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SINGULAR INITIAL CONDITIONS FOR THE HEAT EQUATION WITH A NOISE TERM[1]

BY CARL MUELLER

University of Rochester

We consider the equation

$$\begin{aligned} u_t &= u_{xx} + u^\gamma \dot{W}, \qquad t > 0,\ 0 \le x \le J,\\ u(0,x) &= u_0(x),\\ u(t,0) &= u(t,J) = 0, \end{aligned}$$

where $\dot{W} = \dot{W}(t,x)$ is two-parameter white noise. We show local existence and uniqueness for unbounded initial conditions satisfying certain conditions. Our results are motivated by earlier work, which showed that, for large γ, solutions of this equation can blow up. One would wish to show that solutions can be extended beyond blowup, and our results can be viewed as a step in that direction.

1. Introduction. Consider the equation

$$\begin{aligned} u_t &= u_{xx} + u^\gamma \dot{W}, \qquad t > 0,\ 0 \le x \le J,\\ u(0,x) &= u_0(x), \qquad\qquad (1.1)\\ u(t,0) &= u(t,J) = 0. \end{aligned}$$

Here, $\dot{W} = \dot{W}(t,x)$ is two-parameter white noise. We assume that the initial function $u_0(x)$ is nonnegative and continuous. It was shown in Mueller (1991) that (1.1) has a unique nonnegative solution for $0 \le t < \tau$, where τ is the blowup time described below.

In Mueller and Sowers (1993) and Mueller (1991), it was shown that blowup can occur with positive probability if γ is sufficiently large, but cannot occur if $\gamma < 3/2$. We say that blowup occurs if there exists some random time τ such that $\limsup_{t\uparrow\tau} \sup_{x\in[0,J]} u(t,x) = \infty$. Blowup is a common feature of partial differential equations, but the above references seem to contain the first results for blowup caused by noise. The noise term $\dot{W}$ may push the solution either up or down, so it is not immediately obvious that solutions blow up.

In the case of deterministic PDE's, solutions can sometimes be continued beyond blowup. For example, the papers of Evans and Spruck (1991) and Sethian (1985) deal with this phenomenon. One could ask the same question about SPDE's such as (1.1). We do not answer the question of continuation beyond blowup for (1.1). Instead, we take a step in that direction by showing

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that in some cases (1.1) can be solved if the initial function $u_0(x)$ is unbounded or even a measure. If $\gamma < 3/2$ we show that solutions exist if $u_0(x)\,dx$ is replaced by a finite nonnegative measure. We call the class of such measures $\mathbf{M}$. For larger values of γ, we require that $u_0(x)$ be a nonnegative function in $\mathbf{L}^p$ for some $p > 2\gamma$. If it could be shown that $u(t,x)$ was in the correct $\mathbf{L}^p$-space at the time of blowup, then we could prove continuation after blowup.

Our guess is that solutions can be continued beyond blowup in the case of large γ. In Mueller and Sowers (1993), the function $u(t,x)$ is compared to a smaller function $v(t,x)$, where v is a step function in the variable x. Furthermore, $v(t,x)$ blows up after $u(t,x)$ does. It is not hard to show that $v \in \mathbf{L}^{2\gamma}(dt\,dx)$; (1.1) can be expressed as an integral equation involving $u^\gamma \dot{W}$. The square variation of this integral involves the integral of $u^{2\gamma}$. Therefore, continuation beyond blowup is not ruled out, since $v \in \mathbf{L}^{2\gamma}$.

Next, we discuss the rigorous meaning of (1.1). Indeed, we do not expect solutions to be differentiable in t or x. We regard (1.1) as a shorthand for the following integral equation [we are following the formalism of Walsh (1986)]:

$$\begin{aligned} u(t,x) &= \int_0^J G(t,x,y)u_0(y)\,dy \\ &\quad + \int_0^t \int_0^J G(t-s,x,y)u^\gamma(s,y)W(dy\,ds). \end{aligned} \tag{1.2}$$

Here, $G(t,x,y)$ is the fundamental solution of the heat equation on $[0,J]$, with Dirichlet boundary conditions, and the final integral in (1.2) is an integral with respect to a martingale measure, as in the theory of Walsh (1986). Observe that the first term on the right-hand side of (1.2) is well defined even if $u_0(y)\,dy$ is replaced by a finite measure. For later use, we label the last term in (1.2). Actually, we use a modification of this term which is easier to deal with; later we will see that control of this modified term is all that we need. Let

$$N(t_1,t_2,x) = \int_{t_1}^{t_2} \int_0^J G(t_2-s,x,y)(u(s,y)\wedge L)^\gamma W(dy\,ds), \tag{1.3}$$

where $a \wedge b$ denotes the minimum of a and b.

Let $\mathscr{F}_t$ denote the σ-field generated by the white noise $\dot{W}$ up to time t. In other words, let

$$\mathscr{F}_t = \sigma\left\{\int_0^t \int_0^J \varphi(s,x)W(dx\,ds)\,\middle|\,\varphi \in \mathbf{C}([0,t]\times[0,J])\right\}.$$

In the usual way, we also define $\mathscr{F}_\tau$ for stopping times τ. Later, we will use the strong Markov property of solutions, established in Mueller (1991), to start solutions afresh at a stopping time τ,

$$\begin{aligned} u(\tau+t,x) &= \int_0^J G(t,x,y)u(\tau,y)\,dy \\ &\quad + \int_0^t \int_0^J G(t-s,x,y)u^\gamma(\tau+s,y)W(dy\,ds). \end{aligned} \tag{1.4}$$

Note that in the integral equation (1.2), all of the terms are well defined as long as $u(t,x)$ is bounded. Of course, $u_0(x)$ may not be bounded if it is only a finite nonnegative measure. However, since the heat kernel quickly smooths out singularities, we expect that $u(t,x)$ will soon become bounded. Our goal in this paper is to find solutions $u(t,x)$ of (1.1) which are bounded for $t > 0$ and which decrease at a certain rate, which we specify below. Within this class, we prove uniqueness. From now on, when we discuss solutions to (1.1), we mean solutions in the above sense.

First we deal with the case $\gamma < 3/2$. Let

$$H_1(t) = C_1 t^{-1/2},$$

where C_1 is a constant to be chosen later. We seek solutions $u(t,x)$ of (1.1) such that the following holds:

(A) For some stopping time $\tau > 0$, $t < \tau$ implies $\sup_{0\le x\le J} u(t,x) \le H_1(t)$.

It will turn out that (A) implies that almost surely the function $(s,y) \mapsto G(t-s,x,y)u^\gamma(s,y)$ lies in $\mathbf{L}^2([0,t]\times[0,J])$, and so the right-hand side of (1.2) is well defined.

THEOREM 1.1. *Suppose that* $1 \le \gamma < 3/2$, *and let* u_0 *be a nonnegative finite measure on* $[0,J]$. *If* C_1 *is large enough, then* (1.1) *has a unique solution. More precisely, there exists an almost surely unique random function* $u(t,x)$ *satisfying* (1.2) *and* (A) *with probability* 1.

Second, we consider the case $\gamma \ge 3/2$. For this case, we cannot prove as strong a result. Suppose that $u_0(x) \in \mathbf{L}^p$ for some $p > 2\gamma$, and define

$$H_2(t) = C_2 t^{-1/p}.$$

We seek solutions $u(t,x)$ of (1.1) such that the following holds:

(B) For some stopping time $\tau > 0$, $t < \tau$ implies $\sup_{0\le x\le J} u(t,x) \le H_2(t)$.

It will turn out that (B) implies that almost surely, for $0 \le t < \tau$, the function $(s,y) \mapsto G(t-s,x,y)u^\gamma(s,y)$ lies in $\mathbf{L}^2([0,t]\times[0,J])$, and so the right-hand side of (1.2) is well defined.

