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AN EQUIVALENCE OF H_{-1} NORMS FOR THE SIMPLE EXCLUSION PROCESS[1]

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Resolvent H_{-1} norms with respect to simple exclusion processes play an important role in many problems with respect to additive functionals, tagged particles, and hydrodynamics, among other concerns. Here, general translation-invariant finite-range simple exclusion processes with and without a distinguished particle are considered. For the standard system of indistinguishable particles, it is proved that the corresponding H_{-1} norms are equivalent, in a sense, to the H_{-1} norms of a nearest-neighbor system. The same result holds for systems with a distinguished particle in dimensions $d \geq 2$. However, in dimension $d = 1$, this equivalence does not hold. An application of the H_{-1} norm equivalence to additive functional variances is also given.

1. Introduction. Consider the following formal L^2 setting. Let $\{\eta(t) : t \geq 0\}$ be a Markov process on a state space Σ. Let $T_t : L^2(\Sigma) \to L^2(\Sigma)$ be the semigroup operator, and L the infinitesimal generator. Let also π be an ergodic invariant measure for the process.

A basic problem is to investigate the long term behavior of additive functionals $A_f(t) = \int_0^t f(\eta(s))\,ds$ where $f \in L^2(\Sigma)$ is a function on the state space. For instance, when $f(\eta) = \mathbb{1}_B$, the additive functional $A_f(t)$ is the occupation time of the set $B \subset \Sigma$. When starting under equilibrium π, we have $A_f(t)/t \to E_\pi[f]$ a.s.; π as $t \uparrow \infty$ by the ergodic theorem. Naturally, then, one asks about the diffusive behavior for the centered functionals, in particular, with a view toward central limit theorems, whether the variance,

$$\sigma_t^2(f) = E_\pi\big[\big(A_f(t) - E_\pi[A_f(t)]\big)^2\big]$$

is $O(t)$. To simplify notation, let us assume now that f is a mean-zero function, $E_\pi[f] = 0$. Then, from translation invariance under π, we can rewrite the scaled variance, $\sigma_t^2(f)/t$, as

$$\begin{aligned}\frac{1}{t}E_\pi\big[A_f^2(t)\big] &= 2\int_0^t (1 - s/t)E_\pi\big[f(\eta(0))f(\eta(s))\big]\,ds \\ &= 2\int_0^t (1 - s/t)E_\pi[fT_s f]\,ds.\end{aligned} \tag{1.1}$$

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However, without estimates on T_t, it is not clear that the limit of the last expression even exists, much less is finite as t tends to infinity, but when the process is reversible with respect to π, we have that

$$E_\pi[fT_sf] = E_\pi\big[(T_{s/2}f)^2\big] \geq 0. \tag{1.2}$$

In this case, the scaled variance (1.1) increases mononotically as $t \uparrow \infty$ to the sum of correlations, $2\int_0^\infty E_\pi[fT_sf]\,ds$, well defined as an extended real number. This limit variance can also be identified through resolvent equations. Recall, for $\lambda > 0$, that the resolvent equation

$$\lambda u_\lambda - Lu_\lambda = f$$

has solution

$$\begin{aligned} u_\lambda(\eta) &= (\lambda - L)^{-1}f \\ &= \int_0^\infty e^{-\lambda t}T_tf(\eta)\,dt. \end{aligned}$$

Using (1.2) again, we now have the limit variance equals

$$\begin{aligned} 2\int_0^\infty E_\pi[fT_sf]\,ds &= 2\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda s}E_\pi[fT_sf]\,ds \\ &= 2\lim_{\lambda\downarrow 0}E_\pi[f(\lambda - L)^{-1}f]. \end{aligned}$$

This last expression equals $2E_\pi[f(-L)^{-1}f]$ which can be understood through spectral representation of L.

For the general (nonreversible) process, though, $E_\pi[fT_sf]$ is not necessarily positive, and the above derivations are heuristic. However, the resolvent equation and some martingale computations do lead to a useful variance bound for the general process at time t [cf. Kipnis and Landim (1999), Proposition 11.6.1; Sethuraman (2000), Lemma 3.9]. Indeed, denote by $M_\lambda(t)$ the martingale, $M_\lambda(t) = u_\lambda(\eta(t)) - u_\lambda(\eta(0)) - \int_0^t Lu_\lambda(\eta(s))\,ds$, with respect to process σ-fields and write

$$\begin{aligned} \frac{1}{t}E_\pi\bigg[\bigg(\int_0^t f(\eta(s))\,ds\bigg)^2\bigg] &\leq \frac{3}{t}E_\pi\bigg[\bigg(\int_0^t \lambda u_\lambda\,ds\bigg)^2\bigg] \\ &\quad + \frac{3}{t}E_\pi\big[\big(u_\lambda(\eta(0)) - u_\lambda(\eta(t))\big)^2\big] + \frac{3}{t}E_\pi[M_\lambda^2(t)]. \end{aligned}$$

By stationarity and quadratic variation estimates, the right-hand side is less than

$$\frac{3}{t}[\lambda^2t^2E_\pi[u_\lambda^2] + 2E_\pi[u_\lambda^2]] + 6E_\pi[u_\lambda(-Lu_\lambda)].$$

We are free to choose now $\lambda = 1/t$ so that the above expression is further bounded by

$$6E_\pi\big[((1/t)u_\lambda - Lu_\lambda)u_\lambda\big] + 3E_\pi\big[(1/t)u_\lambda^2\big] \leq 9E_\pi[f(1/t - L)^{-1}f].$$

The point of this derivation is that, at the very least, to get a priori bounds on the variance, it makes sense to study $E_\pi[f(\lambda - L)^{-1} f]$ for small $\lambda > 0$. In this way, "$O(t)$" results can be shown for the variance $\sigma_t^2(f)$.

To this end, it is natural to introduce "H_{-1}" norms and the Hilbert space

$$H_{-1}(\lambda, -L) = \text{completion}\{\phi \text{ test}: E_\pi[f(\lambda - L)^{-1} f] < \infty\}$$

with norm $\|f\|_{-1}(\lambda, -L) = \sqrt{E_\pi[f(\lambda - L)^{-1} f]}$ and associated innerproduct by polarization. In this framework, then, the variances can be bounded,

$$\limsup_{t\uparrow\infty} t^{-1}\sigma_t^2(f) \le 9 \limsup_{\lambda\downarrow 0} \|f\|_{-1}^2(\lambda, -L).$$

However, this is as far as one can go in this abstraction. To proceed further, one must invoke special features of the process considered to handle these H_{-1} norms. In particular, the subject of this paper is an estimate for the H_{-1} norms with respect to two types of simple exclusion particle processes.

The standard simple exclusion process follows the motion of a collection of indistinguishable particles each moving as a random walk on $\mathbb{Z}^d$ with the provision that jumps to already occupied vertices are suppressed. A related exclusion process distinguishes one of the particles, "tagging" it, and follows the motion of the other "environment" particles in its reference frame. These models have proved fruitful in providing a nontrivial setting with a conservation law where some computation for many physical phenomena such as queuing traffic and fluid flow is allowed [Liggett (1999)].

The ergodic invariant measures for these processes depend on the jump-rates of the underlying random walk dynamics $\{p(i, j): i, j \in \mathbb{Z}^d\}$. In this article, we focus upon finite-range translation-invariant jump rates, that is when $p(i, j) = p(j - i)$, and $p(i) = 0$ for $|i| > R$, some R. For the standard and environment exclusion models with finite-range translation-invariant rates, there exist in fact many ergodic invariant measures. For each type of process, one of these measures is fixed for the duration of the article.

In the following, we will say that p and the associated exclusion process are "nearest-neighbor" when the range $R = 1$. Also, we denote by the vector $r(p)$ the drift of the jump rate p, $r(p) = \sum_i i p(i)$.

The purpose of this note is to compare the H_{-1} norms for both types of exclusion processes with finite-range translation-invariant rates with the H_{-1} norms for certain associated nearest-neighbor models. These associated exclusion processes are those with nearest-neighbor jump rates with the same drift as the finite-range processes. That is, let $e^1, e^2, \dots, e^d$ be the standard basis vectors in $\mathbb{Z}^d$, and define the nearest-neighbor jump rate p_1, for $1 \le l \le d$, by

$$p_1(\pm e^l) = \begin{cases} \max[\pm e^l \cdot r(p), 0], & \text{when } e^l \cdot r(p) \neq 0, \\ 1, & \text{when } e^l \cdot r(p) = 0. \end{cases}$$

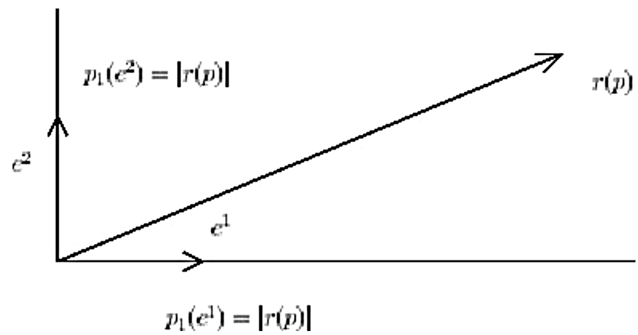

FIG. 1. *Resolution of a possible p to p_1 in $d = 2$.*

Note that the drifts of p_1 and p are equal, $r(p_1) = r(p) = \sum_i ip(i)$ (see Figure 1). Let L_1 denote the generator of standard simple exclusion with rates p_1. Our main result is to show, for some constant C, the equivalence of norms in all dimensions $d \geq 1$ (Theorem 2.1),

$$C^{-1}\|\cdot\|_{-1}(D^{-1}\lambda, -L_1) \leq \|\cdot\|_{-1}(\lambda, -L) \leq C\|\cdot\|_{-1}(D\lambda, -L_1).$$

The same equivalence holds for "environment" exclusion processes in dimensions $d \geq 2$, but interestingly, not in dimension $d = 1$ (Theorem 2.2).

Nearest-neighbor systems are usually much more tractable than finite-range systems with more complicated graph structures. For instance, in one dimension, $d = 1$, the motion in a nearest-neighbor exclusion process can be ordered whereas this is not possible in a general finite-range process due to particles "leap-frogging." In particular, this ordering in one-dimensional nearest-neighbor models allows for a wealth of analysis [DeMasi and Ferrari (1985), Ferrari and Fontes (1994), Kipnis (1986), etc.]. Some analysis also holds for d-dimensional finite-range reversible processes through symmetry arguments and "duality" relations [cf. Liggett (1985), Chapter 8, Sethuraman (2000), Section 2.2.2 for "duality" relations]. For the general d-dimensional finite-range nonreversible process, however, these "dualities" do not hold, and not as much has been proved. In this context, with respect to additive functional variances, say, the motivation for this work is to provide a bridge between nearest-neighbor estimates and finite-range bounds, in particular in the nonreversible situation.

Such an application is made in the last section where we discuss the diffusive behavior of additive functionals $A_f(t)$ with respect to the standard exclusion process. There, we apply the H_{-1} norm equivalence above to show that $\sigma_t^2(f, L) = O(t) \Leftrightarrow \sigma_t^2(f, L_1) = O(t)$ for a class of functions f in all dimensions $d \geq 1$ (Corollary 6.1). In another paper, we show in fact that these nearest-neighbor variances satisfy $\sigma_t^2(f, L_1) = O(t)$, for some one-dimensional nonreversible models using special nearest-neighbor techniques [Seppäläinen and Sethuraman (2003)]. Together, these results imply a relatively complete picture of the variance behavior for one-dimensional nonreversible systems, leading to some central limit theorems. (See Section 6 for more details.)

