of $\mathscr{M}$ with respect to d]. However, neither (1.13) nor (1.14) is a general necessary condition for the continuity of the type of second-order Gaussian chaoses considered in Theorems 2.1 and 2.2. In fact, we do not know necessary and sufficient conditions for the continuity of these chaoses. However, of particular importance in this paper, is that by recent results [15] there are many examples of classes of these chaoses for which a necessary and sufficient condition for continuity is

$$J(d,\mathscr{M}) =_{\text{def}} \int_0^\infty (\log N_d(\mathscr{M},\varepsilon))^{1/2}\, d\varepsilon < \infty. \tag{1.15}$$

REMARK 1.5. The statement in (1.13), but with the log term replaced by its square root, is necessary and sufficient for the continuity of Gaussian chaoses associated with ordinary local times. In this case the set of measures are the unit point masses. This implies that (1.15) is necessary and sufficient for the continuity of Gaussian chaoses associated with local times of Lévy processes. This is discussed in Section 4.

We now specialize to the case of Lévy processes in R^n, or T^n, and for the class of measures we consider the set of translates

$$\{\mu_x,\ x \in R^n \text{ (or } T^n)\} \tag{1.16}$$

of a single finite measure μ; that is, $\mu_x(A) = \mu(x + A)$ for all measurable sets $A \subset R^n$, or T^n. This class of measures includes the point masses which, obviously, are the translations of the point mass at the origin. Thus the family of continuous additive functionals that are determined by the translates of a single finite measure include the local times. It is easy to check that $\mu \in \operatorname{Rev}(X)$ implies that $\mu_x \in \operatorname{Rev}(X)$ for each translate of μ. For a set of measures such as (1.16), we also think of $d_\alpha(x, y) =_{\text{def}} d_\alpha(\mu_x, \mu_y)$ as a metric on R^n, or T^n.

Using the added structure provided by Lévy processes in R^n, or T^n, the basic continuity result, Theorem 1.1, can be simplified as follows.

THEOREM 1.3. *Let $X = \{X_t,\ t \in R^+\}$ be a symmetric Lévy process in R^n. Let $\mu \in \mathscr{G}_1^2$ be a finite measure. If the associated second-order chaos $H_1 = \{H_1(\mu_x),\ x \in R^n\}$ is continuous almost surely, then $\mu \in \operatorname{Rev}(X)$ and $\{L_t^{\mu_x}, (x,t) \in R^n \times R^+\}$ is continuous almost surely. This also holds with R^n replaced by T^n.*

We can do more with continuous additive functionals of Lévy processes in T^n. For a certain class of these processes and for certain smooth measures $\mu \in \mathscr{G}_1^2$, we can show that $\{L_t^{\mu_x}, x \in T^n\}$ is continuous almost surely if and only if the associated second-order chaos $H_1 = \{H_1(\mu_x), x \in T^n\}$ is continuous almost surely. Before presenting this we need to develop some notation and to mention some results about continuity of Gaussian chaoses.

Let $X = \{X_t, t \in R^+\}$ be a symmetric Lévy process in R^n with

$$Ee^{i\lambda X_t} = e^{-t\psi(\lambda)}, \qquad \lambda \in R^n. \tag{1.17}$$

Similarly, let $Y = \{Y(t),\ t \in R^+\}$ be a symmetric Lévy process in T^n with

$$Ee^{ikY_t} = e^{-t\psi(k)}, \qquad k \in Z^n. \tag{1.18}$$

In each case we refer to ψ as the characteristic exponent of the process. One reason for denoting each characteristic exponent by ψ is that for each Lévy process X in R^n as defined in (1.17) we can define a Lévy process Y in T^n with the same function ψ by projecting X onto $[0,2\pi]^n$. This is explained in Section 6.

The isomorphism theorem can also be used to obtain interesting results about Gaussian chaoses. The next theorem, a consequence of the isomorphism theorem, gives necessary conditions for the continuity and boundedness of a class of second-order Gaussian chaoses closely related to the associated chaoses of certain families of continuous additive functionals. It is proved in Section 6. In what follows let $\{g_k\}_{k\in Z^n}$ be independent identically distributed normal random variables with mean 0 and variance 1.

THEOREM 1.4. *Let ψ be the characteristic exponent of a Lévy process in T^n and $\{b(k)\}_{k\in Z^n}$ the Fourier coefficients of a finite measure on T^n. Then*

$$\begin{aligned}(1.19)\qquad &\sup_{x\in T^n}\left|\sum_{k\in Z^n}\frac{b(k)}{1+\psi(k)}e^{-ikx}\right|\\ &\qquad\le CE\left(\sup_{x\in T^n}\left|\sum_{j,k\in Z^n}\frac{(g_jg_k-\delta_{j,k})b(k-j)}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}}e^{i(k-j)x}\right|\right),\end{aligned}$$

where C is a constant independent of ψ and $\{b(k)\}$. Furthermore, a similar result is obtained when the uniform norm is replaced by the Lipshitz norm, that is, $\sup_{|x-y|\le\delta,\ x,y\in T^n}|\cdot|$, with e^{-ikx} replaced by $(e^{-ikx}-e^{-iky})$ for all $k\in Z^n$.

In preparation for the next theorem, we say that a positive function $h(k)$, $k\in Z^n$, is almost regularly varying with index p if there is a regularly varying function $\bar h(x)$, $x\in R^+$, of index p, such that

$$(1.20)\qquad C^{-1}\bar h(|k|)\le h(k)\le C\bar h(|k|)$$

for some constant $0<C<\infty$.

In the next theorem we will assume that the sequences $\{\psi(k)\}_{k\in Z^n}$ and $\{b(k)\}_{k\in Z^n}$ in Theorem 1.4 are symmetric and almost regularly varying with index $n/2<p<n$ and $q<0$, respectively. In this case it follows from Theorem 1.3 of [15] that the two sides of (1.19) are either both finite or both infinite. Note that by the positivity of the Fourier coefficients $\{b(k)\}_{k\in Z^n}$ we have that $U^1\mu(x)$ is bounded if $U^1\mu(0)<\infty$. This allows us to obtain the following equivalence relationships.

THEOREM 1.5. *Let $\{X(t),\ t\in R^+\}$ be a Lévy process in T^n with characteristic sequence $\{\psi(k)\}_{k\in Z^n}$ and let $\{\hat\mu(k)\}_{k\in Z^n}$ be the Fourier coefficients of a finite measure μ on T^n. Assume that $\{\psi(k)\}_{k\in Z^n}$ and $\{\hat\mu(k)\}_{k\in Z^n}$ are symmetric and almost regularly varying with index $n/2<p<n$ and $q<0$, respectively, and that there exists a constant C such that, for all $|j|\ge1$,*

$$(1.21)\qquad \sup_{|k|\ge|j|}\frac{|k|^n\hat\mu(k)}{1+\psi(k)}\le C\frac{|j|^n\hat\mu(j)}{1+\psi(j)}.$$

Then the following are equivalent:

(i) $U^1\mu(0) < \infty$;
(i′) $\sum_{k \in Z^n} \hat{\mu}(k)/(1 + \psi(k)) < \infty$;
(ii) $\mu \in \mathscr{G}_1^2$ *and* $J(d, \mathscr{M}) < \infty$, *where* $\mathscr{M} = \{\mu_x,\ x \in T^n\}$;
(iii) *the Gaussian chaos* $\{H_1(\mu_x),\ x \in T^n\}$ *is continuous almost surely*;
(iv) $\mu \in \operatorname{Rev}(X)$ *and* $\{L_t^{\mu_x},\ (x,t) \in T^n \times R^+\}$ *is continuous almost surely.*

The interesting cases of this theorem are when $p + |q| = n$. The next result is a corollary of this theorem and its proof for Brownian motion in T^3.

COROLLARY 1.1. *Let* $\{X(t),\ t \in R^+\}$ *be Brownian motion in* T^3 *and let* $f(u)$, $u \in [0,\infty)$, *be regularly varying at* 0 *such that* $uf(u)$ *is decreasing on* $(0,1]$ *and* $f(u) \equiv 0$ *for* $u \in (1,\infty)$. *Define*

$$\mathscr{L}(x,t) = \int_0^t f(|X(s) - x|)\,ds, \qquad (x,t) \in T^3 \times R^+. \tag{1.22}$$

Then the following are equivalent:

(i) $\mathscr{L}(0,t)$ *is finite almost surely*;
(ii) $\{\mathscr{L}(x,t),\ (x,t) \in T^3 \times R^+\}$ *is continuous almost surely*;
(iii) $uf(u) \in L^1[0,1]$.

We should mention that $\mu \in \mathscr{G}_1^2$ implies that $n \le 3$. Also Theorem 1.5 does not apply to Brownian motion on T^2. Note that for local times the existence of the 1-potential does not imply the continuity of $\{L_t^{\delta_x},\ x \in T^n\}$ but for Brownian motion in T^3 it does for the measures considered in Theorem 1.5. We do not know whether or not this is true in T^2. The equivalence of the two apparently simple statements (i) and (iv) in Theorem 1.5 suggests that perhaps it can be obtained by a simple direct argument. Our proof is complex and circuitous. Also Theorem 1.5 suggests that the metric d of (1.12), which we have not seen before in potential theory, has a significant role in describing continuity properties of additive functionals of Markov processes.

An analog of Theorem 1.5 and Corollary 1.1 also holds for Lévy processes in R^n. This is given in [11].

Theorem 1.5 depends very strongly on the Fourier coefficients of the measures and on the characteristic sequences of the Lévy processes being smooth. In [13], employing different methods from those used in this paper, we show that there exists a large class of measures and signed measures μ, such that, for arbitrary Lévy processes on T^n, $\{L_t^{\mu_a},\ (a,t) \in T^n \times R^+\}$ is continuous if and only if $J(\tau_\alpha, \mathscr{M}) < \infty$, where

$$\begin{aligned}\tau_\alpha(\mu_a, \mu_b) &\\ &= \left(\iint u^\alpha(x,y)\,d(\mu_a(x) - \mu_b(x))\,d(\mu_a(y) - \mu_b(y))\right)^{1/2}\end{aligned} \tag{1.23}$$

for some $\alpha > 0$ and this is valid for any $n \ge 1$. The metric in (1.23) is associated with the energy integral of X. (In [13] we consider the more general class of continuous additive functionals determined by generalized functions, as defined in [6].)

When the conditions of Theorem 1.5 are not satisfied, we can use Theorem 1.3 which generally allows us to infer continuity of $\{L_t^{\mu_x}, (x,t) \in R^n \times R^+\}$ [or $(x,t) \in T^n \times R^+$] when (1.14) holds and, in fact, (1.14) is not much weaker than (1.15). Here is a concrete application of Theorem 1.3. Let $X = \{X_t,\ t \in R^+\}$ be a symmetric Lévy process in R^n with characteristic exponent ψ and let μ be a finite measure on R^n with characteristic function $\hat{\mu}$. Assume that

$$\gamma(\xi) = \int \frac{d\eta}{(1+\psi(\xi-\eta))(1+\psi(\eta))} < \infty. \tag{1.24}$$

Note that $\gamma(\xi)$ is the Fourier transform of $(u^1(x))^2$ so that

$$\int \gamma(\xi) |\hat{\mu}(\xi)|^2 \, d\xi = \iint (u^1(x,y))^2 \, d\mu(x) \, d\mu(y). \tag{1.25}$$

THEOREM 1.6. *Let $X = \{X_t,\ t \in R^+\}$ be a symmetric Lévy process in R^n with characteristic exponent ψ. If*

$$\int_1^\infty \frac{\left(\int_{|\xi| \geq x} \gamma(\xi) |\hat{\mu}(\xi)|^2 \, d\xi\right)^{1/2}}{x} \, dx < \infty, \tag{1.26}$$

then $\mu \in \mathrm{Rev}(X)$ and $\{L_t^{\mu_x}, (x,t) \in R^n \times R^+\}$ is continuous almost surely. In particular, for Brownian motion in R^2, this is the case when

$$|\hat{\mu}(\xi)| = O\left(\frac{1}{(\log|\xi|)^{2+\varepsilon}}\right) \quad \text{as } |\xi| \to \infty. \tag{1.27}$$

By Theorem 1.5 we have for Brownian motion in T^3 that, if $\mu \in \mathrm{Rev}(X)$, $\{L_t^{\mu_x}, (x,t) \in R^3 \times R^+\}$ is continuous almost surely if and only if

$$\sum_{k \in Z^3} \frac{\hat{\mu}(k)}{(1+|k|^2)} < \infty \tag{1.28}$$

as long as $\{\hat{\mu}(k)\}$ is almost regularly varying in the sense of (1.20) and satisfies (1.21).

The only other papers that we know of that deal with the joint continuity of continuous additive functionals of Markov processes indexed by measures are [2] and [3], which consider this question for Brownian motion in R^n. The material in [3] shows that (1.14), for all compact subsets of measures $\mathscr{M}$ with bounded potential, is sufficient for the continuity of the associated continuous additive functionals $\{L_t^{\mu}, (t,\mu) \in R^+ \times \mathscr{M}\}$ of Brownian motion but with the metric d replaced by

$$\kappa(\mu,\nu) = \sup_{x \in R^n} \left| \int u^\alpha(x-y) d(\mu(y) - \nu(y)) \right|^{1/2}. \tag{1.29}$$

Furthermore, this result is valid in R^n for all $n \ge 1$. However, the metric κ is difficult to estimate. It is not comparable to the metric d_α given in (1.12).

Restricted to R^n for $n = 2$ or 3, the methods of [3] seem to give somewhat weaker results about the continuity of continuous additive functionals of Brownian motion than the ones obtained in this paper. For example, the result in (1.22) is implied by the existence of the 1-potential, whereas the metric κ in (1.29) is a function of the 1-potential, and for the continuity of $L_t^{\mu_x}$ one must also have $I(\kappa, \mathscr{M}) < \infty$. (Pursuing this line, one is off by a factor of the square of a logarithm.) Also in some cases, such as those that occur in Section 7, in the study of moduli of continuity of measure-indexed continuous additive functionals, the metric κ is comparable only to $d^{1/2}$.

A slightly different version of the isomorphism theorem, Theorem 2.2, can be used to obtain the modulus of continuity of measure-indexed continuous additive functionals. Here are two examples of such results for Brownian motion in R^2 where, as above, $\{\mu_a,\ a \in R^2\}$ denotes the set of translates of a fixed finite measure μ on R^2.

Given a set $A \subset R^n$, let $\text{meas}_\alpha A$ denote the Hausdorff measure of A in dimension α. Let $\dim A$ denote the Hausdorff dimension of A. That is, $\dim A \equiv \sup\{\alpha | \text{meas}_\alpha A = \infty\}$. We define the index of a measure μ to be the supremum of the numbers θ such that

$$\sup_{x \in R^n} \mu(B(x,r)) \le Cr^\theta \qquad \forall\, r \le 1, \tag{1.30}$$

where $B(x,r)$ is a Euclidean ball at x of radius r. We note that by Frostman's lemma $\dim A = \beta$ if and only if A carries a finite measure with index β. (See, e.g., Chapter 10 of [8].)

THEOREM 1.7. *Let $B = \{B_t,\ t \in R^+\}$ be Brownian motion in R^2. Let $A \subset R^2$ have Hausdorff dimension β, $0 < \beta \le 2$. Then there exists a finite measure $\mu \in \text{Rev}(X)$, supported on A, for which*

$$\limsup_{\substack{|a-b| \to 0\\ a,b \in [0,1]^2}} \frac{L_t^{\mu_a} - L_t^{\mu_b}}{|a-b|^{(\beta/2)-\varepsilon}} = 0 \tag{1.31}$$

for almost all $t \in R^+$ almost surely for all $\varepsilon > 0$.

Equivalently, let μ be a measure with index β. Then μ is supported on a set A with $\dim A \ge \beta$ *and* (1.31) *holds.*

For the next theorem let us note that it follows from Hölder's inequality that, for Brownian motion in R^2, $U^1 f(x)$ is bounded if $f \in L^p$ for some $1 < p$.

THEOREM 1.8. *Let $B = \{B_t,\ t \in R^+\}$ be Brownian motion in R^2. Let μ be a finite measure on R^2 such that $\mu = f(x)\,dx$, where $f \in L^p$ for some $1 < p \le 2$. Then*

$$\limsup_{\substack{|a-b| \to 0\\ a,b \in [0,1]^2}} \frac{L_t^{\mu_a} - L_t^{\mu_b}}{|a-b|^{2-(2/p)} |\log|a-b||^{3/2}} \le C\|f\|_p \tag{1.32}$$

for almost all $t \in R^+$ *almost surely, where* C *is a finite constant which may depend on* p.

The Gaussian chaos $H(\mu)$ that we have been referring to is carefully defined in Section 2, in which we also give, in Theorems 2.1 and 2.2, two versions of the isomorphism theorem. These are more complicated than the versions given in [14], in which local times are associated with Gaussian processes. Viewing that case in the light of this paper, we see that the measures are point masses and, clearly, integration with respect to point masses is trivial. Here we must carry out the relevant integrations. Thus Section 2 in this paper does not follow easily from the material on the isomorphism theorem in [14].