THEOREM 1.2. *Suppose that* $\gamma \ge 3/2$, $p > 2\gamma$ *and* $u_0(x) \in \mathbf{L}^p$. *If* C_2 *is large enough, then, up to time* τ, (1.1) *has a unique solution, where* τ *is the stopping time mentioned in* (B). *More precisely, for* $0 \le t < \tau$, *there exists an almost surely unique random function* $u(t,x)$ *satisfying* (1.2) *and* (B) *with probability* 1.

We prove Theorem 1.1 in Section 2. The proof of Theorem 1.2 is in Section 3.

2. Proof of Theorem 1.1. In this section we deal with the case where $\gamma < 3/2$ and u_0 is a nonnegative finite measure. This is the easy case, and we use several of the arguments of Mueller (1991). For the purposes of this section, we modify our original equation. Let

$$
\begin{aligned}
&\bar{u}_t = \bar{u}_{xx} + [\bar{u} \wedge H_1(t)]^\gamma \dot{W}, \qquad t > 0,\ 0 \le x \le J,\\
&\bar{u}(0,x) = \bar{u}_0(x), \\
&\bar{u}(t,0) = \bar{u}(t,J) = 0.
\end{aligned} \tag{2.1}
$$

The rigorous meaning of (2.1) is given in terms of an integral equation like (1.2). We leave this to the reader. We will assume that $\bar{u}_0(x)$ is a bounded continuous function. It is easy to modify the proofs of Mueller (1991) to show that long time existence and uniqueness hold for the equation

$$u_t = u_{xx} + f(t,u)\dot{W},$$

where $f(t,u) \le c_0 + c_1 u^\gamma$ for some $\gamma < 3/2$. Therefore, (2.1) possesses a unique nonnegative solution valid for all $t \ge 0$. Furthermore, let

$$\bar{N}(t_1,t_2,x) = \int_{t_1}^{t_2}\int_0^J G(t_2 - s, x, y)(\bar{u}(s,y) \wedge L)^\gamma W(dy\,ds). \tag{2.2}$$

The following lemma is proved exactly as in Mueller [(1991), Lemma 2.1].

LEMMA 2.1. *Suppose that $\delta, \bar{t} > 0$. If $\Delta^2/(L^{2\gamma}t^{1/2-\delta})$ is sufficiently large and $0 < t < \bar{t}$, then, for any $T \ge 0$,*

$$P\left\{\sup_{0\le s\le t}\ \sup_{0\le x\le J} |\bar{N}(T, T+s, x)| > \Delta\right\} \le \exp\left[\frac{-c\Delta^2}{L^{2\gamma}t^{1/2-\delta}}\right],$$

for some constant $c > 0$ depending on $\bar{t}$.

We will use some of the ideas of Mueller (1991), who showed that blowup does not occur if $\gamma < 3/2$. Since this argument is used several times in the sequel, we will summarize it here. For the purposes of this argument, we assume that $\sup_{0\le x\le J} u_0(x) = 1$. Then we inductively define stopping times τ_m as follows. Let $\tau_1 = 0$. Given τ_m, let $M_m = \sup_{0\le x\le J} u(\tau_m, x)$, and let τ_{m+1} be the first time $t > \tau_m$ such that $\sup_{0\le x\le J} u(t,x)$ equals either $2M_m$ or $M_m/2$. We can compare $\log(M_m)$ to a random walk which has negative drift for large values of M_m. Indeed, assume that $M_m = 2^n$, and let $L = 2^{n+1}$. If $\tau_m \le t \le \tau_{m+1}$, then $u(t,x) = u(t,x) \wedge L$, and the assumptions of Lemma 2.1 hold. It was shown in Mueller (1991) that $U(t) \equiv \int_0^J u(t,x)\,dx$ is a continuous nonnegative local martingale, and hence $U(t)$ is bounded with probability 1, say, $U(t) \le K$ for some random variable K. Thus, if a large peak develops, it must be very thin, and then the action of the heat equation will quickly decrease the peak. In particular, one has the estimate

$$\int_0^J G(t,x,y)f(y)\,dy \le \frac{\|f\|_1}{\sqrt{t}}.$$

Therefore, if $\sup_x u(t,x) = c2^n$, then the time required for the heat equation to reduce this supremum by a factor of 2 is on the order of 2^{-2n}. One must show that the peak decreases before enough noise develops to drive it further up. Here are a few of the details. We use the strong Markov property of solutions [see Mueller (1991), Lemma 3.3] to start the process afresh at time τ_m. Let $L = c2^n$, $t = c2^{-2n}$ and $\Delta = c2^n$, where the constant c may be different in each case. One verifies that, with the proper choice of constants c, and if $M_m = 2^n$ is large enough, then $\sup_{0\le s\le t}\sup_{0\le x\le J}|N(\tau_m, \tau_m + s, x)| \le \Delta$ implies $M_{m+1} = M_m/2$. However, Lemma 2.1 implies that if n is large, then the above event has probability close to 1. Thus, if M_m is large, M_{m+1} is much likelier to be below M_m than above, and the process M_m is either recurrent or tends to 0.

We adapt this argument to prove existence for initial conditions $u_0 \in \mathbf{M}$, the class of finite nonnegative measures on $[0, J]$. If u_0 is not a function, we approximate it with functions $u_n(0, x)$ which are bounded. Then we show that any large peaks of $u_n(t, x)$ quickly decrease. Letting $n \to \infty$ gives us a solution.

Now let $U(t) = \int_0^J \bar{u}(t, x)\,dx$.

LEMMA 2.2. *For $t > 0$, $(U(t), \mathscr{F}_t)$ is a nonnegative local supermartingale.*

PROOF. The proof of Lemma 3.1 in Mueller (1991) carries over to Lemma 2.2. $\square$

Now we prove the existence part of Theorem 1.1. Choose a sequence of nonnegative continuous functions $h_n(x) \in \mathbf{C}[0, J]$, such that $h_n(x)\,dx$ tends to $u_0(dx)$ as $n \to \infty$, in the sense of weak convergence of measures. If $u_0(dx)/dx$ is a bounded density, then Theorem 1 of Mueller (1991) implies existence, and uniqueness is also proved in the same paper. Therefore, we henceforth assume that $u_0(dx)/dx$ is not a bounded density. We may choose the $h_n(x)$ such that $\sup_{0\le x\le J} h_n(x) = 2^n$. Since each function h_n is bounded and $\gamma < 3/2$, a slight modification of Theorem 1 of Mueller (1991), and his discussion of uniqueness, shows that (2.1), with initial condition $\bar{u}_0(x) = h_n(x)$, has a unique solution $\bar{u}_n(t, x)$ which is almost surely continuous in (t, x). Furthermore, with probability 1, this solution is valid for all times $t \ge 0$.

Our first goal is to modify the proof of Lemma 2.1 of Mueller (1991) to show that $\bar{u}_n(t, x)$ quickly decreases. For each $n > 0$, we define stopping times $\tau_m = \tau_m^{(n)}$ associated with $\bar{u}_n(t, x)$, as in the outline of the proof of long time existence for $\gamma < 3/2$. To be specific, let $\tau_0 = 0$. Suppose that we have already defined τ_m and that $\sup_{0\le x\le J}\bar{u}_n(\tau_m, x) = 2^k$. Let τ_{m+1} be the first time $t > \tau_m$ that $\sup_{0\le x\le J}\bar{u}_n(t, x)$ equals 2^{k-1} or 2^{k+1}. Let $M_m = M_m^{(n)} = \sup_{0\le x\le J}\bar{u}_n(\tau_m, x)$. We allow the possibility that $\tau_m = \infty$, and Lemma 2.3 gives a bound on the probability of this event.

DEFINITION 2.1. Let $T = T(n, K_0)$ be the first of the times τ_m such that $M_m = 2^{K_0}$.

Finally, let $\mathcal{N}(k,m)$ be the number of indices $i \le m$ such that $M_i = 2^k$, and let $\sigma = \sigma(K_0)$ be the first index $m \ge 0$ such that $M_m = 2^{K_0}$. Therefore, $T = \tau_\sigma$. Assume that $n > K_0$.

