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THE POINT OF VIEW OF THE PARTICLE ON THE LAW OF LARGE NUMBERS FOR RANDOM WALKS IN A MIXING RANDOM ENVIRONMENT[1]

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The point of view of the particle is an approach that has proven very powerful in the study of many models of random motions in random media. We provide a new use of this approach to prove the law of large numbers in the case of one or higher-dimensional, finite range, transient random walks in mixing random environments. One of the advantages of this method over what has been used so far is that it is not restricted to i.i.d. environments.

1. Introduction. Originating from the physical sciences, the subject of random media has gained much interest over the last three decades. One of the fundamental models in the field is random walks in a random environment. The main purpose of this work is to prove the law of large numbers for a certain class of random walks in a mixing random environment. In this model, an environment is a collection of transition probabilities $\omega = (\pi_{ij})_{i,j\in\mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d\times\mathbb{Z}^d}$, with $\sum_{j\in\mathbb{Z}^d}\pi_{ij} = 1$. Let us denote by Ω the space of all such transition probabilities. The space Ω is equipped with the canonical product σ-field $\mathfrak{S}$, and with the natural shift $(T^k\omega)_{i,j} = \omega_{k+i,k+j}$ for $k \in \mathbb{Z}^d$. Here, ω_{ij} stands for the (i,j)th coordinate of $\omega \in \Omega$. We will also use $\omega_i = (\omega_{ij})_{j\in\mathbb{Z}^d}$. On the space of environments $(\Omega, \mathfrak{S})$, we are given a certain T-invariant probability measure $\mathbb{P}$, with $(\Omega, \mathfrak{S}, (T^k)_{k\in\mathbb{Z}^d}, \mathbb{P})$ ergodic. We will say that the environment is i.i.d. when $\mathbb{P}$ is a product measure. Let us now describe the process. First, the environment ω is chosen from the distribution $\mathbb{P}$. Once this is done, it remains fixed for all times. The random walk in environment ω is then the canonical Markov chain $(X_n)_{n\ge0}$ with state space $\mathbb{Z}^d$ and transition probability

$$P_x^\omega(X_0 = x) = 1,$$
$$P_x^\omega(X_{n+1} = j | X_n = i) = \pi_{ij}(\omega).$$

The process P_x^ω is called the *quenched law*. The *annealed law* is then

$$P_x = \int P_x^\omega \, d\mathbb{P}(\omega).$$

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Already, one can see one of the difficulties of the model. When the environment ω is not fixed, that is, under P_0, X_n stops being Markovian.

Many questions arise about the different possible limit theorems, such as the law of large numbers, central limit theorems, large deviation results and so on. In the one-dimensional nearest-neighbor case, the situation has been well understood (see, e.g., [16, 19] and the references therein). The reason for this is the possibility of explicit computations, and the reversibility of the Markov chain. In the higher-dimensional case, however, the amount of results is significantly less (once again, see [16, 19] for an overview).

In the present paper, we are interested in the law of large numbers. In the one-dimensional case, Solomon [15] and Alili [1] proved that the speed of escape of the particle (velocity at large times) is a constant, P_0-a.s., that depends only on the distribution of the environment. Later, Sznitman and Zerner [18] proved that, under some technical transience condition on $\mathbb{P}$ (the so-called Kalikow condition), the law of large numbers still holds in the multidimensional situation with i.i.d. environments. To overcome the non-Markovian character of the walk, they used a renewal-type argument that appeared to be very specific to i.i.d. environments. Still, using the same method, Zeitouni [19] proved the law of large numbers, when i.i.d. environments are replaced by ones that are independent when a gap of size L is allowed. For more general mixing environments, the method seems to be too rigid. However, physically relevant models, such as diffusions with random coefficients, suggest that removing the independent environment hypothesis is an important step toward a further understanding of random walks in a random environment. For this, a different approach is required. One approach that has proven to be very powerful in the study of several other examples of random motions in random media, such as in [9, 4, 13, 14], is termed the "point of view of the particle." In this approach, one considers the process $(T^{X_n}\omega)$ of the environment as seen from the particle. This process is now a Markov process, with initial distribution $\mathbb{P}$. The new inconvenience is that this Markov process has for its state space the huge set Ω. To apply the standard ergodic theorem, Kozlov [11] showed that one needs to find an ergodic measure $\mathbb{P}_\infty$ that is invariant for the process $(T^{X_n}\omega)$ and absolutely continuous relative to $\mathbb{P}$; see also Lemma K below. This approach works perfectly in the one-dimensional case, since one can compute $\mathbb{P}_\infty$ explicitly (see [1]). Moreover, in this case, one does not need the i.i.d. hypothesis. The hard problem, though, is to find such a measure. In the case of balanced walks (see [12]), one can prove the existence of such a measure, without actually computing it. Even though, in the two cases we mentioned above, the method of the point of view of the particle did solve the problem, it seems to have so far been of little help in the more general cases of random walks in random environments.

As one will see in Section 4 below, one cannot always expect to be able to find an invariant measure $\mathbb{P}_\infty$ that is absolutely continuous relative to $\mathbb{P}$, in all of Ω.

However, when studying walks that are transient in some direction $\ell \in \mathbb{R}^d \setminus \{0\}$, one expects the trajectories to stay in some half-plane $H_k = \{x \in \mathbb{Z}^d : x \cdot \ell \geq k\}$ for $k \leq 0$. In this paper we further develop the approach of the point of view of the particle, to be able to use it in the investigation of higher-dimensional random walks in a not necessarily i.i.d. random environment. In Theorem 2, we show that the conclusion of Kozlov's lemma still holds if $\mathbb{P}_\infty$ is absolutely continuous relative to $\mathbb{P}$, in every half-space H_k, instead of all of $\mathbb{Z}^d$. Then, in Theorem 3, we show that Kalikow's condition implies that, after having placed the walker at the origin, the trajectories do not spend "too much" time inside any half-plane H_k, collecting therefore little information about the environment in there, and satisfying the hypotheses of Theorem 2. This implies the law of large numbers we are aiming for.

We will need the following definition. We say that we have a finite range environment, or that the walk has finite range $M < \infty$, when

$$\mathbb{P}(\pi_{ij} = 0 \text{ when } |i - j| > M) = 1.$$

Throughout the rest of this work, we will only consider finite range random walks in a random environment.

Let us now explain the structure of this paper. Section 2 introduces Kalikow's condition. There, we give an effective condition that implies Kalikow's condition, even when $\mathbb{P}$ is not a product measure. By an effective condition, we mean a condition that can be checked directly on the environment.

In Section 3, we start with a warm-up calculation. We consider the one-dimensional finite range situation. We do not assume $\mathbb{P}$ to be a product measure. In Theorem 1, we prove the law of large numbers in the one-dimensional nonnearest-neighbor case, under Kalikow's condition.

In Section 4, we explain why, in general, one cannot use Kozlov's lemma in the multidimensional setup.

In Section 5, we prove Theorem 2, which extends Kozlov's lemma. We show that, to have a law of large numbers, it is enough for the invariant measure $\mathbb{P}_\infty$ to be absolutely continuous relative to $\mathbb{P}$ only in certain "relevant" parts of $\mathbb{Z}^d$.

In Section 6, we introduce the Dobrushin–Shlosman strong mixing condition.

In Section 7, we use Theorem 2 to prove that Kalikow's condition implies the law of large numbers for finite range random walks in a mixing random environment. This is our main theorem (Theorem 3).

2. Kalikow's condition.

Let us start with a definition. We define the drift D to be

$$D(\omega) = E_0^\omega(X_1) = \sum_i i\pi_{0i}(\omega). \tag{2.1}$$

When studying the law of large numbers, one could try to examine first the case when the environment satisfies some condition that guarantees a strong drift in some direction $\ell \in \mathbb{R}^d \setminus \{0\}$. One such condition was introduced by Kalikow [8]:

$$\inf_{U \in \mathscr{U}} \inf_{x \in U} \frac{E_0(\sum_{j=0}^{T_U} \mathbb{1}(X_j = x) D(T^{X_j}\omega) \cdot \ell)}{E_0(\sum_{j=0}^{T_U} \mathbb{1}(X_j = x))} = \varepsilon > 0, \tag{2.2}$$

where $T_U = \inf\{j \geq 0 : X_j \notin U\}$, and $\mathscr{U}$ is ranging over all finite sets that contain 0 and have a path of range M passing through all its points. We will call such sets M-connected. The expectations involved in the above condition are all finite and positive (cf. [8], pages 756–757), if one assumes the following ellipticity condition to hold:

$$\begin{aligned} &\text{There exists } \kappa(\mathbb{P}) \in (0, 1) \text{ such that} \\ &\mathbb{P}(\pi_{i,j} > \kappa \text{ when } |i - j| \leq M) = 1. \end{aligned} \tag{2.3}$$

In some situations, we will assume, instead, the weaker ellipticity condition

$$\mathbb{P}(\forall\, j \text{ s.t. } j \cdot \ell \geq 0 \text{ and } |j| = 1 : \pi_{0j} > 0) = 1. \tag{2.4}$$

In the rest of this work, we will consider condition (2.3) to be part of Kalikow's condition (2.2). Sznitman and Zerner's [18] law of large numbers was established under condition (2.2). As a matter of fact, Kalikow's condition, in the one-dimensional i.i.d. nearest-neighbor case, is equivalent to the condition $\mathbb{E}(\rho) < 1$ (cf. [18], pages 1866–1867). According to Solomon [15], this condition characterizes the situation of walks with a positive speed of escape. This is not the case in higher dimensions. In fact, Sznitman [17] proved that, in the i.i.d. case, Kalikow's condition implies a strictly more general condition (the so-called T' condition), which also implies a law of large numbers with a positive velocity. One way to motivate Kalikow's condition is revealed by Proposition 1 in [8], pages 757–758.