Finally, we mention briefly that exclusion H_{-1} norms, besides being significant in describing additive functional behavior, are also important in other applications with respect to tagged particles, nongradient hydrodynamics, and Green–Kubo formulas [see Kipnis and Landim (1999) as a general reference]. Moreover, other H_{-1} norm properties for exclusion models in various symmetric and asymmetric situations have been derived previously in Kipnis, Landim and Olla (1994), Kipnis and Varadhan (1986), Landim and Yau (1997), Sethuraman (2000), Sethuraman and Xu (1996) and Varadhan (1995) for these and related purposes.

2. Definitions and main result. We now recall the precise definitions of the standard simple exclusion process and the associated tagged exclusion process. Intuitively, the simple exclusion process updates the motion of a collection of indistinguishable random walks on the lattice $\mathbb{Z}^d$ such that jumps to already occupied vertices are suppressed. More carefully, let $\Sigma = \{0, 1\}^{\mathbb{Z}^d}$ be the configuration space and let $\eta(t) \in \Sigma$ be the state of the process at time t. We may represent the configuration in terms of occupation variables $\eta(t) = \{\eta_i(t) : i \in \mathbb{Z}^d\}$ where $\eta_i(t) = 0$ or 1 according to whether the vertex $i \in \mathbb{Z}^d$ is empty or full at time t. Recall that $p(j - i)$ represents the single particle transition rate from i to j. As stated in the Introduction, we concentrate on the translation-invariant finite-range case. In addition, we will assume that the symmetrization $(p(i) + p(-i))/2$ is irreducible to avoid complications.

The evolution of the system $\eta(t)$ is Markovian. As before, let $\{T_t : t \geq 0\}$ denote the process semigroup and let L denote the infinitesimal generator. Define also that a local function $\phi : \Sigma \to \mathbb{R}$ is a function which depends only on a finite number of coordinates.

On local functions ϕ, $(T_t\phi)(\eta) = E_\eta[\phi(\eta(t))]$ and

$$(L\phi)(\eta) = \sum_{i,j} \eta_i(1 - \eta_j)\big(\phi(\eta^{i,j}) - \phi(\eta)\big)p(j - i), \tag{2.1}$$

where $\eta^{i,j}$ is the "exchanged" configuration, $(\eta^{i,j})_i = \eta_j$, $(\eta^{i,j})_j = \eta_i$ and $(\eta^{i,j})_k = \eta_k$ for $k \neq i, j$. The transition rate $\eta_i(1 - \eta_j)p(j - i)$ for $\eta \to \eta^{i,j}$ represents the exclusion property alluded to above.

Now distinguish one of the particles and call it the tagged particle. For simplicity, let us assume initially it is at the origin, and denote its position at time t by $x(t) \in \mathbb{Z}^d$. The position $x(t)$ is not Markovian in general with respect to its own history, as it interacts with the locations of the other particles. A standard technique, however, to compensate for this complication is to form the larger process $\{(x(t), \eta(t)) : t \geq 0\}$, which is Markovian, where the "environment" particles are also followed. In fact, let us also consider the process in the reference frame of the tagged particle, $\{(x(t), \zeta(t)) : t \geq 0\}$ where $\zeta_i(t) = \eta_{i+x(t)}(t)$

for $i \in \mathbb{Z}^d$. This process has semigroup $\tilde{T}_t$ and generator $\tilde{L}$ acting on test functions ϕ, $(\tilde{T}_t\phi)(x,\zeta) = E_{(x,\zeta)}[\phi(x(t),\zeta(t))]$, and

$$(\tilde{L}\phi)(x,\zeta) = \sum_{i,j\neq 0} \zeta_i(1-\zeta_j)\big(\phi(x,\zeta^{ij}) - \phi(x,\zeta)\big)p(j-i) + \sum_j (1-\zeta_j)\big(\phi(x+j,\tau_{-j}\zeta) - \phi(x,\zeta)\big)p(j),$$

where $\tau_{-r}\zeta$ is the configuration obtained by exchanging the values at the origin and r, and then translating in reference to the tagged particle at r,

$$(\tau_{-r}\zeta)_k = \zeta_{k-r} \qquad \text{for } k \neq 0 \text{ or } -r,$$
$$(\tau_{-r}\zeta)_0 = \zeta_0 \quad \text{and} \quad (\tau_{-r}\zeta)_{-r} = \zeta_r.$$

In substituting local functions of the form $\phi(x,\zeta) = \phi(\zeta)$, we can see that the environment process $\{\zeta(t) : t \geq 0\}$ is itself Markovian with semigroup $\mathcal{T}_t$ generated by

$$(\mathcal{L}\phi)(\zeta) = \sum_{i,j\neq 0} \zeta_i(1-\zeta_j)\big(\phi(\zeta^{ij}) - \phi(\zeta)\big)p(j-i) + \sum_j (1-\zeta_j)\big(\phi(\tau_{-j}\zeta) - \phi(\zeta)\big)p(j).$$

We refer to Liggett (1985) for details of the construction of these processes.

The equilibria for these systems have been well studied. As the exclusion model is conservative in that random-walk particles are neither destroyed nor created, one expects a family of invariant measures indexed according to particle density ρ. In fact, let P_ρ, for $\rho \in [0,1]$, be the infinite Bernoulli product measure over $\mathbb{Z}^d$ with marginal $P_\rho\{\eta_i = 1\} = 1 - P_\rho\{\eta_i - 0\} = \rho$. It is shown in Liggett (1985) that $\{P_\rho : \rho \in [0,1]\}$ and $\{P_\rho(\cdot \mid \zeta_0 = 1) : \rho \in [0,1]\}$ are invariant for L and $\mathcal{L}$, respectively. In fact, it is proved in Saada (1987) that the P_ρ and $P_\rho(\cdot \mid \zeta_0 = 1)$ are also extremal in the convex set of invariant measures for L and $\mathcal{L}$, respectively.

Let the path measures for the standard and environment processes with initial distributions P_ρ and $P_\rho(\cdot \mid \zeta_0 = 1)$ be given by $\mathcal{P}_\rho$ and $\mathcal{P}_\rho(\cdot \mid \zeta_0 = 1)$, respectively. Let E_μ be expectation with respect to the measure μ. Also, define $\langle f, g\rangle_\mu = E_\mu[fg]$ with respect to μ. When the context is clear, we will denote E_μ and $\langle f, g\rangle_\mu$ for $\mu = P_\rho$, $P_\rho(\cdot \mid \zeta_0 = 1)$, $\mathcal{P}_\rho$, or $\mathcal{P}_\rho(\cdot \mid \zeta_0 = 1)$ as simply E_ρ and $\langle f, g\rangle_\rho$, respectively.

At this point, we fix $\rho \in [0,1]$ for the remainder of the article.

We now recall and quote some definitions of H_1 and H_{-1} spaces from Sethuraman (2000). First note that L or $\mathcal{L}$ is self-adjoint with respect to P_ρ or $P_\rho(\cdot \mid \zeta_0 = 1)$, respectively, if and only if p is symmetric. In general, however, L and $\mathcal{L}$ may be decomposed into symmetric and antisymmetric parts, respectively, $L = -S - A$ and $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ where $-S = (L + L^*)/2$ and

$-A = (L - L^*)/2$, and $-\mathcal{S} = (\mathcal{L} + \mathcal{L}^*)/2$ and $-\mathcal{A} = (\mathcal{L} - \mathcal{L}^*)/2$; here * refers to adjoint objects. Note that $-S$ and $-\mathcal{S}$, by themselves, generate exclusion processes with symmetric jump rates $(p(j-i)+p(i-j))/2$ and so are reversible. In addition, $-S$ and $-\mathcal{S}$ have nonpositive spectrum.

Explicitly, for local ϕ, these operators take the form

$$
\begin{aligned}
(-S\phi)(\eta) &= \frac{1}{2}\sum_{i,j}\big(p(j-i)+p(i-j)\big)\big(\phi(\eta^{i,j})-\phi(\eta)\big),\\
(-A\phi)(\eta) &= \frac{1}{2}\sum_{i,j}\big(p(j-i)-p(i-j)\big)(\eta_i-\eta_j)\big(\phi(\eta^{i,j})-\phi(\eta)\big),\\
(-\mathcal{S}\phi)(\zeta) &= \frac{1}{2}\sum_{k}\sum_{i\neq 0,-k}\big(p(k)+p(-k)\big)\big(\phi(\zeta^{i,i+k})-\phi(\zeta)\big)\\
&\quad+\frac{1}{2}\sum_{k}\big(p(k)+p(-k)\big)(1-\zeta_k)\big(\phi(\tau_{-k}(\zeta))-\phi(\zeta)\big),\\
(-\mathcal{A}\phi)(\zeta) &= \frac{1}{2}\sum_{k}\sum_{i\neq 0,-k}\big(p(k)-p(-k)\big)(\zeta_i-\zeta_{i+k})\big(\phi(\zeta^{i,i+k})-\phi(\zeta)\big)\\
&\quad+\frac{1}{2}\sum_{k}\big(p(k)-p(-k)\big)(1-\zeta_k)\big(\phi(\tau_{-k}(\zeta))-\phi(\zeta)\big).
\end{aligned}
\tag{2.2}
$$

Note also, in the quadratic (Dirichlet) forms, $\langle\phi,(-L)\phi\rangle_\rho$ and $\langle\phi,(-\mathcal{L})\phi\rangle_\rho$, that only the symmetric part of the generator survives and that we may compute for local ϕ that

$$
\begin{aligned}
\langle\phi,(-L)\phi\rangle_\rho &= \langle\phi,S\phi\rangle_\rho\\
&= \frac{1}{4}\sum_{i,j}E_\rho\big[\big(p(j-i)+p(i-j)\big)\big(\phi(\eta^{i,j})-\phi(\eta)\big)^2\big]
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
\langle\phi,(-\mathcal{L})\phi\rangle_\rho &= \frac{1}{4}\sum_{k}\sum_{i\neq 0,-k}E_\rho\big[\big(p(k)+p(-k)\big)\big(\phi(\zeta^{i,i+k})-\phi(\zeta)\big)^2\big]\\
&\quad+\frac{1}{4}\sum_{k}E_\rho\big[\big(p(k)+p(-k)\big)(1-\zeta_k)\big(\phi(\tau_{-k}(\zeta))-\phi(\zeta)\big)^2\big].
\end{aligned}
$$

To define certain resolvent norms, consider the bounded operators, for $\lambda > 0$, $(\lambda - L)^{-1} : L^2(P_\rho) \to L^2(P_\rho)$ and $(\lambda - \mathcal{L})^{-1} : L^2(P_\rho(\cdot \mid \zeta_0 = 1)) \to L^2(P_\rho(\cdot \mid \zeta_0 = 1))$, given by

$$
(\lambda - L)^{-1} f(\eta) = \int_0^\infty e^{-\lambda t} T_t f(\eta)\, dt
$$

and

$$
(\lambda - \mathcal{L})^{-1} f(\eta) = \int_0^\infty e^{-\lambda t} \mathcal{T}_t f(\zeta)\, dt.
$$

Also, let $[(\lambda - L)^{-1}]_s$ and $[(\lambda - \mathcal{L})^{-1}]_s$ denote the symmetric parts of $(\lambda - L)^{-1}$ and $(\lambda - \mathcal{L})^{-1}$, respectively. From a simple computation, we may compute the inverse

$$\begin{aligned}\big[[(\lambda - L)^{-1}]_s\big]^{-1} &= (\lambda - L^*)(\lambda + S)^{-1}(\lambda - L)\\ &= (\lambda + S) - A(\lambda + S)^{-1}A,\end{aligned}$$

with an analogous calculation for $[[(\lambda - \mathcal{L})^{-1}]_s]^{-1}$. This calculation yields, as both S and $-A(\lambda + S)^{-1}A$ are nonnegative operators, that $[[(\lambda - L)^{-1}]_s]^{-1}$ is also a nonnegative operator.

