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DARLING–ERDŐS THEOREM FOR SELF-NORMALIZED SUMS[1]

BY MIKLÓS CSÖRGŐ, BARBARA SZYSZKOWICZ AND QIYING WANG

Carleton University, Carleton University and Carleton University and Australian National University

Let X, X_1, $X_2, \dots$ be i.i.d. nondegenerate random variables, $S_n = \sum_{j=1}^n X_j$ and $V_n^2 = \sum_{j=1}^n X_j^2$. We investigate the asymptotic behavior in distribution of the maximum of self-normalized sums, $\max_{1\le k\le n} S_k/V_k$, and the law of the iterated logarithm for self-normalized sums, S_n/V_n, when X belongs to the domain of attraction of the normal law. In this context, we establish a Darling–Erdős-type theorem as well as an Erdős–Feller–Kolmogorov–Petrovski-type test for self-normalized sums.

1. Introduction and main results. Let X, X_1, $X_2, \dots$ be a sequence of nondegenerate i.i.d. random variables and put $S_n = \sum_{j=1}^n X_j$ for their partial sums, $n \ge 1$. Darling and Erdős (1956) proved the following remarkable limit theorem for the maximum of the standardized sums.

RESULT A. *If $EX = 0$ and $E|X|^3 < \infty$, then, for every $t \in R$,*

$$\lim_{n\to\infty} P\left\{a(n) \max_{1\le k\le n} S_k/(\sigma\sqrt{k}) \le t + b(n)\right\} = \exp(-e^{-t}), \tag{1}$$

where $\sigma^2 = EX^2$, $a(n) = (2\log\log n)^{1/2}$ and

$$b(n) = 2\log\log n + \tfrac{1}{2}\log\log\log n - \tfrac{1}{2}\log(4\pi).$$

We assume throughout that $\log x = \log(\max\{e, x\})$.

Darling and Erdős (1956) actually established Result A for independent random variables that are not necessarily identically distributed. Several extensions have relaxed their third-moment condition in the i.i.d. case. Oodaira (1976) and Shorack (1979) independently showed that (1) holds when an absolute moment of order $2 + \delta$ is finite for some $\delta > 0$. Einmahl (1989) and Einmahl and Mason (1989) proved that the Darling–Erdős theorem holds for i.i.d. random variables whenever

$$EX^2 I_{(|X|\ge x)} = o\big((\log\log x)^{-1}\big) \qquad \text{as } x \to \infty. \tag{2}$$

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Einmahl (1989) showed the condition (2) to also be necessary in the i.i.d. case. Einmahl (1989) also concluded that if only $EX = 0$ and $EX^2 < \infty$ are assumed, then $\sigma\sqrt{k}$ in Result A needs to be replaced by the less natural B_k as in his theorem that follows.

RESULT B. *If $EX = 0$ and $EX^2 < \infty$, then, for every $t \in R$,*

$$\lim_{n\to\infty} P\left\{a(n) \max_{1\le k\le n} S_k/B_k \le t + b(n)\right\} = \exp(-e^{-t}), \tag{3}$$

where $B_n^2 = \sum_{j=1}^n EX^2 I_{(|X|\le\sqrt{j}/(\log\log j)^2)}$.

For an extension of the Darling–Erdős theorem to stable law, we refer to Bertoin (1998). In this paper, we show that Results A and B can be merged into one result via the use of the natural random normalizer. Furthermore, in this context even the second-moment condition is not required anymore, for it is seen below that a Darling–Erdős-type theorem for self-normalized sums holds true if only X belongs to the domain of attraction of the normal law under some weak additional conditions.

Write $V_n^2 = \sum_{j=1}^n X_j^2$ and $l(x) = EX^2 I_{(|X|\le x)}$. The following is our main theorem.

THEOREM 1. *Suppose that $EX = 0$ and $l(x)$ is a slowly varying function at ∞, satisfying $l(x^2) \le Cl(x)$ for some $C > 0$. Then, for every $t \in R$, we have*

$$\lim_{n\to\infty} P\left\{a(n) \max_{1\le k\le n} S_k/V_k \le t + b(n)\right\} = \exp(-e^{-t}). \tag{4}$$

The proof of Theorem 1 is based on an extension of the truncation techniques of Feller (1946) and Theorem 1 of Einmahl and Mason (1989) for the maximum of normalized sums of bounded independent random variables. Utilizing this method, we also succeed in refining the self-normalized law of the iterated logarithm (LIL). In fact, we obtain the following Erdős–Feller–Kolmogorov–Petrovski (EFKP)-type test for self-normalized sums [cf. Petrovski (1935), Erdős (1942) and Feller (1943, 1946)].

THEOREM 2. *Suppose that $EX = 0$ and $l(x)$ is a slowly varying function at ∞, satisfying $l(x^2) \le Cl(x)$ for some $C > 0$. Then*

$$P(S_n \ge V_n\phi_n,\ i.o.) = 0 \quad or\ = 1 \tag{5}$$

accordingly as

$$J(\phi) \equiv \sum_{n=1}^{\infty} \frac{\phi_n}{n} e^{-\phi_n^2/2} < \infty \quad or\ = \infty, \tag{6}$$

where ϕ_n is a nondecreasing sequence of positive numbers.

