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APPROXIMATION PRICING AND THE VARIANCE-OPTIMAL MARTINGALE MEASURE[1]

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Let X be a semimartingale and let Θ be the space of all predictable X-integrable processes ϑ such that $\int \vartheta \, dX$ is in the space $\mathscr{S}^2$ of semimartingales. We consider the problem of approximating a given random variable $H \in \mathscr{L}^2(P)$ by the sum of a constant c and a stochastic integral $\int_0^T \vartheta_s \, dX_s$, with respect to the $\mathscr{L}^2(P)$-norm. This problem comes from financial mathematics, where the optimal constant V_0 can be interpreted as an approximation price for the contingent claim H. An elementary computation yields V_0 as the expectation of H under the variance-optimal signed Θ-martingale measure $\tilde{P}$, and this leads us to study $\tilde{P}$ in more detail. In the case of finite discrete time, we explicitly construct $\tilde{P}$ by backward recursion, and we show that $\tilde{P}$ is typically not a probability, but only a signed measure. In a continuous-time framework, the situation becomes rather different: we prove that $\tilde{P}$ is nonnegative if X has continuous paths and satisfies a very mild no-arbitrage condition. As an application, we show how to obtain the optimal integrand $\xi \in \Theta$ in feedback form with the help of a backward stochastic differential equation.

Introduction. Let $X = (X_t)$ be a semimartingale on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$, let $0 < T < \infty$ be a fixed time horizon and let Θ be the space of all predictable X-integrable processes ϑ such that the stochastic integral process $G(\vartheta) = \int \vartheta \, dX$ is a semimartingale in $\mathscr{S}^2(P)$. For a given random variable $H \in \mathscr{L}^2(\mathscr{F}_T, P)$, we consider the optimization problem

$$\text{(I.1)} \qquad \text{minimize } E\Big[\big(H - c - G_T(\vartheta)\big)^2\Big] \text{ over all } (c, \vartheta) \in \mathbb{R} \times \Theta$$

and denote its solution by (V_0, ξ) if it exists. This problem arises naturally in financial mathematics where X describes the (discounted) price of a risky asset, H is a contingent claim due at time T and $G(\vartheta)$ gives the cumulative trading gains associated with the self-financing portfolio strategy determined by ϑ. The constant V_0 is then that initial capital which allows the best approximation of H by the terminal wealth $c + G_T(\vartheta)$ achievable by a trading strategy ϑ and thus can be interpreted as an *approximation price* for H. If H is attainable, V_0 is the usual arbitrage-free price of H; hence our

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method provides a *consistent extension* of the familiar pricing concept from a complete to an incomplete market.

The first approaches of this kind are due to Föllmer and Sondermann (1986) and Bouleau and Lamberton (1989), who considered the special case where X is a martingale with respect to P. Extensions to the general semimartingale case were later discussed by Duffie and Richardson (1991), Schweizer (1992) and Hipp (1993) for a geometric Brownian motion, Schäl (1994) and Schweizer (1995a) in discrete time and Schweizer (1994) and Monat and Stricker (1995) in the general continuous-time framework under more or less restrictive additional conditions. While all those papers focussed mainly on the problem of determining the optimal hedging strategy ξ, we are here also interested in the computation of V_0. This leads in turn to some general results on the structure of the solution (V_0, ξ) of (I.1). Hence the present paper partly complements and partly generalizes Schweizer (1994).

An outline of the paper is as follows. A very elementary Hilbert space argument in Section 1 shows that V_0 can be written as the expectation of H under a new signed measure on $(\Omega, \mathscr{F})$, the so-called variance-optimal signed Θ-martingale measure $\tilde{P}$. A *signed* Θ-*martingale measure* is a signed measure $Q \ll P$ whose density dQ/dP is in $\mathscr{L}^2(P)$, has P-expectation 1 and satisfies

$$E\left[\frac{dQ}{dP} G_T(\vartheta)\right] = 0 \quad \text{for all } \vartheta \in \Theta.$$

We call $\tilde{P}$ *variance-optimal* if $\tilde{P}$ minimizes $\|dQ/dP\|_{\mathscr{L}^2(P)}$ over all those Q. After this easy identification of V_0 in terms of $\tilde{P}$, we turn to the study of $\tilde{P}$ and in particular its explicit construction. This problem was discussed in Hansen and Jagannathan (1991) in the simple case of a one-period model, but the multiperiod framework considered here is not so straightforward. Section 2 solves the case of a finite discrete-time index set $\{0, 1, \dots, T\}$ in full generality by first constructing the so-called *adjustment process* β of X by backward recursion and then showing that $\tilde{P}$ is given by

$$\frac{d\tilde{P}}{dP} := \text{const.} \prod_{j=1}^{T} \bigl(1 - \beta_j (X_j - X_{j-1})\bigr) = \text{const.}\, \mathscr{E}\Bigl(-\int \beta \, dX\Bigr)_T.$$

Although this looks elementary, some care has to be taken: since the proofs work recursively backward in time, integrability properties are sometimes rather delicate.

In Section 3, we study the case of a continuous-time index set $[0, T]$. We first provide a characterization of the adjustment process β by means of a *backward stochastic differential equation* and give another criterion for the existence of β if X is continuous. Under a very mild no-arbitrage condition on X, we then show that $\tilde{P}$ is always *nonnegative* if X has *continuous trajectories*. This is in sharp contrast to the discrete-time case, where $\tilde{P}$ is typically a signed measure. By a completely different argument, Delbaen and Schachermayer (1994) have recently proved that $\tilde{P}$ is even equivalent to P if X is continuous and admits an equivalent local martingale measure with square-integrable density. This allows us in turn to give an existence result for the adjustment process β. We conclude Section 3 by discussing the relation between $\tilde{P}$ and the minimal signed local martingale measure $\hat{P}$ for X.

Examples and applications are collected in Section 4. After illustrating various properties of $\tilde{P}$ and β by explicit computations, we show how $\tilde{P}$ can be used to solve quite generally several quadratic optimization problems related to (I.1). In particular, this generalizes results of Hipp (1993), Schäl (1994) and Schweizer (1994). Finally, we provide a *feedback form* description of the optimal strategy ξ, thus extending results of Schweizer (1995a) from discrete to continuous time. This involves the adjustment process β and a second backward stochastic differential equation.

1. Pricing options by $\mathscr{L}^2$-approximation. Consider an $\mathbb{R}^d$-valued stochastic process $X = (X_t)_{t \in \mathscr{T}}$, defined on a probability space $(\Omega, \mathscr{F}, P)$ and adapted to a filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathscr{T}}$, with a time index set $\mathscr{T} \subseteq [0, T]$ for some $T > 0$. We interpret the components of X_t as discounted prices at time t of d risky assets in a financial market and $\mathscr{F}_t$ as information available at time t. We also assume the existence of a riskless asset Y whose discounted price is 1 at all times. Assets X and Y can be traded; we denote by Θ the space of all *trading strategies* ϑ and by $G_T(\vartheta)$ the total *gains* from trade using the strategy $\vartheta \in \Theta$. In addition, we are given a *contingent claim* H representing a payoff to be made or received at time T. Formally, H is a real-valued $\mathscr{F}_T$-measurable random variable; the typical example is $H = (X_T^i - K)^+$, which corresponds to a European call option on the ith stock with strike price K. The problem of *option pricing* is then to associate a price at time 0 with a given H.