LEMMA 2.3. *Suppose that* $M_0 = 2^n$, *and let* $A(m) = A(m,n,K_0,c_1,c_2)$ *be the event that the following hold*:

(a) $\mathcal{N}(k,m) = 0$ *or* 1 *for* $k \ge K_0$;
(b) $\tau_{m+1} - \tau_m \le c_1 2^{-2(n-m)}$;
(c) $\sup_{0\le s\le c_2 2^{-2(n-m)}} \sup_{0\le x\le J} |\bar{N}(\tau_m, \tau_m + s, x)| \le 2^{n-m-3}$.

Then, given $\varepsilon > 0$, *there exists an integer* K_0 *and constants* $c_1, c_2 > 0$ *such that, for all* $m > 0$,

$$\sum_{m=1}^{\sigma} P\{A^c(m)|A(1)\cap\cdots\cap A(m-1)\} < \varepsilon.$$

Before proving Lemma 2.3, we give an immediate consequence. Note that Lemma 2.3 implies that

$$P\left\{\left[\bigcap_{m=1}^{\sigma} A(m)\right]^c\right\} \le \sum_{m=1}^{\sigma} P\{A^c(m)|A(1)\cap\cdots\cap A(m-1)\} < \varepsilon.$$

Now, roughly speaking, conditions (a)–(c) of Lemma 2.3 imply that $\sup_x u(t,x)$ is quickly decreasing. Condition (a) implies that the successive maxima M_i take a given value at most once. Therefore these maxima form either an increasing or decreasing sequence. Condition (c) implies that the sequence is decreasing. Condition (b) implies that the times between maxima are short. All of this is true only up to time τ_σ. Then from the definition of H_1 one deduces the following lemma, which quantifies the rate of decrease of $\sup_x u(t,x)$.

LEMMA 2.4. *Let* $\varepsilon > 0$. *There exists a number* $K_0 > 0$ *depending on all of the constants in Lemma* 2.3, *but not depending on* n, *such that*

$$P\left\{\sup_{0\le x\le J} \bar{u}_n(t,x) > \tfrac{1}{2}H_1(t) \text{ for some } t \in [0, T(n,K_0)]\right\} < \varepsilon.$$

PROOF OF LEMMA 2.3. Assume that $A(1)\cap\cdots\cap A(m-1)$ occurs, and note that the conditions of Lemma 2.3 imply that

$$M_m = 2^{n-m}.$$

In other words, if these events occur and if $M_0 = 2^n$, then M_k is reduced by a factor of 2 each time, for $1 \le k \le m-1$.

We use Lemma 2.1 and the argument in Mueller (1991) which was summarized after Lemma 2.1. In Lemma 2.1, we substitute $L = c_1 2^{(n-m)}$, $t = c_2 2^{-2(n-m)}$ and $\Delta = 2^{n-m-3}$. Let Λ_m be the event that condition (c) of Lemma 2.3 holds. Suppose that c_1 and c_2 are large. Consider the solution $\bar{u}(\tau_m + t, x)$

started at time τ_m. The argument in Mueller (1991), referred to earlier, shows that if c_2 is large enough, then

$$\int_0^J G(t,x,y)\bar{u}(\tau_m,y)\,dy < 2^{n-m-2}.$$

Using (1.4) and the definition of Λ_m, we see that if Λ_m and $A(1)$ through $A(m-1)$ occur, then the noise term $\bar{N}$ is so small that $M_{m+1} = M_m/2$ and $\tau_{m+1} - \tau_m \le c_1 2^{-2(n-m)}$. Thus, if $A(1),\dots,A(m-1)$ occur, then $\Lambda_m \subset A(m)$. By Lemma 2.1, we have that, for $M_m = 2^{n-m}$ sufficiently large,

$$P\{\Lambda_m^c \big| \mathscr{F}_{\tau_m}\} \le \exp\big[-c_1 2^{2(n-m)(3/2-\gamma-\delta)}\big]. \tag{2.3}$$

Our assumptions imply $M_0 \ge 2^{K_0}$. Using (2.3), we find that

$$\begin{aligned}
&\sum_{m=1}^{\sigma} P\{A^c(m)|A(1)\cap\cdots\cap A(m-1)\}\\
&\qquad \le \sum_{m=1}^{\sigma} P\{\Lambda_m^c|A(1)\cap\cdots\cap A(m-1)\}\\
&\qquad \le \sum_{k=K_0}^{\infty} \exp\big[-c2^{2k(3/2-\gamma-\delta)}\big]\\
&\qquad \le c\exp\big[-c2^{2K_0(3/2-\gamma-\delta)}\big]\\
&\qquad < \varepsilon
\end{aligned}$$

if K_0 is large enough. This proves Lemma 2.3. □

Now we return to the proof of Theorem 1.1. Let $h_n(x)$ be as above; that is, as $n \to \infty$, let $h_n(x)\,dx \to u_0(dx)$ in the sense of weak convergence of measures, and let $\sup_{0\le x\le J} h_n(x) = 2^n$. Thus, there is a constant K such that, for all $n \ge 1$, $\int_0^J h_n(x)\,dx \le K$.

Now we proceed with the proof of existence and uniqueness. We use the familiar Picard iteration technique, as described in Walsh [(1986), Theorem 3.2.].

First we give the proof of existence. Let $\bar{u}_n(t,x)$ be a solution of (2.1) with initial condition $\bar{u}_n(0,x) = h_n(x)$. We define a distance $D(t,\cdot,\cdot)$ as follows. For ease of notation, let $u(t,x) = \bar{u}_{n_1}(t,x)$ and $v(t,x) = \bar{u}_{n_2}(t,x)$. Let

$$D(t,f,g) = \int_0^t (t-s)^{-1/2} s^{-\gamma+1} \int_0^J \big|f(s,x) - g(s,x)\big|^2\,dx\,ds. \tag{2.4}$$

Let $D(t) = D(t,u,v)$.

We will show existence of a solution $\bar{u}(t,x)$ to (2.1) and show that $\bar{u}(t,x)$ is also a solution to (1.1) up to some positive stopping time. Then, since $\bar{u}$ is bounded in x at this stopping time, we can use the strong Markov property of solutions to start afresh. The strong Markov property was proved for bounded solutions of (1.1) in Mueller [(1991), Lemma 3.3], but the proof there is easily carried over to our case, seeing that $u(t,x)$ becomes bounded for $t > 0$, with probability 1. Recall that the main result of Mueller (1991) was long time existence of $u(t,x)$ for bounded and continuous initial conditions. Using the strong Markov property to start afresh at time τ_σ, we see that this result implies the existence of the solution $u(t,x)$ for all times $t>0$.

Now we proceed with the Picard iteration, using Theorem 2.5 of Walsh (1986) to compute the expected square of a white noise integral:

$$E\left[\int_0^t\int_0^J f(s,x)W(dx\,ds)\right]^2 = E\int_0^t\int_0^J f(s,x)^2\,dx\,ds.$$

Here, $f(s,x)=f(s,x,\omega)$ is a nonanticipating function. Let

$$\begin{aligned}\delta &= \delta(t,n_1,n_2)\\ &= 2\int_0^J\int_0^t (t-s)^{-1/2}s^{-\gamma+1}\\ &\qquad\times\left(\int_0^J G(s,x,y)[h_{n_1}(y)-h_{n_2}(y)]\,dy\right)^2 ds\,dx.\end{aligned}\tag{2.5}$$

To estimate δ, let $0<a<t$:

$$\begin{aligned}\delta &= 2\int_0^J\int_0^a (t-s)^{-1/2}s^{-\gamma+1}\\ &\qquad\times\left(\int_0^J G(s,x,y)[h_{n_1}(y)-h_{n_2}(y)]\,dy\right)^2 ds\,dx\\ &\quad+2\int_0^J\int_a^t (t-s)^{-1/2}s^{-\gamma+1}\\ &\qquad\times\left(\int_0^J G(s,x,y)[h_{n_1}(y)-h_{n_2}(y)]\,dy\right)^2 ds\,dx\\ &= \delta_{\mathrm{I}}+\delta_{\mathrm{II}}.\end{aligned}\tag{2.6}$$