Of course (2.2) is not very practical, since it is not a condition on the environment. Clearly, if one has a nonnestling environment, that is, if there exists a $\delta > 0$ such that $\mathbb{P}(D \cdot \ell \geq \delta) = 1$, then (2.2) holds. In the nestling case, however, there is a condition that is more concrete than (2.2), that implies it and at the same time follows from many other interesting conditions on the drift [such as $\mathbb{P}(D \cdot \ell < 0) > 0$, but there exists a constant $\varepsilon > 0$, such that $\mathbb{P}(D \cdot \ell \geq -\varepsilon) = 1$, and $\mathbb{P}(D \cdot \ell < 0) < C_\varepsilon$ small enough]. It has already been established in [8], pages 759–760, and [16], pages 36–37, that, under the hypothesis that the environment is i.i.d.,

$$\mathbb{E}(D \cdot \ell^+) > \kappa^{-1} \mathbb{E}(D \cdot \ell^-) \tag{2.5}$$

implies (2.2). In fact, one can relax the i.i.d. hypothesis as follows. Let $\omega_{\not x} = (\omega_y)_{y \neq x}$, and define $Q_{\omega_{\not x}}$ to be the regular conditional probability, knowing $\omega_{\not x}$, Q_x be the marginal of ω_x, and $Q_{\not x}$ the marginal of $\omega_{\not x}$.

PROPOSITION 1. *Suppose that, $Q_{\emptyset}$-almost surely, $Q_{\omega_{\emptyset}} \ll Q_0$, and that there exist two positive constants A and B, such that for $Q_0 \otimes Q_{\emptyset}$-almost every $\omega = (\omega_0, \omega_{\emptyset})$ one has*

$$0 < A \le h(\omega_0, \omega_{\emptyset}) = \frac{dQ_{\omega_{\emptyset}}}{dQ_0}(\omega_0) \le B < \infty. \tag{2.6}$$

Then the ellipticity condition (2.3), *along with*

$$\mathbb{E}(D \cdot \ell^+) > \kappa^{-1} B A^{-1} \mathbb{E}(D \cdot \ell^-), \tag{2.7}$$

implies Kalikow's condition (2.2).

PROOF. Fix $U \subset \mathbb{Z}^d$, with $0 \in U$. Define, for $\omega \in \Omega$, $x, y \in \mathbb{Z}^d$,

$$f_\omega(x) = P_0^\omega\big(\exists k \in [0, T_U) : X_k = x\big),$$
$$g_\omega(x, y) = P_{x+y}^\omega\big(X_k \ne x \ \forall k \in [0, T_U]\big).$$

Note that we have, for $x \in U$,

$$P_x^\omega\big(X_k \ne x \ \forall k \in (0, T_U]\big) = \sum_{|y| \le M} \pi_{x,x+y}(\omega) g_\omega(x, y).$$

Once x is visited before exiting U, the number of returns to x, up to time T_U, is geometrically distributed with the above failure probability. Therefore, for $x \in U$, we have

$$E_0^\omega\left(\sum_{j=0}^{T_U} \mathbb{1}(X_j = x)\right) = \frac{f_\omega(x)}{\sum_{|y| \le M} g_\omega(x, y) \pi_{x,x+y}(\omega)},$$

where the numerator is exactly the probability of visiting x at least once. Since $f_\omega(x)$ and $g_\omega(x, y)$ are $\sigma(\omega_z;\ z \ne x)$-measurable, one has

$$\begin{aligned}
&\int \frac{f_\omega(x) D(T^x \omega) \cdot \ell}{\sum_{|y| \le M} g_\omega(x, y) \pi_{x,x+y}(\omega)}\, d\mathbb{P}(\omega) \\
&\quad = \int dQ_{\not x}(\omega_{\not x}) \int \frac{f_\omega(x) D(T^x \omega) \cdot \ell}{\sum_{|y| \le M} g_\omega(x, y) \pi_{x,x+y}(\omega)} h(\omega_x, \omega_{\not x})\, dQ_x(\omega_x) \\
&\quad \ge \int dQ_{\not x}(\omega_{\not x}) \int \frac{f_\omega(x)}{\max_{|y| \le M} g_\omega(x, y)} \\
&\qquad\qquad \times \big(A D \cdot \ell^+(T^x \omega) - \kappa^{-1} B D \cdot \ell^-(T^x \omega)\big)\, dQ_x(\omega_x) \\
&\quad = \mathbb{E}\big(A D \cdot \ell^+ - \kappa^{-1} B D \cdot \ell^-\big) \int \frac{f_\omega(x)}{\max_{|y| \le M} g_\omega(x, y)}\, d\mathbb{P}(\omega) \\
&\quad \ge \kappa \mathbb{E}\big(A D \cdot \ell^+ - \kappa^{-1} B D \cdot \ell^-\big) \int \frac{f_\omega(x)}{\sum_{|y| \le M} g_\omega(x, y) \pi_{x,x+y}(\omega)}\, d\mathbb{P}(\omega),
\end{aligned}$$

which is Kalikow's condition with $\varepsilon = \kappa\mathbb{E}(AD \cdot \ell^+ - \kappa^{-1} BD \cdot \ell^-) > 0$. $\square$

Notice that in the i.i.d. case, (2.6) clearly holds with $A = B = 1$, and condition (2.7) is the same as (2.5). Condition (2.6) can also be easily checked, in the case of Gibbs specifications, which we will use in the higher-dimensional case; see Section 6. In this case, there exists a C_1 [same as in (A.2)] such that the marginal μ_0, of the reference measure, satisfies

$$C_1^{-2} Q_0 \le C_1^{-1} \mu_0 \le Q_{\omega_\emptyset} \le C_1 \mu_0 \le C_1^2 Q_0.$$

Next, we show two implications of Kalikow's condition (2.2). First, the walk has a ballistic character, in the following sense.

LEMMA 2. *Assume we have a finite range environment for which Kalikow's condition* (2.2) *holds. Let $U \subset \mathbb{Z}^d$ be an M-connected set containing* 0, *for which $E_0(T_U) < \infty$. Then $E_0(X_{T_U} \cdot \ell) \ge \varepsilon E_0(T_U)$.*

PROOF. For a finite U, Kalikow's condition implies that

$$E_0\left(\sum_{j=0}^{T_U} \mathbb{1}(X_j = x) D(T^{X_j}\omega) \cdot \ell\right) \ge \varepsilon E_0\left(\sum_{j=0}^{T_U} \mathbb{1}(X_j = x)\right).$$

Summing over all $x \in U$, and using that $D(T^{X_j}\omega) = E_0^\omega(X_{j+1} - X_j | \mathscr{F}_j)$, and that T_U is a stopping time, one has

$$E_0\left(\sum_{j=0}^{T_U - 1} (X_{j+1} - X_j) \cdot \ell\right) \ge \varepsilon E_0(T_U).$$

The claim follows. For an infinite U, the lemma follows from the monotone convergence theorem, by taking increasing limits of finite sets. $\square$

The other consequence of Kalikow's condition is that, under this condition, the walk almost surely escapes to infinity in direction ℓ. This was originally proved by Kalikow [8], and we reprove it here for the sake of completeness. We also prove that the number of returns to the origin has a finite annealed expectation.