For $f \in L^2$, the quadratic forms $\langle f, (\lambda - L)^{-1} f\rangle_\rho$ may be expressed in variational form over local ϕ as

$$\begin{aligned}&\langle f, [(\lambda - L)^{-1}]_s f\rangle_\rho\\ &\quad = \sup_\phi\big\{2\langle f, \phi\rangle_\rho - \langle\phi, \big[[(\lambda - L)^{-1}]_s\big]^{-1}\phi\rangle_\rho\big\}\\ &\quad = \sup_\phi\big\{2\langle f, \phi\rangle_\rho - \langle\phi, (\lambda + S)\phi\rangle_\rho - \langle A\phi, (\lambda + S)^{-1}A\phi\rangle_\rho\big\}\end{aligned} \tag{2.4}$$

and also in semigroup form by $\int_0^\infty e^{\lambda t}\langle f, T_t f\rangle_\rho\, dt$. Similarly, $\langle f, (\lambda - \mathcal{L})^{-1} f\rangle_\rho$ may also be written in variational or semigroup form.

Define, if the limit exists, the quantity $\langle f, (-L)^{-1} f\rangle_\rho$ by

$$\langle f, (-L)^{-1} f\rangle_\rho = \lim_{\lambda \to 0}\langle f, (\lambda - L)^{-1} f\rangle_\rho$$

and also the analogous object $\langle f, (-\mathcal{L})^{-1} f\rangle_\rho$ when the limit is defined. When the generator L or $\mathcal{L}$ is reversible, it is proved in Sethuraman [(2000), Lemma 3.3] that these limits exist in the extended sense and may be written in form

$$\langle f, S^{-1} f\rangle_\rho = \sup_\phi\big\{2\langle f, \phi\rangle_\rho - \langle\phi, S\phi\rangle_\rho\big\},$$

or $\int_0^\infty \langle f, T_t f\rangle_\rho\, dt$, with similar expressions for $\langle f, \mathcal{S}^{-1} f\rangle_\rho$. However, as alluded to in the Introduction, when the generators are asymmetric, it is not clear these limits exist without additional assumptions on the asymmetries or on f. The difficulty is that in the variational expression (2.4) the second and third terms are of opposite monotonicities—they increase and decrease, respectively, as $\lambda \downarrow 0$. What is known, however, is that when the asymmetries are mean zero $[\sum_i p(i) = 0]$ or when $d \geq 3$, the limit exists [Varadhan (1995), Sethuraman, Varadhan and Yau (2000)]. Also, for a certain class of functions f the limit exists in $d \leq 2$ [Sethuraman (2000)].

Standard Dirichlet form spaces may be defined for symmetric operators, $-L = S$. Let the Hilbert space $H_1(S)$ be the completion with respect to the Dirichlet form $\langle f, Sf\rangle_\rho$,

$$H_1(S) = \text{completion of } \big\{\phi \text{ local}: \langle\phi, S\phi\rangle_\rho < \infty\big\},$$

with norm $\|f\|_1(S) = \langle f, Sf\rangle_\rho^{1/2}$ and inner product by polarization.

Let also $H_{-1}(S)$ denote the completed Hilbert space,

$$H_{-1}(S) = \text{completion of } \{\phi \text{ local} : \langle \phi, S^{-1}\phi\rangle_\rho < \infty\},$$

with norm $\|f\|_{-1}(S) = \langle f, S^{-1}f\rangle_\rho^{1/2}$. Analogously, for reversible $-\mathcal{L} = \mathcal{S}$, we may define Hilbert spaces $H_1(\mathcal{S})$ and $H_{-1}(\mathcal{S})$.

In the same way, for each $\lambda > 0$, Hilbert spaces $H_1(\lambda, -L)$, $H_{-1}(\lambda, -L)$, $H_1(\lambda, -\mathcal{L})$ and $H_{-1}(\lambda, -\mathcal{L})$ may also be defined in terms of completions with respect to local functions ϕ of the corresponding norms:

$$\begin{aligned}
\|\phi\|_1(\lambda, -L) &= \langle (\lambda - L)\phi, (\lambda + S)^{-1}, (\lambda - L)\phi\rangle_\rho^{1/2} \\
&= \langle \phi, (\lambda + S)\phi\rangle_\rho + \langle A\phi, (\lambda + S)^{-1}A\phi\rangle_\rho^{1/2}, \\
\|\phi\|_{-1}(\lambda, -L) &= \langle \phi, (\lambda - L)^{-1}\phi\rangle_\rho^{1/2}, \\
\|\phi\|_1(\lambda, -\mathcal{L}) &= \langle \phi, (\lambda + \mathcal{S})\phi\rangle_\rho + \langle \mathcal{A}\phi, (\lambda + S)^{-1}\mathcal{A}\phi\rangle_\rho^{1/2}, \\
\|\phi\|_{-1}(\lambda, -\mathcal{L}) &= \langle \phi, (\lambda - \mathcal{L})^{-1}\phi\rangle_\rho^{1/2}.
\end{aligned}$$

Observe that these norms and spaces make sense also for exclusion-type operators $-L = S' + A''$ and $-\mathcal{L} = \mathcal{S}' + \mathcal{A}''$ where $-S'$ and $\mathcal{S}'$, and $-A''$ and $-\mathcal{A}''$ are the symmetric and antisymmetric parts of exclusion generators L' or $\mathcal{L}'$, and L'' or $\mathcal{L}''$, respectively. We will use analogous notation, $\|\cdot\|_1(\lambda, S' + A'')$, $\|\cdot\|_1(\lambda, \mathcal{S}' + \mathcal{A}'')$, $\|\cdot\|_{-1}(\lambda, S' + A'')$ and $\|\cdot\|_{-1}(\lambda, \mathcal{S}' + \mathcal{A}'')$ to denote $H_1(\lambda, S' + A'')$, $H_1(\lambda, \mathcal{S}' + \mathcal{A}'')$, $H_{-1}(\lambda, S' + A'')$ and $H_{-1}(\lambda, \mathcal{S}' + \mathcal{A}'')$ norms, respectively.

Define also that symmetric exclusion operators Q and Q' have equivalent quadratic or Dirichlet forms if there are constants $0 < C_1 \le C_2$ such that for all local functions ϕ,

$$C_1\langle \phi, Q'\phi\rangle_\rho \le \langle \phi, Q\phi\rangle_\rho \le C_2\langle \phi, Q'\phi\rangle_\rho.$$

The following definition will be useful in describing processes with nonzero drift. Define that a nearest-neighbor jump rate $p(\cdot)$ with nonzero drift is degenerate on $\mathbb{Z}^d$ if there exists a unit basis vector e^l, for some $1 \le l \le d$, such that either $p(e^l) = 0$ and $p(-e^l) > 0$, or $p(-e^l) = 0$ and $p(e^l) > 0$. Last, a nearest-neighbor system or generator will be called degenerate if its underlying jump rate is degenerate.

We now come to the results.

THEOREM 2.1. *Let L be an exclusion generator with finite-range translation-invariant jump rate p on $\mathbb{Z}^d$ whose symmetrization is irreducible. Let also L' be the generator with nearest-neighbor translation-invariant jump rate p' having the same drift where, for $1 \le l \le d$,*

$$p'(\pm e^l) = \begin{cases} \max[\pm e^l \cdot r(p), 0], & \textit{when } e^l \cdot r(p) \ne 0, \\ 1, & \textit{when } e^l \cdot r(p) = 0. \end{cases}$$

Then, for constants $C = C(p,d)$, $D = D(p,d) > 0$, *and all* $\lambda > 0$, *we have that*

$$C^{-1}\|\cdot\|_{-1}(D^{-1}\lambda, -L') \le \|\cdot\|_{-1}(\lambda, -L) \le C\|\cdot\|_{-1}(D\lambda, -L'). \tag{2.5}$$

Also, the above statement holds, with perhaps different constants $C = C(p,d)$, $D = D(p,d)$, *for environment generators* $\mathcal{L}$ *and* $\mathcal{L}'$ *with the same respective jump rates* p *and* p' *in dimensions* $d \ge 2$.

Possibly, the main impact of the theorem is to asymmetric systems with nonzero drift. In this case, the result suggests reasonably that on a large scale level the motion statistics are not changed if one subtracts "cancelling" or mean-zero jumps but preserves the drift structure. Note also in the theorem that p' is degenerate if and only if p has nonzero drift.

We now turn to the situation in dimension $d = 1$ for environment processes. Things are more delicate due to the restricted geometry of $\mathbb{Z}$. A dichotomy between environment processes with nearest-neighbor rates and those without emerges.

THEOREM 2.2.

(i) *Let* $\mathcal{L}$ *be an environment generator with finite-range, translation-invariant, nonnearest-neighbor rates* p *in* $d = 1$ *whose symmetrization is irreducible, and let* $\mathcal{L}'$ *be the generator with translation-invariant jump rates* p', *of range* 2 *and the same drift, given by* $p'(\pm 2) = 1$, *and*

$$p'(\pm 1) = \begin{cases} \max[\pm r(p), 0], & \text{when } r(p) \neq 0, \\ 1, & \text{when } r(p) = 0. \end{cases}$$

Then, for constants $C = C(p)$, $D = D(p)$ *and all* $\lambda > 0$, *inequalities* (2.5) *hold.*

(ii) *Also, let* $\mathcal{L}$ *be a nearest-neighbor generator in* $d = 1$ *corresponding to translation-invariant rates* p *with nonzero drift, and let* $\mathcal{L}'$ *be the degenerate nearest-neighbor operator with translation-invariant rates* p' *given by* $p'(\pm 1) = \max[\pm(p(1) - p(-1)), 0]$. *Then, for* $C = C(p)$, $D = D(p)$, *and all* $\lambda > 0$, *inequalities* (2.5) *hold. Of course, when* p *is mean zero,* $\mathcal{L}$ *is symmetric and* (2.5) *holds trivially.*

(iii) *However, when* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *generate nearest-neighbor and finite-rate nonnearest-neighbor environment systems in* $d = 1$ *with translation-invariant rates whose symmetrizations are irreducible, there do not exist constants* $C = C(p)$, $D = D(p)$ *such that for all* $\lambda > 0$ *the bound* $\|\cdot\|_{-1}(\lambda, -\mathcal{L}_1) \le C\|\cdot\|_{-1}(D\lambda, -\mathcal{L}_2)$ *holds.*

We also expect that the inequality "$\|\cdot\|_{-1}(\lambda, -\mathcal{L}_2) \le C\|\cdot\|_{-1}(D\lambda, -\mathcal{L}_1)$" in part (iii) does not hold similarly. However, beyond rough indications, we do not have a rigorous example contradicting the inequality.

The intuition for this dichotomy is in that motion through the origin is denied for nearest-neighbor processes, as particles cannot cross the tagged particle, but allowed for nonnearest-neighbor systems due to "leap-frogging" behavior. This leads, respectively, to less and more mixing and therefore larger and smaller variations in configurations. This phenomenon translates to larger and smaller H_{-1} norms which cannot be reconciled.

An important immediate consequence of Theorems 2.1 and 2.2 holds when both $\langle f, (-L)^{-1} f\rangle_\rho = \lim_{\lambda\downarrow 0} \|f\|_{-1}^2(\lambda, L)$ and $\langle f, (-L')^{-1} f\rangle_\rho = \lim_{\lambda\downarrow 0} \|f\|_{-1}^2(\lambda, L')$ exist (and also when corresponding limits with respect to $\mathcal{L}$ and $\mathcal{L}'$ exist).