For a so-called *complete market*, there exists a fairly definitive pricing theory which was originated by Black and Scholes (1973) and Merton (1973) and fully developed in Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983). In the *incomplete* case, the problem is to define a pricing operator on all contingent claims in such a way that it coincides with the usual arbitrage-free price system on the space of attainable claims. By incompleteness, such an extension is no longer uniquely determined from arbitrage arguments alone; additional optimality criteria or preference assumptions have to be imposed. For various approaches in the literature, see, for instance, Bouleau and Lamberton (1989), Barron and Jensen (1990), Cvitanić and Karatzas (1993), Schäl (1994), Davis (1994) or El Karoui and Quenez (1995).

In the present paper, we propose to price options by $\mathscr{L}^2$-*approximation*: we want to determine an initial capital $c \in \mathbb{R}$ and a trading strategy $\vartheta \in \Theta$ such that the achieved terminal wealth $c + G_T(\vartheta)$ approximates H with respect to the distance in $\mathscr{L}^2(P)$. Thus we consider the following *optimization* *problem*:

$$\text{(1.1)} \qquad \begin{array}{l} \text{given } H \in \mathscr{L}^2(P), \text{ minimize } E\Big[\big(H - c - G_T(\theta)\big)^2\Big] \\ \text{over all } (c, \vartheta) \in \mathbb{R} \times \Theta. \end{array}$$

For (1.1) to be well defined, we assume that $G_T(\Theta) \subseteq \mathscr{L}^2(P)$.

DEFINITION. If $(V_0, \xi) \in \mathbb{R} \times \Theta$ solves (1.1), then V_0 is called the Θ-*approximation price* of H and is denoted by $q_\Theta(H)$.

REMARK. If a contingent claim H is *attainable* in the usual sense that it can be written as $H = H_0 + G_T(\xi^H)$ for some $(H_0, \xi^H) \in \mathbb{R} \times \Theta$, then (H_0, ξ^H) obviously solves (1.1) and thus $q_\Theta(H) = H_0$. Hence our approach yields the usual arbitrage-free prices if these exist, and so Θ-approximation pricing is consistent with complete markets. The idea to use an $\mathscr{L}^2$-criterion of the type (1.1) in order to define a price of H is due to Schäl (1994), who called V_0 the "fair hedging price." We refrain from using this terminology since we prefer to view q_Θ as one possible extension of the pricing operator from the space of attainable claims to all of $\mathscr{L}^2(P)$.

It is well known in financial mathematics that option prices can usually be computed as expectations under a suitable *martingale measure* for X. This reflects the duality between martingale measures for X and price systems consistent with the given price process X; see Harrison and Kreps (1979) for a detailed exposition. Our purpose in the rest of this section is to obtain an analogous result for the Θ-approximation price, and to that end, we introduce some terminology.

DEFINITION. A signed measure Q on $(\Omega, \mathscr{F})$ is called a *signed* Θ-*martingale measure* if $Q[\Omega] = 1$, $Q \ll P$ with $dQ/dP \in \mathscr{L}^2(P)$ and

$$E\left[\frac{dQ}{dP} G_T(\vartheta)\right] = 0 \quad \text{for all } \vartheta \in \Theta.$$

We denote by $\mathbb{P}_s(\Theta)$ the set of all signed Θ-martingale measures and by $\mathscr{D}$ the set $\{D = dQ/dP | Q \in \mathbb{P}_s(\Theta)\}$.

Note that the above concept depends in an essential way on the space Θ and the definition of $G_T(\vartheta)$. In many cases of interest, $\mathbb{P}_s(\Theta)$ coincides with the set of so-called signed $\mathscr{L}^2$-martingale measures for X. This more familiar notion, introduced in Müller (1985), is given by the following definition:

DEFINITION. Assume that $X_t \in \mathscr{L}^2(P)$ for every $t \in \mathscr{T}$. A signed measure Q on $(\Omega, \mathscr{F})$ is called a *signed* $\mathscr{L}^2$-*martingale measure for* X if $Q[\Omega] = 1$,

$Q \ll P$ with $dQ/dP \in \mathscr{L}^2(P)$ and

$$E\left[\frac{dQ}{dP}(X_t - X_s)\middle|\mathscr{F}_s\right] = 0 \qquad P\text{-a.s. for all } s, t \in \mathscr{T} \text{ with } s \leq t.$$

The set of all signed $\mathscr{L}^2$-martingale measures for X is denoted by $\mathbb{P}_s^2(X)$.

DEFINITION. A signed Θ-martingale measure $\tilde{P}$ is called *variance-optimal* if $\tilde{P}$ minimizes

$$\operatorname{Var}\left[\frac{dQ}{dP}\right] = E\left[\left(\frac{dQ}{dP} - 1\right)^2\right] = E\left[\left(\frac{dQ}{dP}\right)^2\right] - 1$$

over all $Q \in \mathbb{P}_s(\Theta)$. If $\tilde{P}$ is variance-optimal, we denote its density $d\tilde{P}/dP$ by $\tilde{D}$.

Note that a variance-optimal $\tilde{P}$ is necessarily unique and that $\tilde{P}$ exists whenever $\mathbb{P}_s(\Theta)$ is nonempty, since the density $\tilde{D}$ is obtained by minimizing $\|D\|_{\mathscr{L}^2(P)}$ over the closed convex set $\mathscr{D}$. Throughout the rest of the paper, we shall make the following assumption:

(1.2) standing assumption: $\mathbb{P}_s(\Theta) \neq \varnothing$.

As pointed out by Schachermayer, (1.2) is equivalent to assuming that the closure of $G_T(\Theta)$ in $\mathscr{L}^2(P)$ does not contain the constant 1. In that sense, (1.2) can be viewed as a condition of absence of arbitrage. We denote by π the projection in $\mathscr{L}^2(P)$ on $G_T(\Theta)^\perp$.

LEMMA 1. *Assume* (1.2).

(a) $\tilde{P} \in \mathbb{P}_s(\Theta)$ *is variance-optimal if and only if*

(1.3) $$E\left[\frac{dQ}{dP}\frac{d\tilde{P}}{dP}\right] \text{ is constant over all } Q \in \mathbb{P}_s(\Theta).$$

(b) $\tilde{P}$ *is given by*

(1.4) $$\tilde{D} = \frac{d\tilde{P}}{dP} = \frac{\pi(1)}{E[\pi(1)]} = E[\tilde{D}^2] + R$$

for some $R \in G_T(\Theta)^{\perp\perp}$.

(c) $\tilde{P} \in \mathbb{P}_s(\Theta)$ *is variance-optimal if and only if*

$$\frac{d\tilde{P}}{dP} \in [1, \infty) + G_T(\Theta)^{\perp\perp}.$$

PROOF. (a) The mapping $D \mapsto D^x := xD + (1-x)\tilde{D} = \tilde{D} + x(D - \tilde{D})$ is a bijection of $\mathscr{D} \setminus \{\tilde{D}\}$ onto itself for every $x \neq 0$. Hence (a) follows from

$$E\left[(D^x)^2\right] = E[\tilde{D}^2] + 2xE\left[\tilde{D}(D - \tilde{D})\right] + x^2E\left[(D - \tilde{D})^2\right].$$

(c) The "if" part is immediate from (a), and the "only if" part from (b).