LEMMA 3. *Under Kalikow's condition* (2.2), *we have*

$$P_0\left(\lim_{n\to\infty} X_n \cdot \ell = \infty\right) = 1 \tag{2.8}$$

and

$$\sum_{j \ge 0} P_0(X_j = 0) < \infty. \tag{2.9}$$

PROOF. Let $U \subset \mathbb{Z}^d$ be a finite M-connected set containing 0. Rewriting (2.2), multiplying both sides by $e^{-\lambda x \cdot \ell}$ for $\lambda > 0$, and summing over all $x \in U$, one has

$$E_0\Biggl(\sum_{j=0}^{T_U-1} e^{-\lambda X_j \cdot \ell} D(T^{X_j}\omega) \cdot \ell\Biggr) \geq \varepsilon E_0\Biggl(\sum_{j=0}^{T_U-1} e^{-\lambda X_j \cdot \ell}\Biggr). \tag{2.10}$$

On the other hand, since T_U is a stopping time, one can write

$$\sum_{j=1}^{T_U} E_0^\omega(e^{-\lambda X_j \cdot \ell} | \mathscr{F}_{j-1}) = \sum_{j \geq 1} E_0^\omega(\mathbb{1}(T_U \geq j) e^{-\lambda X_j \cdot \ell} | \mathscr{F}_{j-1}).$$

Hence, we have

$$\begin{aligned}
E_0\Biggl(\sum_{j=1}^{T_U} e^{-\lambda X_j \cdot \ell}\Biggr) &= E_0\Biggl(\sum_{j=1}^{T_U} E_0^\omega(e^{-\lambda X_j \cdot \ell} | \mathscr{F}_{j-1})\Biggr) \\
&= E_0\Biggl(\sum_{j=1}^{T_U} e^{-\lambda X_{j-1} \cdot \ell}(1 - \lambda D(T^{X_{j-1}}\omega) \cdot \ell + O(M^2\lambda^2))\Biggr) \\
&\leq E_0\Biggl(\sum_{j=0}^{T_U-1} e^{-\lambda X_j \cdot \ell}\Biggr)(1 - \lambda\varepsilon + O(M^2\lambda^2)),
\end{aligned}$$

where we have used (2.10) to get the inequality. Taking $\lambda > 0$ small enough, and increasing U to all of $\mathbb{Z}^d$, one has

$$E_0\Biggl(\sum_{j \geq 0} e^{-\lambda X_j \cdot \ell}\Biggr) < \infty \tag{2.11}$$

and, therefore,

$$P_0\Bigl(\liminf_{n \to \infty} X_n \cdot \ell < \infty\Bigr) \leq P_0\Biggl(\sum_{j \geq 0} e^{-\lambda X_j \cdot \ell} = \infty\Biggr) = 0,$$

proving (2.8). Using (2.11), one also proves (2.9),

$$\sum_{j \geq 0} P_0(X_j = 0) = E_0\Biggl(\sum_{j \geq 0} \mathbb{1}(X_j = 0)\Biggr) \leq E_0\Biggl(\sum_{j \geq 0} e^{-\lambda X_j}\Biggr) < \infty. \qquad \square$$

Next, as a warm-up for the method we will use later in the multidimensional situation, we examine the simpler case of one-dimensional random walks.

3. The one-dimensional case. In this section, we will prove the law of large numbers for one-dimensional finite range random walks in a random environment. Let us recall a lemma, also valid for $d \geq 2$, that was proved by Kozlov [11].

LEMMA K [11]. *Assume that the weak ellipticity condition* (2.4) *holds. Suppose also that there exists a probability measure* $\mathbb{P}_\infty$ *that is invariant for the process* $(T^{X_n}\omega)_{n\geq 0}$, *and that is absolutely continuous relative to the ergodic* T*-invariant environment* $\mathbb{P}$. *Then the following hold*:

(i) *The measures* $\mathbb{P}$ *and* $\mathbb{P}_\infty$ *are in fact mutually absolutely continuous.*
(ii) *The Markov process* $(T^{X_n}\omega)_{n\geq 0}$ *with initial distribution* $\mathbb{P}_\infty$ *is ergodic.*
(iii) *There can be at most one such* $\mathbb{P}_\infty$.
(iv) *The following law of large numbers is satisfied*:

$$P_0\left(\lim_{n\to\infty}\frac{X_n}{n} = \mathbb{E}^{\mathbb{P}_\infty}(D)\right) = 1,$$

where D *is the drift defined in* (2.1).

One then has the following theorem.

THEOREM 1. *Under Kalikow's condition* (2.2), *with* $\ell = 1$, *the process* $(T^{X_n}\omega)_{n\geq 0}$ *has an invariant measure* $\mathbb{P}_\infty$ *that is absolutely continuous relative to* $\mathbb{P}$, *and we have a law of large numbers for finite range random walks in the ergodic* T*-invariant environment* $\mathbb{P}$,

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{X_n}{n} = \mathbb{E}^{\mathbb{P}_\infty}(D)\right) = 1.$$

PROOF. Define

$$g_{ij}(\omega) = \sum_{n\geq 0} P_i^\omega(X_n = j) = E_i^\omega(N_j),$$

where N_j is the number of visits of the random walk to site j. The renewal property gives, for $i \neq j$,

$$g_{ij} = E_j^\omega(N_j)P_i^\omega(V_j < \infty) \leq g_{jj},$$

with $V_j = \inf\{n > 0 : X_n = j\}$. Moreover, the g_{jj}'s are all identically distributed in the annealed setting. Thus, according to (2.9), they are all in $L^1(\Omega, \mathbb{P})$. For $i \leq j$ define

$$G_{ij} = \frac{1}{j - i + 1}\sum_{k=i}^{j} g_{kj} \leq g_{jj}.$$

Using the diagonal trick, one can extract a subsequence of the G_{ij}'s that converges weakly, as i decays to $-\infty$, to a limit $\mu_j \in L^1(\Omega, \mathbb{P})$, for all j. Then, for any fixed j, μ_j is a limit point for the g_{ij}'s as well. Using the diagonal trick again, one can find a subsequence of the g_{ij}'s that converges weakly to μ_j, for all j. We will keep referring to both subsequences by G_{ij} and g_{ij}.

Notice that if $k \neq j$, then

$$\sum_i \pi_{ij} g_{ki} = \sum_{n\geq 0} \sum_i \pi_{ij} P_k^\omega(X_n = i) = \sum_{n\geq 0} P_k^\omega(X_{n+1} = j) = g_{kj}.$$

Therefore, for $\mathbb{P}$-a.e. ω, $\sum_i \pi_{ij}\mu_i = \mu_j$. Also,

$$g_{i0} \circ T = \sum_{n\geq 0} P_i^{T\omega}(X_n = 0) = \sum_{n\geq 0} P_{i+1}^\omega(X_n = 1) = g_{i+1,1},$$

and the same holds for the G_{ij}'s. Therefore, for $\mathbb{P}$-a.e. ω,

$$\mu_0(T\omega) = \lim_{i\to-\infty} G_{i0}(T\omega) = \lim_{i\to-\infty} G_{i+1,1}(\omega) = \mu_1(\omega).$$

This shows that $\mu_0\, d\mathbb{P}$ is an invariant measure for the process $(T^{X_n}\omega)_{n\geq 0}$. Next, we need to show that μ_0 is not trivial. To this end, we recall Lemma 3. According to this lemma, Kalikow's condition implies that, for $\mathbb{P}$-a.e. ω, $P_0^\omega(\lim_{n\to\infty} X_n = \infty) = 1$. The finite range character of the walk implies then that for each $i < j$, $\mathbb{P}$-a.s., $\sum_{k=j}^{j+M-1} g_{ik} \geq 1$. Taking the limit in i, we have that, $\mathbb{P}$-a.s., $\sum_{k=j}^{j+M-1} \mu_k \geq 1$. Therefore, by the ergodicity of $\mathbb{P}$, $\mathbb{E}(\mu_0) \geq M^{-1}$.

Defining $\mathbb{P}_\infty$ such that

$$\frac{d\mathbb{P}_\infty}{d\mathbb{P}} = \frac{\mu_0}{\mathbb{E}(\mu_0)}$$

gives an invariant probability measure for the process of the environment, as seen from the particle. This measure is absolutely continuous relative to $\mathbb{P}$, and Lemma K concludes the proof. □

Now, we move to the multidimensional situation. In the following section, we will show why it is quite different from the situation above, and why Kozlov's lemma (Lemma K) cannot be used.

4. Motivation. Consider the case where $d = 2$, the environment is i.i.d., and

$$\mathbb{P}(\pi_{(0,0)(1,0)} = 1) = \mathbb{P}(\pi_{(0,0)(0,1)} = 1) = 0.5.$$

Once the environment is chosen, the walk is determined, following the assigned directions. The annealed process is in fact the same as $0.5(n - S_n, n + S_n)$, with S_n a one-dimensional simple symmetric random walk. Therefore, one obviously has the following law of large numbers:

$$P_0\left(\lim_{n\to\infty} \frac{X_n}{n} = (0.5, 0.5)\right) = 1.$$

Yet, defining $\mathbb{P}_n$ to be the measure on the environment as seen from the particle at time n,

$$\mathbb{P}_n(A) = P_0(T^{X_n}\omega \in A),$$

and $\mathfrak{S}_{-k}$ as the σ-algebra generated by the environment at sites x such that $x \cdot (1,1) \geq -k$, one has the following proposition.

PROPOSITION 4. *There exists a probability measure $\mathbb{P}_\infty$ to which $\mathbb{P}_n$ converges weakly. Moreover, $\mathbb{P}_\infty$ is mutually singular with $\mathbb{P}$, and there is no probability measure that is, at the same time, invariant for $(T^{X_n}\omega)$ and absolutely continuous relative to $\mathbb{P}$. Furthermore, for $k \leq n$, one has $\mathbb{P}_{n|\mathfrak{S}_{-k}} = \mathbb{P}_{k|\mathfrak{S}_{-k}}$, and therefore, $\mathbb{P}_{\infty|\mathfrak{S}_{-k}} = \mathbb{P}_{k|\mathfrak{S}_{-k}} \ll \mathbb{P}_{|\mathfrak{S}_{-k}}$.*

For a complete proof, see [2], Propositions 1.4. and 1.5. Although the ellipticity condition is not satisfied, this model is instructive. It shows us that, to prove a law of large numbers, one need not necessarily look for a $\mathbb{P}_\infty$ that is absolutely continuous relative to $\mathbb{P}$ on the whole space. Instead, maybe one should try to prove that $\mathbb{P}_\infty \ll \mathbb{P}$ in the "relevant" part of the space, that is, all half-spaces $\{x : x \cdot (1,1) \geq -k\}$, for $k \geq 0$. This is still much weaker than absolute continuity in the whole space. We will address this issue in the following section.