Theorem 1.1 and the comments following it are proved in Section 3. In a brief Section 4 we show that when a Markov process has a local time, continuity of the local time and the associated Gaussian chaos are equivalent. Theorem 1.3, for processes in R^n, and Theorem 1.6 are proved in Section 5. In Section 6 we prove Theorem 1.3 for processes in T^n. We also prove Theorems 1.4 and 1.5, Corollary 1.1 and obtain an analog of Theorem 1.6 for Lévy processes in T^n. A concrete description of the Gaussian chaoses associated with continuous additive functionals of Lévy processes on T^n is also given. In Section 7 we briefly consider the moduli of continuity of continuous additive functionals of Markov processes and give the proofs of Theorems 1.7 and 1.8.

Throughout this paper C will denote a constant greater than 0 which is not necessarily the same at each occurrence. Also we use the notation $f(x) \sim g(x)$ as $x \to \infty$ to mean that there exist constants $0 < C_1, C_2 < \infty$ such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all $x \ge x_0$ for some x_0 sufficiently large, and similarly at 0.

2. The isomorphism theorem. In [14] we presented a proof of a version of Dynkin's isomorphism theorem that related the local time of a symmetric Markov process to a mean-zero Gaussian process, which had as its covariance the 1-potential density of the Markov process. In this paper we are interested in Markov processes which may not have local times but for which we can define continuous additive functionals determined by positive measures on the state space. In this section we prove a version of Dynkin's isomorphism theorem which relates these functionals to a Gaussian chaos on the space of measures $\mathscr{G}^2 = \mathscr{G}^2_0$, defined in (1.9). The argument that we give can be used for any $\alpha \ge 0$. However, to keep the notation from becoming too cumbersome, we carry out the argument in detail only in the case $\alpha = 0$.

To define the second-order Gaussian chaos H referred to in Section 1, we first consider the Gaussian process $\{G_\rho,\ \rho \in \mathscr{G}^1\}$, which has mean 0 and covariance

$$E_G(G_\rho G_\phi) = \iint u(x, y)\, d\rho(x)\, d\phi(y). \tag{2.1}$$

Let $\rho_{x,\delta}(dy) \equiv p_\delta(x, y)\, dm(y)$. It is easy to see, by assumption (1.5) for $\alpha = 0$, that $\rho_{x,\delta}(dy) \in \mathscr{G}^1$ for all $\delta > 0$ and $x \in S$ and that

$$E\big(G_{\rho_{x,\delta}} G_{\rho_{y,\delta'}}\big) = u_{\delta+\delta'}(x, y). \tag{2.2}$$

Let $G_{x,\delta} \equiv G_{\rho_{x,\delta}}$. Then, by (2.2) and Lemma 4.5 of [14], we have that

$$E\big(G_{x,\delta}^2 G_{y,\delta'}^2\big) = 2\big(u_{\delta+\delta'}(x, y)\big)^2 + u_{2\delta}(x, x) u_{2\delta'}(y, y). \tag{2.3}$$

Therefore

$$E\Big(\big(G_{x,\delta}^2 - E(G_{x,\delta}^2)\big)\big(G_{y,\delta'}^2 - E\big(G_{y,\delta'}^2\big)\big)\Big) = 2\big(u_{\delta+\delta'}(x, y)\big)^2. \tag{2.4}$$

In order to define the Gaussian chaos which occurs in the isomorphism theorem, we first consider a simpler class of Gaussian chaoses

$$H(\mu, \delta) = \int \big(G_{x,\delta}^2 - E\big(G_{x,\delta}^2\big)\big)\, d\mu(x) \tag{2.5}$$

for $\delta > 0$ and $\mu \in \mathscr{G}^2$. It follows from (2.4) that

$$E(H(\mu, \delta) H(\nu, \delta')) = 2\iint \big(u_{\delta+\delta'}(x, y)\big)^2\, d\mu(x)\, d\nu(y). \tag{2.6}$$

Therefore, for $\mu \in \mathscr{G}^2$, we have that

$$\lim_{\delta\to 0} H(\mu, \delta) \equiv H(\mu) \tag{2.7}$$

exists as a limit in L^2 and satisfies

$$E(H(\mu)) = 0 \tag{2.8}$$

and

$$E(H(\mu) H(\nu)) = 2\iint (u(x, y))^2\, d\mu(x)\, d\nu(y) \qquad \forall\, \mu, \nu \in \mathscr{G}^2. \tag{2.9}$$

Thus we see that

$$\begin{aligned} &\big(E(H(\mu) - H(\nu))^2\big)^{1/2} \\ &\quad = \Big(\iint (u(x, y))^2 (d(\mu(x) - \nu(x)))(d(\mu(y) - \nu(y)))\Big)^{1/2} \\ &\qquad\qquad \forall\, \mu, \nu \in \mathscr{G}^2. \end{aligned} \tag{2.10}$$

We explain why we call $H(\mu, \delta)$ and $H(\mu)$ Gaussian chaoses in Remark 2.1.

We continue to define the terms which appear in the isomorphism theorem. Let f and h be as given in (1.6). Since u is an excessive function in each variable, it follows that h is an excessive function and hence lower semicontinuous. Moreover, $h > 0$ and $1/h$ is locally bounded. For $g \in b\mathscr{S}$ we define

$$P_t^{(h)} g(x) = \frac{1}{h(x)} P_t(gh)(x). \tag{2.11}$$

It is easy to see that $P_t^{(h)}$ is a semigroup. It follows from Theorem 62.19 of [17] that there exists a unique Markov process $(\Omega, \mathscr{F}_t, X_t, P^{x/h})$, called the h-transform of X, with transition operators $P_t^{(h)}$, for which

$$P^{x/h}\big(F(\omega)1_{\{t<\zeta(\omega)\}}\big) = \frac{1}{h(x)}P^x\big(F(\omega)h(X_t(\omega))\big) \tag{2.12}$$

for all $F \in b\mathscr{F}_t$. Let $\rho \in \mathscr{G}^1$ be a compactly supported probability measure. As usual, we set

$$E^{\rho/h}(\cdot) = \int P^{x/h}(\cdot)\, d\rho(x). \tag{2.13}$$

For ν a measure and f a function on S, we denote by $f \cdot \nu$ the measure on S given by $f(x)\nu(dx)$, $x \in S$. Also, if g is a kernel on $S \times S$ and ν is a measure on S, then $g\nu(\cdot) \equiv \int g(\cdot, y)\nu(dy)$ and $gf\nu(\cdot) \equiv \int g(\cdot, y)f(y)\nu(dy)$. Let $\chi \equiv 1/h \cdot \rho$ and $\beta \equiv f \cdot m$. It is easy to verify that $\chi,\ \beta \in \mathscr{G}^1$.

The next theorem is contained in Theorem 6.1 of [5]. We give a more detailed proof for the convenience of the reader. In this theorem, for a given Markov process with 0-potential density $u(x, y)$ and measures $\mu \in \mathscr{G}^2$ with bounded potential, we consider $L_\infty^\mu = \lim_{t\to\infty} L_t^\mu$ and the associated Gaussian chaos $H(\mu)$.

THEOREM 2.1. *Let $\{\mu_i\}_{i=1}^\infty$ be a sequence of measures in $\mathscr{G}^2$ and assume that $U\mu_i$ is a bounded on S for all $1 \le i \le \infty$. Set $L_\infty^{\mu.} = (L_\infty^{\mu_1}, L_\infty^{\mu_2}, \dots)$ and $H(\mu.) = (H(\mu_1), H(\mu_2), \dots)$. Then, for any compactly supported $\rho \in \mathscr{G}^1$ and $\mathscr{C}$-measurable nonnegative function F on R^∞,*

$$E_G E^{\rho/h}\big(F\big(L_\infty^{\mu.} + \tfrac{1}{2}H(\mu.)\big)\big) = E_G\big(F\big(\tfrac{1}{2}H(\mu.)\big)G_\chi G_\beta\big), \tag{2.14}$$

where $\mathscr{C}$ denotes the σ-algebra generated by the cylinder sets of R^∞.

PROOF. This theorem is a generalization of the isomorphism theorem for Example 2 given in Section 4 of [14]. We explain how the proof of that theorem can be modified to prove this one. Our argument is meant to be read in conjunction with the material in [14]. According to (4.38) of [14], we have

$$\begin{aligned} &E_G\left(\prod_{i=1}^n\left(\frac{G_{u_i,\delta}G_{v_i,\delta}}{2}\right)G_\chi G_\beta\right) \\ &\quad= \frac{1}{2^n}\sum_{B\cup C=\{1,2,\dots,n\}}\left(\sum_{\mathscr{P}}\operatorname{cov}_\delta(B_1)\cdots\operatorname{cov}_\delta(B_{|B|})\right) \\ &\qquad\times\Big(\sum u_\delta\chi(y_{\pi(1)})u_{2\delta}(z_{\pi(1)}, y_{\pi(2)})\cdots u_\delta\beta(z_{\pi(|C|)})\Big), \end{aligned} \tag{2.15}$$

where the second sum is taken over the set of all possible pairings $\mathscr{P}$ of $\{u_i\}_{i\in B} \cup \{v_i\}_{i\in B}$. The specific pairs in $\mathscr{P}$ are denoted by $B_1, \dots, B_{|B|}$. If, for example, $B_i = (u_j, v_k)$, then $\operatorname{cov}_\delta(B_i) \equiv E(G_{u_j,\delta}G_{v_k,\delta})$. The last sum is taken over all permutations $(\pi(1), \dots, \pi(|C|))$ of C and over all ways of assigning $\{u_{\pi(i)}, v_{\pi(i)}\}$ to $\{y_{\pi(i)}, z_{\pi(i)}\}$. The explanation of what $\{y_{\pi(i)}, z_{\pi(i)}\}$ are and what we mean by "assigning" is given in the text immediately preceding (4.38) of [14].

For any pairing $\mathscr{P}$ of $\{u_i\}_{i\in B}\cup\{v_i\}_{i\in B}$ and subset $A\subseteq\{1,2,\dots,n\}$, let $\delta_{A,\mathscr{P}}=1$ or 0 depending on whether or not u_i is paired with v_i for all $i\in A$. Applying (2.15) to the simple identity

$$E_G\left(\prod_{i=1}^n\left(\frac{G_{u_i,\delta}G_{v_i,\delta}-E(G_{u_i,\delta}G_{v_i,\delta})}{2}\right)G_\chi G_\beta\right)$$
$$=\sum_{A\subseteq\{1,\dots,n\}}\left(\prod_{i\in A}(-1)^{|A|}E_G\left(\frac{G_{u_i,\delta}G_{v_i,\delta}}{2}\right)E_G\left(\prod_{i\in A^c}\left(\frac{G_{u_i,\delta}G_{v_i,\delta}}{2}\right)G_\chi G_\beta\right)\right),$$

we see that

$$E_G\left(\prod_{i=1}^n\left(\frac{G_{u_i,\delta}G_{v_i,\delta}-E(G_{u_i,\delta}G_{v_i,\delta})}{2}\right)G_\chi G_\beta\right)$$
$$\begin{aligned}(2.16)\qquad &=\frac{1}{2^n}\sum_{B\cup C=\{1,2,\dots,n\}}\sum_{A\subseteq B}(-1)^{|A|}\left(\sum_{\mathscr{P}}\delta_{A,\mathscr{P}}\operatorname{cov}_\delta(B_1)\cdots\operatorname{cov}_\delta(B_{|B|})\right)\\&\qquad\times\left(\sum u_\delta\chi(y_{\pi(1)})u_{2\delta}(z_{\pi(1)},y_{\pi(2)})\cdots u_\delta\beta(z_{\pi(|C|)})\right)\\&=\frac{1}{2^n}\sum_{B\cup C=\{1,2,\dots,n\}}\left(\sum_{\mathscr{P}'}{}'\operatorname{cov}_\delta(B_1)\cdots\operatorname{cov}_\delta(B_{|B|})\right)\\&\qquad\times\left(\sum u_\delta\chi(y_{\pi(1)})u_{2\delta}(z_{\pi(1)},y_{\pi(2)})\cdots u_\delta\beta(z_{\pi(|C|)})\right),\end{aligned}$$

where $\sum'_{\mathscr{P}'}$ is taken over the set of all possible pairings $\mathscr{P}'$ of $\{u_i\}_{i\in B}\cup\{v_i\}_{i\in B}$ such that u_i is not paired with v_i for any i. In the last step we used

$$\begin{aligned}\sum_{A\subseteq B}(-1)^{|A|}\sum_{\mathscr{P}}\delta_{A,\mathscr{P}}w(\mathscr{P})&=\sum_{\mathscr{P}}w(\mathscr{P})\sum_{A\subseteq B}(-1)^{|A|}\delta_{A,\mathscr{P}}\\&=\sum_{\mathscr{P}}w(\mathscr{P})\prod_{i\in B}(1-\delta_{\{i\},\mathscr{P}})=\sum_{\mathscr{P}'}{}'w(\mathscr{P}'),\end{aligned}$$

where $w(\mathscr{P})$ denotes any function of $\mathscr{P}$.

A similar analysis shows that

$$\begin{aligned}(2.17)\qquad &E_G\left(\prod_{i\in B}\left(G_{u_i,\delta}G_{v_i,\delta}-E(G_{u_i,\delta}G_{v_i,\delta})\right)\right)\\&\qquad=\left(\sum_{\mathscr{P}'}{}'\operatorname{cov}_\delta(B_1)\cdots\operatorname{cov}_\delta(B_{|B|})\right).\end{aligned}$$

Therefore if we set $u_i=v_i=x_i$ we get

$$E_G\left(\prod_{i=1}^n\left(\frac{G^2_{x_i,\delta}-E(G^2_{x_i,\delta})}{2}\right)G_\chi G_\beta\right)$$
$$\begin{aligned}(2.18)\qquad &=\sum_{B\cup C=\{1,2,\dots,n\}}E_G\left(\prod_{i\in B}\left(\frac{G^2_{x_i,\delta}-E(G^2_{x_i,\delta})}{2}\right)\right)\\&\qquad\times\left(\sum_{\pi(C)}u_\delta\chi(x_{\pi(1)})u_{2\delta}(x_{\pi(1)},x_{\pi(2)})\cdots u_\delta\beta(x_{\pi(|C|)})\right),\end{aligned}$$

where now the last sum is taken over all permutations π of C. Integrating with respect to the measures $\{\mu_i\}_{i=1}^n$ and recalling (2.5), we get

$$E_G\left(\prod_{i=1}^n\left(\frac{H(\mu_i,\delta)}{2}\right)G_\chi G_\beta\right)$$

$$= \sum_{B\cup C=\{1,2,\dots,n\}} E_G\left(\prod_{i\in B}\left(\frac{H(\mu_i,\delta)}{2}\right)\right) \tag{2.19}$$

$$\times\left(\sum_{\pi(C)}\int u_\delta\chi(x_{\pi(1)})u_{2\delta}(x_{\pi(1)},x_{\pi(2)})\cdots u_\delta\beta(x_{\pi(|C|)})\prod_{i\in C}d\mu_i(x_i)\right).$$

We now show that we can take the limit as $\delta\to 0$ in (2.19) to get

$$E_G\left(\prod_{i=1}^n\left(\frac{H(\mu_i)}{2}\right)G_\chi G_\beta\right)$$

$$= \sum_{B\cup C=\{1,2,\dots,n\}} E_G\left(\prod_{i\in B}\left(\frac{H(\mu_i)}{2}\right)\right) \tag{2.20}$$

$$\times\left(\sum_{\pi(C)}\int u\chi(x_{\pi(1)})u(x_{\pi(1)},x_{\pi(2)})\cdots u\beta(x_{\pi(|C|)})\prod_{i\in C}d\mu_i(x_i)\right).$$

To see this, we note that, since $H(\mu_i,\delta)\to H(\mu_i)$ in L^2, there exists a sequence $\delta_j\to 0$ such that $H(\mu_i,\delta_j)\to H(\mu_i)$ almost surely and $EH^2(\mu_i,\delta_j)\le CEH^2(\mu_i)$ for some constant C, for all $1\le i\le n$. Also, for all integers m, by the hypercontractivity of the Gaussian chaos (see e.g., [1]),

$$\begin{aligned}\left(E_G H^m(\mu_i,\delta_j)\right)^{1/m} &\le C_m\left(E_G H^2(\mu_i,\delta_j)\right)^{1/2}\\ &\le C'_m\left(E_G H^2(\mu_i)\right)^{1/2}.\end{aligned} \tag{2.21}$$

It follows from (2.21) and multiple uses of the Schwarz inequality that the terms involving the Gaussian chaoses in (2.19) are uniformly integrable and we can take the limits as $\delta_j\to 0$. This also shows that the limit as $\delta_j\to 0$ of the last integral in (2.19) exists and since it is monotonically increasing it is equal to the last integral in (2.20). Also, since the terms involving the Gaussian chaoses are bounded, the integrals in (2.20) can be seen to be bounded by induction on n. However, it is easy to see this directly since h and $U\mu_i$, $1\le i<\infty$, are bounded and ρ has compact support.