(b) Due to the standing assumption (1.2), $\pi(1)$ cannot be P-a.s. equal to 0. Thus

$$E[\pi(1)] = E\Big[(\pi(1))^2\Big] > 0 \tag{1.5}$$

shows that $\overline{D} := (\pi(1))/(E[\pi(1)])$ is well defined and in $\mathscr{D}$. Since $\pi(1) = 1 - R^0$ for some $R^0 \in G_T(\Theta)^{\perp\perp}$, we obtain $\overline{D} = c + R$ with $c := 1/(E[\pi(1)]) \geq 1$ and $R := -R^0/(E[\pi(1)]) \in G_T(\Theta)^{\perp\perp}$. Part (a) now implies that $\overline{D} = \tilde{D}$, hence the second equality in (1.4); the third follows from (1.5). □

Proposition 2. *Suppose that $G_T(\Theta) \subseteq \mathscr{L}^2(P)$ is a linear space. If* (1.1) *has a solution (V_0, ξ) for $H \in \mathscr{L}^2(P)$ and if $\tilde{P}$ is variance-optimal, then*

$$q_\Theta(H) = V_0 = \tilde{E}[H].$$

Proof. Since (V_0, ξ) solves (1.1) and $\mathbb{R} \times G_T(\Theta)$ is a linear space, we obtain

$$E\big[H - V_0 - G_T(\xi)\big] = 0$$

and

$$E\big[(H - V_0 - G_T(\xi))G_T(\vartheta)\big] = 0 \quad \text{for all } \vartheta \in \Theta.$$

Hence the signed measure Q with density

$$\frac{dQ}{dP} := \frac{d\tilde{P}}{dP} + H - V_0 - G_T(\xi)$$

is in $\mathbb{P}_s(\Theta)$. However, $\tilde{P}$ is variance-optimal and so (1.3) implies that

$$0 = E\left[\frac{d\tilde{P}}{dP}(H - V_0 - G_T(\xi))\right] = \tilde{E}[H] - V_0,$$

which proves the assertion. □

Proposition 2 shows that the variance-optimal signed Θ-martingale measure $\tilde{P}$ can be interpreted as the price system corresponding to Θ-approximation pricing. Our main interest in the sequel is in the precise structure of $\tilde{P}$.

2. The discrete-time case. In this section, we consider the case of finite discrete time where $\mathscr{T} = \{0, 1, \dots, T\}$ for some $T \in \mathbb{N}$. For notational simplicity, we take X one-dimensional, but the results can be carried over to dimension $d > 1$. More precisely, we shall assume throughout this section that $\mathbb{F} = (\mathscr{F}_k)_{k=0,1,\dots,T}$ is a filtration on $(\Omega, \mathscr{F}, P)$ and that $X = (X_k)_{k=0,1,\dots,T}$ is a real-valued, $\mathbb{F}$-adapted, square-integrable process with increments $\Delta X_k := X_k - X_{k-1}$. Since we want to consider self-financing strategies in a frictionless market, we define the space of all trading strategies by

$$\Theta := \big\{\text{predictable processes } \vartheta \,|\, \vartheta_k \,\Delta X_k \in \mathscr{L}^2(P) \text{ for } k = 1, \dots, T\big\}$$

and take

$$G_T(\vartheta) := \sum_{j=1}^{T} \vartheta_j \, \Delta X_j \quad \text{for } \vartheta \in \Theta$$

so that we clearly have $\mathbb{P}_s^2(X) = \mathbb{P}_s(\Theta)$. In this situation, the variance-optimal $\tilde{P}$ can always be constructed explicitly. With the conventions that a sum over an empty set is 0, a product over an empty set is 1 and $0/0 = 0$, we begin by introducing an auxiliary predictable process associated to X by the following definition:

DEFINITION. The *adjustment process* β of X is defined by

$$\beta_k := \frac{E\left[\Delta X_k \prod_{j=k+1}^{T}(1 - \beta_j \, \Delta X_j)\middle|\mathscr{F}_{k-1}\right]}{E\left[\Delta X_k^2 \prod_{j=k+1}^{T}(1 - \beta_j \, \Delta X_j)^2\middle|\mathscr{F}_{k-1}\right]} \quad \text{for } k = 1, \dots, T. \tag{2.1}$$

LEMMA 3. *β is well defined by* (2.1) *and satisfies for* $k = 1, \dots, T$,

$$\prod_{j=k}^{T}(1 - \beta_j \, \Delta X_j) \in \mathscr{L}^2(P), \tag{2.2}$$

$$\beta_k \, \Delta X_k \prod_{j=k+1}^{T}(1 - \beta_j \, \Delta X_j) \in \mathscr{L}^2(P) \tag{2.3}$$

and

$$\begin{aligned} &E\left[\prod_{j=k}^{T}(1 - \beta_j \, \Delta X_j)^2 \middle| \mathscr{F}_{k-1}\right] \\ &\qquad = E\left[\prod_{j=k}^{T}(1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_{k-1}\right] \le 1 \qquad P\text{-}a.s. \end{aligned} \tag{2.4}$$

PROOF. We argue by backward induction. For $k = T$, β_T is well defined by Jensen's inequality. Since

$$Y_n := \frac{\left(E[\Delta X_T|\mathscr{F}_{T-1}]\right)^2}{E[\Delta X_T^2|\mathscr{F}_{T-1}]} \frac{\Delta X_T^2}{E[\Delta X_T^2|\mathscr{F}_{T-1}]} I_{\{E[\Delta X_T^2|\mathscr{F}_{T-1}] \ge 1/n\}} \ge 0$$

increases to $\beta_T^2 \, \Delta X_T^2$ P-a.s. and ΔX_T^2 and $Y_n \le n \, \Delta X_T^2$ are both integrable,

$$\begin{aligned} E\left[\beta_T^2 \, \Delta X_T^2|\mathscr{F}_{T-1}\right] &= \lim_{n\to\infty} E[Y_n|\mathscr{F}_{T-1}] \\ &= \lim_{n\to\infty} \frac{\left(E[\Delta X_T|\mathscr{F}_{T-1}]\right)^2}{E[\Delta X_T^2|\mathscr{F}_{T-1}]} I_{\{E[\Delta X_T^2|\mathscr{F}_{T-1}] \ge 1/n\}} \le 1 \end{aligned}$$

P-a.s. implies $E[\beta_T^2 \, \Delta X_T^2] \le 1$, which proves (2.3) and (2.2) for $k = T$. Since $\beta_T \, \Delta X_T$ and ΔX_T are both square-integrable, we conclude from the definition of β_T that

$$E[\beta_T \, \Delta X_T|\mathscr{F}_{T-1}] = \beta_T E[\Delta X_T|\mathscr{F}_{T-1}] \ge 0 \qquad P\text{-a.s.}$$

and

$$E\big[\beta_T^2\,\Delta X_T^2|\mathscr{F}_{T-1}\big] = \beta_T^2 E\big[\Delta X_T^2|\mathscr{F}_{T-1}\big] = \beta_T E\big[\Delta X_T|\mathscr{F}_{T-1}\big] \qquad P\text{-a.s.},$$

hence (2.4) for $k = T$. For $k < T$, the argument is almost identical. First of all,

$$0 \le Y_n := \frac{\Big(E\big[\Delta X_k \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)\big|\mathscr{F}_{k-1}\big]\Big)^2}{E\Big[\Delta X_k^2 \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)^2\Big|\mathscr{F}_{k-1}\Big]} \times \frac{\Delta X_k^2 \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)^2}{E\Big[\Delta X_k^2 \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)^2\Big|\mathscr{F}_{k-1}\Big]} \times I_{\{E[\Delta X_k^2 \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)^2|\mathscr{F}_{k-1}] \ge 1/n\}}$$

increases to $\beta_k^2\,\Delta X_k^2 \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)^2$ P-a.s. and therefore as above

$$E\left[\beta_k^2\,\Delta X_k^2 \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)^2 \middle| \mathscr{F}_{k-1}\right] \le 1 \qquad P\text{-a.s.}$$