Here are some facts we will use in the following calculation. Recall that $\int_0^J h_n(x)\,dx\le K$ for all $n\ge 0$. Second, note that $\int_0^J G(s,x,y)^2\,dx = cs^{-1/2}$, for some constant $c>0$. Finally, recall that Minkowski's inequality, with $p=2$, states that, for a nonnegative measure ν and for $b_1,\dots,b_n\in\mathbf{L}^2(d\nu)$,

$$\left[\int\left(\sum_{k=1}^n b_k(y)\right)^2 d\nu(y)\right]^{1/2}\le\sum_{k=1}^n\left[\int b_k(y)^2\,d\nu(y)\right]^{1/2}.$$

Approximating integrals by sums in a standard way, we get Minkowski's inequality for integrals, which states that, for nonnegative measures $d\mu$ and $d\nu$, and for $b(r,y)\in\mathbf{L}^2(d\nu\,d\mu)$, we have

$$\left[\int\left(\int b(r,y)\,d\mu(y)\right)^2 d\nu(r)\right]^{1/2}\le\int\left[\int b(r,y)^2\,d\nu(r)\right]^{1/2} d\mu(y).$$

We will use Minkowski's inequality with $r=(s,x), d\mu(y)=h_k(y)\,dy$ for $k=n_1,n_2$, and $d\nu(r)=d\nu(s,x)=(t-s)^{-1/2}s^{-\gamma+1}\,ds\,dx$. Using these facts, we find

$$\begin{aligned}
\sqrt{\delta_{\mathrm{I}}} &\le \left[2\int_0^J\int_0^a (t-s)^{-1/2}s^{-\gamma+1}\left(\int_0^J G(s,x,y)h_{n_1}(y)\,dy\right)^2 ds\,dx\right]^{1/2}\\
&\quad+\left[2\int_0^J\int_0^a (t-s)^{-1/2}s^{-\gamma+1}\left(\int_0^J G(s,x,y)h_{n_2}(y)\,dy\right)^2 ds\,dx\right]^{1/2}\\
&\le \int_0^J h_{n_1}(y)\left[2\int_0^J\int_0^a (t-s)^{-1/2}s^{-\gamma+1}G(s,x,y)^2\,ds\,dx\right]^{1/2} dy\\
&\quad+\int_0^J h_{n_2}(y)\left[2\int_0^J\int_0^a (t-s)^{-1/2}s^{-\gamma+1}G(s,x,y)^2\,ds\,dx\right]^{1/2} dy \qquad (2.7)\\
&\le cK\left[\int_0^a (t-s)^{-1/2}s^{-\gamma+1}s^{-1/2}\,ds\right]^{1/2}\\
&\le c\left[(t-a)^{-1/2}a^{3/2-\gamma}\right]^{1/2}\\
&\to 0 \qquad \text{as } a\to 0.
\end{aligned}$$

We need the following elementary lemma.

LEMMA 2.5. *Let μ_n be a sequence of finite, nonnegative measures on a compact metric space S. Suppose that the sequence μ_n converges weakly to a finite nonnegative measure μ on S. Let $\mathscr{B}$ be a uniformly bounded, equicontinuous family of functions on S. Then,*

$$\lim_{n\to\infty}\sup_{b\in\mathscr{B}}\left|\int_S b\,d\mu_n-\int_S b\,d\mu\right|=0.$$

PROOF. Suppose that the conclusion is false. Then there exist $\eta>0$ and a sequence $b_n\in\mathscr{B}$ such that

$$\left|\int_S b_n\,d\mu_{n_k}-\int_S b_n\,d\mu\right|>\eta.$$

However, the Arzela–Ascoli theorem states that there is a subsequence n_k and a continuous function b_∞ on S such that $b_{n_k}\to b_\infty$ uniformly. Note that since $\mu_n(S)\to\mu(S)$, we may assume that the $\mu_n(S)$ and $\mu(S)$ are uniformly bounded. Then

$$\begin{aligned}
\eta &< \left|\int_S b_{n_k}\,d\mu_{n_k}-\int_S b_{n_k}\,d\mu\right|\\
&\le \int_S |b_{n_k}-b_\infty|\,d\mu_{n_k}+\int_S|b_{n_k}-b_\infty|\,d\mu\\
&\quad+\left|\int_S b_\infty d\mu_{n_k}-\int_S b_\infty\,d\mu\right|,
\end{aligned}$$

but all the terms on the right-hand side tend to 0 as $k\to\infty$, which contradicts the fact that $\eta>0$. This ends the proof of Lemma 2.5. □

Suppose $t > a > 0$ and $\varepsilon > 0$ are fixed. Note that $\{G(s,x,\cdot)|a \le s \le t, 0 \le x \le J\}$ is a uniformly bounded, equicontinuous family of functions on $[0,J]$. By Lemma 2.5 and since $h_n \to u_0$ weakly as $n \to \infty$, we may choose N so large that $n_1, n_2 > N$ implies that

$$\left|\int_0^J G(s,x,y)[h_{n_1}(y) - h_{n_2}(y)]\,dy\right| < \varepsilon^{1/2},$$

for all $a \le s \le t$, $0 \le x \le J$. Therefore $n_1, n_2 > N$ implies that

$$\begin{aligned}\delta_{\mathrm{II}} &= 2\int_0^J\int_a^t (t-s)^{-1/2}s^{-\gamma+1}\left(\int_0^J G(s,x,y)[h_{n_1}(y) - h_{n_2}(y)]\,dy\right)^2 ds\,dx \\ &\le 2J\varepsilon\int_0^t (t-s)^{-1/2}s^{-\gamma+1}\,ds \\ &\le c\varepsilon.\end{aligned} \tag{2.8}$$

Then

$$\begin{aligned}ED(t) &\le \delta + 2E\int_0^J\int_0^t (t-s)^{-1/2}s^{-\gamma+1} \\ &\qquad \times\left(\int_0^s\int_0^J G(s-r,x,y)[(u(r,y)\wedge H_1(r))^\gamma \right. \\ &\qquad\qquad \left. - (v(r,y)\wedge H_1(r))^\gamma]W(dy\,dr)\right)^2 ds\,dx \\ &\le \delta + 2\int_0^J E\int_0^t\int_0^s\int_0^J (t-s)^{-1/2}s^{-\gamma+1}G(s-r,x,y)^2 \\ &\qquad \times [(u(r,y)\wedge H_1(r))^\gamma - (v(r,y)\wedge H_1(r))^\gamma]^2\,dy\,dr\,ds\,dx \\ &\le \delta + 2\int_0^J E\int_0^t\int_0^s\int_0^J (t-s)^{-1/2}s^{-\gamma+1}G(s-r,x,y)^2\gamma^2 H_1(r)^{2\gamma-2} \\ &\qquad \times [u(r,y) - v(r,y)]^2\,dy\,dr\,ds\,dx \\ &\le \delta + 2\gamma^2\int_0^J\int_0^t\int_0^s\int_0^J (t-s)^{-1/2}s^{-\gamma+1}G(s-r,x,y)^2 r^{-\gamma+1} \\ &\qquad \times \sup_{0\le z\le J} E[u(r,z) - v(r,z)]^2\,dy\,dr\,ds\,dx \\ &\le \delta + C\int_0^t (t-s)^{-1/2}s^{-\gamma+1}\int_0^s (s-r)^{-1/2}r^{-\gamma+1} \\ &\qquad \times \sup_{0\le z\le J} E[u(r,z) - v(r,z)]^2\,dr\,ds \\ &\le \delta + C\int_0^t (t-s)^{-1/2}s^{-\gamma+1}ED(s)\,ds.\end{aligned} \tag{2.9}$$