We will show below that

$$E^{x/h}\left(\prod_{i\in C}L_\infty^{\mu_i}\right)$$

$$= \sum_{\pi(C)}\int\frac{1}{h(x)}u(x,x_{\pi(1)})u(x_{\pi(1)},x_{\pi(2)})\cdots u\beta(x_{\pi(|C|)})\prod_{i\in C}d\mu_i(x_i). \tag{2.22}$$

Integrating (2.22) with respect to ρ gives

$$E^{\rho/h}\Big(\prod_{i\in C} L_\infty^{\mu_i}\Big) = \sum_{\pi(C)} \int u\chi(x_{\pi(1)})u(x_{\pi(1)}, x_{\pi(2)}) \cdots u\beta(x_{\pi(|C|)}) \prod_{i\in C} d\mu_i(x_i). \tag{2.23}$$

Using (2.23) in (2.20), we see that

$$\begin{aligned} E_G\left(\prod_{i=1}^n \left(\frac{H(\mu_i)}{2}\right) G_\chi G_\beta\right) &= \sum_{B\cup C=\{1,2,\dots,n\}} E_G\left(\prod_{i\in B}\left(\frac{H(\mu_i)}{2}\right)\right) \times E^{\rho/h}\Big(\prod_{i\in C} L_\infty^{\mu_i}\Big) \\ &= E_G E^{\rho/h}\left(\prod_{i=1}^n \left(L_\infty^{\mu_i} + \frac{H(\mu_i)}{2}\right)\right). \end{aligned} \tag{2.24}$$

This is equivalent to (4.34) of [14] in the case of Example 2, Section 4 of [14]. Theorem 1.5 now follows from the last paragraph of the proof of Theorem 4.1 of [14].

To obtain (2.22), consider (4.28) of [14] with $L_\infty^{y_i}$ replaced by $L_\infty^{\mu_i}$. Thus H_n in [14] becomes

$$\begin{aligned} H_n &= H_n(\mu_1, \dots, \mu_n) \\ &= \int_0^\infty dL_{r_1}^{\mu_1} \int_{r_1}^\infty dL_{r_2}^{\mu_2} \cdots \int_{r_{n-1}}^\infty dL_{r_n}^{\mu_n}. \end{aligned}$$

To obtain the analog of (4.30) and (4.31) of [14], follow the argument of (4.30) of [14] and use Theorem 3.1, Chapter 6 of [4] to get

$$\begin{aligned} E^{x/h}(L_\infty^{\mu_1}) &= \frac{1}{h(x)} E^x\left(\int_0^\infty h(X_r)\, dL_r^{\mu_1}\right) \\ &= \frac{Uh\mu_1(x)}{h(x)} \\ &= \frac{1}{h(x)} \int u(x, x_1) u\beta(x_1)\, d\mu_1(x_1), \end{aligned} \tag{2.25}$$

since $u\beta(\cdot) = h(\cdot)$. Continuing a proof by induction, we now assume that

$$E^{x/h}(H_{n-1}) = \int \frac{1}{h(x)} u(x, x_1) u(x_1, x_2) \cdots u\beta(x_{n-1}) \prod_{i=1}^{n-1} d\mu_i(x_i) \tag{2.26}$$

and let $H_{n-1,2} \equiv H_{n-1}(\mu_2, \dots, \mu_n)$. [Note that H_n in the line above (4.32) of [14] should be H_{n-1}.] Following (4.33) of [14], we get

$$E^{x/h}(H_n) = \frac{1}{h(x)} E^{x/h}\left(\int_0^\infty E^{X_{r_1}/h}(H_{n-1,2})\, dL_{r_1}^{\mu_1}\right), \tag{2.27}$$

and proceeding as in (2.25), we get

$$(2.28)\quad E^{x/h}(H_n) = \frac{1}{h(x)} \int u(x, x_1) h(x_1) E^{x_1/h}(H_{n-1,2})\, d\mu_1(x_1).$$

Using (2.26) in (2.28), we get (2.22). This completes the proof of Theorem 2.1. □

Let $\mathscr{M}$ be a family of measures contained in $\mathscr{G}_1^2 \cap \mathrm{Rev}(X)$. We present a slightly different version of the isomorphism theorem to use in studying the moduli of continuity of $L = \{L_t^{\mu}, (t, \mu) \in R^+ \times \mathscr{M}\}$. Let λ be an exponential random variable with mean α, which is independent of the Markov process X. We consider L_λ^μ, which is simply L_t^μ, with t replaced by the independent stopping time λ. We associate with L a second-order Gaussian chaos $\{H_\alpha(\mu), \mu \in \mathscr{M}\}$ which is defined exactly the same way as $H(\mu)$ was in the beginning of this section, except that in place of $u(x, y)$ in (2.1) we use $u^\alpha(x, y)$.

THEOREM 2.2. *Let f be a positive function on S such that $f \cdot m \in \mathscr{G}_\alpha^1$. Assume that $U^1\mu$ is bounded for each $\mu \in \mathscr{M}$ and that L_λ^μ and $H_\alpha(\mu)$ are both in $C(\mathscr{M})$, the set of continuous functions on some compact subset $\mathscr{M} \subseteq \mathscr{G}_\alpha^2$. Then, for any compactly supported $\rho \in \mathscr{G}_\alpha^1$ and any nonnegative Borel-measurable function F on $C(\mathscr{M})$, we have*

$$(2.29)\quad E_G E^\rho E_\lambda\big(F\big(L_\lambda^\mu + \tfrac{1}{2}H_\alpha(\mu)\big)f(X_\lambda)\big) = E_G\big(F\big(\tfrac{1}{2}H_\alpha(\mu)\big)G_\rho G_{f\cdot m}\big).$$

PROOF. The proof is similar to the proof of the isomorphism theorem for Example 1 given in Section 4 of [14]. □

REMARK 2.1. We explain why we call $H(\mu, \delta)$ and $H(\mu)$ Gaussian chaoses. Let T be some index set and let $\{g_n\}_{n=1}^\infty$ be independent, identically distributed normal random variables with mean 0 and variance 1. We say that a stochastic process $\{\chi(t), t \in T\}$ is a second-order Gaussian chaos if it can be written in the form

$$(2.30)\qquad \chi(t) = \sum_{j \neq k} g_j g_k \varphi_{j,k}(t) + \sum_j \big(g_j^2 - 1\big)\varphi_{j,j}(t), \qquad t \in T,$$

where we assume that the series converges in L^2 for each $t \in T$. Since we will only be concerned with second-order chaoses, we will not bother to repeat the words "second order" when discussing them. $H(\mu, \delta)$, defined in (2.5), is a Gaussian chaos on $\mathscr{G}^2$. To see this, we note that the Gaussian process $\{G_\rho, \rho \in \mathscr{G}^1\}$ can be written in terms of its Karhunen–Loève expansion. Therefore, in particular, we can consider the Gaussian process

$$(2.31)\qquad G_{x,\delta} = \sum_j g_j \varphi_j(\rho_{x,\delta}), \qquad x \in S.$$

Referring to (2.5), we see that

$$H(\mu, \delta) = \sum_{j \neq k} g_j g_k \int \varphi_j(\rho_{x,\delta}) \varphi_k(\rho_{x,\delta})\, d\mu(x) \tag{2.32}$$
$$+ \sum_j \left(g_j^2 - 1\right) \int \varphi_j^2(\rho_{x,\delta})\, d\mu(x).$$

Since the L^2 limit of a Gaussian chaos is a Gaussian chaos, by Theorem 3.1 of [1], we see that $H(\mu)$ is a Gaussian chaos. In fact, $\{H(\mu),\ \mu \in \mathscr{G}^2\}$ has an expansion as in (2.30) for $\mu \in \mathscr{G}^2$, although we do not know what it is explicitly.

Note that in [1] a Gaussian chaos is defined as the closure in L^2 of expressions of the form (2.30). Here we are using the definition of Gaussian chaos given in [9]. It follows from Theorem 3.1 of [1] that the two definitions are equivalent.

REMARK 2.2. The continuity condition for Gaussian chaoses given in Theorem 1.2 is contained in Theorem 11.22 of [9]. To see this, it is only necessary to note that the metric d_2 is smaller than the metric d_1 (in the notation of [9]).

3. Continuity theorems.

In this section we give the proofs of Theorems 1.1 and its refinements, Theorems 3.1 and 3.2.

PROOF OF THEOREM 1.1. We first consider the case $\alpha = 0$. Recall that we are denoting h_0, u^0, U^0, $\mathscr{G}_0^1$, $\mathscr{G}_0^2$ and so on by h, u, U, $\mathscr{G}^1$, $\mathscr{G}^2$ and so forth. Let us also recall that, by (2.13) and (2.25),

$$E^{x/h}(L_\infty^\mu) = \frac{Uh\mu(x)}{h(x)}, \tag{3.1}$$

and, for $\rho \in \mathscr{G}^1$ a probability measure with compact support,

$$P^{\rho/h}(\cdot) = \int P^{x/h}(\cdot)\, d\rho(x).$$

By working locally it suffices to consider $\mathscr{M}$ compact. Let $\mathscr{D} \subseteq \mathscr{M}$ be a countable dense set. Using the proof of Theorem 6.1 of [14] together with a version of Dynkin's isomorphism theorem, Theorem 2.1 of this paper, we can see that $\{L_t^\mu,\ (t, \mu) \in \overline{R}^+ \times \mathscr{D}\}$ is uniformly continuous almost surely with respect to $P^{\rho/h}$, where $\overline{R}^+ = [0, \infty]$ is the compactification of R^+ obtained by adding the point at ∞. (Thus, e^{-x}: $\overline{R}^+ \mapsto [0,1]$ is an isomorphism of compact sets.)

Let

$$\tilde{\Omega} = \left\{\omega | L_t^\mu(\omega) \text{ is uniformly continuous on } \overline{R}^+ \times \mathscr{D}\right\}. \tag{3.2}$$

We have that

$$\int P^{x/h}(\tilde{\Omega})\, d\rho(x) = P^{\rho/h}(\tilde{\Omega}) = 1 \tag{3.3}$$

for all finite measures $\rho \in \mathscr{G}^1$ with compact support.

Let $Q = \{x | P^{x/h}(\tilde{\Omega}) < 1\}$. By (3.3) and standard arguments (see the discussion on page 285 of [4]), we see that Q is a polar set.

We henceforth restrict our Markov process and measures to $S - Q$, noting that under our assumptions the measures in $\mathscr{M}$ do not charge polar sets. Thus, in effect, we are considering a Markov process defined on $\overline{S} = S - Q$ for which

$$P^{x/h}(\tilde{\Omega}) = 1 \qquad \forall\, x \in \overline{S}. \tag{3.4}$$

We then extend $\{L_t^\mu, (t, \mu) \in \overline{R}^+ \times \mathscr{D}\}$ to $\{\tilde{L}_t^\mu, (t, \mu) \in \overline{R}^+ \times \mathscr{M}\}$ by continuity. Since we may assume that the L_t^μ are perfect continuous additive functionals for all $\mu \in \mathscr{D}$, we immediately see that the same is true for $\tilde{L}_t^\mu$ for each $\mu \in \mathscr{M}$. We now show that $\tilde{L}_t^\mu$ is a version of L_t^μ for the h-transformed process. For this, by Theorem 36.3 of [17], it suffices to show that $\tilde{L}_t^\mu$ has the same potential as L_t^μ, that is, that

$$E^{x/h}\left(\tilde{L}_\infty^\mu\right) = \frac{Uh\mu(x)}{h(x)} \qquad \forall\, x \in \overline{S}. \tag{3.5}$$

By the definition of $\tilde{L}_t^\mu$, (3.5) holds for all $\mu \in \mathscr{D}$, and, for any $\mu \in \mathscr{M}$, if we choose a sequence $\{\mu_i\}_{i=1}^\infty$ of measures in $\mathscr{D}$ such that $\mu_i \to \mu$, then $\tilde{L}_\infty^{\mu_i} \to \tilde{L}_\infty^\mu$ almost surely. Since, by assumption (i) of this theorem, $Uh\mu_i(x) \to Uh\mu(x)$, we can complete the proof of (3.5) by showing that $\{\tilde{L}_\infty^{\mu_i}\}_{i=1}^\infty$ are uniformly integrable. For this it suffices to show that $\{\tilde{L}_\infty^{\mu_i}\}_{i=1}^\infty$ are uniformly bounded in L^2 with respect to $P^{x/h}$. This is easily seen since by (2.22) we have

$$E^{x/h}\left(\left(\tilde{L}_\infty^{\mu_i}\right)^2\right) = \frac{2}{h(x)} U((Uh\mu_i)\mu_i)(x), \tag{3.6}$$

which is uniformly bounded in i by assumption (i).

Redefine $\tilde{L}_t^\mu(\omega)$ by setting it equal to

$$\liminf_{\substack{s \uparrow \zeta(\omega) \\ s \text{ rational}}} \tilde{L}_s^\mu(\omega) \tag{3.7}$$

for all $t \geq \zeta(\omega)$. As in the proof of Theorem 6.1 of [14], we see that $\tilde{L}_t^\mu$ and L_t^μ agree on $[0, \zeta)$, P^x almost surely. We now see that the above limit inferior is a true limit and that $\tilde{L}_t^\mu(\omega) = L_t^\mu(\omega)$ for all $t \in R^+$. Therefore $\tilde{L}_t^\mu$ is a version of L_t^μ for the Markov process X. Finally, as in the proof of Theorem 6.1 of [14], we see that the $P^{x/h}$ almost sure continuity of $\{\tilde{L}_t^\mu, (t, \mu) \in \overline{R}^+ \times \mathscr{M}\}$ implies the P^x almost sure continuity of $\{\tilde{L}_t^\mu, (t, \mu) \in [0, \zeta) \times \mathscr{M}\}$. This completes the proof in the case $\alpha = 0$.

We now give the proof for $\alpha = 1$. The proof for any other $\alpha > 0$ is similar. Let X be a Markov process which satisfies the hypotheses of this theorem in the case $\alpha = 1$. Let Y be the Markov process obtained by killing X at an independent exponential time λ with mean 1. Let $(\Omega, \mathscr{F}, P^x)$ be the probability space of X. As usual, we take $(\Omega \times R^+, \mathscr{F} \times \mathscr{B}, P^x \times \tau)$, where $\tau(dt) = e^{-t}\,dt$, to be the probability space for Y, where $Y_t(\omega, \lambda) = X_t(\omega)$ for $t < \lambda$ and $Y_t(\omega, \lambda) = \Delta$, the cemetary state, for $t \geq \lambda$. It is easy to check that the 0-potential density of Y and the 1-potential density of X are equal and that Y satisfies the hypotheses of this theorem with $\alpha = 0$. Furthermore, we can check that if L_t^μ is a continuous additive functional of X with 1-potential $U^1\mu$, then $L_{t\wedge\lambda}^\mu$ is a continuous additive functional of Y with 0-potential $U^1\mu$.

We have already proved this theorem in the case $\alpha = 0$. Therefore, for

$$V = \left\{(\omega, \lambda) \in \Omega \times R^+ | L_{t\wedge\lambda}^\mu(\omega) \text{ is uniformly continuous on } \overline{R}^+ \times \mathscr{D}\right\},$$

we have

$$(P^{x/h} \times \tau)(V) = 1 \qquad \forall\, x \in S - Q \tag{3.8}$$

for some polar set $Q \subseteq S$. We restrict X, and consequently Y, and the measures $\mathscr{M}$ to $S - Q$ and thus can consider that both X and Y have state space $\overline{S} = S - Q$.

Let

$$\hat{\Omega} = \{\omega | L_t^\mu(\omega) \text{ is locally uniformly continuous on } R^+ \times \mathscr{D}\}$$

and note that $\hat{\Omega} \times R^+ \subseteq V$. Fubini's theorem and a monotonicity argument now show that

$$P^{x/h}(\hat{\Omega}) = 1 \qquad \forall\, x \in \overline{S}. \tag{3.9}$$

From the definition of $\hat{\Omega}$, we see that, for each $\omega \in \hat{\Omega}$, $\{L_t^\mu(\omega), (t, \mu) \in R^+ \times \mathscr{D}\}$ can be extended to a locally uniformly continuous stochastic process $\{\hat{L}_t^\mu(\omega), (t, \mu) \in R^+ \times \mathscr{M}\}$. Set $\hat{L}_t^\mu(\omega) = 0$ for $\omega \in \hat{\Omega}^c$. By taking the limit over sequences of measures in $\mathscr{D}$, we see that $\hat{L}_t^\mu$ is a continuous additive functional for each $\mu \in \mathscr{M}$.