This implies (2.3), and (2.2) follows by the induction hypothesis since

$$\prod_{j=k}^T (1-\beta_j\,\Delta X_j) = \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j) - \beta_k\,\Delta X_k \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j).$$

Conditioning on $\mathscr{F}_{k-1}$ finally yields (2.4) as above and thus completes the proof. □

COROLLARY 4. *The random variable*

$$\tilde{Z}^0 := \prod_{j=1}^T (1-\beta_j\,\Delta X_j) \tag{2.5}$$

is in $\mathscr{L}^2(P)$ *and satisfies* $0 \le E[\tilde{Z}^0] \le 1$, *with* $E[\tilde{Z}^0] = 0$ *if and only if* $\tilde{Z}^0 = 0$ *P-a.s. Furthermore,* $\tilde{Z}^0$ *has the property that*

$$E\big[\tilde{Z}^0\,\Delta X_k|\mathscr{F}_{k-1}\big] = 0 \qquad P\text{-a.s. for } k = 1,\dots,T. \tag{2.6}$$

PROOF. Lemma 3 implies that $\tilde{Z}^0$ is in $\mathscr{L}^2(P)$ and $0 \le E[(\tilde{Z}^0)^2] = E[\tilde{Z}^0] \le 1$, where the first inequality is an equality if and only if $\tilde{Z}^0 = 0$ P-a.s. To prove (2.6), we first note that

$$\begin{aligned} &E\big[\tilde{Z}^0\,\Delta X_k\big|\mathscr{F}_{k-1}\big] \\ &\quad = E\left[(\Delta X_k - \beta_k\,\Delta X_k^2) \prod_{j=k+1}^T (1-\beta_j\,\Delta X_j)\middle|\mathscr{F}_{k-1}\right] \prod_{j=1}^{k-1} (1-\beta_j\,\Delta X_j) \end{aligned} \tag{2.7}$$

since $\Delta X_k \prod_{j=l}^{T}(1 - \beta_j \, \Delta X_j) \in \mathscr{L}^1(P)$ for every l by Lemma 3. Furthermore,

$$U := \beta_k \, \Delta X_k^2 \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j)$$
$$= \Delta X_k \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) - \Delta X_k \prod_{j=k}^{T} (1 - \beta_j \, \Delta X_j)$$

is integrable by Lemma 3 and therefore

$$V := E\left[\beta_k \, \Delta X_k^2 \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_k \right]$$
$$= \beta_k \, \Delta X_k^2 \, E\left[\prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j)^2 \middle| \mathscr{F}_k \right]$$
$$= \beta_k E\left[\Delta X_k^2 \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j)^2 \middle| \mathscr{F}_k \right]$$
$$=: \beta_k W$$

by (2.4). Now $V = E[U | \mathscr{F}_k]$ is integrable since U is, and so is $W \le \Delta X_k^2$ due to (2.4). Thus we obtain

$$E\left[\beta_k \, \Delta X_k^2 \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_{k-1} \right]$$
$$= \beta_k E\left[\Delta X_k^2 \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j)^2 \middle| \mathscr{F}_{k-1} \right]$$
$$= E\left[\Delta X_k \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_{k-1} \right]$$

by the definition of β_k, and this proves (2.6) in view of (2.7). □

REMARKS.

1. From a purely formal point of view, the preceding results are of course straightforward to check. The main difficulty throughout this section is to ensure that all appearing expectations and conditional expectations actually exist, and this is not quite as elementary as it may look. To illustrate the problem, let us rewrite $\tilde{Z}^0$ in (2.5) as

$$\tilde{Z}^0 = 1 - \sum_{k=1}^{T} \beta_k \, \Delta X_k \prod_{j=1}^{k-1} (1 - \beta_j \, \Delta X_j) = 1 - G_T(\bar{\beta}), \tag{2.8}$$

where the predictable process $\bar{\beta}$ is given by

$$\bar{\beta}_k := \beta_k \prod_{j=1}^{k-1} (1 - \beta_j \, \Delta X_j) = \beta_k \mathscr{E}\left(-\int \beta \, dX\right)_{k-1}. \tag{2.9}$$

At first sight, it seems quite plausible that $\bar{\beta}$ should always belong to Θ or, equivalently, that the discrete stochastic exponential $\mathscr{E}(-\int \beta\, dX)_k = \prod_{j=1}^{k}(1-\beta_j\,\Delta X_j)$ for $k = 1,\dots,T$ should always be a square-integrable process. However, this is false: a counterexample (due to Schachermayer) is given in Section 4.

2. It is tempting to conjecture that (2.6) characterizes β among all predictable processes, but this is not true in general. In fact, (2.6) only implies that β_k is given by (2.1) on the set $\{\prod_{j=1}^{k-1}(1-\beta_j\,\Delta X_j)\neq 0\}$, and an easy counter-example shows that this is not enough to determine β. For a similar result in continuous time, see the remark after Proposition 8.

Now here is the promised construction of the variance-optimal $\tilde{P}$.

THEOREM 5. *Assume* (1.2). *Then the signed measure $\tilde{P}$ defined by*

$$\frac{d\tilde{P}}{dP} := \tilde{D} := \frac{\tilde{Z}^0}{E[\tilde{Z}^0]} \tag{2.10}$$

is in $\mathbb{P}_s(\Theta)$ and variance-optimal.

PROOF. If $\tilde{Z}^0$ is not P-a.s. equal to 0, Corollary 4 shows that $\tilde{P}$ is well defined by (2.10) and in $P_s^2(X)$. Since $\mathbb{P}_s(\Theta) = \mathbb{P}_s^2(X)$, Lemma 1 implies that it then only remains to show that

$$E\left[\frac{dQ}{dP}\tilde{Z}^0\right] \text{ is constant over all } Q\in\mathbb{P}_s^2(X). \tag{2.11}$$

Moreover, $\mathbb{P}_s^2(X)\neq\varnothing$ by the standing assumption (1.2), and since the constant in (2.11) will turn out to be 1, (2.11) shows in particular that $\tilde{Z}^0$ cannot be P-a.s. equal to 0. If Ω is finite, (2.11) is easy to prove. We simply use (2.8) and the martingale property of X under Q to obtain $E_Q[\tilde{Z}^0] = 1$; this is straightforward since there are no integrability problems. In the general case, however, $\bar{\beta}_k\,\Delta X_k$ need not be P-integrable. We therefore denote by

$$Z_k := E\left[\frac{dQ}{dP}\middle|\mathscr{F}_{k-1}\right]\in\mathscr{L}^2(P)$$

the density of Q with respect to P on $\mathscr{F}_k$ for $k = 0,1,\dots,T$ and note that

$$E[Z_k\,\Delta X_k|\mathscr{F}_{k-1}] = 0 \qquad P\text{-a.s. for } k = 1,\dots,T, \tag{2.12}$$

since $Q\in\mathbb{P}_s^2(X)$. To prove (2.11), we show by backward induction that

$$E\left[Z_T\prod_{j=k}^{T}(1-\beta_j\,\Delta X_j)\middle|\mathscr{F}_{k-1}\right] = Z_{k-1} \qquad P\text{-a.s. for } k = 1,\dots,T. \tag{2.13}$$