REMARK 1. Giné, Götze and Mason (1997) showed that $S_n/V_n \to_{\mathcal{D}} N(0,1)$ if and only if $EX = 0$ and X belongs to the domain of attraction of the normal law. The latter condition is well known to be equivalent to $l(x)$ being a slowly varying function at ∞. Based on these facts and the corresponding Darling–Erdős theorem in the classical case, that is, that one has (1) in the i.i.d. case if and only if (2) is obtained [cf. Einmahl (1989)], it is not likely that the condition alone that $l(x)$ is a slowly varying function at ∞ is sufficient for establishing (4). We note also that $l(x^2) \le Cl(x)$ is a weak enough assumption, which is satisfied by a large class of slowly varying functions such as $(\log\log x)^\alpha$ and $(\log x)^\alpha$, for example, for some $0 < \alpha < \infty$. However, it remains an open problem to find a necessary condition for establishing (4).

REMARK 2. The EFKP-type test for self-normalized sums was first derived by Griffin and Kuelbs (1991) in case X is symmetric with $EX^2 < \infty$. The symmetricity condition was later eliminated by Wang (1999). Theorem 2 shows that the EFKP-type test for self-normalized sums continues to hold true without assuming the existence of the second moment. This amounts to an essential improvement of the previous results.

REMARK 3. In the past decades, self-normalized sums S_n/V_n have been studied by many researchers. Among them, Giné, Götze and Mason (1997) proved that the tails of S_n/V_n are uniformly sub-Gaussian when the sequence is stochastically bounded. Bentkus and Götze (1996) obtained Berry–Esseen inequalities for self-normalized sums. Wang and Jing (1999) derived exponential nonuniform Berry–Esseen bounds. Shao (1997) showed that no moment conditions are needed for a self-normalized large-deviation result for $P(S_n/V_n \ge x\sqrt{n})$. For a survey of recent developments in this area, we refer to Shao (1998).

In the next section, we prove the main results. Throughout the paper, we shall use A to denote an absolute positive constant whose value may differ at each occurrence.

2. Proofs of the main results. We start with some notation and first prove six lemmas, preliminaries to the proofs of the main results. Put $b = \inf\{x \ge 1 : l(x) > 0\}$ and

$$\eta_n = \inf\left\{s : s \ge b+1, \frac{l(s)}{s^2} \le \frac{(\log\log n)^4}{n}\right\}.$$

Furthermore, we let

$$Z_j = X_j I_{(|X_j| > \eta_j)}, \qquad Y_j = X_j I_{(|X_j| \le \eta_j)}, \qquad Y_j^* = Y_j - EY_j,$$

$$S_n^* = \sum_{j=1}^n Y_j^*, \qquad B_n^2 = \sum_{j=1}^n EY_j^{*2}.$$

Since $l(x)$ is a nondecreasing function, slowly varying at ∞, it follows that

$$P(|X| \geq x) = o\big(l(x)/x^2\big) \quad \text{and} \quad E|X|I_{(|X|\geq x)} = o\big(l(x)/x\big), \tag{7}$$

$$\eta_n \to \infty \quad \text{and} \quad nl(\eta_n) = \eta_n^2 (\log\log n)^4 \qquad \text{for every large enough } n \tag{8}$$

and, under the condition that $l(x^2) \leq Cl(x)$ for some $C > 0$, we also have

$$l(\eta_k) \leq l(n^2) \leq Cl(n) \leq Al(\eta_n) \qquad \text{for } n \leq k \leq n^{3/2} \text{ and } n \text{ large enough,} \tag{9}$$

as well as

$$B_n^2 \sim \sum_{j=1}^n EY_j^2 \sim nl(\eta_n) \sim \eta_n^2 (\log\log n)^4.$$

Consequently, under their respective conditions, we may assume without loss of generality that (8) and (9) hold for $n \geq 1$, as well as that

$$B_n^2 = nl(\eta_n) = \eta_n^2 (\log\log n)^4 \qquad \text{for } n \geq 1. \tag{10}$$

It will be seen that these assumptions will not affect the proofs of the main results, which are based on the following six lemmas.

LEMMA 1. *We have*

$$\sum_{k=1}^\infty P\big(|X| \geq \eta_k (\log\log k)^3\big) < \infty, \tag{11}$$

$$\sum_{k=1}^\infty \frac{1}{(\log\log k)^6} P(|X| \geq \eta_k) < \infty. \tag{12}$$

PROOF. Write $\tau_j = \eta_j (\log\log j)^3$. It follows in terms of (8) and (9) that, for $k \geq 1$,

$$\begin{aligned} \sum_{j=k}^\infty \frac{1}{\tau_j^2 \log j} &= \sum_{j=k}^\infty \frac{1}{jl(\eta_j)(\log j)(\log\log j)^2} \\ &\geq \frac{A}{l(\eta_k)(\log k)(\log\log k)^2} \sum_{j=k}^{k^{3/2}} j^{-1} \\ &\geq \frac{A}{l(\eta_k)(\log\log k)^2} = \frac{Ak}{\tau_k^2}. \end{aligned} \tag{13}$$