5. On the invariant measure for $d \geq 2$. For $k \in \mathbb{Z}$, let $\mathfrak{S}_k = \sigma(\omega_x : x \cdot \ell \geq k)$ be the σ-algebra generated by the part of the environment in the right half-plane $H_k = \{x : x \cdot \ell \geq k\}$. In this section, we will not assume the ellipticity condition (2.3) to hold. Instead, we will assume the weaker ellipticity condition (2.4) we assumed in Lemma K. We modify Lemma K, as suggested by the example in Section 4, and we have the following theorem.

THEOREM 2. *Let $\mathbb{P}$ be ergodic, and T-invariant, with finite range M. Assume that the weak ellipticity condition* (2.4) *holds, and that*

$$P_0\Big(\lim_{n\to\infty} X_n \cdot \ell = \infty\Big) = 1. \tag{5.1}$$

Suppose also that there exists a probability measure $\mathbb{P}_\infty$ that is invariant for the process $(T^{X_n}\omega)_{n\geq 0}$, and that is absolutely continuous relative to $\mathbb{P}$, in every half-space H_k, with $k \leq 0$. Then the following hold:

(i) *The measures $\mathbb{P}$ and $\mathbb{P}_\infty$ are in fact mutually absolutely continuous on every H_k, with $k \leq 0$.*

(ii) *The Markov process $(T^{X_n}\omega)_{n\geq 0}$ with initial distribution $\mathbb{P}_\infty$ is ergodic.*

(iii) *There can be at most one such $\mathbb{P}_\infty$, and if*

$$\tilde{\mathbb{P}}_n(A) = n^{-1} \sum_{m=1}^{n} P_0(T^{X_m}\omega \in A),$$

then $\tilde{\mathbb{P}}_n$ converges weakly to $\mathbb{P}_\infty$.

(iv) *The following law of large numbers is satisfied*:

$$\mathbb{P}\Big(\lim_{n\to\infty}\frac{X_n}{n}=\mathbb{E}^{\mathbb{P}_\infty}(D)\Big)=1.$$

PROOF. ($\forall k\le 0:\mathbb{P}_{\infty|\mathfrak{S}_k}\sim\mathbb{P}_{|\mathfrak{S}_k}$). Fix $k\le 0$, and let $G_k=\frac{d\mathbb{P}_{\infty|\mathfrak{S}_k}}{d\mathbb{P}_{|\mathfrak{S}_k}}$. Then

$$\begin{aligned}0&=\int_{\{G_k=0\}}G_k\,d\mathbb{P}=\int\mathbb{1}_{\{G_k=0\}}\,d\mathbb{P}_\infty=\int\sum_{|e|\le M}\pi_{0e}\mathbb{1}_{\{G_k=0\}}\circ T^e\,d\mathbb{P}_\infty\\&\ge\int\sum_{\substack{|e|=1\\ e\cdot\ell\ge0}}\pi_{0e}\mathbb{1}_{\{G_k=0\}}\circ T^eG_k\,d\mathbb{P}=\int_{\{G_k=0\}}\sum_{\substack{|e|=1\\ e\cdot\ell\le0}}\pi_{e0}G_k\circ T^e\,d\mathbb{P},\end{aligned}$$

where the inequality used the fact that if $e\cdot\ell\ge0$, then $G_k\circ T^e$ is still $\mathfrak{S}_k$-measurable. Using the weak ellipticity condition (2.4), the above inequality implies that $\mathbb{P}$-a.s. we have $\{G_k=0\}\subset T^e\{G_k=0\}$, when $|e|=1$ and $e\cdot\ell\ge0$. Since T is $\mathbb{P}$-preserving, we have $\{G_k=0\}=T^e\{G_k=0\}$, $\mathbb{P}$-a.s. And since $(T^e)_{|e|=1,\,e\cdot\ell\ge0}$ generates the group $(T^x)_{x\in\mathbb{Z}^d}$, we have that $\{G_k=0\}$ is $\mathbb{P}$-a.s. shift-invariant. But $\mathbb{P}$ is ergodic, and thus $\mathbb{P}(G_k=0)$ is 0 or 1. However, $\mathbb{E}(G_k)=1$, and therefore $\mathbb{P}(G_k>0)=1$, and $\mathbb{P}_\infty$ and $\mathbb{P}$ are mutually absolutely continuous on H_k, for any $k\le0$.

Ergodicity of $(T^{X_n}\omega)_{n\ge0}$ *with initial distribution* $\mathbb{P}_\infty$. Consider a bounded local function f on Ω that is $\mathfrak{S}_K$-measurable, for some $K\le0$. Define $g=\mathbb{E}^{\mathbb{P}_\infty}(f|\mathcal{I})$, where $\mathcal{I}$ is the invariant σ-field for the process $(T^{X_n}\omega)_{n\ge0}$. Birkhoff's ergodic theorem implies that, for $\mathbb{P}_\infty$-a.e. ω,

$$P_0^\omega\Big(\lim_{n\to\infty}n^{-1}\sum_{m=1}^n f\big(T^{X_m}\omega\big)=g(\omega)\Big)=1. \tag{5.2}$$

Using the fact that $\mathbb{P}_\infty$ is invariant and that g is harmonic, we have

$$\begin{aligned}&\sum_{|e|\le M}\int\pi_{0e}(g-g\circ T^e)^2\,d\mathbb{P}_\infty\\&\quad=\int g^2\,d\mathbb{P}_\infty-2\int g\sum_{|e|\le M}\pi_{0e}g\circ T^e\,d\mathbb{P}_\infty+\int\sum_{|e|\le M}\pi_{0e}(g\circ T^e)^2\,d\mathbb{P}_\infty\\&\quad=0.\end{aligned}$$

Noticing that π_{0e} is $\mathfrak{S}_0$-measurable we conclude that the above equation, along with the weak ellipticity condition (2.4), implies that, for $|e|=1$ and $e\cdot\ell\ge0$,

$$g=g\circ T^e,\qquad \mathbb{P}_\infty\text{-a.s.} \tag{5.3}$$

Moreover, if we define

$$S = \left\{\omega : \forall\, y \in \mathbb{Z}^d,\ P_y^\omega\left(\inf_{m\ge 0} X_m \cdot \ell < 0\right) = 1\right\},$$

then $\mathbb{P}(S) = 0$. This is because otherwise the renewal property for the quenched walk would imply that $P_0(X_n \cdot \ell < 0 \text{ i.o.}) > 0$, and this contradicts (5.1). Hence, we have that, for $\mathbb{P}$-a.e. ω, there exists a y such that

$$P_y^\omega\left(\inf_{m\ge 0} X_m \cdot \ell \ge 0\right) > 0.$$

In particular, $y \cdot \ell \ge 0$. The weak ellipticity condition (2.4) implies that, for $\mathbb{P}$-a.e. choice of ω, the walk starting at 0 will, with positive probability under P_0^ω, reach y without backtracking below 0. This means that

$$P_0^\omega\left(\inf_{m\ge 0} X_m \cdot \ell \ge 0\right) > 0, \qquad \mathbb{P}\text{-a.s.}$$

However, the above event is $\mathfrak{S}_0$-measurable, and therefore we have

$$P_0^\omega\left(\inf_{m\ge 0} X_m \cdot \ell \ge 0\right) > 0, \qquad \mathbb{P}_\infty\text{-a.s.}$$

Now define

$$\bar{g}(\omega) = P_0^\omega\left(\inf_{m\ge 0} X_m \cdot \ell \ge 0\right)^{-1} \limsup_{n\to\infty} \int_{\{\inf_{m\ge 0} X_m\cdot\ell \ge 0\}} n^{-1} \sum_{m=1}^{n} f\big(T^{X_m}\omega\big)\, dP_0^\omega.$$

Then, because of (5.2), we know that $g = \bar{g}$, $\mathbb{P}_\infty$-a.s. However, it is clear that $\bar{g}$ is $\mathfrak{S}_K$-measurable. Formula (5.3) then implies that $g = g \circ T^e$, $\mathbb{P}$-a.s. and the ergodicity of $\mathbb{P}$ implies that g is constant $\mathbb{P}$-a.s., and thus $\mathbb{P}_\infty$-a.s. This proves that the invariant σ-field $\mathcal{I}$ is trivial, and that concludes the proof of ergodicity of $(T^{X_n}\omega)_{n\ge 0}$ with initial distribution $\mathbb{P}_\infty$.