For $k = T$, we have

$$E[Z_T(1-\beta_T\,\Delta X_T)|\mathscr{F}_{T-1}] = Z_{T-1} - \beta_T E[Z_T\,\Delta X_T|\mathscr{F}_{T-1}] = Z_{T-1} \qquad P\text{-a.s.}$$

by (2.12) since $Z_T\,\Delta X_T$ and $Z_T\beta_T\,\Delta X_T$ are both integrable due to (2.3). For

$k < T$, the induction hypothesis yields

$$\begin{aligned} E\left[Z_T \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_{k-1} \right] \\ = E\left[E\left[Z_T \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_k \right] \middle| \mathscr{F}_{k-1} \right] = Z_{k-1} \qquad P\text{-a.s.} \end{aligned}$$

Furthermore, the induction hypothesis also shows that

$$\begin{aligned} Z_k \, \beta_k \, \Delta X_k &= \beta_k \, \Delta X_k \, E\left[Z_T \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_k \right] \\ &= E\left[Z_T \, \beta_k \, \Delta X_k \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_k \right] \end{aligned}$$

is integrable by (2.3) and therefore

$$\begin{aligned} E\left[Z_T \, \beta_k \, \Delta X_k \prod_{j=k+1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_{k-1} \right] &= E[Z_k \, \beta_k \, \Delta X_k | \mathscr{F}_{k-1}] \\ &= \beta_k E[Z_k \, \Delta X_k | \mathscr{F}_{k-1}] = 0 \qquad P\text{-a.s.} \end{aligned}$$

by (2.12). Taking differences now yields (2.13), and for $k = 1$, (2.13) implies that $E[Z_T \tilde{Z}^0] = E[Z_0] = 1$, hence (2.11). $\square$

Remarks.

1. If $Q \in \mathbb{P}_s^2(X)$ is nonnegative, that is, an absolutely continuous martingale measure for X with square-integrable density, then (2.13) implies

$$E_Q\left[\prod_{j=1}^{T} (1 - \beta_j \, \Delta X_j) \middle| \mathscr{F}_k \right] = \prod_{j=1}^{k} (1 - \beta_j \, \Delta X_j) \tag{2.14}$$

   due to (2.2). By using (2.10) and the definition of $\bar{\beta}$ in (2.9), we conclude that

$$E_Q\left[\frac{d\tilde{P}}{dP} \middle| \mathscr{F}_k \right] = \frac{1}{E[\tilde{Z}^0]} \big(1 - G_k(\bar{\beta}) \big)$$

   can be written as a constant plus a (discrete-time) stochastic integral of X, independently of the choice of Q. For a general version of this fact, see Lemma 2.2 of Delbaen and Schachermayer (1994).
2. The informed reader may wonder at this point how $\tilde{P}$ is related to the minimal signed martingale measure $\hat{P}$ previously studied in the litera-

ture. Recall from Schweizer (1995a) that $\hat{P}$ in discrete time is defined by

$$\frac{d\hat{P}}{dP} := \hat{Z} := \prod_{j=1}^{T}\left(1 - \frac{\tilde{\lambda}_j}{1-\tilde{\lambda}_j\,\Delta A_j}\,\Delta M_j\right) = \prod_{j=1}^{T}\left(1 - \hat{\lambda}_j\,\Delta M_j\right),$$

where $X = X_0 + M + A$ is the Doob decomposition of X, that is,

$$\Delta A_k := E\left[\Delta X_k \middle| \mathscr{F}_{k-1}\right] \quad \text{for } k = 1,\dots,T,$$

and where we assume that

$$\text{(2.15)} \qquad \tilde{\lambda}_k := \frac{\Delta A_k}{E\left[\Delta X_k^2 \middle| \mathscr{F}_{k-1}\right]} \quad \text{for } k = 1,\dots,T$$

satisfies

$$\tilde{\lambda}_k\,\Delta A_k < 1 \qquad P\text{-a.s. for } k = 1,\dots,T.$$

The process $\hat{\lambda}$ is defined by

$$\hat{\lambda}_k := \frac{\tilde{\lambda}_k}{1-\tilde{\lambda}_k\,\Delta A_k} = \frac{\Delta A_k}{\operatorname{Var}\left[\Delta X_k \middle| \mathscr{F}_{k-1}\right]} \quad \text{for } k = 1,\dots,T.$$

If the *mean-variance tradeoff process* of X,

$$\text{(2.16)} \qquad \hat{K}_l := \sum_{j=1}^{l}\hat{\lambda}_j\,\Delta A_j = \sum_{j=1}^{l}\frac{\left(E\left[\Delta X_j \middle| \mathscr{F}_{j-1}\right]\right)^2}{\operatorname{Var}\left[\Delta X_j \middle| \mathscr{F}_{j-1}\right]} \quad \text{for } l = 1,\dots,T,$$

is bounded, then $\hat{P}$ is indeed a signed martingale measure for X; see Schweizer (1995a) for more details. If $\hat{K}$ is *deterministic*, then $\tilde{P} = \hat{P}$ and $\beta = \tilde{\lambda}$; this is proved in Corollary 4.2 of Schweizer (1995a). For a continuous-time analogue of this result, see Example 2 below; Example 3 below shows that, in general, we have $\tilde{P} \neq \hat{P}$ and $\beta \neq \tilde{\lambda}$.

As an amusing consequence of Theorem 5, we obtain the following corollary.

COROLLARY 6. *X is a martingale if and only if $E[\tilde{Z}^0] = 1$.*

PROOF. By Jensen's inequality, $\|dQ/dP\|_{\mathscr{L}^2(P)} \geq 1$ for every $Q \in \mathbb{P}_s^2(X)$, with equality if and only if $dQ/dP = 1$ P-a.s. Hence X is a martingale if and only if

$$\min_{Q \in \mathbb{P}_s^2(X)} \left\| \frac{dQ}{dP} \right\|_{\mathscr{L}^2(P)} = 1,$$

where equality means in particular that the minimum is attained. However, since $\tilde{P}$ is variance-optimal, the minimum is given by $E[\tilde{D}^2] = 1/E[\tilde{Z}^0]$ due to (2.10) and (2.4). □

The adjustment process β plays a very important role in the solution of the optimization problem (1.1). As we have just seen, it allows us to give an explicit construction for the variance-optimal signed Θ-martingale measure $\tilde{P}$ in discrete time. Moreover, β is also crucial for the description of the optimal strategy ξ in the solution of (1.1). For the discrete-time case, this is clearly illustrated in Schweizer (1995a). In the case of continuous time, things (not surprisingly) become more difficult. We provide in the next section some results on the existence of the adjustment process β in continuous time and discuss applications in Section 4. The construction in Section 3 is closely related to the question if the process $\overline{\beta}$ in (2.9) belongs to Θ; see Remark 1 after Corollary 4. As a partial answer, which also serves to motivate the subsequent developments, we provide here the following result:

LEMMA 7. *Suppose that the mean-variance tradeoff process $\hat{K}$ in* (2.16) *is bounded. Then the predictable process $\overline{\beta}$ defined by* (2.9) *is in* Θ.