By using (13) and noting that $\tau_{j+1} \leq j^2$ for j large enough, we get

$$\sum_{j=1}^\infty P(|X| \geq \tau_j) = \sum_{k=1}^\infty kP(\tau_k \leq |X| < \tau_{k+1})$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \tau_k^2 P(\tau_k \le |X| < \tau_{k+1}) \frac{k}{\tau_k^2} \\
&\le A \sum_{k=1}^{\infty} \tau_k^2 P(\tau_k \le |X| < \tau_{k+1}) \sum_{j=k}^{\infty} \frac{1}{\tau_j^2 \log j} \\
&\le A \sum_{j=1}^{\infty} \frac{1}{\tau_j^2 \log j} EX^2 I_{(|X| \le \tau_{j+1})} \\
&\le A \sum_{j=1}^{\infty} \frac{1}{j \log j (\log\log j)^2} < \infty.
\end{aligned}
$$

This proves (11). Similarly, we obtain that

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \frac{1}{(\log\log j)^6} P(|X| \ge \eta_j) \\
&\qquad \le \sum_{k=1}^{\infty} P(\eta_k \le |X| < \eta_{k+1}) \sum_{j=1}^{k} (\log\log j)^{-6} \\
&\qquad \le A \sum_{k=1}^{\infty} \eta_k^2 P(\eta_k \le |X| < \eta_{k+1}) \frac{k}{\tau_k^2} \\
&\qquad \le A \sum_{j=1}^{\infty} \frac{1}{\tau_j^2 \log j} EX^2 I_{(|X| \le \eta_{j+1})} \\
&\qquad \le A \sum_{j=1}^{\infty} \frac{1}{j \log j (\log\log j)^2} < \infty.
\end{aligned}
$$

The proof of Lemma 1 is now complete. □

LEMMA 2. *We have*

(14)
$$
\sum_{j=1}^{n} (|Z_j| + E|Z_j|) \le 5 B_n (\log\log n)^2 \qquad a.s.
$$

PROOF. Let $\tau_j = B_j \log\log j$ and $Z_j^* = X_j I_{(\eta_j < |X_j| < \tau_j)}$. By using (11) and (10), we get

$$
\begin{aligned}
\sum_{j=1}^{\infty} P(Z_j \ne Z_j^*) &= \sum_{j=1}^{\infty} P(|X| \ge \tau_j) \\
&= \sum_{j=1}^{\infty} P\big(|X| \ge \eta_j (\log\log j)^3\big) < \infty.
\end{aligned}
$$

Recalling (7) and (10), on the other hand, it can be easily shown that, for n sufficiently large,

$$\begin{aligned}\sum_{j=1}^{n} E|Z_j^*| &\le \sum_{j=1}^{n} E|Z_j| \\ &\le nE|X|I_{(|X|\ge \eta_n)} + \sum_{j=1}^{n}\sum_{k=j}^{n} E|X|I_{(\eta_{k-1}\le |X|<\eta_k)} \\ &\le nE|X|I_{(|X|\ge \eta_n)} + \sum_{k=1}^{n} kE|X|I_{(\eta_{k-1}\le |X|<\eta_k)} \\ &\le nE|X|I_{(|X|\ge \eta_n)} + l(\eta_n)\max_{1\le k\le n}\frac{k}{\eta_{k-1}} \\ &\le 2B_n(\log\log n)^2.\end{aligned}$$

So, we only need to prove that

$$T_n \equiv \sum_{j=1}^{n}(|Z_j^*| - E|Z_j^*|) \le B_n(\log\log n)^2 \qquad \text{a.s.} \tag{15}$$

Write $\bar{Z}_j = |Z_j^*| - E|Z_j^*|$. Then, for $1 \le j \le 2^{k+1}$,

$$|\bar{Z}_j| \le 2\tau_{2^{k+1}} \le A\tau_{2^k},$$

$$\sum_{j=1}^{2^{k+1}} E(\bar{Z}_j)^2 \le 2^{k+2} l(\tau_{2^{k+1}}) \le \tau_{2^k}^2/64 \qquad \text{for } k \text{ sufficiently large},$$

$$Ee^{8\bar{Z}_j/\tau_{2^k}} \le 1 + \frac{AE(\bar{Z}_j)^2}{\tau_{2^k}^2} \le 1 + A2^{-k}.$$

Therefore, by Kolmogorov's inequality, for k sufficiently large,

$$\begin{aligned}P&\Big(\max_{2^k\le n\le 2^{k+1}} T_n/B_n(\log\log n)^2 \ge 1\Big) \\ &\le P\Big(\max_{2^k\le n\le 2^{k+1}} T_n \ge \tau_{2^k}\log k\Big) \\ &\le 2P\Big(T_{2^{k+1}} \ge \tfrac{1}{2}\tau_{2^k}\log k\Big) \\ &\le 2\exp(-2\log k)\prod_{j=1}^{2^{k+1}} Ee^{8\bar{Z}_j/\tau_{2^k}} \le Ak^{-2}.\end{aligned} \tag{16}$$

Thus, (15) follows from the Borel–Cantelli lemma. This also completes the proof of Lemma 2. □

LEMMA 3. *For any* $0 < \eta < 1/2$ *and* $\theta > 1$, *there exist* $0 < \delta < 1$, $x_0 > 1$ *and* n_0 *such that, for any* $n \ge n_0$ *and* $x_0 < x < \delta\sqrt{n}$,

$$P\Big(\max_{n \le k \le \theta n} S_k/V_k \ge x\Big) \le e^{-\eta x^2}. \tag{17}$$