Uniqueness of $\mathbb{P}_\infty$. Let f be a local bounded $\mathfrak{S}_K$-measurable function, for $K \le 0$. Notice that, due to ergodicity, we have $\mathbb{P}_\infty$-a.s.,

$$E_0^\omega\left(\lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} f(T^{X_m}\omega)\right) = \mathbb{E}^{\mathbb{P}_\infty}(f)$$

and, therefore, for $k \le 0$, we have $\mathbb{P}_\infty$-a.s.,

$$E_0^\omega\left(\lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} f(T^{X_m}\omega);\ \inf_{m\ge 0} X_m \cdot \ell \ge k\right) = \mathbb{E}^{\mathbb{P}_\infty}(f)\, P_0^\omega\left(\inf_{m\ge 0} X_m \cdot \ell \ge k\right).$$

Both functions above are $\mathfrak{S}_{k+K}$-measurable. Therefore, the same equation holds $\mathbb{P}$-a.s. Integrating over ω, one has

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_\infty}(f) &= \lim_{k\to-\infty} E_0\Biggl(\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} f\bigl(T^{X_m}\omega\bigr);\ \inf_{m\ge 0} X_m\cdot\ell\ge k\Biggr)\\ &= \lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} E_0\bigl(f(T^{X_m}\omega)\bigr) = \lim_{n\to\infty}\mathbb{E}^{\tilde{\mathbb{P}}_n}(f),\end{aligned}$$

which uniquely defines $\mathbb{P}_\infty$ as the weak limit of $\tilde{\mathbb{P}}_n$.

The law of large numbers. Taking f to be the drift D, we have, for all $k\le 0$, and $\mathbb{P}_\infty$-a.e. ω,

$$\begin{aligned}&P_0^\omega\Biggl(\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} D\bigl(T^{X_m}\omega\bigr) = \mathbb{E}^{\mathbb{P}_\infty}(D);\ \inf_{m\ge 0} X_m\cdot\ell\ge k\Biggr)\\ &\quad = P_0^\omega\Bigl(\inf_{m\ge 0} X_m\cdot\ell\ge k\Bigr).\end{aligned}$$

Once again, this is also true $\mathbb{P}$-a.s., and taking k to $-\infty$ we have

$$P_0\Biggl(\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} D\bigl(T^{X_m}\omega\bigr) = \mathbb{E}^{\mathbb{P}_\infty}(D)\Biggr) = 1. \tag{5.4}$$

For the rest of the proof, we follow the argument in [16], page 10. To this end, $M_n = X_n - X_0 - \sum_{m=0}^{n-1} D(T^{X_m}\omega)$ is a martingale with bounded increments under P_0^ω. Therefore $P_0^\omega(\lim_{n\to\infty} n^{-1}M_n = 0) = 1$. Combining this with (5.4), one obtains the desired law of large numbers. $\square$

Next, we will relax the absolute continuity condition to a weaker, but sufficient, condition. First, we need some definitions. For a measure $\mathbb{P}_\infty$, and $k\le 0$, define $\mathbb{P}_\infty^{k,\ll}$ (resp. $\mathbb{P}_\infty^{k,\perp}$) to be the absolutely continuous (resp. singular) part of $\mathbb{P}_{\infty|\mathfrak{S}_k}$ relative to $\mathbb{P}_{|\mathfrak{S}_k}$. For $A\in\mathfrak{S}_k$ and $j\le k$, $\mathbb{P}_\infty^{j,\ll}(A)$ [resp. $\mathbb{P}_\infty^{j,\perp}(A)$] is a monotone sequence, and there exists a measure $\mathbb{P}_\infty^{\infty,\ll}$ (resp. $\mathbb{P}_\infty^{\infty,\perp}$) such that $\mathbb{P}_\infty^{\infty,\ll}(A) = \inf_{j\le k}\mathbb{P}_\infty^{j,\ll}(A)$ [resp. $\mathbb{P}_\infty^{\infty,\perp}(A) = \sup_{j\le k}\mathbb{P}_\infty^{j,\perp}(A) = \mathbb{P}_\infty(A) - \mathbb{P}_\infty^{\infty,\ll}(A)$]. Now, we have the following lemma.

LEMMA 5. *If $\mathbb{P}_\infty$ is invariant for the process $(T^{X_n}\omega)_{n\ge 0}$ and $\mathbb{P}_\infty^{\infty,\ll}(\Omega) > 0$, then $\hat{\mathbb{P}}_\infty = \mathbb{P}_\infty^{\infty,\ll}(\Omega)^{-1}\mathbb{P}_\infty^{\infty,\ll}$ is a probability measure that is also invariant. Moreover, $\hat{\mathbb{P}}_\infty$ is absolutely continuous relative to $\mathbb{P}$, in every half-space H_k, with $k\le 0$.*

PROOF. One clearly has $\mathbb{P}_{\infty}^{\infty,\ll}{}_{|\mathfrak{S}_k} \le \mathbb{P}_{\infty}^{k,\ll} \ll \mathbb{P}_{|\mathfrak{S}_k}$. This proves the absolute continuity part of the claim of the lemma. To show the invariance of $\hat{\mathbb{P}}_{\infty}$, it is enough to show the invariance of $\mathbb{P}_{\infty}^{\infty,\ll}$. To this end, denote the transition probability of the process of the environment viewed from the particle by

$$\pi(\omega, A) = \sum_{|e|\le M} \pi_{0e}(\omega)\mathbb{1}_A(T^e\omega),$$

and define the operator Π, acting on measures, as

$$\Pi\mathbb{P}(A) = \int \pi(\omega, A)\, d\mathbb{P}(\omega).$$

Now, consider $A \in \mathfrak{S}_{k+M}$, with $\mathbb{P}(A) = 0$. Since $\Pi\mathbb{P} \ll \mathbb{P}$, we have $\Pi\mathbb{P}(A) = 0$. Therefore, $\pi(\omega, A) = 0$, $\mathbb{P}_{|\mathfrak{S}_k}$-a.s. and thus $\mathbb{P}_{\infty}^{k,\ll}$-a.s. as well. Hence, $\Pi\mathbb{P}_{\infty}^{k,\ll}(A) = 0$. This proves that $\Pi\mathbb{P}_{\infty}^{k,\ll} \ll \mathbb{P}_{|\mathfrak{S}_{k+M}}$ and, since $\Pi\mathbb{P}_{\infty}^{k,\ll} \le \Pi(\mathbb{P}_{\infty|\mathfrak{S}_k}) = \mathbb{P}_{\infty|\mathfrak{S}_{k+M}}$, we have $\Pi\mathbb{P}_{\infty}^{k,\ll} \le \mathbb{P}_{\infty}^{k+M,\ll}$. Taking limits, one has

$$\Pi\mathbb{P}_{\infty}^{\infty,\ll} \le \mathbb{P}_{\infty}^{\infty,\ll}.$$

However, the two measures above give the same mass to Ω, and therefore are equal. □

REMARK 6. Given an invariant measure $\mathbb{P}_{\infty}$, one can decompose it, relative to $\mathbb{P}$, into $\mathbb{P}_{\infty}^{\ll}$ and $\mathbb{P}_{\infty}^{\perp}$. Using the same argument as above, it is easy to see that $\mathbb{P}_{\infty}^{\ll}$ is again invariant, and that $\mathbb{P}_{\infty}^{\ll} \le \mathbb{P}_{\infty}^{\infty,\ll}$. Due to the uniqueness of the measure in Theorem 2, one sees that if $\mathbb{P}_{\infty}^{\ll}$ is not trivial, then $\mathbb{P}_{\infty}^{\ll}$ and $\mathbb{P}_{\infty}^{\infty,\ll}$ are proportional. Therefore, the latter is absolutely continuous, relative to $\mathbb{P}$, in the whole space, and thus $\mathbb{P}_{\infty}^{\infty,\ll} \le \mathbb{P}_{\infty}^{\ll}$, and $\mathbb{P}_{\infty}^{\ll} = \mathbb{P}_{\infty}^{\infty,\ll}$.

Before we move to the discussion of the law of large numbers, we will introduce, and recall some facts about the Dobrushin–Shlosman mixing condition for random fields.

6. The Dobrushin–Shlosman mixing condition. First, we introduce some notation. For a set $V \subset \mathbb{Z}^d$, let us denote by Ω_V the set of possible configurations $\omega_V = (\omega_x)_{x\in V}$, and by $\mathfrak{S}_V$ the σ-field generated by the environments $(\omega_x)_{x\in V}$. For a probability measure $\mathbb{P}$, we will denote by $\mathbb{P}_V$ the projection of $\mathbb{P}$ onto $(\Omega_V, \mathfrak{S}_V)$. For $\omega \in \Omega$, denote by $\mathbb{P}_V^{\omega}$ the regular conditional probability, knowing $\mathfrak{S}_{\mathbb{Z}^d-V}$, on $(\Omega_V, \mathfrak{S}_V)$. Furthermore, for $\Lambda \subset V$, $\mathbb{P}_{V,\Lambda}^{\omega}$ will denote the projection of $\mathbb{P}_V^{\omega}$ onto $(\Omega_\Lambda, \mathfrak{S}_\Lambda)$. Also, we will use the notation $V^c = \mathbb{Z}^d - V$, $\partial_r V = \{x \in \mathbb{Z}^d - V : \mathrm{dist}(x, V) \le r\}$, with $r \ge 0$, and card(V) will denote the cardinality of V. Finally, for $\omega, \bar{\omega} \in \Omega$, $V, W \subset \mathbb{Z}^d$ with $V \cap W = \varnothing$, we will use $(\bar{\omega}_V, \omega_W)$ to denote $\bar{\bar{\omega}}_{V\cup W}$ such that $\bar{\bar{\omega}}_V = \bar{\omega}_V$ and $\bar{\bar{\omega}}_W = \omega_W$. We will also need the following definitions.