COROLLARY 2.1. *Under the conditions of Theorem* 2.1, *when the limits* $\langle f, (-L)^{-1} f\rangle_\rho$ *and* $\langle f, (-L')^{-1} f\rangle_\rho$ *both exist*, *they are equivalent*. *Also*, *under the conditions of Theorems* 2.1 *and* 2.2, *the same statement holds with* $\mathcal{L}$ *and* $\mathcal{L}'$ *replacing* L *and* L', *respectively*.

The strategy of proof for the positive claims made in the theorems above will be to use a sector condition inequality proved for exclusion generators L and $\mathcal{L}$ in Varadhan (1995), and then some work with the variational formulas. For the negative results, counterexamples are presented.

In Section 3, we gather together some supporting lemmas. In Section 4, we prove Theorem 2.1. In Section 5, we prove Theorem 2.2 concerning environment systems in $d = 1$. Finally, in Section 6, we present an application to the diffusive behavior of additive functionals of simple exclusion.

3. Preliminary estimates. We recall some of the lemmas proved in Sethuraman (2000) that which will provide a framework for the proof of the main theorem.

Our starting point will be the important sector condition inequality proved in Varadhan (1995), Theorem 5.1.

LEMMA 3.1. *Let* ϕ *and* ψ *be local functions*. *For simple exclusion processes on* $\mathbb{Z}^d$ *with finite-range*, *translation-invariant*, *mean-zero*, *irreducible* p, *there is a constant* $C = C(d, p, \rho) \geq 1$ *so that a sector condition holds*,

$$\langle \phi, (-L)\psi\rangle_\rho \leq C\langle \phi, S\phi\rangle_\rho^{1/2}\langle \psi, S\psi\rangle_\rho^{1/2}.$$

The same inequality holds with $\mathcal{L}$ *and* $\mathcal{S}$ *in place of* L *and* S, *respectively*.

Note, when L or $\mathcal{L}$ is symmetric, by Schwarz inequality, C may be taken to be $C = 1$.

A corollary of the last lemma proved in Sethuraman (2000), Lemma 4.3, is the following.

LEMMA 3.2. *Let $L = -S - A$ correspond to finite-range, translation-invariant, mean-zero, irreducible p. With respect to local functions ϕ, $\lambda \geq 0$, and C as in Lemma 3.1, we have that*

$$\langle A\phi, (\lambda + S)^{-1} A\phi\rangle_\rho \leq (C+1)^2 \langle \phi, (\lambda + S)\phi\rangle_\rho.$$

The same inequality holds with $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ replacing L.

We now state a useful comparison. The next lemma is Sethuraman [(2000), Lemma 3.6] with only notational changes.

LEMMA 3.3. *Let $L = -S - A$ and $L' = -S' - A$ be generators with the same antisymmetric part. If S and S', with respect to some constants $0 < C_1 < C_2$ have equivalent Dirichlet forms, then for $f \in L^2(P_\rho)$ and $\lambda > 0$ we have*

$$\begin{aligned}\big[\max(C_1^{-1}, C_2)\big]^{-1} \|f\|_{-1}^2 (C_1^{-1}\lambda, -L') &\leq \|f\|_{-1}^2(\lambda, -L)\\ &\leq \big[\min(C_1, C_2^{-1})\big]^{-1} \|f\|_{-1}^2 (C_2^{-1}\lambda, -L').\end{aligned}$$

Also, the same result is true with L and L' replaced by $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ and $\mathcal{L}' = -\mathcal{S}' - \mathcal{A}'$, respectively.

The next two results are key estimates used to prove the main theorems.

LEMMA 3.4. *Let $L = -S - A$ and $L_m = -S_m - A_m$ correspond to finite-range translation-invariant rates p and p_m, respectively. Assume that p_m is mean zero and irreducible, and that S and S_m have equivalent Dirichlet forms. Then there are constants $C, D > 0$ such that for all $\lambda > 0$,*

$$\begin{aligned}C^{-1}\|\cdot\|_{-1}&\big(D^{-1}\lambda, (S + S_m) + (A + A_m)\big)\\ &\leq \|\cdot\|_{-1}(\lambda, -L)\\ &\leq C\|\cdot\|_{-1}\big(D\lambda, (S + S_m) + (A + A_m)\big).\end{aligned}$$

The same statement holds with $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ in place of L.

PROOF. We give the proof for the generator L; the arguments for $\mathcal{L}$ are identical. We show first the upper bound. From the assumption on S and S_m and Lemma 3.3, there is C_1 and D_1 such that for $f \in L^2(P_\rho)$,

$$\begin{aligned}\|f\|_{-1}^2(\lambda, -L) &\leq C_1 \|f\|_{-1}^2 (D_1\lambda, S_m + A)\\ &= C_1 \sup_\phi \big\{2\langle f, \phi\rangle_\rho - \langle \phi, (D_1\lambda + S_m)\phi\rangle_\rho\\ &\qquad - \langle A\phi, (D_1\lambda + S_m)^{-1} A\phi\rangle_\rho\big\}.\end{aligned} \tag{3.1}$$

Note from Lemma 3.2 and the inequality $(a+b)^2 \le 2a^2 + 2b^2$ that

$$
\begin{aligned}
&-2^{-1}\langle \phi, (D_1\lambda + S_m)\phi\rangle_\rho - \langle A\phi, (D_1\lambda + S_m)^{-1} A\phi\rangle_\rho \\
&\qquad \le -C_2\langle A_m\phi, (D_1\lambda + S_m)^{-1} A_m\phi\rangle_\rho - \langle A\phi, (D_1\lambda + S_m)^{-1} A\phi\rangle_\rho \\
&\qquad \le -C_3\langle (A + A_m)\phi, (D_1\lambda + S_m)^{-1}(A + A_m)\phi\rangle_\rho
\end{aligned}
$$

for some positive constant C_2 and $C_3 = \min\{C_2, 1\}$.

Substituting this last bound into (3.1) gives

$$
\|f\|_{-1}(\lambda, -L) \le \sqrt{C_1/\min(1/2, C_3)}\,\|f\|_{-1}\big(D_1\lambda, S_m + (A + A_m)\big).
$$

Observe now that $S + S_m$ and S_m have equivalent Dirichlet forms from assumption. Therefore, by invoking once more Lemma 3.3, the upper bound is proved.

The lower bound follows similarly using that $A = (A + A_m) - A_m$. $\square$

LEMMA 3.5. *Let $-S$ and $-S'$ generate symmetric exclusion processes on $\mathbb{Z}^d$ with finite-range translation-invariant irreducible jump rates p and p', respectively. Then S and S' have equivalent Dirichlet forms. Analogously, environment generators $-\mathcal{S}$ corresponding to rates p and p' under the same assumptions have equivalent forms in $d \ge 2$ and in $d = 1$ when, in addition, p and p' are both not nearest neighbor.*

Of course, in $d = 1$, when both $\mathcal{S}$ and $\mathcal{S}'$ are nearest neighbor, one is the multiple of the other.

PROOF OF LEMMA 3.5. For operators S and S', the result is in Sethuraman (2000), Lemma 3.7. However, for environment generators some modifications are required. To simplify notation, we prove the result in $d = 1$, as arguments for $d \ge 2$ are similar and in fact easier.

It is enough to show that the quadratic forms for $\mathcal{S}$ and $\mathcal{S}'$ are both equivalent to the Dirichlet form of an operator with transitions of range 2. We prove [as in Sethuraman (2000), Lemma 3.7] the lower and upperbounds for local ϕ,

$$
C^{-1}\langle \phi, \mathcal{S}_2\phi\rangle_\rho \le \langle \phi, \mathcal{S}\phi\rangle_\rho \le C\langle \phi, \mathcal{S}_2\phi\rangle_\rho,
$$

where $\mathcal{S}_2$ is the environment generator with symmetric rates $p(1) = p(-1) = 1$, $p(-2) = p(2) = 1$, and $p(i) = 0$ for $|i| \ge 3$. The explicit Dirichlet form $\langle \phi, \mathcal{S}_2\phi\rangle_\rho$ for local ϕ takes the form

$$
\begin{aligned}
&\tfrac{1}{2}\sum_{i,i+1\ne 0} E_\rho\big[\big(\phi(\zeta^{i,i+1}) - \phi(\zeta)\big)^2\big] + \tfrac{1}{2}\sum_{i,i+2\ne 0} E_\rho\big[\big(\phi(\zeta^{i,i+2}) - \phi(\zeta)\big)^2\big] \\
&\qquad + \tfrac{1}{2}\big\{E_\rho\big[(1-\zeta_1)\big(\phi(\tau_{-1}\zeta) - \phi(\zeta)\big)^2\big] + E_\rho\big[(1-\zeta_{-1})\big(\phi(\tau_1\zeta) - \phi(\zeta)\big)^2\big] \\
&\qquad\quad + E_\rho\big[(1-\zeta_2)\big(\phi(\tau_{-2}\zeta) - \phi(\zeta)\big)^2\big] + E_\rho\big[(1-\zeta_{-2})\big(\phi(\tau_2\zeta) - \phi(\zeta)\big)^2\big]\big\}.
\end{aligned}
$$

By looking at (2.3) and the above expression, one sees that the Dirichlet forms for $\mathcal{S}$ and $\mathcal{S}_2$ fall into two parts, a sum corresponding to environment particle jumps and a sum associated to the reference frame shifts due to the tagged particle motion. The required bounds are shown by treating these two types of jumps separately.

For the upper bounds, we bound each expectation in the Dirichlet form for $\mathcal{S}$ by a finite number of expectations in the form for $\mathcal{S}_2$. We first bound the shift-type jumps. Note that the configuration $\tau_{-k}\zeta$, which exchanges the values at 0 and k and then shifts the labels backward by k, can also be realized as the sequence which shifts the reference frame to the left in k nearest-neighbor steps and then moves the value at -1 by $k-1$ nearest-neighbor exchanges to $-k$. This gives, by the invariance of P_ρ to exchanges and frame shifts τ_r, the inequality $(a_1+\cdots+a_m)^2 \le m(a_1^2+\cdots+a_m^2)$, adding and subtracting $2k-2$ terms, and $1-\zeta_r \le 1$, that $E_\rho[(1-\zeta_k)(\phi(\tau_{-k}\zeta)-\phi(\zeta))^2]$ is less than

$$2k\sum_{m=1}^{k-1} E_\rho\big[\big(\phi(\zeta^{-m,-(m+1)})-\phi(\zeta)\big)^2\big] + 2k\sum_{m=1}^{k} E_\rho\big[(1-\zeta_m)\big(\phi(\tau_{-1}\zeta)-\phi(\zeta)\big)^2\big].$$

(For instance, $E_\rho[(1-\zeta_2)(\phi(\tau_{-2}\zeta)-\phi(\zeta))^2] = E_\rho[(1-\zeta_2)(\{\phi((\tau_{-1}(\tau_{-1}\zeta))^{-1,-2})-\phi(\tau_{-1}(\tau_{-1}\zeta))\} + \{\phi(\tau_{-1}(\tau_{-1}\zeta))-\phi(\tau_{-1}\zeta)\} + \{\phi(\tau_{-1}\zeta)-\phi(\zeta)\})^2]$ and then P_ρ-invariance properties and inequalities are used.) The expectations in the first sum above all appear in the form for $\mathcal{S}_2$. For the second sum, as $\tau_{-1}(\zeta^{1,m}) = (\tau_{-1}\zeta)^{-1,m-1}$, one obtains that $E_\rho[(1-\zeta_m)(\phi(\tau_{-1}\zeta)-\phi(\zeta))^2]$ equals

$$\begin{aligned} &E_\rho\big[(1-\zeta_1)\big(\phi(\tau_{-1}(\zeta^{1,m}))-\phi(\zeta^{1,m})\big)^2\big] \\ &\quad\le 3E_\rho\big[\big(\phi(\zeta^{-1,m-1})-\phi(\zeta)\big)^2\big] + 3E_\rho\big[\big(\phi(\zeta^{1,m})-\phi(\zeta)\big)^2\big] \\ &\qquad + 3E_\rho\big[(1-\zeta_1)\big(\phi(\tau_{-1}\zeta)-\phi(\zeta)\big)^2\big], \end{aligned}$$

of which only the last expectation appears explictly in the $\mathcal{S}_2$ form.