PROOF. See Remark 4.2 of Shao (1997) with an obvious modification. □

LEMMA 4. *We have that*

$$P\Bigg(S_k \ge V_k\sqrt{\log\log k},\ \sum_{j=1}^{k} |Z_j| \ge B_k/\log\log k,\ i.o.\Bigg) = 0, \tag{18}$$

$$P\Bigg(S_k \ge V_k\sqrt{\log\log k},\ \sum_{j=1}^{k} E|Z_j| \ge B_k/\log\log k,\ i.o.\Bigg) = 0, \tag{19}$$

$$P\Bigg(S_k^* \ge B_k\sqrt{\log\log k},\ \sum_{j=1}^{k} |Z_j| \ge B_k/\log\log k,\ i.o.\Bigg) = 0, \tag{20}$$

$$P\Bigg(S_k^* \ge B_k\sqrt{\log\log k},\ \sum_{j=1}^{k} E|Z_j| \ge B_k/\log\log k,\ i.o.\Bigg) = 0. \tag{21}$$

PROOF. We first prove (18). Put

$$m_k = \min\{n : B_n^2 \ge 2^{k-1}/(\log k)^8\}, \qquad n_k = \min\{n : B_n^2 \ge 2^k\}.$$

Also, we write

$$F_k = \bigcup_{n=n_{k-1}}^{n_k-1} \Bigg\{S_n \ge V_n\sqrt{\log\log n},\ \sum_{j=1}^{n} |Z_j| \ge B_n/\log\log n\Bigg\},$$

$$G_k = \bigcup_{j=m_k}^{n_k-1} \{|Z_j| \ne 0\}, \qquad H_k = \bigcup_{n=n_{k-1}}^{n_k-1} \Big\{S_n \ge V_n\sqrt{\log\log n}\Big\},$$

$$R_k = \bigcup_{n=n_{k-1}}^{n_k-1} \Bigg\{\sum_{j=1}^{n} |Z_j| \ge B_n/\log\log n,\ Z_j = 0,\ j = m_k, \dots, n_k - 1\Bigg\}.$$

Using $B_{m_k}^2 \sim 2^{k-1}/(\log k)^8$, $B_{n_k}^2 \sim 2^k$ and Lemma 2, if $n_{k-1} \le n \le n_k - 1$, then

$$\sum_{j=1}^{m_k} |Z_j| \le B_{m_k}(\log\log m_k)^2 \le \tfrac{1}{2} B_n/\log\log n \qquad \text{a.s.}$$

Hence, $P(R_k, \text{i.o.}) = 0$. Since $F_k \subset (G_k \cap H_k) \cup R_k$, to prove (18), on account of the Borel–Cantelli lemma, one only needs to show that

$$\sum_{k=1}^{\infty} P(G_k \cap H_k) < \infty. \tag{22}$$

Define $S_n^{(j)} = S_n - X_j$ and $V_n^{(j)} = (V_n^2 - X_j^2)^{1/2}$. Noting that, for any $s, t \in R$, $c \geq 0$ and $x \geq 1$,

$$\begin{aligned} x\sqrt{c+t^2} &= \sqrt{(x^2-1)c + t^2 + c + (x^2-1)t^2} \\ &\geq \sqrt{(x^2-1)c + t^2 + 2t\sqrt{(x^2-1)c}} \\ &= t + \sqrt{(x^2-1)c}, \end{aligned}$$

we have

$$\{s + t \geq x\sqrt{c+t^2}\} \subset \{s \geq (x^2-1)^{1/2}\sqrt{c}\}.$$

Hence, for any $1 \leq j \leq n$,

$$\{S_n \geq V_n\sqrt{\log\log n}, Z_j \neq 0\} \subseteq \{S_n^{(j)} \geq (\log\log n - 1)^{1/2} V_n^{(j)}, Z_j \neq 0\}. \tag{23}$$

Recalling that $B_n^2 = nl(\eta_n)$, it is easy to see that $n_k \geq \theta n_{k-1}$ for some $\theta > 1$ and k sufficiently large. Thus, it follows from (23), the independence of X_j and Lemma 3 that

$$\begin{aligned} P(G_k \cap H_k) &\leq \sum_{j=m_k}^{n_k-1} P\left(\bigcup_{n=n_{k-1}}^{n_k-1} \{S_n \geq V_n\sqrt{\log\log n},\ Z_j \neq 0\}\right) \\ &\leq \sum_{j=m_k}^{n_k-1} P(Z_j \neq 0) P\left(\bigcup_{n=n_{k-1}}^{n_k-1} \{S_n^{(j)} \geq V_n^{(j)}\sqrt{\log\log n - 1}\}\right) \\ &\leq \sum_{j=m_k}^{n_k-1} P(Z_j \neq 0) P\left(\max_{n_{k-1} \leq n \leq n_k - 1} S_n^{(j)}/V_n^{(j)} \geq \tfrac{1}{2}\sqrt{\log k}\right) \\ &\leq k^{-1/10} \sum_{j=m_k}^{n_k-1} P(|X| \geq \eta_j) \qquad \text{for } k \text{ sufficiently large.} \end{aligned} \tag{24}$$

Write $k_1(j) = \max\{k : n_k \leq j\}$ and $k_2(j) = \max\{k : m_k \leq j\}$. Using $B_n^2 = nl(\eta_n)$ again, it can be easily shown that $k_2(j) \leq k_1(j) + A\log\log j$ for some constant $A > 0$, and, hence, for j large enough,

$$\sum_{k=k_1(j)}^{k_2(j)} k^{-1/10} \leq (\log j)^{-1/20}.$$

This, together with (12) and (24), implies that

$$\sum_{k=1}^{\infty} P(G_k \cap H_k) \le \sum_{k=1}^{\infty} k^{-1/10} \sum_{j=m_k}^{n_k-1} P(|X| \ge \eta_j)$$

$$\le \sum_{j=1}^{\infty} P(|X| \ge \eta_j) \sum_{k=k_1(j)}^{k_2(j)} k^{-1/10}$$

$$\le A \sum_{j=1}^{\infty} \frac{1}{(\log\log j)^6} P(|X| \ge \eta_j) < \infty.$$

This completes the proof of (18).