PROOF. Since $\hat{K}$ is bounded, Theorem 2.1 of Schweizer (1995a) implies that $G_T(\Theta)$ is closed in $\mathscr{L}^2(P)$; hence the projection of 1 in $\mathscr{L}^2(P)$ on $G_T(\Theta)$ exists and equals $G_T(\psi)$ for some $\psi \in \Theta$. Moreover, Section 4.2 of Schweizer (1995a) shows that $\overline{\beta}$ coincides with ψ and hence belongs to Θ. $\square$

3. On the structure of $\tilde{P}$ in continuous time. In this section, we provide some results on the variance-optimal signed Θ-martingale measure $\tilde{P}$ in the continuous-time case where $\mathscr{T} = [0, T]$ for some $T > 0$. We shall assume throughout this section that X is a *semimartingale* with respect to P and $\mathbb{F}$ and that

$$\Theta = \left\{ \vartheta \in L(X) \middle| G(\vartheta) := \int \vartheta \, dX \in \mathscr{S}^2(P) \right\}. \tag{3.1}$$

In (3.1), $L(X)$ denotes the space of all $\mathbb{R}^d$-valued X-integrable predictable processes, and $\mathscr{S}^2 = \mathscr{S}^2(P)$ is the space of semimartingales admitting a decomposition $X = X_0 + M + A$ with $M \in \mathscr{M}_0^2(P)$ and A of square-integrable variation. We want to consider self-financing trading strategies in a frictionless market with continuous trading and so we take

$$G_T(\vartheta) = \int_0^T \vartheta_s \, dX_s. \tag{3.2}$$

Without special mention, all stochastic processes will be defined for $t \in [0, T]$. For any $\psi \in L(X)$, we denote by $\mathscr{E}^{\psi}$ the stochastic exponential of $-\int \psi \, dX$, that is, the unique strong solution $Z = \mathscr{E}(-\int \psi \, dX)$ of the stochastic differential equation

$$dZ_t = -Z_{t-} \psi_t \, dX_t, \qquad Z_0 = 1.$$

Finally we recall the standing assumption (1.2) and the notation π for the projection in $\mathscr{L}^2(P)$ on the closed subspace $G_T(\Theta)^{\perp}$.

DEFINITION. A process $\beta \in L(X)$ is called an *adjustment process for X* if the process $\overline{\beta} := \beta \mathscr{E}^{\beta}_{-}$ is in Θ and if the random variable $\tilde{Z}^0 := \mathscr{E}(-\int \beta \, dX)_T$

is in $G_T(\Theta)^{\perp}$. That is,

$$E\Big[\tilde{Z}^0 G_T(\vartheta)\Big] = 0 \quad \text{for all } \vartheta \in \Theta. \tag{3.3}$$

Note that this definition is motivated by the properties of β in discrete time; see Theorem 5 and Lemma 7.

PROPOSITION 8. *Assume* (1.2). *If β is an adjustment process for X, then $\tilde{P}$ defined by*

$$\frac{d\tilde{P}}{dP} := \frac{\tilde{Z}^0}{E[\tilde{Z}^0]} \tag{3.4}$$

is in $\mathbb{P}_s(\Theta)$ and variance-optimal.

PROOF. By the definition of the stochastic exponential,

$$\tilde{Z}^0 = 1 - \int_0^T \mathscr{E}^{\beta}_{s-}\beta_s \, dX_s = 1 - G_T(\bar{\beta}) \tag{3.5}$$

is in $\mathscr{L}^2(P)$ since $\bar{\beta} \in \Theta$. For any $Q \in \mathbb{P}_s(\Theta)$, (3.5) and the fact that $\bar{\beta} \in \Theta$ imply that

$$E\left[\frac{dQ}{dP}\tilde{Z}^0\right] = 1,$$

and since $\mathbb{P}_s(\Theta) \neq \varnothing$ by the standing assumption (1.2), $\tilde{Z}^0$ cannot be P-a.s. equal to 0. Moreover,

$$E[\tilde{Z}^0] = E\Big[\tilde{Z}^0\big(1 - G_T(\bar{\beta})\big)\Big] = E\Big[(\tilde{Z}^0)^2\Big] > 0$$

by (3.5) and (3.3), and this shows that $\tilde{P}$ is well defined by (3.4), in $\mathbb{P}_s(\Theta)$ by (3.3) and variance-optimal by Lemma 1. $\square$

REMARK. If $\mathbb{P}_s(\Theta)$ contains a probability measure Q equivalent to P, the adjustment process for X is *unique* in the following sense: the set $N := \{\mathscr{E}^{\beta}_{-} \neq 0\} \subseteq \Omega \times [0, T]$ does not depend on the choice of adjustment process β, and all adjustment processes coincide on N. To see this, choose adjustment processes β^1, β^2 and use Proposition 8 to write

$$\frac{d\tilde{P}}{dP} = \frac{\mathscr{E}^{\beta^i}_T}{E\Big[\mathscr{E}^{\beta^i}_T\Big]} =: c_i \mathscr{E}^{\beta^i}_T \quad \text{for } i = 1, 2.$$

From (3.5) and (3.3), we deduce that $c := \tilde{E}[d\tilde{P}/dP] = c_i$ for $i = 1, 2$ and therefore

$$\frac{d\tilde{P}}{dP} = c\mathscr{E}^{\beta^i}_T = c\big(1 - G_T(\bar{\beta}^i)\big) \quad \text{for } i = 1, 2.$$

However, since $G(\bar{\beta}^i)$ is a Q-martingale for $i = 1, 2$, this implies that $G(\bar{\beta}^1)$ and $G(\bar{\beta}^2)$ are indistinguishable, hence $\beta^1 \mathscr{E}^{\beta^1}_{-} = \bar{\beta}^1 = \bar{\beta}^2 = \beta^2 \mathscr{E}^{\beta^2}_{-}$, and the assertion follows. In particular, $N = \Omega \times [0, T]$ P-a.s. if X is continuous.

For the discrete-time case, we have seen in Section 2 how the adjustment process β can be explicitly constructed by backward recursion. The analogue in continuous time is a characterization of β as the solution of a *backward stochastic differential equation.*

THEOREM 9. *Assume* (1.2). *Then there exists an adjustment process* β *for* X *if and only if there exists a solution* $(\beta, U) \in L(X) \times \mathscr{S}^2$ *of the backward stochastic differential equation*

$$dU_t = -U_{t-}\beta_t \, dX_t, \qquad U_T = \pi(1) \tag{3.6}$$

with U_0 *deterministic. More precisely,* $\beta \in L(X)$ *is an adjustment process for* X *if and only if* $U := \mathscr{E}^{\beta}$ *is in* $\mathscr{S}^2$ *and* (β, U) *solves* (3.6).

PROOF. If there exists an adjustment process β, then $\bar{\beta} \in \Theta$ implies that $U := \mathscr{E}^{\beta}$ is in $\mathscr{S}^2$, U satisfies

$$dU_t = -U_{t-}\beta_t \, dX_t$$

and $U_0 = 1$ is deterministic. Moreover, Proposition 8 implies that $U_T = E[U_T]\, d\tilde{P}/dP$ is in $G_T(\Theta)^{\perp}$, and since $1 - U_T = G_T(\bar{\beta})$ is in $G_T(\Theta) \subseteq G_T(\Theta)^{\perp\perp}$, we have $U_T = \pi(1)$.