We now show that $\hat{L}_t^\mu$ has 1-potential $(U^1 h\mu(x))/h(x)$ with respect to the h-transform of X and consequently is a continuous version of $\{L_t^\mu(\omega), (t, \mu) \in R^+ \times \mathscr{M}\}$. According to the proof in the case $\alpha = 0$ applied to the h-transform of Y, $\{L_{t\wedge\lambda}^\mu(\omega), (t, \mu) \in \overline{R}^+ \times \mathscr{D}\}$ has a continuous extension to $\overline{R}^+ \times \mathscr{M}$ for all $(\omega, \lambda) \in \hat{\Omega} \times R^+$. We denote this extension by $\{\tilde{L}_t^\mu(\omega, \lambda), (t, \mu) \in \overline{R}^+ \times \mathscr{M}\}$ and recall that it has 0-potential $(U^1 h\mu(x))/h(x)$. Integrating by parts and using Fubini's theorem, we have

$$\begin{aligned} E^{x/h}\left(\int_0^\infty e^{-s}\, d\tilde{L}_\infty^\mu(\omega, s)\right) &= E^{x/h}\left(\int_0^\infty e^{-s} \tilde{L}_\infty^\mu(\omega, s)\, ds\right) \\ &= (E^{x/h} \times \tau)\big(\tilde{L}_\infty^\mu(\omega, \lambda)\big) \\ &= \frac{U^1 h\mu(x)}{h(x)}. \end{aligned} \tag{3.10}$$

Clearly, $L_\infty^\mu(\omega, s) = \hat{L}_s^\mu(\omega)$ for all $\omega \in \hat{\Omega}$ and $s \in R^+$, since both sides are continuous extensions of $L_{\infty\wedge s}^\mu(\omega) = L_s^\mu(\omega)$ for $\mu \in \mathscr{D}$. Thus we see that $\hat{L}_s^\mu$ has 1-potential $(U^1 h\mu(x))/h(x)$ for the h-transform of X. The transition from the h-transformed process to the original process is the same as in the case $\alpha = 0$. This completes the proof of Theorem 1.1. $\square$

The next theorem relates to Remark 1.2.

THEOREM 3.1. *Let X be a Markov process as in Theorem* 1.1 *and let $\mathscr{M} \subseteq \mathscr{G}_\alpha^2$ be a set of measures with common compact support. Assume that we are given a topology $\mathscr{O}$ for $\mathscr{M}$ under which $\mathscr{M}$ is locally compact and has a countable base. Assume also that*:

(i) *$\mu \mapsto U^\alpha h_{(\alpha)} \mu$ is a continuous map from $\mathscr{M}$ to $\mathscr{B}(S)$;*

(ii) *the associated second-order Gaussian chaos $H_\alpha(\mu)$ is continuous almost surely on $\mathscr{M}$.*

Then there exists a polar set $Q \subseteq S$ such that, if we restrict X and $\mathscr{M}$ to $S - Q$, we can find a continuous version of $\{L_t^\mu, (t, \mu) \in R^+ \times \mathscr{M}\}$.

PROOF. The condition that $\mu \mapsto U^\alpha \mu$ is a continuous map from $\mathscr{M}$ to $\mathscr{B}(S)$ of Theorem 1.1 was used only to enable us to satisfy the requirement of the isomorphism theorem that $U^\alpha \mu$ is bounded for each $\mu \in \mathscr{M}$ and to obtain upper bounds in (3.6). Actually, we only used the weaker condition:

(i′) $\mu \mapsto U^\alpha h_{(\alpha)} \mu$ is a continuous map from $\mathscr{M}$ to $\mathscr{B}(S)$ and $\mu \mapsto U^\alpha \mu$ is a bounded map from $\mathscr{C}$ to $\mathscr{B}(S)$ for all compact sets $\mathscr{C} \subset \mathscr{M}$; that is, $\sup_{\mu \in \mathscr{C}} \sup_{x \in S} U^1 \mu(x) < \infty$.

The first condition of (i′) implies that $\sup_{\mu \in \mathscr{C}} \sup_{x \in S} U^\alpha h_{(\alpha)} \mu(x) < \infty$. Then the second condition in (i′) follows from this, since $\inf_K h_{(\alpha)}(x) > 0$, where $K \subseteq S$ denotes a compact neighborhood containing the supports of all the measures $\mu \in \mathscr{M}$. The rest of the proof follows from the proof of Theorem 6.3 of [14] and the fact that none of the continuous additive functionals L_t^μ are increasing unless $X_t \in K$. (See Chapter 6, Theorem 3.1, of [4].) $\square$

We now develop the material to explain Remark 1.1. Let Γ be a separable locally compact group and let X be a Lévy process in Γ, with α-potential density $u^\alpha(x, y) = u^\alpha(xy^{-1})$. We will use the canonical representation for X in which Ω is the set of cadlag paths ω: $R^+ \mapsto \Gamma$, $X_t = \omega(t)$ and

$$E^x(f(\omega)) = E^0(f(\omega x)).$$

For these processes L_t^μ denotes the continuous additive functional of X with α-potential $U^\alpha \mu(x) = \int u^\alpha(xy^{-1})\, d\mu(y)$. For each measure μ on Γ and $x \in \Gamma$, we define the measure μ_x to be the unique measure on Γ for which

$$\int g(z)\, d\mu_x(z) = \int g(zx^{-1})\, d\mu(z)$$

for all bounded continuous functions g on Γ. Note that $(\mu_x)_y = \mu_{yx}$ and $\mu(Ax) = \mu_x(A)$ for all Borel sets $A \subset \Gamma$. Let T_x denote the bijection on the space of measures defined by $T_x(\mu) = \mu_x$. We say that a set $\mathscr{M}$ of measures on Γ is translation invariant if it is invariant under T_x for each $x \in \Gamma$ and that a topology $\mathscr{O}$ on such a set $\mathscr{M}$ is homogeneous if T_x is an isomorphism for each $x \in \Gamma$.

The next theorem relates to Remark 1.1.

THEOREM 3.2. *Let X be a symmetric Lévy process in Γ and let $\mathscr{M} \subseteq \mathscr{G}_\alpha^2$ be a translation-invariant set of measures on Γ. Assume that there is a homogeneous topology $\mathscr{O}$ for $\mathscr{M}$ under which $\mathscr{M}$ is locally compact and has a countable base and that*:

(i) $\mu \mapsto U^\alpha \mu$ *and* $\mu \mapsto U^\alpha h_{(\alpha)} \mu$ *are continuous maps from* $\mathscr{M}$ *to* $\mathscr{B}(\Gamma)$;
(ii) *the associated second-order Gaussian chaos* $H_\alpha(\mu)$ *is continuous almost surely on* $\mathscr{M}$.

Then there exists a continuous version of $\{L_t^\mu, (t, \mu) \in R^+ \times \mathscr{M}\}$.

PROOF. We give the proof in the case $\alpha = 1$. The same proof is valid for all $\alpha > 0$ and also for $\alpha = 0$ for transient processes. For any $x \in \Gamma$ and $\mu \in \mathscr{M}$, set

$$A_t(\omega) = L_t^\mu(\omega x).$$

Clearly, A_t is a continuous additive functional. Computing its 1-potential

$$\begin{aligned} E^y\left(\int_0^\infty e^{-t}\, dA_t(\omega)\right) &= E^y\left(\int_0^\infty e^{-t}\, dL_t^\mu(\omega x)\right) \\ &= E^{yx}\left(\int_0^\infty e^{-t}\, dL_t^\mu(\omega)\right) \\ &= \int u^1(yxz^{-1})\, d\mu(z) \qquad (3.11) \\ &= \int u^1\left(y(zx^{-1})^{-1}\right) d\mu(z) \\ &= \int u^1(yz^{-1})\, d\mu_x(z), \end{aligned}$$

we see that

$$L_t^\mu(\omega x) = L_t^{\mu_x}(\omega) \quad \text{a.s.} \tag{3.12}$$

for each $x \in \Gamma$ and $\mu \in \mathscr{M}$.

Let $\mathscr{D} \subseteq \mathscr{M}$ be the countable dense set of measures that enters into the proof of Theorem 1.1. The proof of Theorem 1.1 shows that $\{L_t^\mu, (t, \mu) \in R^+ \times \mathscr{D}\}$ is locally uniformly continuous, P^a almost surely, for all $a \in Q^c$, where $Q \in \Gamma$ is a polar set. As in Theorem 1.1, we extend $L_t^\mu(\omega)$ by continuity to $\{\tilde{L}_t^\mu(\omega), (t, \mu) \in R^+ \times \mathscr{M}\}$ for all paths ω starting in Q^c. However, this procedure provides no way to extend $\{\tilde{L}_t^\mu, (t, \mu) \in R^+ \times \mathscr{M}\}$ to paths ω starting in Q. That is why, in Theorem 1.1, we found it necessary to restrict the Markov process to Q^c. In this theorem we use the translation invariance of $\mathscr{M}$ to extend $\{\tilde{L}_t^\mu, (t, \mu) \in R^+ \times \mathscr{M}\}$ to paths ω starting in Q.

We can assume that (3.12) holds for all $\mu \in \mathscr{D}$, almost surely. Therefore, by continuity, we have that

$$\tilde{L}_t^{\mu}(\omega x) = \tilde{L}_t^{\mu_x}(\omega) \qquad \forall\, \mu \in \mathscr{M}, \quad P^y \text{ a.s.} \tag{3.13}$$

for each $x \in \Gamma$ and $y \in Q^c$ such that $yx \in Q^c$. This suggests how we can extend $\{\tilde{L}_t^{\mu}, (t, \mu) \in R^+ \times \mathscr{M}\}$ to paths ω starting in Q. Fix $a \in Q^c$ and for each path ω starting at a and each $y \in Q$ set

$$\tilde{L}_t^{\mu}(\omega a^{-1} y) =_{\text{def}} \tilde{L}_t^{\mu_{a^{-1}y}}(\omega). \tag{3.14}$$

We note that $\omega \mapsto \omega a^{-1} y$ is a bijection from the set of cadlag paths in Γ starting at a to those paths starting at y.

We must verify that with this definition $\{\tilde{L}_t^{\mu}, (t, \mu) \in R^+ \times \mathscr{M}\}$ satisfies the requirements of this theorem, that is, it is continuous almost surely, and that, for each $\mu \in \mathscr{M}$, $\tilde{L}_t^{\mu}$ is a continuous additive functional of X with 1-potential $U^1\mu$.

We first show that $\{\tilde{L}_t^{\mu}, (t, \mu) \in R^+ \times \mathscr{M}\}$ are continuous additive functionals for the Lévy process X. The only part requiring proof is the additivity:

$$\tilde{L}_{t+s}^{\mu}(\omega) = \tilde{L}_t^{\mu}(\omega) + \tilde{L}_s^{\mu}(\theta_t \omega), \quad P^y \text{ a.s.} \tag{3.15}$$

for all $y \in \Gamma$ and $\mu \in \mathscr{M}$.

If $y \in Q^c$, (3.15) follows by continuity since it holds for all $\mu \in \mathscr{D}$. Note that, since Q is a polar set, $\theta_t \omega(0) = \omega(t) \in Q^c$, P^y almost surely.

Consider now that $y \in Q$. By definition (3.14) we have that, P^a almost surely,

$$\begin{aligned} \tilde{L}_{t+s}^{\mu}(\omega a^{-1} y) &= \tilde{L}_{t+s}^{\mu_{a^{-1}y}}(\omega) \\ &= \tilde{L}_t^{\mu_{a^{-1}y}}(\omega) + \tilde{L}_s^{\mu_{a^{-1}y}}(\theta_t \omega) \\ &= \tilde{L}_t^{\mu}(\omega a^{-1} y) + \tilde{L}_s^{\mu_{a^{-1}y}}(\theta_t \omega). \end{aligned} \tag{3.16}$$

Assume that (3.13) holds without restriction on $x \in \Gamma$, that is, that

$$\tilde{L}_t^{\mu}(\omega x) = \tilde{L}_t^{\mu_x}(\omega) \qquad \forall\, \mu \in \mathscr{M} \quad P^y \text{ a.s.} \tag{3.17}$$

for each $x \in \Gamma$ and $y \in Q^c$. Then, using the Markov property, we see that

$$\begin{aligned} &P^a\big(\tilde{L}_s^{\mu}(\theta_t \omega a^{-1} y) = \tilde{L}_s^{\mu_{a^{-1}y}}(\theta_t \omega)\big) \\ &\quad = P^a\big(P^{X_t}\big(\tilde{L}_s^{\mu}(\omega a^{-1} y) = \tilde{L}_s^{\mu_{a^{-1}y}}(\omega)\big)\big) \\ &\quad = 1. \end{aligned} \tag{3.18}$$

The final equality follows from (3.17), since, by the polarity of Q, we have $X_t \in Q^c$, P^a almost surely. Using (3.16) and (3.18), we see that

$$\tilde{L}_{t+s}^{\mu}(\omega a^{-1} y) = \tilde{L}_t^{\mu}(\omega a^{-1} y) + \tilde{L}_s^{\mu}(\theta_t \omega a^{-1} y), \quad P^a \text{ a.s.}, \tag{3.19}$$

which is equivalent to (3.15).

To show that $\tilde{L}_t^{\mu}$ has 1-potential $U^1\mu$, we note that, by Theorem 1.1,

$$E^y\left(\int_0^{\infty} e^{-t}\, d\tilde{L}_t^{\mu}(\omega)\right) = U^1\mu(y) \tag{3.20}$$

for all $y \in \Gamma - Q$, and, since both sides of (3.20) are excessive functions, they will be equal for all $y \in \Gamma$ by Theorem 3.2, Chapter 2 of [4]. [Alternately, we can use the calculation in (3.11).] Again, by Theorem 1.1, since $\zeta = \infty$ for Lévy processes, $\{\tilde{L}_t^{\mu}, (t, \mu) \in R^+ \times \mathscr{M}\}$ is jointly continuous, P^y almost surely, for all $y \in \Gamma - Q$ and hence for all $y \in \Gamma$ by definition (3.14) and the fact that T_x is an isomorphism for each $x \in \Gamma$.

We now return to the proof of (3.17). If $yx \in Q^c$, this is precisely the content of (3.13). If $yx \in Q$, then by (3.14) we have

$$\tilde{L}_t^{\mu}(\omega a^{-1} yx) = \tilde{L}_t^{\mu_{a^{-1}yx}}(\omega), \quad P^a \text{ a.s.} \tag{3.21}$$

Now, since both $a \in Q^c$ and $a(a^{-1}y) = y \in Q^c$, it follows from (3.13) that

$$\tilde{L}_t^{\mu_x}(\omega a^{-1} y) = \tilde{L}_t^{(\mu_x)_{a^{-1}y}}(\omega) = \tilde{L}_t^{\mu_{a^{-1}yx}}(\omega), \quad P^a \text{ a.s.} \tag{3.22}$$

Combining (3.21) and (3.22), we obtain

$$\tilde{L}_t^{\mu}(\omega a^{-1} yx) = \tilde{L}_t^{\mu_x}(\omega a^{-1} y), \quad P^a \text{ a.s.},$$

which is equivalent to (3.17). □

4. Local times. In this section we explain how local times fit into our framework. Suppose that the Markov process X has a local time. Consider the set of measures $\tilde{\mathscr{M}} = \{\delta_a, a \in S\}$, where δ_a denotes the unit point mass at a. We set $L_t^{\delta_a} = L_t^a$ and note that it is the ordinary local time of X at a. Let $\{H_1(\delta_a), \delta_a \in \tilde{\mathscr{M}}\}$ be the associated Gaussian chaos. When a Markov process has a local time, continuity of the local time and the associated Gaussian chaos are equivalent, and we obtain the results discussed in Remark 1.5.

Let

$$T_\sigma(d, \mathscr{M}) =_{\text{def}} \lim_{\eta \to 0} \sup_{t \in \mathscr{M}} \int_0^\eta \left(\log \frac{1}{\sigma(B_d(t, \varepsilon))} \right)^{1/2} d\varepsilon, \tag{4.1}$$

where d, σ, $\mathscr{M}$ and B are defined in Theorem 1.2.

THEOREM 4.1. *Let X be a Markov process as in Theorem* 1.1 *with state space S and assume that a local time exists for X at all points $a \in S$. Then the following are equivalent*:

(i) $\{L_t^a, a \in S\}$ *has a continuous version almost surely*;
(ii) $\{H_1(\delta_a), \delta_a \in \tilde{\mathscr{M}}\}$ *has a continuous version almost surely*;
(iii) *for all compact subsets $\mathscr{M}$ of $\tilde{\mathscr{M}}$, there exists a probability measure σ on $\mathscr{M}$ such $T_\sigma(d, \mathscr{M}) = 0$.*

Furthermore, if X is a Lévy process in R^n, or T^n, (iii) *can be replaced by*

(iii′) $J(d, \mathscr{M}) < \infty$ *for all compact subsets $\mathscr{M}$ of $\tilde{\mathscr{M}}$.*

PROOF. It is enough to prove this theorem for S compact. In this case we can take $\tilde{\mathscr{M}}$ to be compact and we denote it by $\mathscr{M}$. If X has a local time for all $x \in S$, then $u^1(x, x) < \infty$ for all $x \in S$. (See, e.g., Theorem 3.2 of [14].) In this case the Gaussian process $\{G(\delta_a), \delta_a \in \mathscr{M}\}$, defined in Section 2, is the same as the Gaussian process $\{G(a),\ a \in S\}$ with covariance $u^1(x,y)$. (As we remarked in Section 2 the construction can be carried out for all α.) It follows from Theorem 1 of [14] that $\{L_t^a,\ a \in S\}$ has a continuous version almost surely if and only if $\{G(a),\ a \in S\}$ is continuous. We see from the construction of the chaos $H_1(\delta_a)$ associated with $L_t^{\delta_a} = L_t^a$ that $H_1(\delta_a) = G^2(a) - EG^2(a)$. Thus, obviously, $\{H_1(\delta_a),\ \delta_a \in \mathscr{M}\}$ is continuous if and only if $\{G(a),\ a \in S\}$ is continuous. Thus we see that (i) and (ii) are equivalent.