We next prove (19). Let $N_1 = \{k : \sum_{j=1}^{n} E|Z_j| \le \frac{1}{2} B_n / \log\log n$ for all $n_{k-1} \le n \le n_k\}$ and $N_2 = N - N_1$. Using (12), we have

$$\sum_{k \in N_2} k^{-1/10} \le A \sum_{k \in N_2} \sum_{n=n_{k-1}}^{n_k-1} \frac{1}{n(\log\log n)^3}$$

$$\le A \sum_{n=1}^{\infty} \frac{1}{n B_n (\log\log n)^2} \sum_{j=1}^{n} E|Z_j|$$

$$\le A \sum_{j=1}^{\infty} E|Z_j| \sum_{n=j}^{\infty} \frac{1}{n^{3/2} l^{1/2}(\eta_n)(\log\log n)^2}$$

$$\le A \sum_{j=1}^{\infty} \frac{1}{j^{1/2} l^{1/2}(\eta_j)(\log\log j)^2} \sum_{k=j}^{\infty} E|X| I_{(\eta_k \le |X| < \eta_{k+1})}$$

$$\le A \sum_{k=1}^{\infty} \frac{k^{1/2}\eta_{k+1}}{l^{1/2}(\eta_k)(\log\log k)^2} P(\eta_k \le |X| < \eta_{k+1})$$

$$\le A \sum_{k=1}^{\infty} \frac{k}{(\log\log k)^6} P(\eta_k \le |X| < \eta_{k+1})$$

$$\le A \sum_{k=1}^{\infty} \frac{1}{(\log\log k)^6} P(|X| \ge \eta_k) < \infty.$$

Therefore, similarly to the proof of (24),

$$\sum_{k=1}^{\infty} P\left(\bigcup_{n=n_{k-1}}^{n_k-1} \left\{ S_n \ge V_n \sqrt{\log\log n},\ \sum_{j=1}^{n} E|Z_j| \ge B_n / \log\log n \right\} \right)$$

$$\le \sum_{k \in N_2} P\left(\bigcup_{n=n_{k-1}}^{n_k-1} \{ S_n \ge V_n \sqrt{\log\log n} \} \right)$$

$$\leq \sum_{k\in N_2} P\left(\max_{n_{k-1}\leq n\leq n_k-1} S_n/V_n \geq \tfrac{1}{2}\sqrt{\log k}\right)$$
$$\leq \sum_{k\in N_2} k^{-1/10} < \infty,$$

and hence (19) follows from the Borel–Cantelli lemma.

We next prove (20) and (21). Since, for any $1 \leq j \leq n$,

$$\{S_n^* \geq B_n\sqrt{\log\log n},\, Z_j \neq 0\} \subseteq \{S_n^* - Y_j^* \geq (\log\log n - 1)^{1/2}B_n,\, Z_j \neq 0\},$$

by tracking the proof of (18) and (19), it is clear that (20) and (21) will follow if we can prove that

$$P\left(\max_{n_{k-1}\leq n\leq n_k-1} S_n^*/B_n \geq \tfrac{1}{2}\sqrt{\log k}\right) \leq Ak^{-\delta} \qquad \text{for some } 0<\delta<\tfrac{1}{2}. \tag{25}$$

Recalling $|Y_j^*| \leq 2\eta_j \leq 4B_{n_k}/(\log k)^2$ for $1 \leq j \leq n_k$, if $|t| \leq \frac{1}{8}\sqrt{\log k}$, we get

$$\left|Ee^{tY_j^*/B_{n_k}} - 1 - \frac{t^2}{2B_{n_k}^2}EY_j^{*2}\right| \leq \frac{|t|^3}{6B_{n_k}^3}E|Y_j^*|^3 e^{|tY_j^*|/B_{n_k}} \leq \frac{t^2}{6B_{n_k}^2}E|Y_j^*|^2.$$

This implies that, for all $|t| \leq \frac{1}{8}\sqrt{k}$,

$$Ee^{tY_j^*/B_{n_k}} \leq 1 + \frac{1+t^2}{2B_{n_k}^2}EY_j^{*2} \leq \exp\left(\frac{1+t^2}{2B_{n_k}^2}EY_j^{*2}\right).$$