By an r-specification ($r \geq 0$) we mean a system of functions $Q = \{Q_V^{\cdot}(\cdot): V \subset \mathbb{Z}^d, \mathrm{card}(V) < \infty\}$, such that, for all $\omega \in \Omega$, Q_V^ω is a probability measure on $(\Omega_V, \mathfrak{S}_V)$, and, for all $A \in \mathfrak{S}_V$, $Q_V^{\cdot}(A)$ is $\mathfrak{S}_{\partial_r V}$-measurable. Sometimes, for notational convenience, $Q_V^{\cdot}(A)$ will be thought of as a function on $\Omega_{\partial_r V}$. For $\Lambda \subset V$, we will denote by $Q_{V,\Lambda}^\omega$ the projection of Q_V^ω onto $(\Omega_\Lambda, \mathfrak{S}_\Lambda)$.

A specification Q is self-consistent if, for any finite Λ, V, $\Lambda \subset V \subset \mathbb{Z}^d$, one has, for Q_V^ω-a.e. $\bar{\omega}_V$, $(Q_V^\omega)_\Lambda^{\bar{\omega}_V} = Q_\Lambda^{(\omega_{V^c}, \bar{\omega}_V)}$. We will say that a probability measure $\mathbb{P}$ is consistent with a specification Q if $\mathbb{P}_V^\omega$ coincides with Q_V^ω, for every finite $V \subset \mathbb{Z}^d$ and $\mathbb{P}$-a.e. ω. Notice that this can only happen when Q is self-consistent. In this case, Q is uniquely determined by $\mathbb{P}$. The question is, however, whether Q determines $\mathbb{P}$, and whether it does so uniquely. To this end, Dobrushin and Shlosman [5] gave a sufficient condition to answer the above questions positively.

THEOREM DS [5]. *Let Q be a self-consistent r-specification, and assume the Dobrushin–Shlosman strong decay property holds; that is, there exist $G, g > 0$ such that for all $\Lambda \subset V \subset \mathbb{Z}^d$ finite, $x \in \partial_r V$ and $\omega, \bar{\omega} \in \Omega$, such that $\omega_y = \bar{\omega}_y$ when $y \neq x$, we have*

$$\mathrm{Var}\left(Q_{V,\Lambda}^\omega, Q_{V,\Lambda}^{\bar{\omega}}\right) \leq G e^{-g\,\mathrm{dist}(x,\Lambda)}, \tag{6.1}$$

where $\mathrm{Var}(\cdot,\cdot)$ *is the variational distance* $\mathrm{Var}(\mu,\nu) = \sup_{E \in \mathfrak{S}}(\mu(E) - \nu(E))$. *Then there exists a unique $\mathbb{P}$ that is consistent with Q. Moreover, we have, for all $\omega \in \Omega$,*

$$\lim_{\mathrm{dist}(\Lambda, V^c) \to \infty} \mathrm{Var}(Q_{V,\Lambda}^\omega, \mathbb{P}_\Lambda) = 0. \tag{6.2}$$

The main example of self-consistent specifications are Gibbs specifications. For the precise definition of a Gibbs specification with inverse temperature $\beta > 0$, see [5]. Moreover, if the interaction is translation-invariant, and the specification satisfies (6.1), then the unique field $\mathbb{P}$ is also shift-invariant; see [7], Section 5.2. One should note that the conditions of Theorem DS are satisfied when one considers Gibbs fields in the high-temperature region, that is, when β is small; see [6].

We will need the following lemma. The proof depends on another lemma and will be outlined in the Appendix.

LEMMA 7. *Let $(\mathbb{P}_V^\omega)$ be a Gibbs r-specification satisfying* (6.1), *and let $\mathbb{P}$ be the unique translation-invariant Gibbs field, consistent with $(\mathbb{P}_V^\omega)$. Consider $H \subset \mathbb{Z}^d$ and $\Lambda \subset H^c$ with* $\mathrm{dist}(\Lambda, H) > r$. *Then*

$$\sup_{F \in \mathfrak{F}} \sup_{\omega} \frac{\mathbb{E}(F|\mathfrak{S}_H)(\omega)}{\mathbb{E}(F)} \leq \exp\left(C \sum_{x \in \partial_r(H^c),\, y \in \partial_r(\Lambda^c)} e^{-g\,\mathrm{dist}(x,y)}\right),$$

where $\mathfrak{F} = \{F \geq 0, \mathfrak{S}_\Lambda$-measurable, s.t. $\mathbb{E}(F) > 0\}$.

7. The law of large numbers. We now need to find an invariant measure $\mathbb{P}_\infty$ that is absolutely continuous relative to $\mathbb{P}$, in each half-plane. The reason for which such a measure would exist is a strong enough transience condition. We will consider an environment that either satisfies the Dobrushin–Shlosman mixing condition (6.1) or is L-dependent in direction ℓ, that is, there exists $L > 0$, such that

$$\sigma(\omega_x;\ x\cdot\ell \le 0) \text{ and } \sigma(\omega_x;\ x\cdot\ell \ge L) \text{ are independent.} \tag{7.1}$$

The following is our main theorem.

THEOREM 3. *Suppose that $\mathbb{P}$ is of finite range, is T-invariant, is ergodic and satisfies one of the mixing conditions* (6.1) *or* (7.1). *Suppose also that the strong κ-ellipticity condition* (2.3) *holds, and that Kalikow's condition* (2.2), *in direction $\ell \in S^{d-1}$, is satisfied. Then the process $(T^{X_n}\omega)_{n\ge 0}$ admits an invariant probability measure $\hat{\mathbb{P}}_\infty$ that is absolutely continuous relative to $\mathbb{P}$, in every half-space H_k with $k \le 0$, and we have a law of large numbers for the finite range random walk in environment $\mathbb{P}$, with a nonzero limiting velocity*:

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{X_n}{n} = \mathbb{E}^{\hat{\mathbb{P}}_\infty}(D) \ne 0\right) = 1.$$

Moreover, if $\mathbb{P}_n(A) = P_0(T^{X_n}\omega \in A)$, that is, $\mathbb{P}_n$ is the measure on the environment as seen from the particle at time n, then $N^{-1}\sum_{n=1}^N \mathbb{P}_n$ converges weakly to $\hat{\mathbb{P}}_\infty$.

PROOF. Define the spaces

$$W_n = \{\text{paths } w, \text{ of range } M, \text{ length } n+1 \text{ and ending at } 0\}$$

and the space W of paths w, of range M, ending at 0 and of either finite or infinite length. Being a closed subspace of $(\{e \in \mathbb{Z}^d : |e| \le M\} \cup \{\text{'Stop'}\})^{\mathbb{N}}$, endowed with the product topology, W is compact. If we now consider the space $W_\infty \subset W$ of paths of range M and of infinite length that end at 0, then W_∞ is again a compact space.

Let us now define a sequence of measures R_n on $W \times \Omega$ as follows. Clearly, R_n will be supported on $W_n \times \Omega$, and for $w = (x_0, x_1, x_2, \ldots, x_n = 0) \in W_n$, $A \in \mathfrak{S}$,

$$R_n(\{w\} \times A) = P_0\big((-X_n, X_1 - X_n, \ldots, X_{n-1} - X_n, 0) = w, T^{X_n}\omega \in A\big).$$

Notice that $\mathbb{P}_n$ is the marginal of R_n, and therefore the disintegration lemma implies that

$$\mathbb{P}_n(A) = \int \mathbb{P}_w(A)\, dQ_n(w),$$

where Q_n is the marginal of R_n over W_n. It assigns probability $\mathbb{E}(\pi_w)$ to paths w of length $n+1$, and ending at 0. Here,

$$\pi_w = \prod_{i=0}^{n-1} \pi_{x_i, x_{i+1}}.$$

In fact, one can compute $\mathbb{P}_w$ explicitly. Indeed,

$$\begin{aligned}\mathbb{P}_n(A) &= P_0(T^{X_n}\omega \in A) = \sum_{x\in\mathbb{Z}^d} P_0(X_n = x, T^x\omega \in A) = \sum_{x\in\mathbb{Z}^d} P_x(X_n = 0, \omega \in A) \\ &= \int_A \sum_{x\in\mathbb{Z}^d} P_x^\omega(X_n = 0)\, d\mathbb{P}(\omega) = \int_A \sum_{w\in W_n} \pi_w(\omega)\, d\mathbb{P}(\omega).\end{aligned}$$

Using Fubini's theorem, we have

$$\mathbb{P}_n(A) = \int \mathbb{P}_w(A)\, dQ_n(w) \qquad \text{with } \frac{d\mathbb{P}_w}{d\mathbb{P}} = \frac{\pi_w}{\mathbb{E}(\pi_w)}.$$

The measure $\mathbb{P}_w$ could be thought of as the a posteriori measure on the environment, after having taken the path w.