Our efforts go now to bound the jump-type terms of the form $E_\rho[(\phi(\zeta^{i,j})-\phi(\zeta))^2]$ for $i,j\neq 0$, thereby finishing the upper bounds. If both $i<j$ are of the same sign, then $\zeta^{i,j}$ is the same as moving the value at i in $j-i$ nearest-neighbor exchanges to j and then moving the value ζ_j now at $j-1$ back to i in $j-i-1$ nearest-neighbor exchanges. We have then that the expectation is bounded above by

$$2(j-i)\sum_{m=1}^{j-i} E_\rho\big[\big(\phi(\zeta^{m,m+1})-\phi(\zeta)\big)^2\big].$$

Otherwise, if say $i < 0 < j$, by the same argument, we obtain that the expectation is bounded by

$$\begin{aligned}2(j-i)\sum_{m=1}^{|i|-1} E_\rho\big[\big(\phi(\zeta^{-m,-(m+1)})-\phi(\zeta)\big)^2\big]\\ +2(j-i)\sum_{m=1}^{|j|-1} E_\rho\big[\big(\phi(\zeta^{m,m+1})-\phi(\zeta)\big)^2\big]\\ +2(j-i)E_\rho\big[\big(\phi(\zeta^{-1,1})-\phi(\zeta)\big)^2\big].\end{aligned}$$

All of these terms appear explicitly in the form for $\mathcal{S}_2$. This completes the proof of the upper bounds as p is finite range. At this point, it will be useful to remark for the proof of the next lemma that we have actually shown that the $\mathcal{S}$-form is bounded above by a multiple of the form

$$\langle\phi, \mathcal{S}_1\phi\rangle_\rho + \tfrac{1}{2}E_\rho\big[\big(\phi(\zeta^{-1,1})-\phi(\zeta)\big)^2\big], \tag{3.2}$$

where $-\mathcal{S}_1$ is the $d = 1$ nearest-neighbor translation-invariant environment generator with rates $p(1) = p(-1) = 1/2$.

For the lower bounds, we bound the expectations in the Dirichlet form for $\mathcal{S}_2$ in terms of expectations in the form for $\mathcal{S}$. First, as p is irreducible, there exists a finite path r from the origin 0 to 1, $0 = r(0), r(1), \dots, r(n_l)$ in the support of p such that a p-random walker may jump from 0 to $r(1)$, then to $r(1) + r(2)$, and so on to $r(1) + \cdots + r(n_l) = 1$. Similarly, there is a finite path s from the origin 0 to 2. Denote $j(k) = r(0) + \cdots + r(k)$ and bound

$$E_\rho\big[\big(\phi(\zeta^{i,i+1})-\phi(\zeta)\big)^2\big] \le 2n_l \sum_{m=0}^{n_l-1} E_\rho\big[\big(\phi(\zeta^{i+j(m),i+j(m+1)})-\phi(\zeta)\big)^2\big],$$

and also $E_\rho[(1-\zeta_1)(\phi(\tau_{-1}\zeta)-\phi(\zeta))^2]$ less than

$$\begin{aligned}2n_l\sum_{m=1}^{n_l-1} E_\rho\big[\big(\phi(\zeta^{-j(n_l-m),-j(n_l-(m+1))})-\phi(\zeta)\big)^2\big]\\ +2n_l\sum_{m=0}^{n_l-1} E_\rho\big[(1-\zeta_{j(m)+1})\big(\phi(\tau_{-r(m+1)}\zeta)-\phi(\zeta)\big)^2\big].\end{aligned}$$

The expectations in the last sum are estimated in terms of the $\mathcal{S}$ form by similar arguments as in the upper bound proof. The expectation $E_\rho[(1-\zeta_{-1})(\phi(\tau_1\zeta) - \phi(\zeta))^2]$ is handled analogously. Analogously, using the path s we may bound terms in the form for $\mathcal{S}_2$ corresponding to jumps of size 2. The desired lower bounds now follow because p is finite range. $\square$

Let now $-\mathcal{S}'$ be a symmetric nearest-neighbor environment generator in $d=1$ with translation-invariant irreducible jump rates p. Define now an exclusion-type process by the generator $-\mathcal{S}'-\mathcal{B}$ where

$$(-\mathcal{B}\phi)(\zeta) = \zeta_{-1}\big(\phi(\tau_1\zeta)-\phi(\zeta)\big)+\zeta_1\big(\phi(\tau_{-1}\zeta)-\phi(\zeta)\big). \tag{3.3}$$

In other words, $-\mathcal{S}'-\mathcal{B}$ generates an exclusion process where the tagged particle may also exchange places with particles in nearest-neighbor positions. The Bernoulli product measures $P_\rho[\cdot \mid \zeta_0 = 1]$ are reversible for $-\mathcal{B}$ and therefore also for the $-\mathcal{S}'-\mathcal{B}$ process. We observe also that the quadratic form $\langle \phi, \mathcal{B}\phi\rangle_\rho$ is given by

$$\tfrac{1}{2}E_\rho\big[\zeta_{-1}\big(\phi(\tau_1\zeta)-\phi(\zeta)\big)^2\big]+\tfrac{1}{2}E_\rho\big[\zeta_1\big(\phi(\tau_{-1}\zeta)-\phi(\zeta)\big)^2\big].$$

LEMMA 3.6. *Let $-\mathcal{S}'$ be as above and $-\mathcal{S}''$ be an environment generator in $d=1$ with symmetric finite-range nonnearest-neighbor translation-invariant irreducible rates p. Then $\mathcal{S}''$ and $\mathcal{S}'+\mathcal{B}$ have equivalent quadratic forms.*

PROOF. We first make a reduction. Recall the environment generators $-\mathcal{S}_1$ and $-\mathcal{S}_2$ from the proof of Lemma 3.5. As both $\mathcal{S}'$ and $\mathcal{S}_1$ are $d=1$ symmetric nearest-neighbor operators, they are multiples of each other and so their forms are equivalent. Therefore, $\mathcal{S}'+\mathcal{B}$ and $\mathcal{S}_1+\mathcal{B}$ have equivalent forms also. In the same vein, we have from Lemma 3.5 that the $\mathcal{S}''$ and $\mathcal{S}_2$ forms are equivalent. Hence, to prove the lemma, we need only show for local functions ϕ the equivalence of $\mathcal{S}_2$ and $\mathcal{S}_1+\mathcal{B}$ forms,

$$C^{-1}\langle\phi, \mathcal{S}_2\phi\rangle_\rho \le \langle \phi, (\mathcal{S}_1+\mathcal{B})\phi\rangle_\rho \le C\langle\phi,\mathcal{S}_2\phi\rangle_\rho,$$

for some constant C.

In the proof of Lemma 3.5 [cf. near (3.2)], we showed that the $\mathcal{S}_2$ form is bounded above by a multiple of the form of $\mathcal{S}_1+\mathcal{N}$ where $(-\mathcal{N}\phi)(\zeta) = (\phi(\zeta^{-1,1})-\phi(\zeta))$ and

$$\langle\phi,\mathcal{N}\phi\rangle_\rho = \tfrac{1}{2}E_\rho\big[\big(\phi(\zeta^{-1,1})-\phi(\zeta)\big)^2\big].$$

Now, the configuration $\zeta^{-1,1}$ which exchanges values at -1 and 1 is the same as shifting the reference frame to -1, exchanging values at 1 and 2, and then shifting back the frame to 1. This gives that $E_\rho[(\phi(\zeta^{-1,1})-\phi(\zeta))^2]$ is less than

$$3E_\rho\big[\big(\phi(\tau_1\zeta)-\phi(\zeta)\big)^2\big]+3E_\rho\big[\big(\phi(\tau_{-1}\zeta)-\phi(\zeta)\big)^2\big]+3E_\rho\big[\big(\phi(\zeta^{1,2})-\phi(\zeta)\big)^2\big].$$

However, we may write $E_\rho[(\phi(\tau_{\pm 1}\zeta)-\phi(\zeta))^2]$ as

$$E_\rho\big[(1-\eta_{\pm 1})\big(\phi(\tau_{\pm 1}\zeta)-\phi(\zeta)\big)^2\big]+E_\rho\big[\eta_{\pm 1}\big(\phi(\tau_{\pm 1}\zeta)-\phi(\zeta)\big)^2\big],$$

so that all terms in the $\mathcal{S}_1+\mathcal{N}$ form are bounded by multiples of terms in the $\mathcal{S}_1+\mathcal{B}$ form. This establishes the desired upper bound.

The lower bound follows similarly. $\square$

4. Proof of Theorem 2.1. The proof of Theorem 2.1 is in three parts. The first part is to bound the norm $\|\cdot\|_{-1}^2(\lambda, -L)$ [respectively, $\|\cdot\|_{-1}^2(\lambda, \mathcal{L})$] by the negative norm of a process which makes no nonaxes jumps (Lemma 4.1). The second part is to get a further bound in terms of a negative norm of a nearest-neighbor system (Lemma 4.2). The final part is to obtain another bound in terms of a degenerate nearest-neighbor model (Lemma 4.3).

In order to simplify notation, we concentrate on $d = 2$ which typifies the general situation. Also, as all arguments for the environment process generated by $\mathcal{L}$, in $d \geq 2$, differ only notationally from those for the process associated to L, we present proofs only for the models driven by L. For the remainder of the section, let ϕ be a local function, and $f \in L^2(P_\rho)$.

LEMMA 4.1. *Let $L = -S - A$ be a generator corresponding to finite-range translation-invariant p whose symmetrization is irreducible. Let also $L' = -S' - A'$ be the generator with translation-invariant rates p' which have the same drift as p and involve only jumps on the axes given by*

$$p'((i_1, i_2)) = \begin{cases} \sum_{j_2} p((i_1, j_2)), & \textit{for } i_1 \neq 0,\ i_2 = 0, \\ \sum_{j_1} p((j_1, i_2)), & \textit{for } i_1 = 0,\ i_2 \neq 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Then, for $C = C(p)$, $D = D(p) > 0$ and $\lambda > 0$, we have that

$$C^{-1}\|\cdot\|_{-1}(D^{-1}\lambda, -L') \leq \|\cdot\|_{-1}(\lambda, -L) \leq C\|\cdot\|_{-1}(D\lambda, -L').$$

The same result holds with $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ and $\mathcal{L}' = -\mathcal{S}' - \mathcal{A}'$ replacing L and L' respectively.

PROOF. First, note that without loss of generality, we may assume that both $e^1 = (1, 0)$ and $e^2 = (0, 1)$ are in the support of $p(i) + p(-i)$, as otherwise, form the operator

$$(-\bar{S}\phi)(\eta) = (-S\phi)(\eta) + \sum_i (\phi(\eta^{i,i+e^1}) - \phi(\eta)) + \sum_i (\phi(\eta^{i,i+e^2}) - \phi(\eta)).$$

Clearly, jump rates for $-\bar{S}$ are irreducible, and so S and $\bar{S}$ have equivalent quadratic forms from Lemma 3.5. Therefore, from Lemma 3.3,

$$C^{-1}\|\cdot\|_{-1}(D^{-1}\lambda, \bar{S} + A) \leq \|\cdot\|_{-1}(\lambda, S + A) \leq C\|\cdot\|_{-1}(D\lambda, \bar{S} + A)$$

and so we could start with $L = -\bar{S} - A$, if necessary.