Therefore, by using $B_{n_k}^2 \sim 2^k$ and Kolmogorov's inequality, we obtain

$$\begin{aligned}
&P\left(\max_{n_{k-1}\leq n\leq n_k-1} \frac{S_n^*}{B_n} \geq \frac{1}{2}\sqrt{\log k}\right)\\
&\quad\leq P\left(\max_{n_{k-1}\leq n\leq n_k-1} S_n^* \geq \frac{1}{4}B_{n_k}\sqrt{\log k}\right)\\
&\quad\leq 2P\left(S_{n_k}^* \geq \frac{1}{8}B_{n_k}\sqrt{\log k}\right)\\
&\quad\leq 2\exp\left(-\frac{t}{8}\sqrt{\log k}\right)\prod_{j=1}^{n_k} E\exp\left(\frac{tY_j^*}{B_{n_k}}\right)\\
&\quad\leq 2\exp\left(-\frac{t}{8}\sqrt{\log k} + \frac{1+t^2}{2}\right)
\end{aligned}$$

for all $|t| \leq \frac{1}{8}\sqrt{\log k}$. Now (25) follows by choosing $t = \frac{1}{8}\sqrt{\log k}$. This also completes the proof of Lemma 4. $\square$

LEMMA 5. *We have that, for all $t \in R$,*

$$\lim_{n\to\infty} P\left\{a(n)\max_{1\le k\le n} S_k^*/B_k \le t + b(n)\right\} = \exp(-e^{-t}). \tag{26}$$

PROOF. It follows from (10) that

$$|Y_j^*| \le 2\eta_j \le \varepsilon_j B_j/(\log\log B_j^2)^{3/2}, \tag{27}$$

where $\varepsilon_j \to 0$ as $j \to \infty$. By using Theorem 1 of Einmahl and Mason (1989) and the method of its proof, we get, for all $t \in R$,

$$P\left(a(B_n^2)\max_{1\le k\le n} S_k^*/B_k \le t + b(B_n^2)\right) = \exp(-e^{-t}).$$

On the other hand, $\log\log B_n^2 = \log\log n + o(1)$ by (10) again. Therefore,

$$\begin{aligned}
a(n)&\max_{1\le k\le n} S_k^* B_k - b(n)\\
&= \frac{a(n)}{a(B_n^2)}\left(a(B_n)\max_{1\le k\le n} S_k^*/B_k - b(B_n)\right) + \frac{a(n)}{a(B_n^2)} b(B_n^2) - b(n)\\
&= (1+o(1))\left(a(B_n)\max_{1\le k\le n} S_k^*/B_k - b(B_n)\right) + o(1).
\end{aligned}$$

The assertion (26) immediately follows by an application of the continuous mapping theorem. □

LEMMA 6. *Let X_j, $j \ge 1$, be independent normal random variables with $EX_j = 0$ and $EX_j^2 < \infty$. Let $S_n = \sum_{j=1}^n X_j$ and $s_n^2 = \sum_{j=1}^n EX_j^2$. Assume that there exist positive constants α, β_1 and β_2 such that*

$$\beta_1 n^\alpha h(n) \le s_n^2 \le \beta_2 n^\alpha h(n) \qquad \textit{for } n \textit{ sufficiently large},$$

where $h(x)$ is a slowly varying function at ∞. Then

$$P(S_n \ge s_n\phi_n,\ i.o.) = 0 \quad or \ = 1 \tag{28}$$

according as

$$J(\phi) \equiv \sum_{n=1}^\infty \frac{\phi_n}{n} e^{-\phi_n^2/2} < \infty \quad or \ = \infty, \tag{29}$$

where ϕ_n is a nondecreasing sequence of positive numbers.

PROOF. We may assume that $2\log_2 n \le \phi_n^2 \le 3\log_2 n$ [cf., e.g., Bai (1989), Lemma 1]. Let θ be an integer satisfying $\beta_1\theta^\alpha \ge 4\beta_2$. Since $h(x)$ is a slowly varying function at ∞, it can be easily shown that

$$s_{\theta^{k+1}}^2 \le A s_{\theta^k}^2, \quad s_{\theta^k}^2 \le \tfrac{1}{2} s_{\theta^{k+1}}^2 \qquad \text{for } k \text{ sufficiently large} \tag{30}$$

and (29) is correspondingly equivalent to

$$\sum_{k=1}^{\infty} \log^{1/2} k \exp\left(-\tfrac{1}{2}\phi_{\theta^k}^2\right) < \infty \quad \text{or} = \infty. \tag{31}$$

Put

$$a_n = s_n \phi_n \quad \text{and} \quad b_n = s_n / \phi_n.$$

It is well known that, for $x > 0$,

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) \le 1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}\,x}\exp\left(-\frac{x^2}{2}\right). \tag{32}$$

Noting that S_n/s_n is a standard normal random variable, from (30)–(32) we easily obtain: if $J(\phi) < \infty$, then

$$\begin{aligned} &\sum_{n=1}^{\infty} \min[1, (a_{n+1} - a_n)/b_n] P(S_n \ge s_n \phi_n) \\ &\qquad \le A \sum_{k=1}^{\infty} \frac{1}{s_{\theta^k}} \sum_{n=\theta^k}^{\theta^{k+1}} (a_{n+1} - a_n) \exp\left(-\frac{1}{2}\phi_n^2\right) \\ &\qquad \le A \sum_{k=1}^{\infty} \frac{s_{\theta^{k+1}}}{s_{\theta^k}} \log^{1/2} k \exp\left(-\frac{1}{2}\phi_{\theta^k}^2\right) < \infty; \end{aligned} \tag{33}$$

if $J(\phi) = \infty$, then

$$\begin{aligned} &\sum_{n=1}^{\infty} \min[1, (a_{n+1} - a_n)/b_n] P(S_n \ge s_n \phi_n) \\ &\qquad \ge A \sum_{k=1}^{\infty} \frac{1}{s_{\theta^{k+1}}} \sum_{n=\theta^k}^{\theta^{k+1}} (a_{n+1} - a_n) \exp\left(-\frac{1}{2}\phi_n^2\right) \\ &\qquad \ge A \sum_{k=1}^{\infty} \frac{s_{\theta^{k+1}} - s_{\theta^k}}{s_{\theta^{k+1}}} \log^{1/2} k \exp\left(-\frac{1}{2}\phi_{\theta^{k+1}}^2\right) = \infty. \end{aligned} \tag{34}$$