Define $\tilde{R}_N = N^{-1}\sum_{n=1}^N R_n$, with marginals $\tilde{\mathbb{P}}_N$ and $\tilde{Q}_N$. Then, since $W \times \Omega$ is compact, one can find a subsequence of the $\tilde{R}_N$'s that converges weakly to a probability measure R_∞ on $W \times \Omega$. In fact, R_∞ will be supported on $W_\infty \times \Omega$.

Now define $\mathbb{P}_\infty$, Q_∞ to be the marginals of R_∞ on Ω and W_∞, respectively. Notice that

$$\begin{aligned}\int P_0^\omega(T^{X_1}\omega \in A)\, d\mathbb{P}_n &= \int \sum_{|e|\le M} \mathbb{1}(T^e\omega \in A)\pi_{0e}(\omega) \sum_{x\in\mathbb{Z}^d} P_x^\omega(X_n = 0)\, d\mathbb{P} \\ &= \int_A \sum_{x\in\mathbb{Z}^d} \sum_{|e|\le M} \pi_{e0}(\omega) P_x^\omega(X_n = e)\, d\mathbb{P} = \int_A d\mathbb{P}_{n+1}.\end{aligned}$$

This implies that $\mathbb{P}_\infty$ is an invariant measure for the process $(T^{X_n}\omega)_{n\ge 0}$.

Let $\mathbb{P}_w$ be given by the disintegration formula

$$\mathbb{P}_\infty = \int \mathbb{P}_w\, dQ_\infty(w).$$

We would like to show that the conditions of Lemma 5 are in effect. For this, define, for $k \le 0$ and $w \in \bigcup_{n\ge 1} W_n$,

$$A_k(w) = \sup_{\omega\in\Omega} \frac{d\mathbb{P}_{w|\mathfrak{S}_k}}{d\mathbb{P}_{|\mathfrak{S}_k}}(\omega).$$

Also, define, for $a > 0$, the measure

$$\tilde{\theta}_N^{a,k} = \int_{A_k \le a} \mathbb{P}_w\, d\tilde{Q}_N(w).$$

Then, one has that $\frac{d\tilde{\theta}^{a,k}_{N\,|\mathfrak{S}_k}}{d\mathbb{P}_{|\mathfrak{S}_k}} \le a$ and, therefore, one can find a further subsequence of the $\tilde{\theta}^{a,k}_N$'s that converges to a measure $\theta^{a,k}_\infty$, with $\frac{d\theta^{a,k}_{\infty\,|\mathfrak{S}_k}}{d\mathbb{P}_{|\mathfrak{S}_k}} \le a$. Moreover, one clearly has, for each N, $\tilde{\theta}^{a,k}_N \le \tilde{\mathbb{P}}_N$. Letting N pass to infinity, one has $\theta^{a,k}_{\infty\,|\mathfrak{S}_k} \le \mathbb{P}_{\infty\,|\mathfrak{S}_k}$. Thus, using the same notation as in Lemma 5, it follows that

$$\mathbb{P}^{k,\ll}_\infty(\Omega) \ge \theta^{a,k}_\infty(\Omega) \ge \liminf_{N\to\infty} \tilde{\theta}^{a,k}_N(\Omega) = \liminf_{N\to\infty} \tilde{Q}_N(A_k \le a). \tag{7.2}$$

So, according to Lemma 5, to use Theorem 2 for the purpose of proving a law of large numbers, one needs to show that

$$\inf_k \sup_a \liminf_{N\to\infty} \tilde{Q}_N(A_k \le a) > 0. \tag{7.3}$$

Assume now that the mixing condition (6.1) holds. Then, due to Lemma 7, one has that, for $w \in W_n$,

$$\begin{aligned}
\frac{d\mathbb{P}_{w\,|\mathfrak{S}_k}}{d\mathbb{P}_{|\mathfrak{S}_k}} &= \mathbb{E}\left(\frac{\pi_w}{\mathbb{E}(\pi_w)}\middle|\mathfrak{S}_k\right) \le \mathbb{E}\left(\frac{\pi_{w\cap H^c_{k-r}}}{\mathbb{E}(\pi_w)}\middle|\mathfrak{S}_k\right) \\
&\le \frac{\mathbb{E}(\pi_{w\cap H^c_{k-r}})}{\mathbb{E}(\pi_w)} \exp\left(C \sum_{x\in\partial_r H^c_k,\, y\in w\cap H^c_{k-r}} e^{-g\,\mathrm{dist}(x,y)}\right) \\
&\le \kappa^{-\mathrm{card}(w\cap H_{k-r})} \exp\left(\tilde{C} \sum_{y\in w\cap H^c_{k-r}} e^{-0.5g\,\mathrm{dist}(y,H_k)}\right) \\
&\le \kappa^{-\mathrm{card}(w\cap H_{k-r})} \exp\left(\tilde{C} \sum_{i\ge r} V_{k-i}(w) e^{-0.5gi}\right) = Z_k(w),
\end{aligned}$$

where $w \cap H^c_{k-r} = \{x_i \in H^c_{k-r}, 0 \le i \le n\}$, $V_j(w) = \mathrm{card}\,(w \cap (H_{j-1}\backslash H_j))$, and

$$\pi_{w\cap H^c_{k-r}} = \prod_{i=0,\, x_i\in H^c_{k-r}}^{n} \pi_{x_i,x_{i+1}}.$$

Clearly, the left-hand side in (7.3) is bounded from below by

$$\inf_k \sup_a \liminf_{N\to\infty} \tilde{Q}_N(Z_k \le a).$$

For a path $(X_n)_{n\ge0}$, define $\tilde{Z}_{k,n}$ to be

$$\tilde{Z}_{k,n} = Z_k(X_0 - X_n, X_1 - X_n, \dots, X_{n-1} - X_n, 0).$$

Also, let $\tau_s = \inf\{n \geq 0 : X_n \cdot \ell \geq s\}$. Then, for any $\delta \in (0, 1)$, one has

$$
\begin{aligned}
&\tilde{Q}_N(Z_k \leq a) \\
&\quad = N^{-1} \sum_{n=1}^{N} P_0(\tilde{Z}_{k,n} \leq a) \geq N^{-1} E_0\Biggl(\sum_{1 \leq j \leq \delta N} \mathbb{1}_{\tilde{Z}_{k,\tau_j} \leq a} \mathbb{1}_{\tau_{\delta N} \leq N} \Biggr) \\
&\quad = N^{-1} \sum_{1 \leq j \leq \delta N} P_0\bigl(\tilde{Z}_{k,\tau_j} \leq a\bigr) - N^{-1} \sum_{1 \leq j \leq \delta N} P_0\bigl(\tilde{Z}_{k,\tau_j} \leq a, \tau_{\delta N} > N\bigr) \\
&\quad \geq N^{-1} \sum_{1 \leq j \leq \delta N} P_0\bigl(\tilde{Z}_{k,\tau_j} \leq a\bigr) - \delta P_0(\tau_{\delta N} > N).
\end{aligned}
\tag{7.4}
$$

On one hand, we have,

$$
P_0(\tau_{\delta N} > N) \leq N^{-1} E_0(\tau_{\delta N}) \leq (N\varepsilon)^{-1} E_0(X_{\tau_{\delta N}}) \leq \frac{\delta N + M}{N\varepsilon}, \tag{7.5}
$$

where we have used Lemma 2. On the other hand,

$$
P_0\bigl(\tilde{Z}_{k,\tau_j} \leq a\bigr) \geq 1 - a_1^{-1} E_0\bigl(\hat{V}^j_{j+k-r,j+M}\bigr) - a_2^{-1} \sum_{i \geq r} E_0\bigl(\hat{V}^j_{j+k-i,j+M}\bigr) e^{-0.5gi},
$$

where $a_1 = 0.5 \operatorname{Log} a / \operatorname{Log}(\kappa^{-1})$, $a_2 = 0.5 \operatorname{Log} a / \tilde{C}$ and

$$
\hat{V}^j_{i_1 i_2} = \operatorname{card}\{n : 0 \leq n \leq \tau_j,\ i_1 \leq X_n \cdot \ell < i_2\}.
$$

We had to enlarge the V_j's we had before, to take into account the fact that the position of X_{τ_j} is not known precisely. To estimate the above expectations, notice that one has, path by path,

$$
\sum_{0 \leq n \leq \tau_j - 1,\, i \leq X_n \cdot \ell < j} (X_{n+1} - X_n) \cdot \ell \leq (j - i) + M.
$$