We now remove nonaxes jumps from the antisymmetric part of L. Let $E \subset \mathbb{Z}^2$ denote the set of nonaxes jumps present in A:

$$E = \{k = (k_1, k_2) : p(k) - p(-k) > 0 \text{ and } k_1, k_2 \neq 0\}.$$

Now suppose that $a = (a_1, a_2) \in E$, if E is nonempty. Then define the jump rate $m(\cdot)$ by

$$m\big((a_1, 0)\big) = m\big((0, a_2)\big) = m(-a) = p(a) - p(-a),$$
$$m\big((-a_1, 0)\big) = m\big((0, -a_2)\big) = m(a) = 0$$

and

$$m(k) = m(-k) = \big(p(k) + p(-k)\big)/2 \qquad \text{for } k \neq \pm a,\ (\pm a_1, 0) \text{ and } (0, \pm a_2).$$

Let $L_m = -A_m - S_m$ be the generator corresponding to $m(\cdot)$. As the symmetrizations of both p and m contain e^1 and e^2 in their support, the quadratic forms of S and S_m are equivalent by Lemma 3.5. Then, from Lemma 3.4, there are constants C and D such that $C^{-1}\|f\|_{-1}(D^{-1}\lambda, (S + S_m) + (A + A_m)) \le \|f\|_{-1}(\lambda, -L) \le C\|f\|_{-1}(D\lambda, (S + S_m) + (A + A_m))$. Recalling the explicit calculation (2.2), note that the operator $-(S + S_m) - (A + A_m)$ is an exclusion generator whose antisymmetric part $A + A_m$ is constructed to contain no a-jumps [the compensation is in terms corresponding to $(a_1, 0)$ and $(0, a_2)$]. In this way, as $|E| < \infty$ from the finite-range assumption on p, we remove all such nonaxes points $a \in E$ and obtain, at the end of these iterations, a generator $\tilde{L} = -\tilde{S} - \tilde{A}$, with no nonaxes jumps in the antisymmetric part, such that, with worse constants C, D, we have $C^{-1}\|f\|_{-1}(D^{-1}\lambda, -\tilde{L}) \le \|f\|_{-1}(\lambda, -L) \le C\|f\|_{-1}(D\lambda, -\tilde{L})$.

Finally, note that A', the antisymmetric part of $-L' = S' + A'$ is the same as $\tilde{A}$ from construction. Observe also that the symmetric part S' corresponding to the symmetrization of p' is irreducible. Therefore, the quadratic forms of S' and $\tilde{S}$ are equivalent by Lemma 3.5, and consequently, we may apply Lemma 3.3 to finish the proof. $\square$

LEMMA 4.2. *Let $L = -S - A$ be an exclusion generator with finite-range translation-invariant jump rates p whose support contains only axes points and whose symmetrization is irreducible. Let also L' be a nearest-neighbor generator with translation-invariant rates p' having the same drift as p given by*

$$p'(i) = \begin{cases} \sum_j \max[\pm(j \cdot e^l), 0]\, p(j), & \textit{for } i = \pm e^l,\ l = 1, 2, \\ 0, & \textit{otherwise}. \end{cases}$$

Then, for $C = C(p)$, $D = D(p) > 0$ and $\lambda > 0$, we have that

$$C^{-1}\|\cdot\|_{-1}(D^{-1}\lambda, -L') \le \|\cdot\|_{-1}(\lambda, -L) \le C\|\cdot\|_{-1}(D\lambda, -L').$$

The same statement holds with $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ and $\mathcal{L}'$ replacing L and L', respectively.

PROOF. Without loss of generality, as in the previous lemma, we assume that e^1 and e^2 are in the support of $p(i) + p(-i)$. Classify now the support of p such that the points $(i_l, 0)$ for $l = 1, \dots, n$ and $(0, j_k)$ for $k = 1, \dots, r$ are exactly those vertices for which $p((i_l, 0)) - p((-i_l, 0)) > 0$ and $p((0, j_k)) - p((0, -j_k)) > 0$. For each nonnearest-neighbor jump, say in direction $(i_l, 0)$ for $i_l > 1$ to make a choice, we construct the corresponding mean-zero jump rate m:

$$\begin{aligned} m\big((-i_l, 0)\big) &= p\big((i_l, 0)\big) - p\big((-i_l, 0)\big), \\ m\big((1, 0)\big) &= i_l\big(p\big((i_l, 0)\big) - p\big((-i_l, 0)\big)\big), \\ m\big((i_l, 0)\big) &= m\big((-1, 0)\big) = 0 \end{aligned}$$

and

$$m(k) = \big(p(k) + p(-k)\big)/2 \qquad \text{for all } k \neq (\pm i_l, 0), (\pm 1, 0).$$

Mean-zero jump rates m are constructed similarly for directions $(i_l, 0)$ for $i_l < -1$ and $(0, j_k)$ for $|j_k| > 1$. We now follow the method of the previous lemma to finish the argument. $\square$

Denote the nearest-neighbor generators $L = -S - A$ and $\mathcal{L} = -\mathcal{S} - \mathcal{A}$, corresponding to jumps up, down, right and left with respective translation-invariant rates $u, d, r,$ and l, by $L = L(u, d, r, l)$ and $\mathcal{L} = \mathcal{L}(u, d, r, l)$, respectively. Observe that the antisymmetric parts A and $\mathcal{A}$ may be reduced from (2.2) to simpler expressions,

$$\begin{aligned} -A\phi &= \sum_i a(\eta_i - \eta_{i+e^1})\big(\phi(\eta^{i,i+e^1}) - \phi(\eta)\big) \\ &\quad + \sum_i b(\eta_i - \eta_{i+e^2})\big(\phi(\eta^{i,i+e^2}) - \phi(\eta)\big), \\ -\mathcal{A}\phi &= \sum_{i,i+e^1 \neq 0} a(\zeta_i - \zeta_{i+e^1})\big(\phi(\zeta^{i,i+e^1}) - \phi(\zeta)\big) \\ &\quad + \sum_{i,i+e^2 \neq 0} b(\zeta_i - \zeta_{i+e^2})\big(\phi(\zeta^{i,i+e^2}) - \phi(\zeta)\big) \\ &\quad + a\big[(1 - \zeta_{e^1})\big(\phi(\tau_{-e^1}\zeta) - \phi(\zeta)\big) - (1 - \zeta_{-e^1})\big(\phi(\tau_{e^1}\zeta) - \phi(\zeta)\big)\big] \\ &\quad + b\big[(1 - \zeta_{e^2})\big(\phi(\tau_{-e^2}\zeta) - \phi(\zeta)\big) - (1 - \zeta_{-e^2})\big(\phi(\tau_{e^2}\zeta) - \phi(\zeta)\big)\big], \end{aligned}$$

where $a = (u - d)/2$ and $b = (r - l)/2$. Note that when the drift is nonzero, both a and b cannot vanish.

LEMMA 4.3. *Let $L = L(u, d, r, l)$ generate a nearest-neighbor translation-invariant exclusion with nonzero drift. Then, there corresponds a degenerate* *nearest-neighbor generator L' with translation-invariant rates such that, for $C = C(p)$, $D = D(p) > 0$ and $\lambda > 0$,*

$$C^{-1}\|\cdot\|_{-1}(D^{-1}\lambda, -L') \leq \|\cdot\|_{-1}(\lambda, -L) \leq C\|\cdot\|_{-1}(D\lambda, -L').$$

The same result applies when $\mathcal{L}$ replaces L.

PROOF. Let $a' = u - d$ and $b' = r - l$ and let us consider the case when the drift is directed into the first quadrant, as other cases are similar. Then, either $a', b' > 0$, or $a' > 0$ and $b' = 0$, or $b' > 0$ and $a' = 0$. In these three cases, let $L' = L'(a', 0, b', 0)$, $L'(a', 0, u, u)$ and $L'(r, r, b', 0)$, respectively. Write $-L = S + A$ and $-L' = S' + A'$. Note that A' is arranged so that $A' = A$. Clearly, S' and S are equivalent in the sense of quadratic forms. The proof now follows from Lemma 3.3. $\square$

Finally, we consider the proof of Theorem 2.1.

PROOF OF THEOREM 2.1. The proof in $d \geq 1$ is the same as in $d = 2$ with only notational changes. The two-dimensional argument follows from Lemmas 4.1, 4.2 and 4.3. $\square$

5. Proof of Theorem 2.2. The proof of Theorem 2.2(i) and (ii) follow the same strategy as for Theorem 2.1. The proof of part (iii), however, is demonstrated by presenting a counterexample.

PROOF OF THEOREM 2.2(i) AND (ii). The proof of part (ii) is given in Lemma 4.3 with only changes in notation.

The proof of part (i) is in three steps. Write the generators as $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ and $\mathcal{L}' = -\mathcal{S}' - \mathcal{A}'$. Step 1 is to construct a finite-range irreducible mean-zero jump rate p_m and form the corresponding operator $\mathcal{L}_m = -\mathcal{S}_m - \mathcal{A}_m$ so that $\mathcal{A} + \mathcal{A}_m$ is the antisymmetric operator of a nearest-neighbor jump rate. This is done as in the first part of Lemma 4.2. Denote $\mathcal{A}_1 = \mathcal{A} + \mathcal{A}_m$. Step 2 is to conclude, as $\mathcal{S}$ and $\mathcal{S} + \mathcal{S}_m$ have equivalent quadratic forms from Lemma 3.5, that the H_{-1} norms of $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ and $-(\mathcal{S} + \mathcal{S}_m) - \mathcal{A}_1$ are equivalent by Lemma 3.4. At this point, note that all jumps of size $k \geq 2$ are reflected only in $\mathcal{S} + \mathcal{S}_m$. Step 3 is to observe first that $\mathcal{A}_1$ and $\mathcal{A}'$, the antisymmetric operator corresponding to p' in the statement of the theorem, are the same. Observe next that $\mathcal{S} + \mathcal{S}_m$ and $\mathcal{S}'$ have equivalent forms by Lemma 3.5. Part (i) now follows from applying Lemma 3.3. $\square$

The following result may also be of interest in that H_{-1} norms with respect to one-dimensional environment systems with long-range rates may be bounded by the H_{-1} norm of an "environment-type" system with, in fact, only nearest-neighbor rates. Recall the definition of the operator $\mathcal{B}$ from (3.3).