On the other hand, by using (32) and again that S_n/s_n is a standard normal random variable, it is easily seen that, for fixed $u < v$,

$$1 < \lim_{n\to\infty} \frac{P(S_n > a_n + b_n u)}{P(S_n > a_n + b_n v)} = e^{v-u} < \infty. \tag{35}$$

Therefore, via (33)–(35), Lemma 6 follows immediately from Theorem 2 of Feller (1970). □

We are now ready to prove the main results.

PROOF OF THEOREM 1. Write $K_n = \exp(\log^{1/3} n)$,

$$\Omega_1 = \left\{k : \sum_{j=1}^{k} |Z_j| \le B_k / \log\log k\right\},$$

$$\Omega_2 = \left\{k : \sum_{j=1}^{k} E|Z_j| \le B_k / \log\log k\right\}$$

and $\Omega = \{k : K_n \le k \le n\} \cap \Omega_1 \cap \Omega_2$. Recalling (27), we obtain from Theorem 3.1 in Shao (1995) that $\limsup_{n\to\infty} S_n^*/(2B_n^2 \log\log n)^{1/2} = 1$ a.s. Thus,

$$\max_{1\le k\le K_n} S_k^*/B_k \le \sqrt{2\log\log K_n} \le \sqrt{2/3 \log\log n} \qquad \text{a.s.} \tag{36}$$

for n sufficiently large. Since $b(n) > a(n)\sqrt{2\log\log n}$, using (20), (21) and (36), it is easy to see that

$$a(n) \max_{k\notin\Omega} S_k^*/B_k - b(n) \to -\infty \qquad \text{a.s.}$$

Similarly, using (18), (19) and

$$\limsup_{n\to\infty} S_n/(2V_n^2 \log\log n)^{1/2} = 1 \qquad \text{a.s.} \tag{37}$$

[cf. Griffin and Kuelbs (1989)], we get

$$a(n) \max_{k\notin\Omega} S_k/V_k - b(n) \to -\infty \qquad \text{a.s.}$$

These facts combined with Lemma 7 of Einmahl (1989) and Lemma 5, together imply that Theorem 1 will follow if we can prove

$$L_n \equiv a(n) \max_{k\in\Omega} \left| \frac{S_k}{V_k} - \frac{S_k^*}{B_k} \right| = o(1) \qquad \text{a.s.} \tag{38}$$

Recalling (12), we get

$$\begin{aligned} \sum_{j=1}^{\infty} & \frac{(\log\log j)^2}{B_j^4} EX^4 I_{(|X|\le\eta_j)} \\ &\le A \sum_{k=1}^{\infty} EX^4 I_{(\eta_k<|X|\le\eta_{k+1})} \sum_{j=k}^{\infty} \frac{(\log\log j)^2}{j^2 l^2(\eta_j)} \\ &\le A \sum_{k=1}^{\infty} \frac{k}{(\log\log k)^6} P(\eta_k < |X| \le \eta_{k+1}) \\ &\le A \sum_{k=1}^{\infty} \frac{1}{(\log\log k)^6} P(|X| \ge \eta_k) < \infty. \end{aligned} \tag{39}$$

Thus, via the Borel–Cantelli lemma and the Kronecker lemma, we easily obtain

$$\frac{\log\log k}{B_k^2}\sum_{j=1}^{k}(Y_j^2 - EY_j^2) \to 0 \qquad \text{a.s.},$$

and hence,

$$\max_{k\in\Omega}\frac{\log\log k}{B_k^2}\left|\sum_{j=1}^{k}(Y_j^2 - EY_j^2)\right| \le \max_{K_n\le k\le n}\frac{\log\log k}{B_k^2}\left|\sum_{j=1}^{k}(Y_j^2 - EY_j^2)\right| \to 0 \qquad \text{a.s.} \tag{40}$$

On the other hand, if $k \in \Omega$, then $\log\log k \ge \frac{1}{3}\log\log n$ and

$$\sum_{j=1}^{k}\bigl(Z_j^2 + (E|Z_j|)^2\bigr) \le \left(\sum_{j=1}^{k}|Z_j| + E|Z_j|\right)^2 \le 36(B_k/\log\log n)^2. \tag{41}$$