Conversely, let (β, U) be a solution of (3.6) with U_0 deterministic. Then (3.6) yields $U = U_0 \mathscr{E}^{\beta} = U_0(1 - G(\bar{\beta}))$, and thus $\bar{\beta}$ is in Θ since U is in $\mathscr{S}^2$ and U_0 is deterministic. Note that $U_0 \neq 0$ by the standing assumption (1.2); more precisely, $\pi(1) = U_T = U_0(1 - G_T(\bar{\beta})) \in G_T(\Theta)^{\perp}$ implies that

$$\begin{aligned} U_0 E\left[1 - G_T(\bar{\beta})\right] &= E[\pi(1)] = E\left[(\pi(1))^2\right] \\ &= U_0^2 E\left[\left(1 - G_T(\bar{\beta})\right)^2\right] = U_0^2 E\left[1 - G_T(\bar{\beta})\right] \end{aligned}$$

and therefore $U_0 = 1$. Thus $\mathscr{E}_T^{\beta} = U_T = \pi(1)$ is in $G_T(\Theta)^{\perp}$, and so β is an adjustment process. $\square$

The next result gives another criterion for the existence of β in the case where X is continuous.

THEOREM 10. *Assume* (1.2). *If X is continuous, the following statements are equivalent*:

(a) *There exists an adjustment process* β *for* X.
(b) $1 - \pi(1)$ *is in* $G_T(\Theta)$ *and*

$$\pi(1) > 0 \qquad P\text{-}a.s. \tag{3.7}$$

PROOF.
Step 1. If there exists an adjustment process β for X, Theorem 9 yields $\pi(1) = \mathscr{E}_T^{\beta} > 0$ P-a.s. by the continuity of X, and $1 - \pi(1) = G_T(\bar{\beta})$ is in $G_T(\Theta)$.

Step 2. Conversely, suppose that $1 - \pi(1) = G_T(\psi)$ for some $\psi \in \Theta$. We first show that (3.7) implies the stronger result that

(3.8) the process $1 - G(\psi)$ is P-a.s. strictly positive.

To that end, define

$$\tau := \inf\{t \in [0, T] | G_t(\psi) \geq 1\}$$

with $\inf \varnothing := \infty$ and set $\hat{\psi} := \psi I_{]\!]0, \tau \wedge T]\!]}$. Since X is continuous, we have

$$G_{\tau \wedge T}(\psi) = 1 \quad \text{on } C := \{\tau \leq T\}$$

and

$$G(\psi) < 1 \quad \text{on } C^c = \{\tau = \infty\}. \tag{3.9}$$

Since $\psi \in \Theta$, so is $\hat{\psi}$, and

$$G_T(\hat{\psi}) = G_{\tau \wedge T}(\psi) = I_C + G_T(\psi) I_{C^c}$$

implies that

$$E\Big[\big(1 - G_T(\hat{\psi})\big)^2\Big] = E\Big[I_{C^c}\big(1 - G_T(\psi)\big)^2\Big] \leq E\Big[\big(1 - G_T(\psi)\big)^2\Big].$$

However, $G_T(\psi)$ is the projection in $\mathscr{L}^2(P)$ of 1 on $G_T(\Theta)$; hence we must have $G_T(\hat{\psi}) = G_T(\psi)$ P-a.s. and therefore

$$G_T(\psi) = 1 \qquad P\text{-a.s. on } C.$$

By (3.7), this implies $P[C] = 0$ and therefore (3.8) in view of (3.9).

Step 3. Thanks to (3.8), the process $(1 - G(\psi))^{-1}$ is continuous and locally bounded so that $\beta := \psi(1 - G(\psi))^{-1}$ is in $L(X)$. Moreover,

$$1 - G(\psi) = 1 - \int (1 - G(\psi)) \beta \, dX = \mathscr{E}^{\beta}$$

shows that $\beta \mathscr{E}^{\beta} = \psi$ is in Θ, and β satisfies (3.3) since $1 - G_T(\psi) = \pi(1)$ is in $G_T(\Theta)^{\perp}$. Hence β is an adjustment process for X. $\square$

COROLLARY 11. *Assume* (1.2) *and suppose that X is continuous. If $G_T(\Theta)$ is closed in $\mathscr{L}^2(P)$, the following statements are equivalent*:

(a) *There exists an adjustment process β for X.*

(b) $\pi(1) > 0$ *P-a.s.*

(c) *The variance-optimal signed Θ-martingale measure $\tilde{P}$ is equivalent to P.*

PROOF. Due to (1.5) and part (b) of Lemma 1, (c) implies (b), and (a) implies (c) since $\mathscr{E}^{\beta} > 0$ by the continuity of X. Finally (b) implies (a) by Theorem 9, since $G_T(\Theta) = G_T(\Theta)^{\perp\perp}$ for $G_T(\Theta)$ closed. $\square$

In general, the variance-optimal $\tilde{P}$ is not a measure, but only a signed measure; this is illustrated by an explicit example in Section 4. At first sight, this might seem to indicate that Theorem 10 and Corollary 11 are of little use. Moreover, a signed measure $\tilde{P}$ is not very attractive for the characterization of the Θ-approximation price in Proposition 2, since it might assign a negative price to a nonnegative random variable. However, the situation becomes different if X is continuous and satisfies in addition a no-arbitrage-type condition. Following Schweizer (1994), we say that a process $X \in \mathscr{S}^2_{\mathrm{loc}}(P)$ *satisfies the structure condition* (SC) if in the canonical decomposition $X = X_0 + M + A$, we have

$$A^i \ll \langle M^i \rangle \quad \text{for } i = 1, \dots, d \tag{3.10}$$

and if there exists a predictable $\mathbb{R}^d$-valued process $\hat{\lambda}$ in $L^2_{\mathrm{loc}}(M)$ such that

$$\sigma_t \hat{\lambda}_t = \gamma_t \qquad P\text{-a.s. for } t \in [0, T]. \tag{3.11}$$

The predictable processes σ and γ in (3.11) are defined by

$$A^i_t = \int_0^t \gamma^i_s \, dB_s \quad \text{for } i = 1, \dots, d$$

and

$$\langle M^i, M^j \rangle_t = \int_0^t \sigma^{ij}_s \, dB_s \quad \text{for } i, j = 1, \dots, d,$$

where B is a fixed increasing predictable RCLL process null at 0 such that $\langle M^i \rangle \ll B$ for each i. The increasing predictable process $\hat{K}$ defined as an RCLL version of

$$\hat{K}_t := \int_0^t \hat{\lambda}^{\mathrm{tr}}_s \, dA_s = \int_0^t \hat{\lambda}^{\mathrm{tr}}_s \sigma_s \hat{\lambda}_s \, dB_s = \left\langle \int \hat{\lambda} \, dM \right\rangle_t \tag{3.12}$$

is then called the *mean-variance tradeoff process* of X.