To get the third inequality in (2.9), we have used the mean value theorem to deduce that, for nonnegative numbers a, b, and H, there exists a number

c between $a \wedge H$ and $b \wedge H$ such that

$$|(a \wedge H)^\gamma - (b \wedge H)^\gamma| = \gamma c^{\gamma-1}|a \wedge H - b \wedge H|$$

and, therefore,

$$|(a \wedge H)^\gamma - (b \wedge H)^\gamma| \le \gamma H^{\gamma-1}|a - b|.$$

Note that

$$\int_0^t (t-s)^{-1/2} s^{-\gamma+1}\, ds \le \left(2 + \frac{1}{2-\gamma}\right) t^{(3/2-\gamma)} 2^{\gamma-3/2}.$$

Using Gronwall's lemma, we conclude that

$$\sup_{0\le s\le t} ED(s) \le 2\delta \exp\left[\frac{C(2+1/(2-\gamma))t^{(3/2-\gamma)}}{2^{\gamma-3/2}}\right].$$

Now we choose a subsequence n_k such that

$$\sum_{k=1}^\infty \delta(t, n_k, n_{k+1}) < \infty.$$

It follows that

$$\sum_{k=1}^\infty D(t, \bar{u}_{n_k}, \bar{u}_{n_{k+1}}) < \infty,$$

so that $\bar{u}_{n_k}$ converges in the $\mathbf{L}^2$-norm which we have chosen. Call the limit $\bar{u}(t,x)$. It is standard, as in Shiga (1994), that $\bar{u}(t,x)$ satisfies (2.1).

However, we are looking for solutions of (1.1), not (2.1). Let $\xi(t_i, t, u)$ be the first time $s \in [t_i, t]$ such that $\sup_{0\le x\le J} u(s,x) > H_1(s)$. If there is no such time, let $\xi(t_i,t,u) = t$. Since $u_{n_k} \to u$ weakly, using Lemma 2.4, the reader can easily check that

$$P(\xi(t_i,t,u) > r) = \lim_{k\to\infty} P(\xi(t_i,t,u_k) > r) \ge 1 - \varepsilon(t,r),$$

where $\varepsilon(t,r) \to 0$ as $t \downarrow 0$ and where $\varepsilon(t,r)$ does not depend on i or k. However, $\xi(t_i,t,u) \downarrow \xi(0,t,u)$ as $t_i \downarrow 0$, so

$$P(\xi(0,t,u) > r) \ge 1 - \varepsilon(r).$$

Thus, with probability 1, $\sup_{0\le x\le J} \bar{u}(t,x) \le H_1(t)$ for $t > 0$ and for t less than some almost surely positive random time. This proves existence, in the sense of Theorem 1.1.

Next we give the proof of uniqueness, which follows the lines of our existence proof. Let $u(t,x)$ and $v(t,x)$ be two solutions to (1.1). According to our definition of solutions to (1.1), this means that the following hold:

1. $u(t,x)$ and $v(t,x)$ satisfy (1.2);
2. for $t > 0$, $u(t,x)$ and $v(t,x)$ are bounded functions of x;
3. for some constant $C > 0$, and for some stopping time $T > 0$ a.s., $0 < t < T$ implies that $\sup_{x\in\mathbb{R}} \max[u(t,x), v(t,x)] \le CH_1(t)$.

Again, we use the integral equation (1.2) to estimate the difference of solutions $u(t,x)$ and $v(t,x)$. Then the first integral in (1.2) makes sense, because u_0 is a finite measure. Define $D(t,f,g)$ as in (2.4), and let $D(t)=D(t,u,v)$.

Observe that the difference $u(t,x)-v(t,x)$ is simpler than in the case of existence, since the terms $\int_0^J G(t,x,y)u_0(dy)$ cancel out. Therefore,

$$\begin{aligned} u(t,x)-v(t,x) &= \int_0^t\int_0^J G(t-s,x,y)u^\gamma(s,y)W(dy\,ds) \\ &\quad - \int_0^t\int_0^J G(t-s,x,y)v^\gamma(s,y)W(dy\,ds), \end{aligned}$$

and (2.9) is simpler, because there is no δ term. However, everything else is exactly the same, so we obtain

$$ED(t) \le C\int_0^t (t-s)^{-1/2}s^{-\gamma+1}ED(s)\,ds,$$

and Gronwall's lemma implies that

$$ED(t)=0.$$

It follows that, with probability 1, $u(t,x)=v(t,x)$ for all $t\in[0,T]$, $0\le x\le J$. Here, T is the minimum of the τ_σ associated with u and v. We note that $T>0$ almost surely. Then, using the strong Markov property of solutions, we can start afresh at $t=T$. Since, at $t=T$, u and v are bounded, uniqueness beyond $t=T$ follows from Mueller (1991). This proves uniqueness for all $t\ge 0$, in the sense of Theorem 1.1.

3. The case $\boldsymbol{\gamma \ge 3/2}$. If $\gamma\ge 3/2$, then some of the techniques we used in Section 2 break down. In particular, Lemma 2.1, while it is still true, is no longer useful. Recall that, for $\gamma<3/2$, $L=c2^n$, $t=2^{-2n}$ and $\Delta=c2^n$, Lemma 2.1 showed that a certain probability tended to 0 as $n\to\infty$. For $\gamma\ge 3/2$, the bound on this probability would tend to ∞ rather than 0.

Instead of assuming that u_0 is a nonnegative finite measure, we make the more stringent assumption that u_0 is a nonnegative function in $\mathbf{L}^p$ for $p>2\gamma$. Let

$$K=K(p)\equiv\int_0^J u_0^p(x)\,dx. \tag{3.1}$$

As in the previous section, we consider a modified equation for $u(t,x)$. Recall from the statement of Theorem 1.2 that $H_2(t)=C_2t^{-1/p}$. Let $\bar u_n(t,x)$ satisfy

$$\begin{aligned} &\bar u_t=\bar u_{xx}+[\bar u\wedge H_2(t)]^\gamma\dot W, \qquad t>0,\ 0\le x\le J, \\ &\bar u(0,x)=f_n(x), \\ &\bar u(t,0)=\bar u(t,J)=0, \end{aligned} \tag{3.2}$$

where

$$f_n(x)=u_0(x)\wedge 2^n.$$

Again, let

$$\bar{N}(t_1, t_2, x) = \int_{t_1}^{t_2} \int_0^J G(t_2 - s, x, y)(\bar{u}(s, y) \wedge L)^\gamma W(dy\, ds).$$

Now we introduce an auxiliary function which will allow us to estimate some integrals. For a nonnegative function $f(x)$ on $\mathbb{R}$, we define a function $\lambda_f(y)$ by

$$\lambda_f(y) = m\{x \in \mathbb{R} | f(x) > y\},$$

where $m(dx)$ is Lebesgue measure. To compare definitions, recall that the classical distribution function of f is equal to $J - \lambda_f(y)$. In order to define λ_{u_0}, we extend u_0 to be 0 on $\mathbb{R} \setminus [0, J]$. Arguing as in Markov's inequality, we find

$$\begin{aligned} \lambda_{u_0}(y) &\le \frac{\int_0^J u_0(x)^p\, dx}{y^p} \\ &= \frac{K}{y^p}. \end{aligned} \tag{3.3}$$

Our strategy is to do computations with a sum or maximum of Gaussian functions, whose λ-function is greater than or equal to the λ-function of $u(t, \cdot)$. It will turn out that the sum of Gaussians with which we deal is almost equal to the maximum of the same Gaussians. It is easy to convolve Gaussians with each other, so the computations in the integral equation (1.2) are easy to perform. For the purposes of this section, let

$$G(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

If $G(t, x, y)$ has three arguments rather than two, we give G its old meaning as the fundamental solution of the heat equation on $[0, J]$, with Dirichlet boundary conditions. The maximum principle for the heat equation implies that for, $t > 0$ and $x, y \in [0, J]$,

$$G(t, x, y) \le G(t, x - y). \tag{3.4}$$

First, we construct a sum $\bar{g}(x)$ of Gaussians such that $\lambda_{u_0}(y) \le 2\lambda_{7\bar{g}}(y)$ for all $y > 0$. Let

$$\begin{aligned} c_k &= K 2^{-k(p-1)}, \\ t_k &= K^2 2^{-2kp}, \end{aligned} \tag{3.5}$$

and, for $x \in \mathbb{R}$, let

$$\begin{aligned} \bar{g}(x) &= \sum_{k=1}^{\infty} c_k G(t_k, x), \\ g(x) &= \sup_{1 \le k < \infty} c_k G(t_k, x). \end{aligned}$$

The following lemma relates $u_0(x)$ to $\bar{g}(x)$.