A necessary and sufficient condition for the continuity of $\{G(a),\ a \in S\}$ is that there exists a probability measure σ on S such that $T_\sigma(\rho, S) = 0$, where

$$\rho(x,y) = \left(u^1(x,x) + u^1(y,y) - 2u^1(x,y)\right)^{1/2}. \tag{4.2}$$

For the measures in $\mathscr{M}$, the metric defined in (1.12) is

$$\begin{aligned} d(\delta_x, \delta_y) &= \left(\left(u^1(x,x)\right)^2 + \left(u^1(y,y)\right)^2 - 2\left(u^1(x,y)\right)^2\right)^{1/2} \\ &\le \sqrt{2}\left(\left(u^1(x,x) - u^1(x,y)\right)u^1(x,x)\right. \\ &\qquad \left. + \left(u^1(y,y) - u^1(x,y)\right)u^1(y,y)\right)^{1/2} \\ &\le \sqrt{2}\sup_{x\in S}\left(u^1(x,x)\right)^{1/2}\rho(x,y) \\ &\le C_1\,\rho(x,y). \end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned} d(\delta_x, \delta_y) &\ge \inf_{x \in S}\left(u^1(x,x)\right)^{1/2}\rho(x,y) \\ &\ge C_2\,\rho(x,y). \end{aligned}$$

We know that $C_2 > 0$. (See, e.g., Lemma 3.6 of [14].) Thus, if there exists a probability measure σ on S such that $T_\sigma(\rho, S) \neq 0$, then $T_\sigma(d, \mathscr{M}) \neq 0$. On the other hand, if there exists a probability measure σ on S such that $T_\sigma(\rho, S) = 0$, then $\{L_t^a, (a,t) \in S \times R^+\}$ is continuous and hence, by Theorem 3.7 of [14], $u^1(x,x)$ is continuous on S. Hence $C_1 < \infty$ and consequently $T_\sigma(d, \mathscr{M}) = 0$. Since $\{G(a),\ a \in S\}$ is continuous if and only if there exists a probability measure σ on S such that $T_\sigma(\rho, S) = 0$, it follows that (ii) and (iii) are equivalent.

It is clear from the first line of (4.3) that d is also a metric on R^n, or T^n. Thus we can write (4.1) as

$$\tilde{T}_\lambda(d, [0, 2\pi]) = \lim_{\eta \to 0}\ \sup_{x \in [0, 2\pi]} \int_0^\eta \left(\log \frac{1}{\lambda(B_d(t, \varepsilon))}\right)^{1/2} d\varepsilon, \tag{4.4}$$

where λ is a probability measure on $[0, 2\pi]$. Furthermore, if X is a Lévy process d is translation invariant. It is well known that for translation-invariant metrics on R^n and T^n we can take the measure λ in (4.4) to be normalized Lebesgue measure and that $\tilde{T}_\lambda(d, [0, 2\pi]) = 0$ if and only if $J(d, [0, 2\pi]) < \infty$. However, now considering d as a metric on $\mathscr{M}$, we see that $J(d, [0, 2\pi]) = J(d, \mathscr{M})$. Thus, when X is a Lévy process in R^n, or T^n, (iii) can be replaced by (iii′). □

5. Lévy processes in R^n. We now specialize to Lévy processes in R^n and families of measures $\mathscr{M} = \{\mu_x,\ x \in R^n\}$ which consist of translates of a fixed measure μ on R^n. We first develop material leading to the proof of Theorem 1.3. We then prove Theorem 1.6 and lastly consider the relationship between the 1-potential and the metric entropy integral (1.15).

Let us define

$$V_h(x_1,\dots,x_k)(z) \tag{5.1}$$
$$= \sum_{\pi} \int u^1(z,y_1)u^1(y_1,y_2)\cdots u^1(y_{k-1},y_k)h(y_k)\prod_{i=1}^{k} d\mu_{x_{\pi(i)}}(y_i),$$

where the sum runs over all permutations π of $\{1,\dots,k\}$. Note that

$$V_h(x)(z) = U^1 h\mu_x(z)$$

and also that functions such as $V_h(x_1,\dots,x_k)(z)$ arise in the proof of the isomorphism theorem, as in (2.26). The next theorem is used in the proof of Theorem 1.3.

THEOREM 5.1. *Let $X = \{X_t,\ t \in R^+\}$ be a symmetric Lévy process in R^n. Let $\mu \in \mathscr{G}_1^2$ be a finite measure on R^n. If $\{H_1(\mu_x),\ x \in R^n\}$ is continuous almost surely, then for any $h = U^1 f$, where $f \in \mathscr{S}(R^n)$ is strictly positive, $x \mapsto U^1 h\mu_x$ is a bounded and uniformly continuous map from R^n to $\mathscr{B}(R^n)$, and, more generally,*

$$\{x_1, x_2,\dots,x_k\} \mapsto V_h(x_1,\dots,x_k)(z) \tag{5.2}$$

is a bounded and uniformly continuous map from $(R^n)^k$ to $\mathscr{B}(R^n)$.

PROOF. We will first show that $x \mapsto U^1 h\mu_x$ is a bounded and uniformly continuous map from R^n to $\mathscr{B}(R^n)$. That is,

$$\sup_{x,z\in R^n} U^1 h\mu_x(z) < \infty \tag{5.3}$$

for each $x \in R^n$, and

$$\lim_{\delta\to 0}\ \sup_{|x-y|\le\delta}\ \sup_{z\in R^n} \left|U^1 h\mu_x(z) - U^1 h\mu_y(z)\right| = 0. \tag{5.4}$$

Let us assume first that $\mu = g(x)\,dx$, where $g(x)$ is bounded and uniformly continuous. In particular, this guarantees that

$$L_s^{\mu_x} = \int_0^s g(X_r + x)\,dr, \qquad x \in R^n, \tag{5.5}$$

is continuous almost surely and has bounded 1-potential $U^1 g_x(\cdot)$, where $g_x(\cdot) =_{\text{def}} g(x + \cdot)$. We now obtain bounds on $U^1 h\mu_{\cdot}$ in terms of $H_1(\mu_{\cdot})$ which will extend to all finite $\mu \in \mathscr{G}_1^2$.

We begin by noting that

$$U^1 h\mu(z) = E_\lambda^z(L_\lambda^\mu f(X_\lambda)) = E_\lambda^y\big(L_\lambda^{\mu_{z-y}} f_{z-y}(X_\lambda)\big). \tag{5.6}$$

The first equality is particularly easy to see for L_t^μ of the form (5.5). The second inequality follows by a change of variables.

Let $B_r = \{x \in R^n | \, |x| \le r\}$. As in (5.6), for $x, z, v \in B_1$ we have that

$$U^1 h \mu_x(z) = E_\lambda^v \big(L_\lambda^{\mu_{x+z-v}} f_{z-v}(X_\lambda) \big). \tag{5.7}$$

Therefore

$$\sup_{z \in B_1} U^1 h \mu_x(z) \le E_\lambda^v \Big(\sup_{z \in x + B_2} L_\lambda^{\mu_z} \tilde{f}(X_\lambda) \Big), \tag{5.8}$$

where

$$\tilde{f}(x) = \sup_{z \in B_2} f_z(x). \tag{5.9}$$

We note that, since $f \in \mathscr{S}(R^d)$, $\tilde{f}$ has the property that

$$\iint u^1(x, y) \tilde{f}(x) \tilde{f}(y)\, dx\, dy < \infty. \tag{5.10}$$

Therefore, integrating (5.8) with respect to $dm(v)$ restricted to B_1 and using the isomorphism theorem, Theorem 2.2, and the stationarity of H_1, we see that

$$\sup_{z \in B_1} U^1 h \mu_x(z) \le C \Big\| \sup_{z \in B_2} H_1(\mu_z) \Big\|_2, \tag{5.11}$$

uniformly in $x \in R^n$. Finally, using the stationarity of H_1 again and the fact that

$$U^1 h_y \mu_{x+y}(z) = U^1 h \mu_x(z + y) \tag{5.12}$$

and that (5.10) is unchanged if we replace $\tilde{f}$ by $\tilde{f}_y$, we see that (5.11) gives

$$U^1 h \mu_x(z) \le C \Big\| \sup_{z \in B_2} H_1(\mu_z) \Big\|_2 \tag{5.13}$$

for all $x, z \in R^n$.

Similarly, for $x, y, z, v \in B_1$ we have, as in (5.6), that

$$U^1 h \mu_x(z) - U^1 h \mu_y(z) = E_\lambda^v \big((L_\lambda^{\mu_{x+z-v}} - L_\lambda^{\mu_{y+z-v}}) f_{z-v}(X_\lambda) \big). \tag{5.14}$$

Hence

$$\sup_{\substack{|x-y| \le \delta \\ x, y, z \in B_1}} U^1 h \mu_x(z) - U^1 h \mu_y(z) \le E_\lambda^v \Bigg(\sup_{\substack{|x-y| \le \delta \\ x, y \in B_3}} (L_\lambda^{\mu_x} - L_\lambda^{\mu_y}) \tilde{f}(X_\lambda) \Bigg),$$

where $\tilde{f}$ is defined in (5.9). As above, the isomorphism theorem, Theorem 2.2, now shows that

$$\sup_{\substack{|x-y| \le \delta \\ x, y, z \in B_1}} \big| U^1 h \mu_x(z) - U^1 h \mu_y(z) \big| \le C \Bigg\| \sup_{\substack{|x-y| \le \delta \\ x, y \in B_3}} H_1(\mu_x) - H_1(\mu_y) \Bigg\|_2 \tag{5.15}$$

and that (5.15) implies that

$$\sup_{|x-y|\le\delta}\left|U^1h\mu_x(z)-U^1h\mu_y(z)\right|\le C\left\|\sup_{\substack{|x-y|\le\delta\\ x,y\in B_3}}H_1(\mu_x)-H_1(\mu_y)\right\|_2 \tag{5.16}$$

for all $z\in R^n$.

We now prove the assumption, that μ has a bounded uniformly continuous density, in (5.13) and (5.16). Let $b(x)$ be a positive continuous and symmetric function supported on B_1 with $\int b(x)\,dx=1$. Let

$$b^\gamma(x)=\frac{1}{\gamma^d}b\left(\frac{x}{\gamma}\right)$$

and set $\mu^\gamma=\mu*b^\gamma=g^\gamma(x)\,dx$. Note that $g^\gamma(x)=\int b^\gamma(x-y)\mu(dy)$ is bounded and uniformly continuous when μ is a finite measure. We apply (5.13) with μ replaced by μ^γ to obtain

$$\begin{aligned}\sup_{x\in R^n}U^1h\mu_x^\gamma(z)&\le C\left\|\sup_{x\in B_2}H_1(\mu_x^\gamma)\right\|_2\\ &=C\left\|\sup_{x\in B_2}H_1(\mu_\cdot)*b^\gamma(x)\right\|_2\\ &\le C\left\|\sup_{x\in B_3}H_1(\mu_x)\right\|_2.\end{aligned}\tag{5.17}$$

In the second line of (5.17) we use the fact that

$$H_1(\mu_x^\gamma)=H_1(\mu_\cdot)*b^\gamma(x),\tag{5.18}$$

which follows easily from (2.9).

We now take the limit in (5.17) as $\gamma\to0$. We show below that, for any $x\in R^n$,

$$U^1h\mu_x^\gamma(z)\to U^1h\mu_x(z)\tag{5.19}$$

in $L^1(R^n,dz)$ as $\gamma\to0$. Therefore for some subsequence $\gamma_k\to0$ we have

$$U^1h\mu_x^{\gamma_k}(z)\to U^1h\mu_x(z)\tag{5.20}$$

for almost all z with respect to Lebesgue measure. This implies that

$$U^1h\mu_x(z)\le C\left\|\sup_{x\in B_3}H_1(\mu_x)\right\|_2\tag{5.21}$$

for almost all z. However, since $U^1h\mu_x(z)$ is 1-excessive, it follows that (5.21) holds for all z.

Let us now prove (5.19). It suffices to consider the case $x=0$. Since

$$U^1h\mu^\gamma(z)=\int U^1h\mu_x(z)b^\gamma(x)\,dx,\tag{5.22}$$

it is enough to show that $x\mapsto U^1h\mu_x(z)$ is a bounded and uniformly continuous map from R^n to $L^1(R^n,dz)$. To see this, it suffices to note that

$$\|U^1h\mu_x\|_1\le\int h(y)\mu_x(dy)\le\|h\|_\infty\mu(R^n)\tag{5.23}$$

and

$$
\begin{aligned}
&\|U^1 h\mu_x - U^1 h\mu\|_1 \\
&\quad = \int \left| \int u^1(z-y)h(y)\mu_x(dy) - \int u^1(z-y)h(y)\mu(dy) \right| dz \\
&\quad = \int \left| \int u^1(z-y-x)h(y+x)\mu(dy) - \int u^1(z-y)h(y)\mu(dy) \right| dz \qquad (5.24) \\
&\quad \le \left(\sup_{y\in R^n} \int \left| u^1(z-y-x)h(y+x) - u^1(z-y)h(y) \right| dz \right) \mu(R^n) \\
&\quad \le \left(\|u^1_x - u^1\|_1 \|h\|_\infty + \|u^1\|_1 \|h_x - h\|_\infty \right) \mu(R^n).
\end{aligned}
$$

Similarly, we can now remove the assumption, that μ has a bounded uniformly continuous density in (5.16), by arguing exactly as above, with respect to the L^1-function $z \mapsto U^1 h\mu_x(z) - U^1 h\mu_y(z)$. This shows that (5.16) holds for all finite $\mu \in \mathscr{G}_1^2$. This completes the proof of (5.13) and (5.16) in the general case, and verifies (5.3) and (5.4).

We now prove that (5.2) is bounded and uniformly continuous. As above, let us assume first that $\mu = g(x)\,dx$, where $g(x)$ is bounded and uniformly continuous. Note that

$$
\begin{aligned}
V_h(x_1,\dots,x_k)(z) &= E^z_\lambda \left(\prod_{i=1}^k L_\lambda^{\mu_{x_i}} f(X_\lambda) \right) \\
&= E^y_\lambda \left(\prod_{i=1}^k L_\lambda^{\mu_{x_i+z-y}} f_{z-y}(X_\lambda) \right).
\end{aligned} \qquad (5.25)
$$

The first equality is straightforward for $L_t^{\mu_\cdot}$ of the form (5.5). [See also (2.22).] The second equality follows by a change of variables. Therefore

$$
\sup_{z\in B_1} V_h(x_1,\dots,x_k)(z) \le E^v_\lambda \left(\sup_{z_i\in x_i+B_2} \prod_{i=1}^k L_\lambda^{\mu_{z_i}} \tilde{f}(X_\lambda) \right),
$$

where, as before,

$$
\tilde{f}(x) = \sup_{z\in B_2} f_z(x). \qquad (5.26)
$$

As above, Theorem 2.2, Hölder's inequality and the stationarity of H_1 show us that

$$
\sup_{z\in B_1} V_h(x_1,\dots,x_k)(z) \le C \left\| \sup_{z\in B_2} H_1^k(\mu_x) \right\|_2, \qquad (5.27)
$$

uniformly in $\{x_1,\dots,x_k\} \in (R^n)^k$. Again, by stationarity, as above, we see that

$$
\sup_{z\in R^n} V_h(x_1,\dots,x_k)(z) \le C \left\| \sup_{z\in B_2} H_1^k(\mu_x) \right\|_2, \qquad (5.28)
$$

uniformly in $\{x_1,\dots,x_k\} \in (R^n)^k$. The assumption that μ has a bounded uniformly continuous density can be removed exactly as before. This shows that (5.2) is bounded. The proof of uniform continuity is similar. □

PROOF OF THEOREM 1.3. By Theorem 5.1, $U^1 h_{(1)}\mu(z)$ is bounded so that $\mu \in \mathrm{Rev}(X)$. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in R^n. We will first show that the version of the isomorphism theorem, Theorem 2.1, still holds if we take $\{\mu_{x_i}\}_{i=1}^{\infty}$ for the sequence of measures $\{\mu_i\}_{i=1}^{\infty}$, even without the assumption that the $U^1\mu_{x_i}(z)$ are bounded on R^n. We first note that, for each i,

$$U^1\mu_{x_i}(x) < \infty \quad \text{q.e. } x, \tag{5.29}$$

which follows from the fact, (1.10), that $\mu_{x_i} \in \mathscr{G}_1^1$ so that $\int U^1\mu_{x_i}(x)\,d\rho(x) < \infty$ for all $\rho \in \mathscr{G}_1^1$. (See, e.g., Theorem 3.3.2 of [6].) Furthermore,

$$L_t^{\mu} = \int_0^t \frac{1}{h_{(1)}(X_s)}\,dL_s^{h_{(1)}\cdot\mu}, \tag{5.30}$$

since both sides are continuous additive functionals with the same Revuz measure μ. Together with (5.29) and Theorem 3.1, Chapter 6, of [4], this shows that

$$E^x\left(L_{\infty}^{\mu_{x_i}}\right) = U^1\mu_{x_i}(x) < \infty \quad \text{q.e. } x. \tag{5.31}$$

This, together with Theorem 5.1 which enables us to control the integrals in (2.26), allows us to establish (2.22) for q.e. x which is sufficient to establish the isomorphism theorem, Theorem 2.1.