Combining (37), (40), (41) and $EX = 0$, it can be easily shown that

$$\begin{aligned}
L_n &\le a(n)\max_{k\in\Omega}\left|\frac{S_k}{V_k} - \frac{S_k}{B_k}\right| + a(n)\max_{k\in\Omega}\frac{1}{B_k}\sum_{j=1}^{k}(|Z_j| + E|Z_j|)\\
&\le a(n)\max_{k\in\Omega}\frac{|S_k|\,|V_k^2 - B_k^2|}{V_k B_k(V_k + B_k)} + 6(\log\log n)^{-1/2}\\
&\le 2(\log\log n)\max_{k\in\Omega}\frac{1}{B_k^2}\left|V_k^2 - \sum_{j=1}^{k}EY_j^{*2}\right| + 6(\log\log n)^{-1/2}\\
&\le 6\max_{k\in\Omega}\frac{\log\log k}{B_k^2}\left|\sum_{j=1}^{k}(Y_j^2 - EY_j^2)\right|\\
&\quad + 2(\log\log n)\max_{k\in\Omega}\frac{1}{B_k^2}\sum_{j=1}^{k}\bigl(|Z_j|^2 + (E|Z_j|)^2\bigr) + 6(\log\log n)^{-1/2}\\
&\to 0 \qquad \text{a.s. as } n\to\infty.
\end{aligned}$$

This proves (38) and, hence, the proof of Theorem 1 is complete. □

PROOF OF THEOREM 2. In terms of (39), by using Theorem 4.1 of Shao (1995) with $b_n = \eta_n$ and $H_n = B_n/\sqrt{\log\log n}$, we get that there exists a sequence of independent normal random variables $\{W_j, j \ge 1\}$, $W_j \stackrel{\mathcal{D}}{=} N(0, EY_j^{*2})$, such that

$$\sum_{j=1}^{n} Y_j^* - \sum_{j=1}^{n} W_j = o\bigl(B_n/\sqrt{\log\log n}\bigr) \qquad \text{a.s.} \tag{42}$$

As before, we assume without loss of generality that

$$2\log_2 n \le \phi_n^2 \le 3\log_2 n$$

[cf., e.g., Bai (1989), Lemma 1]. Put $U_n = \sum_{j=1}^n W_j$. Recalling (38) and (42), it is clear that if $\sum_{j=1}^n |Z_j| \le B_n/\log\log n$ and $\sum_{j=1}^n E|Z_j| \le B_n/\log\log n$, then

$$\begin{aligned}\left|\frac{S_n}{V_n} - \frac{U_n}{B_n}\right| &\le \left|\frac{S_n}{V_n} - \frac{S_n^*}{B_n}\right| + \left|\frac{|S_n^* - U_n|}{B_n}\right| \\ &\le \frac{1}{4}(\log\log n)^{-1/2} \qquad \text{a.s.}\end{aligned}$$

Thus, by using Lemma 4, one concludes easily that

$$\begin{aligned} &P(S_n \ge V_n\phi_n,\ \text{i.o.}) \\ &\quad \le P\big(U_n \ge B_n(\phi_n + 1/\phi_n),\ \text{i.o.}\big) \\ &\qquad + P\left(S_n \ge V_n\sqrt{\log\log n},\ \sum_{j=1}^n |Z_j| \ge B_n/\log\log n,\ \text{i.o.}\right) \\ &\qquad + P\left(S_n \ge V_n\sqrt{\log\log n},\ \sum_{j=1}^n E|Z_j| \ge B_n/\log\log n,\ \text{i.o.}\right) \\ &\quad = P\big(U_n \ge B_n(\phi_n + 1/\phi_n),\ \text{i.o.}\big)\end{aligned} \tag{43}$$

and

$$\begin{aligned} &P(U_n \ge B_n\phi_n,\ \text{i.o.}) \\ &\quad \le P\big(S_n \ge V_n(\phi_n + 1/\phi_n),\ \text{i.o.}\big) \\ &\qquad + P\left(U_n \ge B_n\phi_n,\ \sum_{j=1}^n |Z_j| \ge B_n/\log\log n,\ \text{i.o.}\right) \\ &\qquad + P\left(U_n \ge B_n\phi_n,\ \sum_{j=1}^n E|Z_j| \ge B_n/\log\log n,\ \text{i.o.}\right) \\ &\quad \le P\big(S_n \ge V_n(\phi_n + 1/\phi_n),\ \text{i.o.}\big) \\ &\qquad + P\left(S_n^* \ge B_n\sqrt{\log\log n},\ \sum_{j=1}^n |Z_j| \ge B_n/\log\log n,\ \text{i.o.}\right) \\ &\qquad + P\left(S_n^* \ge B_n\sqrt{\log\log n},\ \sum_{j=1}^n E|Z_j| \ge B_n/\log\log n,\ \text{i.o.}\right) \\ &\quad = P\big(S_n \ge V_n(\phi_n + 1/\phi_n),\ \text{i.o.}\big).\end{aligned} \tag{44}$$

Since, for any fixed real constant D, replacing ϕ_n by $\phi_n + D/\phi_n$ does not change the convergence of the series (6), Theorem 2 follows immediately from (43), (44) and Lemma 6. □

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M. CSÖRGŐ
B. SZYSZKOWICZ
SCHOOL OF MATHEMATICS
AND STATISTICS
CARLETON UNIVERSITY
1125 COLONEL BY DRIVE
OTTAWA K1S 5B6
CANADA
E-MAIL: mcsorgo@math.carleton.ca
bszyszko@math.carleton.ca

Q. WANG
CENTRE FOR MATHEMATICS
AND ITS APPLICATIONS
MATHEMATICAL SCIENCES INSTITUTE
AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, ACT 0200
AUSTRALIA
E-MAIL: qiying@wintermute.anu.edu.au