LEMMA 3.1. *For all* $y > 0$, *we have*

$$\lambda_{u_0}(y) \le 2\lambda_{7\bar{g}}(y).$$

PROOF. First, we prove Lemma 3.1 for $y = 2^k$ and $k \ge 1$. We easily compute that, for $|x| \le \sqrt{t}$,

$$G(t,x) \ge \frac{\exp(-\frac{1}{2})}{\sqrt{2\pi t}} = \frac{0.3}{\sqrt{t}}. \tag{3.6}$$

Thus, if $|x| \le \sqrt{t_k}$,

$$c_k G(t_k, x) > \frac{0.3c_k}{\sqrt{t_k}} \ge 0.3 \times 2^k,$$

and so,

$$\begin{aligned}\lambda_{\bar{g}}(0.3 \times 2^k) &= m\{x \in \mathbb{R} \big| \bar{g}(x) > 0.3 \times 2^k\}\\ &\ge m\{x \in \mathbb{R} \big| c_k G(t_k, x) > 0.3 \times 2^k\}\\ &\ge 2\sqrt{t_k}\\ &= K2^{-kp+1}.\end{aligned}$$

Thus, by (3.3),

$$\begin{aligned}\lambda_{u_0}(2^k) &\le K2^{-kp}\\ &\le 2\lambda_{\bar{g}}(0.3 \times 2^k).\end{aligned}$$

Finally, note that the definition of λ implies that $\lambda_g(cy) = \lambda_{g/c}(y)$. Therefore, if $2^k \le y \le 2^{k+1}$, then

$$\begin{aligned}\lambda_{u_0}(y) &\le \lambda_{u_0}(2^k)\\ &\le 2\lambda_{\bar{g}}(0.3 \times 2^k)\\ &\le 2\lambda_{\bar{g}}(0.15 \times y)\\ &\le 2\lambda_{7\bar{g}}(y).\end{aligned}$$

This proves Lemma 3.1. □

Next, we show an inequality between $g(x)$ and $\bar{g}(x)$. Of course, it follows from the definition that, for all $x \in \mathbb{R}$, $g(x) \le \bar{g}(x)$. Here is an inequality in the other direction.

LEMMA 3.2. *There is a constant* $c > 0$ *such that, for all* $x \in \mathbb{R}$, *we have*

$$\bar{g}(x) \le cg(x).$$

PROOF. Note that

$$c_k G(t_k, x) \le c_k G(t_k, 0) = \frac{2^k}{\sqrt{2\pi}}$$

$$c_k G(t_k, x) \le \frac{2^k}{\sqrt{2\pi}} \exp(-2^{2p(k-k_0-1)}) \quad \text{if } |x| \ge t_{k_0+1}.$$

By (3.6), if $|x| \le \sqrt{t}_{k_0}$, then

$$c_{k_0} G(t_{k_0}, x) \ge 0.3 \times 2^{k_0}.$$

Suppose that $t_{k_0+1} \le |x| \le t_{k_0}$. Using the previous two inequalities and the definitions of g and $\bar{g}$, we have

$$\begin{aligned}
\bar{g}(x) &= \sum_{k=1}^{\infty} c_k G(t_k, x) \\
&= \sum_{k=1}^{k_0} c_k G(t_k, x) + \sum_{k=k_0+1}^{\infty} c_k G(t_k, x) \\
&\le \sum_{k=1}^{k_0} \frac{2^k}{\sqrt{2\pi}} + \sum_{k=k_0+1}^{\infty} \frac{2^k}{\sqrt{2\pi}} \exp(-2^{2p(k-k_0-1)}) \\
&\le c 2^{k_0} \\
&\le c \cdot c_{k_0} G(t_{k_0}, x) \\
&\le c g(x).
\end{aligned}$$

This proves Lemma 3.2. □

Now we continue with the proof of Theorem 1.2.

LEMMA 3.3. *If c_k and t_k are as in* (3.5), *then there is a constant C_0 such that*

$$\lambda_{u_0}(y) \le 2\lambda_{C_0 g}(y),$$

for all $y > 0$. Furthermore, if $\hat{g}(x) = C_0 g(x/2)$, then

$$\lambda_{u_0}(y) \le \lambda_{\hat{g}}(y),$$

for all $y > 0$.

PROOF. Note that if $r(x) \le s(x)$, for all x, then $\lambda_r(y) \le \lambda_s(y)$. The first assertion of Lemma 3.3 then follows from Lemmas 3.1 and 3.2. The second assertion follows from the first assertion and a simple scaling argument. This proves Lemma 3.3. □

We keep track of the time necessary for the heat flow to reduce the maximum of g to some lower level. Note that the maximum of $G(t, x)$ over $x \in \mathbb{R}$ occurs at $x = 0$.

LEMMA 3.4. *We have*

$$\sup_{x\in\mathbb{R}}[G(4s,\cdot)*\hat{g}](x)\le C_0[s(p-1)]^{-1/(2p)}\frac{p}{p-1}.$$

PROOF. Note that $G(4s,y)=(1/2)G(s,y/2)$ and, therefore,

$$\begin{aligned}\sup_{x\in\mathbb{R}}[G(4s,\cdot)*\hat{g}](x)&=C_0\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty}G(s,x/2-y/2)g(y/2)d(y/2)\\&\le C_0\sup_{x\in\mathbb{R}}\sup_{k\ge1}c_kG(t_k+s,x)\\&\le C_0\sup_{k\ge1}\frac{c_k}{\sqrt{t_k+s}}\\&=C_0\sup_{k\ge1}\frac{2^k}{\sqrt{1+s2^{2kp}}}\\&\le C_0\sup_{y\ge0}\frac{2^y}{\sqrt{1+s2^{2yp}}}.\end{aligned}$$

Now let

$$S(y)=\frac{2^y}{\sqrt{1+s2^{2yp}}}.$$

Note that $\lim_{y\to\infty}S(y)=0$ and that $S(0)=1/\sqrt{1+s}$. We will differentiate to find the critical points of $S(y)$ in $(0,\infty)$:

$$\begin{aligned}S'(y)&=\frac{2^y\ln 2}{\sqrt{1+s2^{2py}}}-\frac{2^{y-1}s2^{2py}2p\ln 2}{(1+s2^{2py})^{3/2}}\\&=0\end{aligned}$$

if

$$y=y_{\max}\equiv-\frac{1}{2p}\log_2[s(p-1)].$$

Evaluating $S(y)$ at $y=y_{\max}$, we find that

$$S(y_{\max})=[s(p-1)]^{-1/(2p)}\frac{p}{p-1}.$$

This proves Lemma 3.4. □

LEMMA 3.5. *Suppose that $f(x)$ and $h(x)$ are nonnegative even functions on $\mathbb{R}$, and are nonincreasing for $x\in[0,\infty)$. Let $\underline{f}(x)$ and $\underline{h}(x)$ be functions on $\mathbb{R}$ such that $\lambda_f(y)=\lambda_{\underline{f}}(y)$, and $\lambda_h(y)=\lambda_{\underline{h}}(y)$, for all $y>0$. Then, for all $t>0$, we have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x)G(t,x-y)h(y)\,dy\,dx\ge\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underline{f}(x)G(t,x-y)\underline{h}(y)\,dy\,dx.$$

PROOF. Note that we may approximate $f(x)$ by step functions of the form

$$\sum_{n=1}^{\infty} \alpha_n 1(-a_n \le x \le a_n), \tag{3.7}$$

where α_n and b_n are nonnegative constants. Since $\underline{f}(x)$ is a rearrangement of $f(x)$, we may approximate $\underline{f}(x)$ by simple functions of the form $\sum_{n=1}^{\infty} \alpha_n A_n$, where $m(A_n) = 2a_n$, and $\overline{a_n}$ and α_n are the same constants as above. We leave these details to the reader. Here, $m(dx)$ is Lebesgue measure. The same statement would be true for $h(x)$ and $\underline{h}(x)$.