We then follow the proof of Theorem 1.1. Here, the assumptions of that theorem concerning $U^1\mu_x(z)$ are used only in bounding (3.6), and, once again, Theorem 5.1 allows us to do this. Finally, in the proof of Theorem 3.2 we can use the relationship (5.30) between L_t^{μ} and $L_t^{h_{(1)}\cdot\mu}$ to establish (3.12) and to identify $\tilde{L}_t^{\mu}$. Putting all this together completes the proof of Theorem 1.3. □

PROOF OF THEOREM 1.6. By Theorem 1.2, Remark 1.4 and Theorem 1.3, we need only show that (1.14) holds with $d = d_1$. By (1.12) and (1.25),

$$d_1(y+h, y) = 2\left(\int_{\xi \in R^n} \sin^2 \frac{\xi h}{2}\,\gamma(\xi)\,|\hat{\mu}(\xi)|^2\,d\xi\right)^{1/2}. \tag{5.32}$$

[Recall that $d_1(y+h, y) = d_1(\mu_y, \mu_{y+h})$.] For $x \geq 0$ define

$$F(x) = \int_{|\xi| \leq x} \gamma(\xi)\,|\hat{\mu}(\xi)|^2\,d\xi \tag{5.33}$$

and note that

$$\begin{aligned} d_1(y+h, y) &\leq C\left(|h|^2 \int_0^{1/|h|} u^2\,dF(u) + (1 - F(1/|h|))\right)^{1/2} \\ &\equiv \phi(|h|). \end{aligned} \tag{5.34}$$

A slight modification of the argument on pages 152 of [7] [take $\zeta(u) = 1/u$] or of the proof of Lemma 1.1, Chapter 7, of [10] shows that (1.26) implies that

$$\int_0^1 \frac{\phi(|u|)}{u}\,du < \infty. \tag{5.35}$$

It is easy to see that (5.35) implies that (1.14) holds. (See, e.g., Lemma 5.3, Chapter 4, of [7] or the proof of Lemma 3.6, Chapter 2, of [10].) □

The equivalence of (i) and (ii) in Theorem 1.5 is a surprising relationship between the 1-potential of a Lévy process in T^n and the square-root metric entropy of the Gaussian chaos associated with certain of its continuous additive functionals. In the next theorem we see that a similar result holds for processes and measures in R^n.

THEOREM 5.2. *Let X be a Lévy process in R^n with characteristic sequence ψ and let μ be a finite measure on R^n, $n = 2, 3$. Assume that*:

(i) $\hat{\mu}(\xi) \ge 0$;
(ii) *$\psi(\xi)$ and $\hat{\mu}(\xi)$ are radially symmetric*;
(iii) *there exist constants $0 < C_1, C_2 < \infty$ such that*

$$C_1 \frac{\gamma(\xi)}{|\xi|^n} \le \frac{1}{(1 + \psi(\xi))^2} \le C_2 \frac{\gamma(\xi)}{|\xi|^n}$$

for $|\xi| \ge 1$;
(iv) *there exist a decreasing sequence $\{a_j\}_{j=1}^{\infty}$ of positive numbers and constants $0 < C_1', C_2' < \infty$ such that*

$$C_1' a_j \le \frac{|\xi|^n \hat{\mu}(\xi)}{1 + \psi(\xi)} \le C_2' a_j$$

for $2^{j-1} \le |\xi| \le 2^j$.

Then

$$U^1\mu(0) < \infty \Leftrightarrow \int \left(\log N_{d_1}(R^3, \varepsilon)\right)^{1/2} d\varepsilon < \infty. \tag{5.36}$$

We use Boas's lemma, Lemma 2.2, Chapter 4, of [7].

LEMMA 5.1. *Let $\{s_j\}_{j=1}^{\infty}$ be a sequence of positive real numbers. Suppose $s_j \downarrow$ as $j \to \infty$. Then there exist constants $0 < C_1, C_2 < \infty$ such that*

$$C_1 \sum_{j=1}^{\infty} s_j \le \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=n}^{\infty} s_j^2 \right)^{1/2} \le C_2 \sum_{j=1}^{\infty} s_j. \tag{5.37}$$

Furthermore, the left-hand side of (5.37) *remains valid without the condition that $s_j \downarrow$ as $j \to \infty$.*

PROOF OF THEOREM 5.2. By (iv) there exist constants $0 < C_1, C_2 < \infty$ such that

$$C_1 a_j \le \int_{2^{j-1} < |\xi| \le 2^j} \frac{\hat{\mu}(\xi)}{1 + \psi(\xi)} \, d\xi \le C_2 a_j, \qquad j \ge 1. \tag{5.38}$$

Therefore it follows from Lemma 5.1 that

$$U^1\mu(0) = \int_{\xi\in R^n} \frac{\hat{\mu}(\xi)}{1+\psi(\xi)}\,d\xi < \infty \tag{5.39}$$

if and only if

$$\sum_{k=2}^{\infty}\left(\frac{1}{k}\sum_{j=k}^{\infty}\frac{2^{2nj}|\hat{\mu}(2^j)|^2}{(1+\psi(2^j))^2}\right)^{1/2} < \infty. \tag{5.40}$$

By (iii), (5.40) holds if and only if

$$\sum_{k=2}^{\infty}\left(\frac{1}{k}\sum_{j=k}^{\infty}2^{nj}\gamma(2^j)|\hat{\mu}(2^j)|^2\right)^{1/2} < \infty \tag{5.41}$$

or, equivalently, if and only if

$$\int_2^{\infty}\frac{\left(\int_{|\xi|\ge x}\gamma(\xi)|\hat{\mu}(\xi)|^2\,d\xi\right)^{1/2}}{x(\log x)^{1/2}}\,dx < \infty. \tag{5.42}$$

An argument similar to the one used in the proof of Theorem 1.6 shows that (5.42) implies that the integral on the right-hand side of (5.36) is finite. Thus we get the implication to the right in (5.36). To get the reverse implication, by Lemma 6.2, Chapter 4, of [7], it is enough to show that

$$C\int_{|\xi|\ge 1/|h|}\gamma(\xi)|\hat{\mu}(\xi)|^2\,d\xi \le d_1^2(y+h,y) \tag{5.43}$$

for some $C > 0$. Note that, by (ii), $\gamma(\xi)$ is also radially symmetric. Therefore

$$\begin{aligned} &d_1^2(y+h,y)\\ &\quad = \int_{\xi\in R^n}(\sin^2 \xi h)\gamma(\xi)|\hat{\mu}(\xi)|^2\,d\xi\\ &\quad = \int_0^{\infty}\left(\int\sin^2(v|h|(h/|h|)u)\,d\sigma(u)\right)\gamma(|v|)(\hat{\mu}(|v|))^2 v^{n-1}\,dv, \end{aligned}\tag{5.44}$$

where $\sigma(u)$ is uniform measure on the unit sphere in R^n. It is easy to verify that

$$\int\sin^2(v|h|(h/|h|)u)\,d\sigma(u) \ge C > 0 \tag{5.45}$$

for $v|h| \ge 1$. Thus

$$d_1^2(y+h,y) \ge C\int_{1/|h|}^{\infty}\gamma(v)|\hat{\mu}(v)|^2 v^{n-1}\,dv, \tag{5.46}$$

which gives us (5.43). □

The next result gives an upper bound for the supremum of the 1-potential of a Lévy process under much weaker conditions than those required in Theorem 5.2. The term on the right-hand side of (5.48) is an upper bound for the metric entropy integral in (5.36) and is equivalent to it when the Fourier coefficients of the measure and the characteristic exponent of the Lévy process are sufficiently smooth.

LEMMA 5.2. *Suppose that*

$$\frac{1}{(1+\psi(\xi))^2} \le C\frac{\gamma(\xi)}{|\xi|^n} \qquad \forall\, \xi \in R^n. \tag{5.47}$$

Then

$$\sup_{x\in R^n} \left|U^1\mu(x)\right| \le C\int_1^\infty \frac{\left(\int_{|\xi|\ge x}\gamma(\xi)|\hat{\mu}(\xi)|^2\,d\xi\right)^{1/2}}{x(\log 2x)^{1/2}}\,dx, \tag{5.48}$$

where $\gamma(\xi)$ *is given in* (1.24).

PROOF. This is a simple application of Lemma 5.1 and the Schwarz inequality. Without loss of generality, we assume that

$$\int_{|\xi|>1}\frac{|\hat{\mu}(\xi)|}{1+\psi(\xi)}\,d\xi > 0 \tag{5.49}$$

and obtain (5.48) from the following sequence of inequalities:

$$\begin{aligned}
\int_{\xi\in R^n}\frac{|\hat{\mu}(\xi)|}{1+\psi(\xi)}\,d\xi &\le C\sum_{j=1}^\infty \int_{2^{j-1}<|\xi|\le 2^j}\frac{|\hat{\mu}(\xi)|}{1+\psi(\xi)}\,d\xi\\
&\le C\sum_{k=1}^\infty\left(\frac{1}{k}\sum_{j=k}^\infty\left(\int_{2^{j-1}<|\xi|\le 2^j}\frac{|\hat{\mu}(\xi)|}{1+\psi(\xi)}\,d\xi\right)^2\right)^{1/2}\\
&\le C\sum_{k=1}^\infty\left(\frac{1}{k}\sum_{j=k}^\infty 2^{jn}\left(\int_{2^{j-1}<|\xi|\le 2^j}\frac{|\hat{\mu}(\xi)|^2}{(1+\psi(\xi))^2}\,d\xi\right)\right)^{1/2}\\
&\le C\sum_{k=1}^\infty\left(\frac{1}{k}\sum_{j=k}^\infty 2^{jn}\left(\int_{2^{j-1}<|\xi|\le 2^j}\frac{\gamma(\xi)|\hat{\mu}(\xi)|^2}{|\xi|^n}\,d\xi\right)\right)^{1/2}\\
&\le C\sum_{k=1}^\infty\left(\frac{1}{k}\sum_{j=k}^\infty\left(\int_{2^{j-1}<|\xi|\le 2^j}\gamma(\xi)|\hat{\mu}(\xi)|^2\,d\xi\right)\right)^{1/2}.
\end{aligned} \tag{5.50}$$

□

The next lemma gives conditions under which (5.47) holds.

LEMMA 5.3. *If* $\psi(\xi) = \psi(|\xi|)$ *is regularly varying at* ∞, *then* (5.47) *holds.*

PROOF. By the assumption of regular variation,

$$\begin{aligned}\gamma(\xi) &\geq \int_{\eta \geq 2|\xi|} \frac{d\eta}{(1+\psi(|\xi-\eta|))(1+\psi(|\eta|))} \\ &\geq C \int_{\eta \geq 2|\xi|} \frac{|\eta|^{n-1}\, d|\eta|}{(1+\psi(|\eta|))^2} \\ &\geq C \frac{|\xi|^n}{(1+\psi(|\xi|))^2},\end{aligned} \tag{5.51}$$

which is (5.47). □

6. Lévy processes in T^n. We begin by showing how to obtain a large and interesting class of Lévy processes taking values in an n-dimensional torus. Let X be a symmetric Lévy process in R^n, $n \geq 1$, with transition probability density $p_t(x, 0) =_{\text{def}} p_t(x)$ and characteristic function

$$Ee^{i\lambda X_t} = e^{-t\psi(\lambda)}. \tag{6.1}$$

Let $\pi\colon R^n \mapsto T^n$ denote the natural projection, $\pi(x) = x (\operatorname{mod} 2\pi)$, onto the n-dimensional torus. Consider

$$Y_t = \pi(X_t).$$

It is easy to see that $Y = \{Y_t,\ t \in T^n\}$ is a Markov process with transition probability density

$$q_t(x,y) = (2\pi)^n \sum_{j \in Z^n} p_t(x - y + 2\pi j) \qquad \forall\, x, y \in T^n, \tag{6.2}$$

with respect to the normalized Lebesgue measure $dx/(2\pi)^n$ on T^n. Since X is a Lévy process, $q_t(x, y) = q_t(x - y, 0) =_{\text{def}} q_t(x - y)$, where $x - y$ is subtraction in T^n. We have

$$\begin{aligned}\hat{q}_t(j) &= \frac{1}{(2\pi)^n} \int_{T^n} e^{ijx} q_t(x)\, dx \\ &= \int_{T^n} e^{ijx} \sum_{k \in Z^n} p_t(x + 2\pi k)\, dx \\ &= \int_{R^n} e^{ijx} p_t(x)\, dx \\ &= \hat{p}_t(j) = e^{-t\psi(j)}.\end{aligned} \tag{6.3}$$

Therefore, in particular, Y is determined by its characteristic sequence as in (1.18).

Lévy processes on the torus are recurrent. Therefore we consider the 1-potential density of Y denoted by

$$v^1(x) = \int_0^\infty e^{-t} q_t(x)\, dt = (2\pi)^n \sum_{j \in Z^n} u^1(x + 2\pi j) \tag{6.4}$$

from which it follows that

$$\hat{v}^1(j) = \hat{u}^1(j) = \frac{1}{1 + \psi(j)}. \tag{6.5}$$

The following useful inequalities relating the 1-potential of Lévy processes in T^n with the associated Gaussian chaos is an immediate consequence of Theorem 5.1.

THEOREM 6.1. *Let $X = \{X_t,\ t \in R^+\}$ be a symmetric Lévy process in T^n. Let $\mu \in \mathscr{G}_1^2$ be a finite measure on T^n. Then*

$$\sup_{x \in T^n} U^1\mu(x) \le \Big\| \sup_{x \in T^n} H_1(\mu_2) \Big\|_2 \tag{6.6}$$

and, for any $\delta > 0$,

$$\sup_{\substack{|x-y| \le \delta \\ x, y \in T^n}} U^1\mu(x) - U^1\mu(y) \le \Big\| \sup_{\substack{|x-y| \le \delta \\ x, y \in T^n}} H_1(\mu_x) - H_1(\mu_y) \Big\|_2. \tag{6.7}$$

In particular, if H_1 is continuous almost surely, then $U^1\mu(x)$ is continuous on T^n.

Note that we can always assume that we are working with a measurable and separable version of H_1. Inequalities such as (6.6) and (6.7) should be understood to be applying to such versions.

PROOF. Following the proof of Theorem 5.1 but with R^n replaced by T^n, we see that f can be taken to be identically 1 in (1.6) and hence $h = 1$. This theorem now follows immediately from (5.13) and (5.16) since we can take B_2 or B_3 equal to T^n. [Doing this carefully, one sees that the constant can be taken to be 1 in (6.6) and (6.7).] $\square$

We now obtain concrete sufficient conditions for the continuity of continuous additive functionals of Lévy processes in T^n, which are analogous to those obtained in Theorem 1.6 for Lévy processes in R^n. Let μ be a finite measure on T^n with Fourier coefficients $\{\hat{\mu}(k),\ k \in Z^n\}$; that is,

$$\hat{\mu}(j) = \int_{T^n} e^{ijx}\, d\mu(x), \qquad j \in Z^n.$$

Assume that

$$\beta(j) = \sum_{k \in Z^n} \frac{1}{(1 + \psi(k-j))(1 + \psi(k))} < \infty. \tag{6.8}$$

Note that $\beta(j)$ are the Fourier coefficients of $(u^1(x))^2$ so that

$$\sum_{j\in Z^n} \beta(j)|\hat{\mu}(j)|^2 = \iint (u^1(x,y))^2\, d\mu(x)\, d\mu(y). \tag{6.9}$$

THEOREM 6.2. *Let $Y = \{Y_t,\ t \in R^+\}$ be a symmetric Lévy process in T^n with characteristic exponent ψ. Let μ be a finite measure on T^n. If*

$$\sum_{n=1}^{\infty} \frac{\left(\sum_{|j|\ge n} \beta(j)|\hat{\mu}(j)|^2\right)^{1/2}}{n} < \infty, \tag{6.10}$$

then $\mu \in \mathrm{Rev}(X)$ and $\{L_t^{\mu_a},\ (a,t) \in T^n \times R^+\}$ has a continuous version. In particular, for Brownian motion on T^2, this is the case when

$$|\hat{\mu}(k)| = O\left(\frac{1}{(\log|k|)^{2+\varepsilon}}\right) \quad \text{as } |k| \to \infty \tag{6.11}$$

for any $\varepsilon > 0$.

PROOF. Writing (1.12) in terms of its Fourier series, we have

$$\begin{aligned} d(\mu_a, \mu_b) &= \left(E(H(\mu_a) - H(\mu_b))^2\right)^{1/2} \\ &= 2\sqrt{2}\left(\sum_{j\in Z^n} \beta(j)|\hat{\mu}(j)|^2 \sin^2\frac{j(a-b)}{2}\right)^{1/2}, \end{aligned} \tag{6.12}$$

since $\hat{\mu}_a(j) = \hat{\mu}(j)e^{ija}$. Let $\mathscr{M} = \{\mu_a,\ a \in T^n\}$. It follows from Lemma 3.6, Chapter 2, of [10] and Lemma 1.1, Chapter 7, of [10] that (6.10) implies (1.14). Therefore we see by Theorem 1.2 that $\{H(\mu_a),\ a \in T^n\}$ has a continuous version almost surely. □

The next lemma describes the Gaussian chaos associated with a continuous additive functional of a Lévy process on T^n by the isomorphism theorem. This is interesting in its own right and also leads, in Theorem 1.4, to remarkable necessary conditions for the continuity and boundedness of certain second-order Gaussian chaoses.