COROLLARY 5.1. *Let $\mathcal{L} = -\mathcal{S} - \mathcal{A}$ be an environment generator with finite-range nonnearest-neighbor translation-invariant rates p in $d = 1$ whose symmetrization is irreducible. Let also $\mathcal{L}_1 = -\mathcal{S}_1 - \mathcal{A}_1$ be the nearest-neighbor generator corresponding to translation-invariant rates p_1 with the same drift given by*

$$p_1(\pm 1) = \begin{cases} \max[\pm r(p), 0], & \text{when } r(p) \neq 0, \\ 1, & \text{when } r(p) = 0. \end{cases}$$

Form now the environment-type generator $\mathcal{L}' = -\mathcal{L}_1 - \mathcal{B}$. Then, for constants $C = C(p)$, $D = D(p)$, and $\lambda > 0$, inequalities (2.5) *hold.*

PROOF. The proof follows from the proof of Theorem 2.2(i). Replace only the second to last sentence by "Observe next that $\mathcal{S} + \mathcal{S}_m$ and $\mathcal{S}_1 + \mathcal{B}$ have equivalent quadratic forms from Lemma 3.6." $\square$

We now turn to the proof of part (iii) of Theorem 2.2. The argument is best expressed in terms of the "zero-range process." It is now a standard fact that in one dimension the nearest-neighbor environment exclusion system, with rates $p(1) = p$ and $p(-1) = q$, is in one–one correspondence with a zero-range system [see Kipnis (1986) for instance]: Order and label the exclusion particle positions on $\mathbb{Z}$ initially at time $t = 0$ as $\dots, x_{-2}, x_{-1}, x_0, x_1, \dots$ where x_0, say, is the tagged particle at the origin. As the dynamics is nearest-neighbor, this ordering is preserved at later times $t \geq 0$. Consider the well-defined interparticle spacings at times $t \geq 0$,

$$\dots, \xi_{-1}(t) = x_{-1}(t) - x_{-2}(t), \qquad \xi_0(t) = x_0(t) - x_{-1}(t), \dots.$$

Translating the exclusion rules in terms of the spacing process, $\xi(t)$, where $\xi_i(t) = x_i(t) - x_{i-1}(t)$, yields that $\xi(t)$ is a nearest-neighbor zero-range process with rate function $g(k) = \mathbb{1}_{(k \geq 1)}$ and reversed rates $p(1) = q$ and $p(-1) = p$. Also, the Bernoulli product invariant measures $P_\rho\{\cdot \mid \zeta_0 = 1\}$ transform into the product geometric invariant measures $Z_{\alpha(\rho)}$ for the zero-range process, with $\alpha(\rho) = 1 - \rho$ and $Z_{\alpha(\rho)}(\xi_i = k) = (1 - \alpha(\rho))\alpha(\rho)^k$. Here again ρ is the density, $\rho = E_{Z_{\alpha(\rho)}}[\xi_1]$. For simplicity, we will denote $\alpha(\rho)$ as α.

More carefully, the $d = 1$ nearest-neighbor zero-range process with translation-invariant rates p is a system of particles on $\{0, 1, 2, \dots\}^{\mathbb{Z}}$ governed by the generator

$$L^{\mathrm{ZR}}\phi(\xi) = \sum_{i \in \mathbb{Z}} \bigl[\bigl(\phi(\xi^{i,i+1}) - \phi(\xi)\bigr)g(\xi_i)p(1) + \bigl(\phi(\xi^{i,i-1}) - \phi(\xi)\bigr)g(\xi_i)p(-1)\bigr]$$

acting on local functions ϕ where $\xi^{i,j}$ is the configuration obtained from ξ by moving a particle from i to j,

$$\xi_k^{i,j} = \begin{cases} \xi_i - 1, & \text{for } k = i, \\ \xi_j + 1, & \text{for } k = j, \\ \xi_k, & \text{otherwise.} \end{cases}$$

As for exclusion generators, L^{ZR} can be decomposed into a symmetric and antisymmetric part, $L^{\mathrm{ZR}} = -S^{\mathrm{ZR}} - A^{\mathrm{ZR}}$ where $-S^{\mathrm{ZR}} = (L^{\mathrm{ZR}} + L^{\mathrm{ZR}*})/2$ and $-A^{\mathrm{ZR}} = (L^{\mathrm{ZR}} - L^{\mathrm{ZR}*})/2$. Explicitly, these are

$$-S^{\mathrm{ZR}}\phi = \frac{p(1)+p(-1)}{2}\sum_{i\in\mathbb{Z}}\big[\big(\phi(\xi^{i,i+1}) - \phi(\xi)\big)g(\xi_i) + \big(\phi(\xi^{i,i-1}) - \phi(\xi)\big)g(\xi_i)\big],$$

$$-A^{\mathrm{ZR}}\phi = \frac{p(1)-p(-1)}{2}\sum_{i\in\mathbb{Z}}\big[\big(\phi(\xi^{i,i+1}) - \phi(\xi)\big)g(\xi_i) - \big(\phi(\xi^{i,i-1}) - \phi(\xi)\big)g(\xi_i)\big].$$

Also, the quadratic form may be computed as

$$E_{Z_\alpha}[\phi S^{\mathrm{ZR}}\phi] = \frac{p(1)+p(-1)}{4}\sum_i E_{Z_\alpha}\big[g(\xi_i)\big(\phi(\xi^{i,i+1}) - \phi(\xi)\big)^2\big].$$

We defer to Andjel (1982) and Sethuraman [(2001), Section 2] for more details and discussion of zero-range processes, including construction of the process on "Lipschitz" and on L^2 functions.

It will help us now to rewrite the environment-type process $\mathcal{L}' = \mathcal{L}_1 + \mathcal{B}$, from Corollary 5.1, also in the "zero-range" context. We must simply understand what motion is added to the usual zero-range process by $\mathcal{B}$. The operator $\mathcal{B}$, in the exclusion model, is a symmetric operator governing "shifts" τ_1 and τ_{-1} with rates ζ_{-1} and ζ_1, respectively. In other words, when -1 or 1 is occupied, then at rate 1, the tagged particle at the origin exchanges places with its neighboring particle and then shifts the reference frame. Note that the exclusion functions ζ_{-1} and ζ_1 correspond to $\mathbb{1}_{(\xi_0=0)}$ and $\mathbb{1}_{(\xi_1=0)}$, respectively. Therefore, after a moment's thought, the zero-range motions associated to $\mathcal{B}$ are coordinate shifts to the left and right at rate 1 when vertices 0 and 1 are empty. More carefully, let $s_r\xi$ be the r-shifted configuration, $(s_r\xi)_k = \xi_{k+r}$ for $k \in \mathbb{Z}$. Then $\mathcal{B}$ transforms to the operator B^{ZR},

$$B^{\mathrm{ZR}}\phi(\xi) = \mathbb{1}_{(\xi_0=0)}\big(\phi(s_1\xi) - \phi(\xi)\big) + \mathbb{1}_{(\xi_1=0)}\big(\phi(s_{-1}\xi) - \phi(\xi)\big).$$

The quadratic form of $-B^{\mathrm{ZR}}$ is given by

$$\begin{aligned} &E_{Z_\alpha}[\phi(-B^{\mathrm{ZR}})\phi] \\ &\quad = \tfrac{1}{2}E_{Z_\alpha}\big[\mathbb{1}_{(\xi_0=0)}\big(\phi(s_1\xi) - \phi(\xi)\big)^2\big] + \tfrac{1}{2}E_{Z_\alpha}\big[\mathbb{1}_{(\xi_1=0)}\big(\phi(s_{-1}\xi) - \phi(\xi)\big)^2\big]. \end{aligned}$$

We now consider a sequence of functions that will form the basis of our counterexample. Let $J_n : [0,1]^n \to \mathbb{R}$ be a smooth function for each $n \ge 1$. Let also $i = (i_1, \dots, i_n) \in \mathbb{R}^n$ and define

$$h_{n,m}(\xi) = \frac{1}{\sqrt{nm^{n-2}}}\sum_{|i|\le m} J_n\left(\frac{i}{m}\right)(\xi_{i_1} - \rho)\cdots(\xi_{i_n} - \rho).$$

We now compute the quadratic forms of $-L^{\mathrm{ZR}}$ and $-L^{\mathrm{ZR}} - B^{\mathrm{ZR}}$ on the sequence $\{h_{n,m}\}$.

LEMMA 5.1. *We have for constants $C = C(\alpha, p)$ and $D = D(\alpha)$ that*

$$E_{Z_\alpha}\big[h_{n,m}(-S^{\mathrm{ZR}})h_{n,m}\big] = \frac{C}{nm^n}\sum_{|i|\le m} |\nabla J_n|^2\left(\frac{i}{m}\right) + o\left(\frac{1}{m}\right)$$

and

$$E_{Z_\alpha}\big[h_{n,m}(-B^{\mathrm{ZR}})h_{n,m}\big] = \frac{D}{nm^n}\sum_{|i|\le m} (\operatorname{div} J_n)^2\left(\frac{i}{m}\right) + o\left(\frac{1}{m}\right).$$

PROOF. These calculations follow from the definitions, orthogonality of $(\xi_k - \rho)$ and $(\xi_j - \rho)$ for $k \ne j$ with respect to Z_α, summation-by-parts and straightforward approximations, and are left to the reader. $\square$

LEMMA 5.2. *Let $J(x) = \sqrt{2(\exp(2)-1)^{-1}}\exp(x)$, and for $\theta = (\theta_1, \dots, \theta_n) \in [0,1]^n$ let $J_n(\theta) = J(\theta_1)\cdots J(\theta_n)$. Then, with respect to constants C and D as in Lemma* 5.1, *we have*

$$E_{Z_\alpha}[h_{n,m}S^{\mathrm{ZR}}h_{n,m}] = \frac{C}{m^n}\left[\sum_{|i|\le m}(J')^2\left(\frac{i}{m}\right)\right]^n + o\left(\frac{1}{m}\right),$$

where $\lim_{n\to\infty}\lim_{m\to\infty} E_{Z_\alpha}[h_{n,m}(-S^{\mathrm{ZR}})h_{n,m}] = C < \infty$. *However*,

$$E_{Z_\alpha}\big[h_{n,m}(-B^{\mathrm{ZR}})h_{n,m}\big] = \frac{Dn}{m^n}\left[\sum_{|i|\le m}(J')^2\left(\frac{i}{m}\right)\right]^n + o\left(\frac{1}{m}\right),$$

where $\lim_{n\to\infty}\lim_{m\to\infty} E_{Z_\alpha}[h_{n,m}(-B^{\mathrm{ZR}})h_{n,m}] = \infty$.

In addition, for constants $C_1 = C_1(\alpha, p)$ and $D_1 = D_1(\alpha)$, we have

$$E_{Z_\alpha}[h_{n,m}^2] = \frac{D_1}{nm^{n-2}}\left[\sum_{|i|\le m} J^2\left(\frac{i}{m}\right)\right]^n$$

and

$$E_{Z_\alpha}\big[(A^{\mathrm{ZR}}h_{n,m})^2\big] = \frac{C_1 n}{m^{n+2}}\left[\sum_{|i|\le m}(J'')^2\left(\frac{i}{m}\right)\right]^n + o\left(\frac{1}{m}\right).$$

PROOF. As in the previous lemma, these computations follow straigtforwardly from definitions, summation-by-parts, orthogonality of $(\xi_k - \rho)$ and $(\xi_j - \rho)$ for $k \ne j$ with respect to Z_α, and usual approximations and are left to the reader. $\square$

PROPOSITION 5.1. *Let $-\mathcal{S}_1$ and $-\mathcal{S}_2$ generate symmetric nearest-neighbor and finite-range nonnearest-neighbor environment processes in $d = 1$ with* *translation-invariant irreducible rates p_1 and p_2, respectively. Then, there exists a constant $C = C(p_1, p_2)$ such that for all local ϕ,*

$$\langle \phi, (-\mathcal{S}_1)\phi\rangle_\rho \le C\langle \phi, (-\mathcal{S}_2)\phi\rangle_\rho.$$

However, there exists no constant such that the reverse inequality is true.

PROOF. By Lemma 3.6, $\mathcal{S}_2$ and $\mathcal{S}_1 + \mathcal{B}$ have equivalent forms. The inequality in the proposition now follows as the quadratic form of $\mathcal{B}$ is nonnegative.