Because the double integral involved in Lemma 3.5 is bilinear, it suffices to show the following. Let $A, B \subset \mathbb{R}$, with $m(A) = 2a$ and $m(B) = 2b$. Let $\bar{A} = [-a, a]$ and let $\bar{B} = [-b, b]$. We must show that

$$\int_A \int_B G(t, x-y)\, dy\, dx \le \int_{\bar{A}} \int_{\bar{B}} G(t, x-y)\, dy\, dx. \tag{3.8}$$

In fact, since $G(t, x-y)$ is continuous in (x, y), it is enough to show (3.8) for sets A and B of the form

$$A = \bigcup_{k=1}^{m} \left[\frac{a_{k-1}}{N}, \frac{a_k}{N}\right],$$
$$B = \bigcup_{k=1}^{n} \left[\frac{b_{k-1}}{N}, \frac{b_k}{N}\right],$$

where the sets $\{a_k\}_{k=0}^{m}$ and $\{b_k\}_{k=0}^{n}$ contain $m+1$ and $n+1$ integers, respectively. By subdividing each interval $[a_{k-1}/N, a_k/N]$ if necessary, we may assume that m is even, and likewise that n is even.

We claim that elementary geometry gives us the following. For $1 \le k \le m/2$, let

$$R_k = \left(\left[\frac{a_{k-1}}{n}, \frac{a_k}{n}\right] \cup \left[\frac{a_{m-k}}{n}, \frac{a_{m-k+1}}{n}\right]\right) \times B.$$

Let $N_k(z)$ be the number of $1/N \times 1/N$ squares in R_k whose distance from the diagonal $\{x = y\}$ is less than or equal to z. We claim that it is clear that, for each k, z, the maximum value of $N_k(z)$ is attained when $\bar{A} = A$ and $\bar{B} = B$. Letting $N \to \infty$, we see that the maximum value of

$$M(z) \equiv m\big\{(x, y) \in \bar{A} \times \bar{B} \big|\ |x-y| < z\big\}$$

is attained when $\bar{A} = A$ and $\bar{B} = B$. Because $G(t, x-y)$ is a nonincreasing function of $|x-y|$, we see that

$$\int_{\bar{A}} \int_{\bar{B}} G(t, x-y)\, dy\, dx = \int_0^{\infty} G(t, z)\, dM(z)$$

is minimized when $\bar{A} = A$ and $\bar{B} = B$. We have reduced Lemma 3.5 to this case, so the lemma is proved. □

By taking f and $\underline{f}$ to be indicator functions, we may deduce the following corollary of Lemma 3.5.

COROLLARY 3.6. *Let h and $\underline{h}$ be nonnegative functions on $\mathbb{R}$ such that h is an even function, nonincreasing on $[0,\infty)$ and such that $\lambda_{\underline{h}}(y) \le \lambda_h(y)$ for all $y > 0$. Then,*

$$\lambda_{G(t,\cdot)*\underline{h}}(y) \le \lambda_{G(t,\cdot)*h}(y),$$

for all $t > 0$, $y > 0$.

Now we use Lemmas 2.1 and 3.5 and Corollary 3.6 to show that, in the time necessary for the heat kernel to take down $u_0(x)$, the noise term $N(t,s,x)$ cannot push up the solution very much. This idea is similar to the proof of long time existence for $\gamma \le 3/2$, as found in Mueller (1991).

Consider the initial function $f_\infty = u_0 \in \mathbf{L}^p$. Recall that $f_n(x) = u_0(x) \wedge 2^n$. Note that

$$\lambda_{f_n}(y) \le \lambda_{u_0}(y) \le \lambda_{\hat{g}}(y),$$

for $y > 0$ and $1 \le n \le \infty$. Let

$$s_k = C2^{-2kp}.$$

Now we come to the main point of all our calculations with the λ functions.

LEMMA 3.7. *There is a constant $C_1 > 0$ such that, for $k \ge 1$ and $1 \le n \le \infty$,*

$$\sup_{0\le x\le J} \int_0^J G(s_k,x,y) f_n(y)\,dy \le C_1 2^k.$$

PROOF. Using Lemma 3.4 and Corollary 3.6, we have

$$\begin{aligned}
\sup_{0\le x\le J} \int_0^J G(s_k,x,y) f_n(y)\,dy &\le \sup_{x\in\mathbb{R}} [G(s_k,\cdot) * f_n](x) \\
&= \sup\{y \ge 0 | \lambda_{G(s_k,\cdot)*f_n}(y) > 0\} \\
&\le \sup\{y \ge 0 | \lambda_{C_0 G(s_k,\cdot)*\hat{g}}(y) > 0\} \\
&= \sup_{x\in\mathbb{R}} [G(s_k,\cdot) * \hat{g}](x) \\
&\le C_0 \left[\frac{s_k}{4}(p-1)\right]^{-1/(2p)} \frac{p}{p-1} \\
&\le C_1 2^k.
\end{aligned}$$

This proves Lemma 3.7. □

Now we let (1) $L = L(k) = 2^{k+1}$ and (2) $\Delta = \Delta(k) = c_0 2^{-k\varepsilon}$.
Let A_k denote the following event:

$$A_k = \left\{ \sup_{0 \le s \le s_k - s_{k+1}} \sup_{0 \le x \le J} |\bar{N}(s_{k+1}, s_{k+1} + s, x)| > \Delta \right\}.$$

Lemma 2.1 asserts that, for c_0 large enough,

$$\begin{aligned} P\{A_k\} &\le \exp\left[\frac{-c\Delta^2}{L^{2\gamma}(s_k - s_{k+1})^{1/2-\varepsilon}} \right] \\ &\le \exp\left[-c 2^{k(p-2\gamma-2(1+p)\varepsilon)} \right]. \end{aligned}$$

Note that, by the Markov property of solutions, if

$$p > 2\gamma$$

and $\varepsilon < (p - 2\gamma)/3$, then, for large enough k, we have

$$\sum_{k=K_0}^{n} P\{A_k \mid A_{k+1} \cap \cdots \cap A_n\} < \exp[-c 2^{K_0(p-2\gamma-2(1+p)\varepsilon)}]. \tag{3.9}$$

We remark that the right-hand side of (3.9) does not depend on n. □

LEMMA 3.8. *Fix $\varepsilon > 0$. If C, C_2 and K_0 are sufficiently large, then, for all $n > K_0$, we have*

$$P = \left\{ \sup_{0 \le x \le J} \bar{u}_n(t, x) \ge H_2(t) \text{ for some } t \le s_{K_0} \right\} < \varepsilon.$$

PROOF. The lemma follows from (3.9), once we note that

$$\begin{aligned} \bar{u}_n(t, x) \le &\int_0^J G(t, x, y) f_n(y)\, dy \\ &+ \sum_{k=K_0}^{n} \sup_{0 \le s \le s_k - s_{k+1}} \sup_{0 \le x \le J} |\bar{N}(s_{k+1}, s_{k+1} + s, x)|, \end{aligned}$$

for $0 \le t \le s_{K_0}$. □

Next, note that if $p > 1$, then

$$\int_0^t s^{-1/p}\, ds = c t^{1-1/p}.$$

We will use the above integral as an error term if t is very small.