LEMMA 6.1. *Let $Y = \{Y_t,\ t \in R^+\}$ be a symmetric Lévy process with characteristic exponent ψ. For all $\mu \in \mathscr{G}_1^2$, the Gaussian chaos $H_1(\mu)$ associated with L_t^{μ}, the continuous additive functional of Y determined by μ, is given by*

$$\begin{aligned} H_1(\mu) = \frac{1}{2} \sum_{j,k\in Z^n} & \frac{1}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}} \\ & \times\{(a(j+k) + a(j-k))(g_j g_k - \delta_{j,k}) \\ & \quad -2(b(j+k) - b(j-k))g_j g_k' \\ & \quad +(a(j-k) - a(j+k))(g_j' g_k' - \delta_{j,k})\}, \end{aligned} \tag{6.13}$$

where $\{g_j\}$ *and* $\{g'_j\}$, $j \in Z^n$, *are independent normal random variables with mean* 0 *and variance* 1, $\{a(j)\}$ *and* $\{b(j)\}$ *denote the real and imaginary parts of* $\hat{\mu}(j)$ *and* $\delta_{j,k} = 1$ *if* $j = k$ *and is equal to* 0 *otherwise. Furthermore, for* $\nu \in \mathcal{G}_1^2$,

$$(6.14)\qquad \begin{aligned} E(H_1(\mu)H_1(\nu)) &= 2\iint \left(v^1(x-y)\right)^2 d\mu(x)\,d\nu(y) \\ &= 2\sum_{j\in Z^n} \widehat{\left(v^1\right)^2}(j)\,\hat{\mu}(j)\,\hat{\nu}(-j), \end{aligned}$$

where

$$(6.15)\qquad \widehat{\left(v^1\right)^2}(j) = \sum_{k\in Z^n} \frac{1}{(1+\psi(k-j))(1+\psi(k))}.$$

PROOF. If μ, ν are measures in $\mathcal{G}_1^2$,

$$(6.16)\qquad \iint v^1(x-y)\,d\mu(x)\,d\nu(y) = \sum_{j\in Z^n} \frac{1}{1+\psi(j)}\hat{\mu}(j)\,\hat{\nu}(-j).$$

Therefore the Gaussian process that is used to construct the Gaussian chaos associated with Y (see Section 2) can be represented by

$$(6.17)\qquad \begin{aligned} G_\mu &= \sum_{j\in Z^n} \frac{1}{\sqrt{1+\psi(j)}} \operatorname{Re}\left\{\hat{\mu}(j)(g_j + ig'_j)\right\} \\ &= \sum_{j\in Z^n} \frac{1}{\sqrt{1+\psi(j)}} \left(a(j)g_j - b(j)g'_j\right). \end{aligned}$$

Letting $c(j)$ and $d(j)$ denote the real and imaginary parts of $\hat{\nu}(j)$, we have

$$(6.18)\qquad \begin{aligned} E(G_\mu G_\nu) &= \sum_{j\in Z^n} \frac{1}{1+\psi(j)}(a(j)c(j) + b(j)d(j)) \\ &= \sum_{j\in Z^n} \frac{1}{1+\psi(j)}\hat{\mu}(j)\,\hat{\nu}(-j), \end{aligned}$$

where we use the facts that $\psi(j)$ is real and even, $a(j)$ and $c(j)$ are even and $b(j)$ and $d(j)$ are odd, for all j.

In particular, if $\mu_{x,\delta} = q_\delta(y-x)\,dy/(2\pi)^n$ [see (6.2)]

$$(6.19)\qquad \hat{\mu}_{x,\delta} = e^{ijx}e^{-\delta\psi(j)} = \cos(jx)e^{-\delta\psi(j)} + i\sin(jx)e^{-\delta\psi(j)}.$$

Therefore

$$(6.20)\qquad \begin{aligned} G_{x,\delta} &=_{\text{def}} G_{\mu_{x,\delta}} \\ &= \sum_{j\in Z^n} e^{-\delta\psi(j)} \frac{1}{\sqrt{1+\psi(j)}} \left(\cos(jx)g_j - \sin(jx)g'_j\right) \end{aligned}$$

and

$$G_{x,\delta}^2 - E\left(G_{x,\delta}^2\right) = \frac{1}{2} \sum_{j,k \in Z^n} \frac{e^{-\delta(\psi(j)+\psi(k))}}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}}$$
$$\times\big\{\cos((j+k)x) + \cos((j-k)x)(g_j g_k - \delta_{j,k})$$
$$-2(\sin((j+k)x) - \sin((j-k)x)) g_j g_k'$$
$$+(\cos((j-k)x) - \cos((j+k)x))(g_j' g_k' - \delta_{j,k})\big\}.$$

Integrating with respect to μ, we get

$$H_1(\mu, \delta) = \frac{1}{2} \sum_{j,k \in Z^n} \frac{e^{-\delta(\psi(j)+\psi(k))}}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}}$$
$$\times\big\{(a(j+k) + a(j-k))(g_j g_k - \delta_{j,k}) \tag{6.21}$$
$$-2(b(j+k) - b(j-k)) g_j g_k'$$
$$+(a(j-k) - a(j+k))(g_j' g_k' - \delta_{j,k})\big\}.$$

Taking the limit in (6.21), as $\delta \to 0$, we get (6.13).

The first line in (6.14) follows from the construction of $H_1(\mu)$. It is given in (2.9) and, of course, the same analysis holds for the 1-potential as for the 0-potential. Taking its Fourier transform gives the rest of (6.14) and (6.15). [We will verify the transition between the first and second lines of (6.14) by a computation in the proof of Lemma 6.2.] □

The next lemma enables us to study questions of continuity and boundedness of Gaussian chaoses associated with continuous additive functionals of Lévy processes on T^n in terms of simpler Gaussian chaoses.

LEMMA 6.2. *Let $\mathscr{M} \subset \mathscr{G}_1^2$ and assume that $\hat{\mu}(0) = C$ for all $\mu \in \mathscr{M}$. Let $H_1 = \{H_1(\mu), \mu \in \mathscr{M}\}$ be as given in* (6.13). *Then H_1 has a continuous version if and only if $\{\chi(\mu), \mu \in \mathscr{M}\}$ or $\{\chi'(\mu), \mu \in \mathscr{M}\}$ has a continuous version, where*

$$\chi(\mu) = \sum_{\substack{j \neq k \\ j,k \in Z^n}} \frac{g_j g_k \hat{\mu}(j-k)}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}} \tag{6.22}$$

and

$$\chi'(\mu) = \sum_{j,k \in Z^n} \frac{g_j g_k' \hat{\mu}(j-k)}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}}. \tag{6.23}$$

Furthermore,

$$\left\| \sup_{\mu \in \mathscr{M}} H_1(\mu) \right\|_2 \le C_1 \left\| \sup_{\mu \in \mathscr{M}} \sum_{j,k \in Z^n} \frac{(g_j g_k - \delta_{j,k}) \hat{\mu}(j-k)}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}} \right\|_2 \tag{6.24}$$

and

$$\begin{aligned} &\left\| \sup_{\mu, \nu \in \mathscr{M}} H_1(\mu) - H_1(\nu) \right\|_2 \\ &\quad \le C_2 \left\| \sup_{\mu, \nu \in \mathscr{M}} \sum_{j,k \in Z^n} \frac{(g_j g_k - \delta_{j,k})(\hat{\mu}(j-k) - \hat{\nu}(j-k))}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}} \right\|_2, \end{aligned} \tag{6.25}$$

where C_1 and C_2 are constants independent of ψ and μ.

Proof. Define

$$\tilde{G}_\mu = \sum_{j \in Z^n} \frac{1}{\sqrt{1+\psi(j)}} \hat{\mu}(j) \tilde{g}_j, \tag{6.26}$$

where $\tilde{g}_j = g_j + i g'_j$. Note that

$$\tilde{G}_{\mu_{x,\delta}} =_{\mathrm{def}} \tilde{G}_{x,\delta} = G_{x,\delta} + i G'_{x,\delta}, \tag{6.27}$$

where $G'_{x,\delta}$ is an independent copy of $G_{x,\delta}$ given in (6.20). Using the fact that $\psi(\cdot)$ is symmetric, one can check that $E(G_{x,\delta} G'_{y,\delta}) = 0$, that is, that $\{G_{x,\delta}, x \in T^n\}$ and $\{G'_{x,\delta}, x \in T^n\}$ are independent, identically distributed Gaussian processes. Therefore

$$\begin{aligned} \tilde{H}_1(\mu) &=_{\mathrm{def}} \lim_{\delta \to 0} \int \left(|\tilde{G}_{x,\delta}|^2 - E|\tilde{G}_{x,\delta}|^2 \right) d\mu(x) \\ &= H_1(\mu) + H'(\mu), \end{aligned} \tag{6.28}$$

where $H_1(\mu)$ and $H'(\mu)$ are independent copies of $H_1(\mu)$ given in (6.13). We now see that

$$E\tilde{H}_1(\mu)\tilde{H}_1(\nu) = 2EH_1(\mu)H_1(\nu). \tag{6.29}$$

Furthermore, by (6.19) and (6.26), we have that

$$\begin{aligned} \tilde{G}_{x,\delta} &=_{\mathrm{def}} \tilde{G}_{\mu_{x,\delta}} \\ &= \sum_{j \in Z^n} e^{-\delta\psi(j)} \frac{1}{\sqrt{1+\psi(j)}} e^{ijx} \tilde{g}_j \end{aligned} \tag{6.30}$$

and so, by (6.28),

$$\tilde{H}_1(\mu) = \sum_{j,k \in Z^n} \frac{\hat{\mu}(j-k)}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}} \left(\tilde{g}_j \bar{\tilde{g}}_k - 2\delta_{j,k} \right). \tag{6.31}$$

Using the facts that $E(\tilde{g}_j)^2 = 0$, $E|\tilde{g}_j|^2 = 2$ and $E|\tilde{g}_j|^4 = 8$, we see that

$$E\tilde{H}_1(\mu)\tilde{H}_1(\nu) = 4\sum_{j,k\in Z^n}\frac{\hat{\mu}(j-k)\hat{\nu}(k-j)}{(1+\psi(j))(1+\psi(k))}. \tag{6.32}$$

Note that (6.29) and (6.32) together give an independent verification of (6.14) and (6.15).

We can rewrite (6.31) as

$$\begin{aligned}\tilde{H}_1(\mu) = &\sum_{j\neq k}\frac{\hat{\mu}(j-k)}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}}\tilde{g}_j\bar{\tilde{g}}_k\\ &+\sum_{k\in Z^n}\frac{C}{1+\psi(k)}\left(|\tilde{g}_k|^2-2\right).\end{aligned} \tag{6.33}$$

Note that the condition that $\mu \in \mathscr{G}_1^2$ implies, by (6.14) and (6.15), that $\sum_{k\in Z^n}(1+\psi(k))^{-2} < \infty$. Thus we see that the last term in (6.33) is a fixed random variable and hence it plays no role in the question of the continuity of $H_1(\mu)$ on $\mathscr{M}$. Letting $a(j)$ and $b(j)$ denote the real and imaginary parts of $\hat{\mu}(j)$, as above, the first term to the right of the equal sign in (6.33) can be written as

$$\tilde{H}_0(\mu) = \sum_{j\neq k}\frac{a(j-k)(g_jg_k+g'_jg'_k)+b(j-k)(g_jg'_k-g'_jg_k)}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}}. \tag{6.34}$$

It is clear by (6.28) and the above remark on the diagonal terms of $\tilde{H}_1(\mu)$ that the continuity of $H_1(\mu)$ and $\tilde{H}_0(\mu)$ are equivalent. Consider

$$\chi_1(\mu) = \sum_{j\neq k}\frac{a(j-k)g_jg_k}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}} \tag{6.35}$$

and

$$\chi_2(\mu) = \sum_{j\neq k}\frac{b(j-k)g_jg'_k}{\sqrt{1+\psi(j)}\sqrt{1+\psi(k)}} \tag{6.36}$$

for $\mu \in \mathscr{G}_1^2$. It is obvious that if $\{\chi_1(\mu),\ \mu\in\mathscr{G}_1^2\}$ and $\{\chi_2(\mu),\ \mu\in\mathscr{G}_1^2\}$ are continuous, then so is $\tilde{H}_0(\mu)$ and hence $H_1(\mu)$. It is also easy to see, taking into account the equivalence of continuity for the coupled and decoupled Gaussian chaoses, that $\chi_1(\mu)$ and $\chi_2(\mu)$ are continuous if and only if $\chi(\mu)$ is continuous. Thus we see that the continuity of $\chi(\mu)$ implies the continuity of $H_1(\mu)$. The equivalence of the continuity of $\chi(\mu)$ and $\chi'(\mu)$ follows from the decoupling property and the above remark on the diagonal of $H_1(\mu)$. One can also show that the continuity of $\tilde{H}_0(\mu)$ implies the continuity of $\chi_1(\mu)$ and $\chi_2(\mu)$ and hence of $\chi(\mu)$ and $\chi'(\mu)$.

Inequalities (6.24) and (6.25) follow from (6.28) and (6.31) and simple comparison theorems between the norms of coupled and decoupled Gaussian chaoses. (See, e.g., Section 2 of [1].) This completes the proof of Lemma 6.2. □

PROOF OF THEOREM 1.4. The proof follows immediately from the inequalities in Theorem 6.1 and Lemma 6.2, replacing the 1-potential by its Fourier series. We use the fact that all the moments of a norm of a Gaussian chaos are equivalent to express (1.19) in terms of the first moment of the norm. □

PROOF OF THEOREM 1.5. The expression in (i′) is, except for a constant multiple, the Fourier series of $U^1\mu(x)$ at $x = 0$. Since $\hat{\mu}(k) > 0$ for all $k \in Z^n$, the Fourier series of $U^1\mu(x)$ converges uniformly. Thus (i) and (i′) are equivalent. That (iii) implies (i) follows from Theorem 1.4 and (ii) implies (iii) follows from Theorem 1.2 of [15]. That (i) implies (ii) is a simple manipulation of the metric entropy integral $J(d, \mathscr{M})$. It is done for processes in R^n and measures on R^n, $n = 2, 3$, in Theorem 5.2; however, the upper bound in this theorem is also valid when $n = 1$. [Note that the conditions on $\{\psi(k)\}$ imply that (iii) of Theorem 5.2 holds.]

Theorem 1.3 shows that (iii) implies (iv). We now complete the proof by showing that (iv) implies (i). Assume that $\{L_t^{\mu_x}, (x,t) \in T^n \times R^+\}$ is continuous almost surely. Let

$$b^{\gamma}(x) = \sum_k e^{ikx} e^{-\gamma|k|}$$

denote the Poisson kernel on T^n. Set $\mu^{\gamma} = \mu * b^{\gamma} =_{\text{def}} g^{\gamma}(x)\,dx$ and note that $g^{\gamma}(x) = \int b^{\gamma}(x-y)\mu(dy)$. A straightforward calculation shows that

$$L_t^{\mu_x^{\gamma}} = \int L_t^{\mu_y} b^{\gamma}(x-y)\,dy, \tag{6.37}$$

since, in view of the continuity of $\{L_t^{\mu_x}, (x,t) \in T^n \times R^+\}$, the right-hand side of (6.37) is a continuous additive functional with Revuz measure μ_x^{γ}. Furthermore, $L_\lambda^{\mu_x^{\gamma}}$ converges almost surely, as $\gamma \downarrow 0$, to L_λ^{μ} which is finite. Hence for any $c > 0$ we can find a finite constant K such that

$$E^0\left(L_\lambda^{\mu^{\gamma}} \ge K\right) \le c/2. \tag{6.38}$$

We show that (6.38) implies (i). Assume, to the contrary, that $U^1\mu(0) = \infty$. Then

$$U^1\mu^{\gamma}(0) = \sum_k \frac{\hat{\mu}(k) e^{-\gamma|k|}}{1 + \psi(k)} \uparrow U^1\mu(0) = \infty. \tag{6.39}$$

By the Paley–Zygmund inequality we see that

$$E^0\left(L_\lambda^{\mu^{\gamma}} \ge \delta E^0\left(L_\lambda^{\mu^{\gamma}}\right)\right) \ge (1-\delta)^2 \frac{\left(E^0\left(L_\lambda^{\mu^{\gamma}}\right)\right)^2}{E^0\left(\left(L_\lambda^{\mu^{\gamma}}\right)^2\right)} \tag{6.40}$$

for any $0 < \delta < 1$. Also,

$$E^0\left(L_\lambda^{\mu^{\gamma}}\right) = U^1\mu^{\gamma}(0) \tag{6.41}$$

and

$$E^0\Big(\big(L_\lambda^{\mu^\gamma}\big)^2\Big) = 2\int u^1(0,y)\big(U^1\mu^\gamma(y)\big)\,d\mu^\gamma(y) \le 2\big(U^1\mu^\gamma(0)\big)^2, \tag{6.42}$$

where, in the last inequality, we use the fact that $\hat{\mu}(k) \ge 0$ for all $k \in Z^n$ which implies that $U^1\mu^\gamma(0) \ge U^1\mu^\gamma(x)$ for all $x \in T^n$.