Using Kalikow's condition (2.2), one has

$$
\begin{aligned}
E_0\bigl(\hat{V}^j_{i,j+M}\bigr) &= 1 + E_0\Biggl(\sum_{0 \leq n \leq \tau_j - 1,\, i \leq x \cdot \ell < j} \mathbb{1}_{X_n = x} \Biggr) \\
&\leq 1 + \varepsilon^{-1} E_0\Biggl(\sum_{0 \leq n \leq \tau_j - 1,\, i \leq x \cdot \ell < j} \mathbb{1}_{X_n = x} D(T^x \omega) \cdot \ell \Biggr) \\
&= 1 + \varepsilon^{-1} E_0\Biggl(\sum_{0 \leq n \leq \tau_j - 1,\, i \leq X_n \cdot \ell < j} (X_{n+1} - X_n) \cdot \ell \Biggr) \\
&\leq 1 + \varepsilon^{-1}\bigl((j - i) + M\bigr).
\end{aligned}
$$

This implies that

$$P_0(\tilde{Z}_{k,\tau_j} \le a) \\ \ge 1 - a_1^{-1}(1+\varepsilon^{-1}(M+r-k)) - a_2^{-1}\sum_{i\ge r}(1+\varepsilon^{-1}(M+i-k))e^{-0.5gi}.$$

Combining this with (7.4) and (7.5), and taking $\delta = 0.5\varepsilon$, one has

$$\inf_k \sup_a \liminf_{N\to\infty} \tilde{Q}_N(Z_k \le a) \ge 0.25\varepsilon > 0.$$

Recalling (7.2) and using Lemma 5, one has the existence of the invariant measure $\hat{\mathbb{P}}_\infty$ that satisfies the conditions of Theorem 2. The transience condition (5.1) is implied by Kalikow's condition (2.2), due to Lemma 3. The law of large numbers, along with the weak convergence of the Cesaro mean of $\mathbb{P}_n$ to $\hat{\mathbb{P}}_\infty$, follows from Theorem 2.

If the environment is L-dependent, instead of mixing, then we have

$$\frac{d\mathbb{P}_{w|\mathfrak{S}_k}}{d\mathbb{P}_{|\mathfrak{S}_k}} \le \kappa^{-\operatorname{card}(w\cap H_{k-L})},$$

and the rest of the proof is essentially the same as above.

Once one has a law of large numbers, one can use Lemma 2, with $U_L = \{x \in \mathbb{Z}^d : x\cdot\ell \le L\}$, and Fatou's lemma, to show that $T_{U_L}^{-1} X_{T_{U_L}} \cdot \ell \ge L T_{U_L}^{-1}$ cannot converge to 0, proving that the limiting velocity is nonzero. □

REMARK 8. In the course of preparing of this paper, we learnt of [10], where the authors prove the law of large numbers for L-dependent nonnestling environments. Their approach is a first step toward the method we use. They, nevertheless, make use of the regeneration times, introduced in [18]. Apart from the ellipticity condition, our results include those of [10]. We also learnt of [3], where the authors use the regeneration times to prove a result very similar to our Theorem 3. However, they require moment controls on the regeneration times, which we do not need in our approach. Working with cones instead of hyperplanes, our method should be able to handle mixing on cones, as in [3].

APPENDIX

First, we prove a consequence of the Dobrushin–Shlosman mixing property (6.1), in the case of Gibbs fields.

LEMMA 9. *Let* $(\mathbb{P}_V^\omega)$ *be a Gibbs* r*-specification, corresponding to a translation-invariant bounded* r*-interaction* U *and satisfying* (6.1). *Then there exists a constant* C *such that, for all* $\Lambda \subset V \subset \mathbb{Z}^d$ *finite, with* $\operatorname{dist}(\Lambda, V^c) > r$, *and* *for all* $x \in V^c$, *we have*

$$\sup_{\sigma_\Lambda,\omega,\bar\omega:\,(\omega_y)_{y\neq x}=(\bar\omega_y)_{y\neq x}} \left|\frac{d\mathbb{P}^{\omega}_{V,\Lambda}}{d\mathbb{P}^{\bar\omega}_{V,\Lambda}}(\sigma_\Lambda) - 1\right| \le C \sum_{y\in\partial_r(\Lambda^c)} e^{-g\,\mathrm{dist}(x,y)}.$$

PROOF. Fix $x \in V^c$, and consider $\omega, \bar\omega \in \Omega$, such that $\omega_y = \bar\omega_y$, for all $y \neq x$. Also, let $\sigma_\Lambda, \bar\sigma_\Lambda \in \Omega_\Lambda$. We have then

$$\frac{d\mathbb{P}^{\omega}_{V,\Lambda}}{d\mathbb{P}^{\bar\omega}_{V,\Lambda}}(\sigma_\Lambda) = \mathbb{E}^{\mathbb{P}^{\bar\omega}_V}\left(\frac{d\mathbb{P}^{\omega}_{V}}{d\mathbb{P}^{\bar\omega}_{V}}\Big|\mathfrak{S}_\Lambda\right)(\sigma_\Lambda).$$

Notice that, for $\xi_V \in \Omega_V$, we have

$$\frac{d\mathbb{P}^{\omega}_{V}}{d\mathbb{P}^{\bar\omega}_{V}}(\xi_V) = \frac{\exp(-\beta\sum_{A:\,A\cap V\neq\varnothing,\,x\in A} U_A(\omega_{V^c},\xi_V))}{\exp(-\beta\sum_{A:\,A\cap V\neq\varnothing,\,x\in A} U_A(\bar\omega_{V^c},\xi_V))}. \tag{A.1}$$

So we see that $\frac{d\mathbb{P}^{\omega}_{V}}{d\mathbb{P}^{\bar\omega}_{V}}$ is $\mathfrak{S}_{V^r_x}$-measurable, where $V^r_x = \{y \in V : \mathrm{dist}(x,y) \le r\}$. Therefore,

$$\mathbb{E}^{\mathbb{P}^{\bar\omega}_V}\left(\frac{d\mathbb{P}^{\omega}_{V}}{d\mathbb{P}^{\bar\omega}_{V}}\Big|\mathfrak{S}_\Lambda\right)(\sigma_\Lambda) = \mathbb{E}^{\mathbb{P}^{\eta}_{V-\Lambda,V^r_x}}\left(\frac{d\mathbb{P}^{\omega}_{V}}{d\mathbb{P}^{\bar\omega}_{V}}\right),$$

where $\eta = (\bar\omega_{\Lambda^c}, \sigma_\Lambda)$. Moreover, clearly

$$\left|\frac{d\mathbb{P}^{\omega}_{V}}{d\mathbb{P}^{\bar\omega}_{V}}\right| \le \exp\left(2^{\mathrm{card}(\{y\in\mathbb{Z}^d:\,\|y\|\le r\})+1}\beta\|U\|\right) = C_1. \tag{A.2}$$

Then, setting $\bar\eta = (\bar\omega_{\Lambda^c}, \bar\sigma_\Lambda)$, we have

$$\begin{aligned}
&\left|\frac{d\mathbb{P}^{\omega}_{V,\Lambda}}{d\mathbb{P}^{\bar\omega}_{V,\Lambda}}(\sigma_\Lambda) - \frac{d\mathbb{P}^{\omega}_{V,\Lambda}}{d\mathbb{P}^{\bar\omega}_{V,\Lambda}}(\bar\sigma_\Lambda)\right| \\
&\quad\le C_1 \operatorname{Var}\left(\mathbb{P}^{\eta}_{V-\Lambda,V^r_x}, \mathbb{P}^{\bar\eta}_{V-\Lambda,V^r_x}\right) \\
&\quad\le C_1 G \sum_{y\in\partial_r(\Lambda^c)} e^{-g\,\mathrm{dist}(y,V^r_x)} \\
&\quad\le C_1 G e^{gr} \sum_{y\in\partial_r(\Lambda^c)} e^{-g\,\mathrm{dist}(y,x)}.
\end{aligned} \tag{A.3}$$

The conclusion of the lemma follows from integrating out $\bar\sigma_\Lambda$. □

Notice now that if (6.1) is satisfied, one can define $\mathbb{P}^{\omega}_{V}$, even for an infinite V, as the limit of $\mathbb{P}^{\omega}_{V_n}$, for an increasing sequence of finite volumes V_n. It is easy then to see that (A.2) still holds, and that, due to the lower semicontinuity of the variational distance, computation (A.3) goes through for all $\Lambda \subset V \subset \mathbb{Z}^d$. Therefore Lemma 9 still holds for infinite $\Lambda \subset V$.

PROOF OF LEMMA 7. Using $V = H^c$ and applying Lemma 9, we have

$$\begin{aligned}\frac{\mathbb{E}(F|\mathfrak{S}_H)(\omega)}{\mathbb{E}(F|\mathfrak{S}_H)(\bar\omega)} &= \frac{\mathbb{E}^{\mathbb{P}^{\omega}_{V,\Lambda}}(F)}{\mathbb{E}^{\mathbb{P}^{\bar\omega}_{V,\Lambda}}(F)} \\ &\le \prod_{x\in\partial_r V}\left(1 + C\sum_{y\in\partial_r(\Lambda^c)} e^{-g\,\mathrm{dist}(x,y)}\right) \\ &\le \exp\left(C\sum_{x\in\partial_r(H^c),\,y\in\partial_r(\Lambda^c)} e^{-g\,\mathrm{dist}(x,y)}\right).\end{aligned}$$

Once again, the conclusion follows by averaging over $\bar\omega$. □

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