Now we show existence, at least with positive probability, up to some random time $T_0 > 0$. The probability of existence is arbitrarily close to 1 for T_0

small enough. Inspired by the proof of existence for $\gamma < 3/2$, we define

$$D(t,f,g) = \int_0^t (t-s)^{-1/2} s^{1-1/\gamma} \int_0^J \left|f(s,x) - g(s,x)\right|^2 \, dx\, ds.$$

Let $D(t) = D(t, \bar{u}_{n_1}, \bar{u}_{n_2})$. From here on, much of the analysis is similar to that in Section 2, but we give most of the details for completeness. The main difference is that in Section 2 we used $-\gamma+1$ instead of $1-1/\gamma$. Let

$$\begin{aligned}\delta &= \delta(t, n_1, n_2)\\ &= 2\int_0^J \int_0^{t\wedge\tau_\sigma} (t-s)^{-1/2} s^{1-1/\gamma} \left(\int_0^J G(s,x,y)[f_{n_1}(y) - f_{n_2}(y)]\, dy\right)^2 ds\, dx.\end{aligned}$$

To estimate δ, let $0 < a < t$.

$$\begin{aligned}\delta = {}& 2\int_0^J \int_0^a (t-s)^{-1/2} s^{1-1/\gamma}\\ &\quad\times \left(\int_0^J G(s,x,y)[f_{n_1}(y) - f_{n_2}(y)]\, dy\right)^2 ds\, dx\\ &+ 2\int_0^J \int_a^t (t-s)^{-1/2} s^{1-1/\gamma}\\ &\quad\times \left(\int_0^J G(s,x,y)[f_{n_1}(y) - f_{n_2}(y)]\, dy\right)^2 ds\, dx\\ = {}& \delta_{\mathrm{I}} + \delta_{\mathrm{II}}.\end{aligned}$$

Let $M = \int_0^J f(x)\,dx$, and note that $\int_0^J f_n(x)\,dx \le M$ for all $n \ge 1$. The next two equations are similar to (2.7) and (2.8). Using Minkowski's inequality, we have

$$\begin{aligned}\sqrt{\delta_{\mathrm{I}}} &\le \left[2\int_0^J \int_0^a (t-s)^{-1/2} s^{1-1/\gamma} \left(\int_0^J G(s,x,y) f_{n_1}(y)\,dy\right)^2 ds\,dx\right]^{1/2}\\ &\quad + \left[2\int_0^J \int_0^a (t-s)^{-1/2} s^{1-1/\gamma} \left(\int_0^J G(s,x,y) f_{n_2}(y)\,dy\right)^2 ds\,dx\right]^{1/2}\\ &\le \int_0^J f_{n_1}(y) \left[2\int_0^J \int_0^a (t-s)^{-1/2} s^{1-1/\gamma} G(s,x,y)^2\, ds\,dx\right]^{1/2} dy\\ &\quad + \int_0^J f_{n_2}(y) \left[2\int_0^J \int_0^a (t-s)^{-1/2} s^{1-1/\gamma} G(s,x,y)^2\, ds\,dx\right]^{1/2} dy\\ &\le cMJ\left[\int_0^a (t-s)^{-1/2} s^{1-1/\gamma} s^{-1/2}\right]^{1/2}\\ &\le c(t-a)^{-1/2} a^{3/2-1/\gamma}\\ &\to 0 \quad \text{as } a \to 0.\end{aligned}$$

Suppose $a > 0$ and $\varepsilon > 0$ are fixed. Because $f_n \to f$ weakly as $n \to \infty$, we may choose N so large that $n_1, n_2 > N$ implies that

$$\begin{aligned}\delta_{\mathrm{II}} &= 2\int_0^J \int_a^t (t-s)^{-1/2} s^{1-1/\gamma} \left(\int_0^J G(s,x,y)[f_{n_1}(y) - f_{n_2}(y)]\,dy \right)^2 ds \\ &\le 2J\varepsilon \int_0^t (t-s)^{-1/2} s^{1-1/\gamma}\,ds \\ &\le c\varepsilon.\end{aligned}$$

Then

$$\begin{aligned}ED(t) &\le \delta + 2E\int_0^J \int_0^t (t-s)^{-1/2} s^{1-1/\gamma} \\ &\qquad \times \left(\int_0^s \int_0^J G(s-r,x,y)\big[(u(r,y)\wedge H_2(r))^\gamma \right. \\ &\qquad\qquad \left. - (v(r,y)\wedge H_2(r))^\gamma\big] W(dy\,dr) \right)^2 ds\,dx \\ &= \delta + 2\int_0^J E\int_0^t \int_0^s \int_0^J (t-s)^{-1/2} s^{1-1/\gamma} G(s-r,x,y)^2 \\ &\qquad \times [(u(r,y)\wedge H_2(r))^\gamma - (v(r,y)\wedge H_2(r))^\gamma]^2\,dy\,dr\,ds\,dx \\ &\le \delta + 2\int_0^J E\int_0^t \int_0^s \int_0^J (t-s)^{-1/2} s^{1-1/\gamma} G(s-r,x,y)^2 \gamma H_2(r)^{2\gamma-2} \qquad (3.10) \\ &\qquad \times [u(r,y) - v(r,y)]^2\,dy\,dr\,ds, dx \\ &= \delta + 2\gamma \int_0^J \int_0^t \int_0^s \int_0^J (t-s)^{-1/2} s^{1-1/\gamma} G(s-r,x,y)^2 r^{1-1/\gamma} \\ &\qquad \times \sup_{0\le z\le J} E[u(r,z) - v(r,z)]^2\,dy\,dr\,ds\,dx \\ &\le \delta + C\int_0^t (t-s)^{-1/2} s^{-\gamma+1} \int_0^s (s-r)^{-1/2} r^{1-1/\gamma} \\ &\qquad \times \sup_{0\le z\le J} E[u(r,z) - v(r,z)]^2\,dr\,ds \\ &\le \delta + C\int_0^t (t-s)^{-1/2} s^{1-\gamma} D(s)\,ds.\end{aligned}$$

Here, we have used the mean value theorem as in (2.9). Note that

$$\int_0^t (t-s)^{-1/2} s^{1-1/\gamma} ds \le \left(2 + \frac{1}{2-1/\gamma}\right) t^{(3/2-1/\gamma)} 2^{1/\gamma-3/2}.$$

Using Gronwall's lemma, we conclude that

$$\sup_{0\le s\le t} ED(s) \le \delta \exp\left[\frac{C(2+1/(2-1/\gamma))t^{(3/2-1/\gamma)}}{2^{1/\gamma-3/2}} \right].$$

Next, as in Section 2, we choose a subsequence $\bar{u}_{n_k}(t,x)$ which is convergent in $\mathbf{L}^2$, by choosing n_k such that

$$\sum_{k=1}^{\infty} \delta(t, n_k, n_{k+1}) < \infty.$$

It follows that

$$\sum_{k=1}^{\infty} D(t, \bar{u}_{n_k}, \bar{u}_{n_{k+1}}) < \infty,$$

and $\bar{u}_{n_k}$ is convergent as claimed. Call the limit $u(t,x)$.

However, $\bar{u}(t,x)$ does not satisfy (1.1), but rather (3.2). We claim that, as in Section 2, if C_2 is large enough, then, by Lemma 3.8, $\sup_{0\le x\le J} u(t,x) < H_2(t)$ for all t less than some stopping time, with probability arbitrarily close to 1. This proves existence.

The proof of uniqueness again follows the lines of Section 2, with the same modifications as the above existence proof. We leave uniqueness to the reader. Of course, as with the existence proof, we have uniqueness with arbitrarily high probability. In other words, given two solutions in the sense of Theorem 1.2, and $\varepsilon > 0$, there exists a stopping time $\tau > 0$ a.s. such that, with probability at least ε, the two solutions agree until time τ.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ROCHESTER
ROCHESTER, NEW YORK 14627
E-mail: cmlr@troi.cc.rochester.edu