Using these inequalities in (6.40), we see that for $0 < \gamma \le \gamma_0$, for some γ_0 sufficiently small,

$$E^0\left(L_\lambda^{\mu^\gamma} \ge \frac{1}{2}U^1\mu^\gamma(0)\right) \ge c \tag{6.43}$$

for the same constant c as in (6.38). This contradiction proves that (iv) implies (i). □

PROOF OF COROLLARY 1.1. It is convenient to take $T^3 =_{\text{def}} (-\pi,\pi]^3$. Let $\{h(\xi),\ \xi \in T^3\}$ be radially symmetric such that $h(\xi) = f(|\xi|)$ and let μ be a measure on T^3 with density $h(\xi)$. Clearly, $U^1\mu(0) < \infty$ if and only if $uf(u) \in L^1([0,1])$.

We first show that (iii) implies (ii). Let us assume that (iii) holds. This implies that f is regularly varying at 0 with index greater than or equal to -2 and that $\{\mathscr{L}(x,t),\ (x,t) \in T^3 \times R^+\}$ is stochastically equivalent to $L = \{L_t^{\mu_x},\ (x,t) \in T^3 \times R^+\}$. We show that the Gaussian chaos associated with L is continuous almost surely which implies, by Theorem 1.3, that L is continuous almost surely. To begin, we estimate the Fourier coefficients $\{\hat{\mu}(n)\}_{n \in Z^3}$ of μ. We have

$$\begin{aligned}\hat{\mu}(n) &= (2\pi)^{-3}\int_{|\xi|\le 1}(\cos \xi n)f(|\xi|)\,d\xi \\ &= (2\pi)^{-3}\int_0^1\left(\int \cos(v|n|(n/|n|)u)\,d\sigma(u)\right)v^2 f(v)\,dv,\end{aligned} \tag{6.44}$$

where $\sigma(u)$ is a uniform measure on the unit sphere in R^3. Let $n \ne 0$. Since the last integral in (6.44) is independent of $n/|n|$, we take $n/|n| = (0,0,1)$. Then

$$\begin{aligned}\hat{\mu}(n) &= (2\pi)^{-2}\int_0^1\int_0^\pi \cos(v|n|\cos\phi)\sin\phi\,d\phi\, v^2 f(v)\,dv \\ &= \frac{1}{2\pi^2|n|}\int_0^1 \sin(v|n|)vf(v)\,dv.\end{aligned} \tag{6.45}$$

Since $vf(v)$ is decreasing and f is regularly varying at 0 with index greater than or equal to -2, we see that

$$0 \le \hat{\mu}(n) \le \frac{1}{2\pi^2}\int_0^{\pi/|n|} v^2 f(v)\,d(v) \le C\frac{f(1/|n|)}{|n|^3}. \tag{6.46}$$

If f is regularly varying at 0 with index greater than -2, it follows from (6.46) and Theorem 6.2 that L is continuous almost surely. Thus we need only consider the case when f is regularly varying at 0 with index -2. By (6.46) and the fact that in Lemma 2.2 of [15] we can work with a majorant of $\hat{\mu}$, we see that the Gaussian chaos

$$\sum_{j,k\in Z^n} \frac{(g_j g_k - \delta_{j,k})\hat{\mu}(j-k)e^{i(j-k)x}}{\sqrt{1+|j|^2}\sqrt{1+|k|^2}}, \qquad x \in T^3, \tag{6.47}$$

is continuous almost surely. Therefore, by Lemma 6.2, the Gaussian chaos associated with L is continuous almost surely. Thus we see that (iii) implies (ii).

Obviously, (ii) implies (i). Now assume that (i) holds. For $v \in (0,1]$ define

$$f_m(v) = \begin{cases} 1/v, & 1/v > f(1/m), \\ f(v), & 1/v \le f(1/m). \end{cases}$$

Let μ_m be the measure on T^3 with density $h_m(\xi) = f_m(|\xi|)$. Since $vf_m(v)$ is nonincreasing on $(0,\infty)$, it follows from (6.46) that $\hat{\mu}_m(n) \ge 0$ for all $n \in Z^3$. This implies that $U^1\mu_m(0) \ge U^1\mu_m(x)$ for all $x \in T^3$. Note that $U^1\mu_m(0)$ is finite for all $m \in Z^3$ and $U^1\mu_m(0)\uparrow U^1\mu(0)$ as $m \to \infty$. Also,

$$\int_0^\lambda f_m(X_s)\,ds \uparrow \int_0^\lambda f(X_s)\,ds \quad \text{as } m \to \infty. \tag{6.48}$$

Using these facts and the Paley–Zygmund inequality, as in the preceding proof, we see that (i) implies (iii). □

7. Moduli of continuity. Let X be a Markov process with 1-potential density $u^1(x,y)$. Let $\mathscr{M} \subset \mathscr{G}_1^2 \cap \mathrm{Rev}(X)$ be compact with respect to the metric d_1 given in (1.12). We consider $L = \{L_t^\mu, (t,\mu) \in R^+ \times \mathscr{M}\}$ and the associated Gaussian chaos $H_1 = \{H_1(\mu), \mu \in \mathscr{M}\}$ defined just before Theorem 2.2 and assume that both of these processes are continuous almost surely. As in [12], we can use the isomorphism theorem to carry over some results about the moduli of continuity of H_1 to L.

THEOREM 7.1. *Let H_1 and L be as defined above and let τ be a real-valued function on $\mathscr{M} \times \mathscr{M}$. Assume that*

$$\lim_{\delta\to 0} \sup_{\tau(\mu,\nu)\le\delta} \frac{|H_1(\mu) - H_1(\nu)|}{\tau(\mu,\nu)} \le C \quad a.s. \tag{7.1}$$

for some constant $0 \le C < \infty$. Then

$$\lim_{\delta\to 0} \sup_{\tau(\mu,\nu)\le\delta} \frac{|L_t^\mu - L_t^\nu|}{\tau(\mu,\nu)} \le C, \quad P^x\ a.s. \tag{7.2}$$

for almost all $t \in [0,\zeta)$, for all $x \in S - Q$, for some polar set Q.

PROOF. We assume for simplicity that $\zeta = \infty$; essentially the same proof works for general ζ. Take f to be strictly positive and $F = 1_{B^c}$ in Theorem 2.2, where

$$B = \left\{ g \in C(\mathscr{M}) \,\middle|\, \lim_{\delta \to 0} \sup_{\tau(\mu,\nu) \le \delta} \frac{|g(\mu) - g(\nu)|}{\tau(\mu,\nu)} \le \frac{C}{2} \right\}.$$

Assumption (7.1) implies that

$$F\left(\tfrac{1}{2} H_1(\mu)\right) = 0, \quad P_G \text{ a.s.,} \tag{7.3}$$

where P_G is the probability measure on the probability space of H_1. Hence the left-hand side of (2.29) is equal to 0. Since f is strictly positive, this implies that

$$F\left(L_\lambda^\mu + \tfrac{1}{2} H_1(\mu)\right) = 0, \quad P_G \times P^\rho \times P_\lambda \text{ a.s.} \tag{7.4}$$

or, equivalently, that

$$\begin{aligned} \lim_{\delta \to 0} \sup_{\tau(\mu,\nu) \le \delta} & \frac{\left| L_\lambda^\mu + \frac{1}{2} H_1(\mu) - \left(L_\lambda^\nu + \frac{1}{2} H_1(\nu) \right) \right|}{\tau(\mu,\nu)} \\ & \le \frac{C}{2}, \quad P_G \times P^\rho \times P_\lambda \text{ a.s.} \end{aligned} \tag{7.5}$$

Using (7.1) again, we see that

$$\lim_{\delta \to 0} \sup_{\tau(\mu,\nu) \le \delta} \frac{|L_\lambda^\mu - L_\lambda^\nu|}{\tau(\mu,\nu)} \le C, \quad P^\rho \times P_\lambda \text{ a.s.} \tag{7.6}$$

Using the argument in the paragraph following (3.3), in the proof of Theorem 1.1, we see that

$$\lim_{\delta \to 0} \sup_{\tau(\mu,\nu) \le \delta} \frac{|L_\lambda^\mu - L_\lambda^\nu|}{\tau(\mu,\nu)} \le C, \quad P^x \times P_\lambda \text{ a.s.} \tag{7.7}$$

for all $x \in S$ outside some polar set $Q \subset S$. It is clear that (7.7) implies (7.2). This completes the proof of Theorem 7.1. □

Similar to Theorem 3.2, for continuous additive functionals of a Lévy process on a group determined by a translation-invariant set of measures, we can remove the restriction on the starting point of the process. For simplicity, we consider Lévy processes on R^n and measures $\mathscr{M} = \{\mu_a,\ a \in R^n\}$, a set of translations of a fixed finite measure μ on R^n.

THEOREM 7.2. *Let X be a Lévy process on R^n. Let μ be a finite measure in $\mathscr{G}_\alpha^2$ and let $\mu_a = \mu(A + a)$ for all $a \in R^n$ and Borel sets $A \subset R^n$. Let $\mathscr{M} = \{\mu_a, a \in R^n\}$ with $\mu \in \mathrm{Rev}(X)$ and let H_1 and L be as defined in the beginning of this section with respect to X and $\mathscr{M}$. Assume that, for some real-valued function τ,*

$$\limsup_{\substack{|a-b| \to 0 \\ a, b \in [0,1]^n}} \frac{H_1(\mu_a) - H_1(\mu_b)}{\tau(|a-b|)} \le C \quad a.s. \tag{7.8}$$

for some constant $0 \le C < \infty$. *Then*

$$\limsup_{\substack{|a-b|\to 0\\ a,b\in[0,1]^n}} \frac{L_t^{\mu_a} - L_t^{\mu_b}}{\tau(|a-b|)} \le C \tag{7.9}$$

for almost all $t \in R^+$ *almost surely.*

PROOF. This follows immediately from Theorem 7.1 and the fact that (3.12) holds for these processes. □

In order to use Theorem 7.2, we need to know the moduli of continuity for $\{H_1(\mu_a),\ a \in [0,1]^n\}$. Actually, not much is known about the moduli of continuity of second-order Gaussian chaoses. We make do here with some simple consequences of the well-known fact that for any second-order Gaussian chaos, say $\{\tilde{H}(t),\ t \in T\}$, where T is some index set,

$$E\left(\exp \lambda \frac{\tilde{H}(a) - \tilde{H}(b)}{\theta(a,b)}\right) < C \qquad \forall\, a,b \in T \tag{7.10}$$

for some $\lambda > 0$, where

$$\theta(a,b) = \left(E\left(\tilde{H}(a) - \tilde{H}(b)\right)\right)^{1/2}. \tag{7.11}$$

LEMMA 7.1. *Let* $\{\tilde{H}(a),\ a \in R^n\}$ *be a second-order Gaussian chaos satisfying*

$$\left(E\left(\tilde{H}(a) - \tilde{H}(b)\right)^2\right)^{1/2} \le \rho(|a-b|), \tag{7.12}$$

where ρ *is a regularly varying function, at* 0, *with index greater than* 0. *Then there exists a constant* $0 \le C < \infty$ *such that*

$$\limsup_{\substack{|a-b|\to 0\\ a,b\in[0,1]^n}} \frac{\tilde{H}(a) - \tilde{H}(b)}{\rho(|a-b|)|\log|a-b||} \le C \quad a.s. \tag{7.13}$$

PROOF. There are many well-known techniques for obtaining a result such as this one. We will use a lemma of Garcia, Rodemich and Rumsey. Since ρ is regularly varying with index greater than 0, without loss of generality we can assume that $\rho(|u|)\downarrow 0$ and $|u|\downarrow 0$ and that it is continuous in some neighborhood of 0. It is well known that

$$E\exp\left(\frac{\left|\tilde{H}(a) - \tilde{H}(b)\right|}{4\rho(|a-b|)}\right) < C \qquad \forall\, a,b \in R^n \tag{7.14}$$

for some constant $C < \infty$. (See, e.g., Corollary 3.9 of [9].) Furthermore, since ρ is regularly varying with index greater than 0, it is easy to see that (1.14) holds. Thus $\{\tilde{H}(a),\ a \in R^n\}$ has a continuous version. Let $\zeta(x) = \exp(|x|/4)$. It is clear from (7.14) that

$$\int_{[0,1]^n}\int_{[0,1]^n} \zeta\left(\frac{\left|\tilde{H}(a) - \tilde{H}(b)\right|}{4\rho(|a-b|)}\right) da\, db < C < \infty \tag{7.15}$$

on a set of measure 1. The statement in (7.13) now follows from Lemma 4.1, Chapter 4, of [7]. (See also Lemma 3.3.13 of [18].) □

PROOF OF THEOREM 1.7. By the hypotheses on the dimension of A, there exists a finite measure μ supported on A such that

$$\iint \frac{d\mu(x)\,d\mu(y)}{|x-y|^{\alpha}} < \infty \qquad \forall\, \alpha < \beta. \tag{7.16}$$

This implies that

$$\int \frac{|\hat{\mu}(\xi)|^2}{|\xi|^{2-\alpha}}\,d\xi < \infty. \tag{7.17}$$

Let $\alpha < \beta$. Consider the Gaussian chaos $\{H_1(\mu_a),\ a \in [0,1]^n\}$ associated with B. Taking the Fourier transform in (1.12), we get

$$\begin{aligned}
d_1(\mu_a, \mu) &\le C\left(\int \frac{(\sin^2 \xi a)\log(1+|\xi|)\,|\hat{\mu}(\xi)|^2}{1+|\xi|^2}\,d\xi\right)^{1/2}\\
&\le C\left(\sup_{\xi\in R^2} \frac{(\sin^2 \xi a)\log(1+|\xi|)}{|\xi|^{\alpha}} \int \frac{|\hat{\mu}(\xi)|^2}{(1+|\xi|^2)^{(2-\alpha)/2}}\,d\xi\right)^{1/2}\\
&\le C(|a|^{\alpha}\log(1+1/|a|))^{1/2}\\
&\le |a|^{\beta/2-\varepsilon}
\end{aligned}\tag{7.18}$$

for all $\varepsilon > 0$, for $|a| \le a_0$, for some a_0 sufficiently small. Since $d_1(\mu_{a-b}, \mu) = d_1(\mu_a, \mu_b)$, we can use (7.18) in Lemma 7.1 to get

$$\limsup_{\substack{|a-b|\to 0\\ a,b\in[0,1]^2}} \frac{H_1(\mu_a) - H_1(\mu_b)}{|a-b|^{(\beta/2)-\varepsilon}} = 0 \quad \text{a.s.} \tag{7.19}$$

Therefore (1.31) will follow from Theorem 7.2 once we show that H_1 and L are continuous. Obviously, (7.19) implies that H_1 is continuous. Theorem 1.3 now completes the proof of Theorem 1.7. □

PROOF OF THEOREM 1.8. As in the above proof, everything follows from an upper bound for $d_1(\mu_a, \mu)$. In this case, for $p = 2$, we have

$$\begin{aligned}
d_1(\mu_a, \mu) &\le C\left(\int \frac{|\sin^2 \xi a|\log(1+|\xi|)\,|\hat{f}(\xi)|^2}{1+|\xi|^2}\,d\xi\right)^{1/2}\\
&\le C\left(\sup_{|\xi|} \frac{|\sin^2 \xi a|\log(1+|\xi|)}{1+|\xi|^2}\right)^{1/2}\|\hat{f}\|_2\\
&\le C|a|(\log(1+1/|a|))^{1/2}\|\hat{f}\|_2
\end{aligned}\tag{7.20}$$

and, for $1 < p < 2$, we have

$$d_1(\mu_a, \mu) \le C \left(\int \frac{|\sin \xi a|^{2r} (\log(1 + |\xi|))^r}{(1 + |\xi|^2)^r} \, d\xi \right)^{1/(2r)} \|\hat{f}\|_q, \tag{7.21}$$

where $1/p + 1/q = 1$ and $1/r + 2/q = 1$. By Young's inequality $\|\hat{f}\|_q \le C\|f\|_p$. Also, by a simple estimate,

$$\begin{aligned} &\int \frac{|\sin \xi a|^{2r} (\log(1 + |\xi|))^r}{(1 + |\xi|^2)^r} \, d\xi \\ &\quad \le |a|^{2r} \int_{|\xi| \le 1/|a|} (\log(1 + |\xi|))^r \, d\xi + \int_{|\xi| \ge 1/|a|} \frac{(\log(1 + |\xi|))^r}{|\xi|^{2r}} \, d\xi. \end{aligned} \tag{7.22}$$

Thus

$$d_1(\mu_a, \mu) \le C|a|^{1-1/r} (\log(1 + 1/|a|))^{1/2} \|f\|_p. \tag{7.23}$$

Using this in Lemma 7.1, we get

$$\limsup_{\substack{|a-b| \to 0 \\ a,b \in [0,1]^2}} \frac{H(\mu_a) - H(\mu_b)}{|a - b|^{2-(2/p)} |\log|a - b||^{3/2}} \le C\|f\|_p \quad \text{a.s.} \tag{7.24}$$

Therefore, as in the proof of Theorem 1.7, (1.32) follows from Theorem 7.2. This completes the proof of Theorem 1.8. □

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