For the negative statment, we invoke Lemma 5.2 to show that the $\mathcal{S}_1$ form, or after transform the S^{ZR} form, cannot bound above the the $\mathcal{B}$ form, or B^{ZR} form, for all local or L^2 functions. $\square$

PROOF OF THEOREM 2.2(iii). To deduce a contradiction, suppose there were constants C and D such that for all $\lambda > 0$ the bound

$$\|\cdot\|_{-1}(\lambda, -\mathcal{L}_1) \le C\|\cdot\|_{-1}(D\lambda, -\mathcal{L}_2)$$

holds. Then, as there exists a nearest-neighbor generator $\mathcal{L}_1' = -\mathcal{S}_1' - \mathcal{A}_1'$ such that the H_{-1} norms of $\mathcal{L}_2$ and $\mathcal{L}_1' + \mathcal{B}$ are equivalent by Corollary 5.1, we would have that

$$\|\cdot\|_{-1}(\lambda, -\mathcal{L}_1) \le C\|\cdot\|_{-1}(D\lambda, -\mathcal{L}'_1 - \mathcal{B})$$

for perhaps different constants. Then, by duality arguments, we would have that the quadratic form of $(D\lambda + \mathcal{S}_1' - \mathcal{B}) - \mathcal{A}_1'(D\lambda + \mathcal{S}_1' - \mathcal{B})^{-1}\mathcal{A}_1'$ would be bounded above by a multiple of the form for $(\lambda + \mathcal{S}_1) - \mathcal{A}_1(\lambda + \mathcal{S}_1)^{-1}\mathcal{A}_1$.

We now focus on the zero-range setting and functions $h_{n,m}$ with J_n as in Lemma 5.2. Let us rewrite $\mathcal{L}_1$ as $L_1^{\mathrm{ZR}} = -S_1^{\mathrm{ZR}} - A_1^{\mathrm{ZR}}$ and $\mathcal{L}'$ as $L_2^{\mathrm{ZR}} = -S_2^{\mathrm{ZR}} - A_2^{\mathrm{ZR}}$. Choose now λ in the form $\lambda_{n,m} = n/m^2$. We will show that the terms

$$\lambda_{n,m}E_{Z_\alpha}[h_{n,m}^2],\ E_{Z_\alpha}[h_{n,m}S_1^{\mathrm{ZR}}h_{n,m}],\ E_{Z_\alpha}[h_{n,m}S_2^{\mathrm{ZR}}h_{n,m}],$$
$$E_{Z_\alpha}[A_1^{\mathrm{ZR}}h_{n,m}(\lambda_{n,m} + S_1^{\mathrm{ZR}})^{-1}A_1^{\mathrm{ZR}}h_{n,m}]$$

and

$$E_{Z_\alpha}[A_2^{\mathrm{ZR}}h_{n,m}(\lambda_{n,m} + S_2^{\mathrm{ZR}} - B^{\mathrm{ZR}})^{-1}A_2^{\mathrm{ZR}}h_{n,m}],$$

are all uniformly bounded as $m \uparrow \infty$ and then $n \uparrow \infty$. However, by Lemma 5.2, we have that $\lim_{n\uparrow\infty}\lim_{m\uparrow\infty} E_{Z_\alpha}[h_{n,m}(-B^{\mathrm{ZR}})h_{n,m}]$ diverges. This would furnish the contradiction.

From Lemma 5.2, we have that $\lim_{n\uparrow\infty}\lim_{m\uparrow\infty}\lambda_{n,m}E_{Z_\alpha}[h_{n,m}^2] < \infty$ and also that $\lim_{n\uparrow\infty}\lim_{m\uparrow\infty} E_{Z_\alpha}[h_{n,m}S_1^{\mathrm{ZR}}h_{n,m}] < \infty$. As $\mathcal{S}_1$ and $\mathcal{S}'$, and therefore S_1^{ZR} and S_2^{ZR}, have equivalent forms, we have that $\lim_{n\uparrow\infty}\lim_{m\uparrow\infty} E_{Z_\alpha}[h_{n,m} \times \mathcal{S}_2^{\mathrm{ZR}}h_{n,m}] < \infty$. Now we bound the term

$$E_{Z_\alpha}[A_1^{\mathrm{ZR}}h_{n,m}(\lambda_{n,m} + S_1^{\mathrm{ZR}})^{-1}A_1^{\mathrm{ZR}}h_{n,m}] \le \frac{1}{\lambda_{n,m}}E_{Z_\alpha}[(A_1^{\mathrm{ZR}}h_{n,m})^2]$$

by the standard resolvent bound. From Lemma 5.2 once more,

$$\lim_{n\uparrow\infty}\lim_{m\uparrow\infty}\frac{1}{\lambda_{n,m}}E_{Z_\alpha}\big[(A_1^{\mathrm{ZR}}h_{n,m})^2\big]<\infty.$$

Similarly, the term $E_{Z_\alpha}[A_2^{\mathrm{ZR}}h_{n,m}(\lambda_{n,m}+S_1^{\mathrm{ZR}}-B^{\mathrm{ZR}})^{-1}A_2^{\mathrm{ZR}}h_{n,m}]$ is bounded also. $\square$

6. Application to additive functionals. We present here an example of how Theorem 2.1 can be used. Consider the standard simple exclusion model in dimension $d\geq 1$ with translation-invariant finite-range jump rates p whose symmetrization is irreducible. Let L denote the generator. Let also the initial distribution be P_ρ, and let $f(\eta)$ be a local function on the state space Σ. As alluded to in the Introduction, a basic question is to ask about the diffusive behavior of additive functionals with kernels f under P_ρ. More specifically, in anticipation of central limit theorems, a first step is to investigate for which functions f, the variance,

$$\sigma_t^2(f,\rho,L)=E_\rho\bigg[\bigg(\int_0^t\big(f(\eta(s))-E_\rho[f]\big)\,ds\bigg)^2\bigg],$$

is $O(t)$.

The answer depends on whether the jump rates p are mean zero or not. A rough reasoning is that under mean-zero rates the particle dynamics is in local balance. While under rates with nonzero drift, the dynamics is with some velocity. Heuristically then, individual particles under mean-zero dynamics stay put more and interact more with each other than under dynamics with drift, leading to larger and smaller additive functional variances, respectively.

To give more precise answers, let us observe that any local function f, by distinguishing the finite number of configurations of its support, can be decomposed in terms of a finite number of "centered k-point" functions,

$$\begin{aligned}f(\eta)=E_\rho[f]&+\sum_i c_i(\eta_i-\rho)+\sum_{i,j}c_{i,j}(\eta_i-\rho)(\eta_j-\rho)\\&+\sum_{i,j,k}c_{i,j,k}(\eta_i-\rho)(\eta_j-\rho)(\eta_k-\rho)+\cdots.\end{aligned}$$

When p is mean zero, it was proved in Sethuraman and Xu (1996) and Sethuraman (2000), using in part duality relations, that

$$\sigma_t^2(f,\rho,L)=O(t)\quad\Longleftrightarrow\quad\begin{cases}\sum_{i,j}c_{i,j},\ \sum_i c_i,\ E_\rho[f]=0, & \text{when } d=1,\\ \sum_i c_i,\ E_\rho[f]=0, & \text{when } d=2,\\ E_\rho[f]=0, & \text{when } d\geq 3.\end{cases}$$

In fact, the orders of $\sigma^2(f, \rho, L)$ when f does not satisfy these conditions were also described in Sethuraman (2000). Namely, in $d = 1$ and 2, it was shown that $\sigma_t^2(f, \rho, L) = O(t^{3/2})$ and $O(t \log(t))$, respectively, for mean-zero f when $\sum_i c_i \neq 0$. Also, in $d = 1$, it was proved that $0 < \limsup(t \log(t))^{-1} \sigma_t^2(f, \rho, L) < \infty$ for mean-zero f when $\sum_i c_i = 0$ but $\sum_{i,j} c_{i,j} \neq 0$.

Analogously, when p is with drift, and dimension $d \geq 3$, it was shown in Sethuraman, Varadhan and Yau (2000) through in part duality relations again, that $\sigma_t^2(f, \rho, L) = O(t)$ exactly when f is mean zero, $E_\rho[f] = 0$.

However, when p has drift in dimension $d \leq 2$, duality relations are not useful anymore, and the variance behavior is of a different character. In fact, some facets of the problem are still open.

To describe the progress on the problem, it will be convenient to define

$$\sigma^2(f, \rho, L) = \lim_{t \uparrow \infty} t^{-1} \sigma_t^2(f, \rho, L),$$

when the limit exists. Also, it will be useful to consider another decomposition of a local function f in terms of finite number of "monotone k-point" functions,

$$f(\eta) = E_\rho[f] + \sum_i d_i \eta_i + \sum_{i,j} d_{i,j} \eta_i \eta_j + \sum_{i,j,k} d_{i,j,k} \eta_i \eta_j \eta_k + \cdots.$$

Here, "monotone" refers to the fact that the k-point function $\eta_{i_1} \eta_{i_2} \cdots \eta_{i_k}$ is an increasing function of η coordinatewise. We also observe, by grouping together positive and negative terms in this decomposition, that f can be written as the difference of two increasing functions, $f = f_+ - f_-$, where f_+ is the sum of the positive terms and $-f_-$ is the sum of the negative terms. Therefore, to show $\sigma_t^2(f, \rho, L) = O(t)$, a possible strategy is to show both $\sigma_t^2(f_+, \rho, L)$ and $\sigma_t^2(f_-, \rho, L)$ are $O(t)$.

In Sethuraman (2000), this strategy was pursued and in Sethuraman (2000), Theorem 1.1, it was shown that when f is an increasing mean-zero function, limits $\sigma^2(f, \rho, L)$ and $\langle f, (\lambda - L)^{-1} f \rangle_\rho$ exist and moreover,

$$\sigma^2(f, \rho, L) = 2\langle f, (\lambda - L)^{-1} f \rangle_\rho. \tag{6.1}$$

Also, it was proved when $f = f_+ - f_-$ is the difference of two local increasing mean-zero functions f_+ and f_- for which both $\sigma^2(f_+, \rho, L)$ and $\sigma^2(f_-, \rho, L)$ are finite, then also $\sigma^2(f, \rho, L)$ exists and (6.1) holds, and also $\sigma^2(f, \rho, L) < \infty$.

One of the remaining questions then in $d = 1, 2$ when p has drift is to specify for which increasing mean-zero functions f the bound $\sigma^2(f, \rho, L) < \infty$ is valid. Now, given the H_{-1} norm equivalence statements in Theorem 2.1, verification of this bound can be reduced from a problem for finite-range systems to one for nearest-neighbor processes. As remarked in the introduction, the significance of such a reduction is that in the nearest-neighbor case calculations can be made sometimes through special methods.

COROLLARY 6.1. *Let L and L' be as in Theorem* 2.1. *When f is a local increasing mean-zero function*, $\sigma^2(f, \rho, L) < \infty$ *if and only if* $\sigma^2(f, \rho, L') < \infty$.

The proof follows directly from relation (6.1) and Corollary 2.1.

Finally, we remark that in recent work [Seppäläinen and Sethuraman (2002)], desired nearest-neighbor bounds have been computed in $d = 1$ when p is with drift. Namely, it is shown that $\sigma^2(f, \rho, L') < \infty$ for all local mean-zero increasing functions f in $d = 1$ when $\rho \neq 1/2$. Therefore, following the strategy mentioned earlier, we have $\sigma^2(f, \rho, L') < \infty$ for all local mean-zero functions f in this setting. Related central limit theorems are also proved in this paper.

Open questions remain when p has drift in the cases $\rho = 1/2$ in $d = 1$, and for all $\rho \in (0, 1)$ in $d = 2$. Namely, it is not clear if the variance $\sigma_t^2(f, \rho, L) = O(t)$ in these cases. Perhaps, it is surprising that there are heuristics where $\sigma^2(f, 1/2, L') = \infty$ for the function $f(\eta) = \eta_0 - 1/2$ in $d = 1$, reminiscent of the variance behavior with respect to symmetric rates p. We refer to Seppäläinen and Sethuraman (2003) for more details.

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