Although it may look rather special at first sight, condition SC appears quite naturally in applications to financial mathematics. It is a very mild formulation of the assumption that X should not admit arbitrage opportunities, that is, riskless profit strategies. Sufficient conditions for SC are given for instance in Ansel and Stricker (1992) or Schweizer (1995b). As an example, every adapted continuous process X admitting an equivalent martingale measure satisfies SC. We remark that for $d = 1$, condition SC reduces to the combination of (3.10), that is,

$$X = X_0 + M + \int \alpha \, d\langle M \rangle,$$

with the assumption that $\alpha \in L^2_{\mathrm{loc}}(M)$; (3.11) is then satisfied with $\hat{\lambda} = \alpha$, and the mean-variance tradeoff process is given by $\hat{K} = \int \alpha \, dA = \int \alpha^2 d\langle M \rangle$.

LEMMA 12. (a) *If* $X \in \mathscr{S}^2_{\mathrm{loc}}(P)$ *satisfies* (3.10), *then* $\Theta = L^2(M) \cap L^2(A)$, *where*

$$L^2(A) := \left\{ \text{predictable } \mathbb{R}^d\text{-valued } \vartheta \,\middle|\, \int_0^T |\vartheta^{\mathrm{tr}}_s| \, d|A|_s = \int_0^T |\vartheta^{\mathrm{tr}}_s \gamma_s| \, dB_s \in \mathscr{L}^2(P) \right\}.$$

If in addition X satisfies the structure condition SC and $\hat{K}_T$ is P-a.s. bounded, then $\Theta = L^2(M)$.

(b) *If $X \in \mathscr{S}^2(P)$ satisfies the structure condition SC, then $\mathbb{P}_s(\Theta) = \mathbb{P}_s^2(X)$.*

PROOF. Since (a) is proved in Lemma 2 of Schweizer (1994), we only show (b). First of all, it is easy to see that Θ contains all bounded predictable processes if and only if $X - X_0 \in \mathscr{S}^2(P)$, and in that case, we clearly have $\mathbb{P}_s(\Theta) \subseteq \mathbb{P}_s^2(X)$. To obtain the reverse inclusion, take any $Q \in \mathbb{P}_s^2(X)$ and denote by Z an RCLL version of the density process of Q with respect to P. Then $Z \in \mathscr{M}^2(P)$ and ZX is a P-martingale. For any $\vartheta \in \Theta$, the product rule yields

$$\begin{aligned} d(ZG(\vartheta)) = \bigg\{ & G_-(\vartheta)\, dZ + Z_- d\left(\int \vartheta\, dM\right) + d\left[Z, \int \vartheta^{\mathrm{tr}}\, dA\right] \\ & + d\left[Z, \int \vartheta\, dM\right] - d\left\langle Z, \int \vartheta\, dM\right\rangle\bigg\} \\ & + d\left\langle Z, \int \vartheta\, dM\right\rangle + Z_- \vartheta^{\mathrm{tr}}\, dA, \end{aligned}$$

and by part (a) and Yoeurp's lemma, the term in curly brackets on the right-hand side is (the differential of) a local P-martingale. Since X satisfies SC, Proposition 2 of Schweizer (1995b) implies that

$$dZ = -Z_- d\left(\int \hat{\lambda}\, dM\right) + dR$$

for some $R \in \mathscr{M}^2_{0,\mathrm{loc}}(P)$ strongly P-orthogonal to each M^i, and so we get

$$d\left\langle Z, \int \vartheta\, dM\right\rangle + Z_- \vartheta^{\mathrm{tr}}\, dA = -Z_- \hat{\lambda}^{\mathrm{tr}} \sigma \vartheta\, dB + Z_- \vartheta^{\mathrm{tr}} \gamma\, dB = 0$$

from (3.10) and (3.11). This shows that $ZG(\vartheta)$ is a local P-martingale, and because Z is in $\mathscr{M}^2(P)$ and $G(\vartheta)$ is in $\mathscr{S}^2(P)$, $ZG(\vartheta)$ is even a P-martingale. Since $\vartheta \in \Theta$ was arbitrary, we conclude that $Q \in \mathbb{P}_s(\Theta)$. $\square$

The next result shows that for a *continuous* process X satisfying SC, the variance-optimal $\tilde{P}$ is in fact a probability measure. From the point of view of possible applications, this is very important: it implies by Proposition 2 that the Θ-approximation price of any nonnegative contingent claim H is also nonnegative. This is clearly a highly desirable property of any reasonable price system.

THEOREM 13. *Assume* (1.2). *If X is a continuous adapted process satisfying the structure condition SC, then the variance-optimal signed Θ-martingale measure $\tilde{P}$ is a measure, that is, is nonnegative.*

PROOF.

Step 1. Suppose first that $\hat{K}_T$ is P-a.s. bounded. Then Theorem 2.4 of Monat and Stricker (1994) shows that $G_T(\Theta)$ is closed in $\mathscr{L}^2(P)$. If we denote by $G_T(\psi)$ the projection of 1 on $G_T(\Theta)$, the same argument as in Step 2 of the proof of Theorem 10 shows that

$$G_T(\psi) = G_T(\hat{\psi}) = I_C + G_T(\psi) I_{C^c} \le 1 \qquad P\text{-a.s.}$$

Note that this is the only place where the continuity of X is used. Moreover, the standing assumption (1.2) implies that $P[C] < 1$ and so $G_T(\psi) < 1$ with positive probability by (3.9). By part (b) of Lemma 1, $\tilde{P}$ is given by

$$\frac{d\tilde{P}}{dP} = \frac{1 - G_T(\psi)}{E[1 - G_T(\psi)]} \ge 0 \qquad P\text{-a.s.},$$

and this proves the assertion in the case where $\hat{K}_T$ is bounded.

Step 2. The process $\hat{K}$ is predictable and RCLL, hence locally bounded. Take a localizing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ such that each $\hat{K}^{T_n}$ is bounded and define the spaces

$$\Theta^n := \{\vartheta I_{]\!]0, T_n]\!]} | \vartheta \in \Theta\} \subseteq \Theta^{n+1} \subseteq \Theta$$

and

$$\mathscr{V}_n := G_T(\Theta^n) \subseteq \mathscr{V}_{n+1} \subseteq G_T(\Theta) \subseteq \mathscr{L}^2(P).$$

Then we claim that each $\mathscr{V}_n$ is a closed subspace of $\mathscr{L}^2(P)$. To see this, we note that

$$\mathscr{V}_n = \left\{ \int_0^T \vartheta_s \, dX_s^{T_n} \middle| \vartheta \in \Theta \right\} = \left\{ \int_0^T \xi_s \, dX_s^{T_n} \middle| \xi \in L^2(M^{T_n}) \right\}$$

by part (a) of Lemma 12, and so we can apply Theorem 2.4 of Monat and Stricker (1994) to X^{T_n} instead of X. If we denote by V_n the projection of 1 on $\mathscr{V}_n$, the sequence $(1 - V_n)_{n \in \mathbb{N}}$ converges in $\mathscr{L}^2(P)$ to some $\tilde{Z}^0$. By Step 1, $1 - V_n \ge 0$ P-a.s. for every n and so $\tilde{Z}^0 \ge 0$ P-a.s. For each $\vartheta \in \Theta$, $G_T(\vartheta I_{]\!]0, T_n]\!]})$ converges to $G_T(\vartheta)$ in $\mathscr{L}^2(P)$ [see, for instance, Schweizer (1994), Lemma 5] and this implies that

$$E[\tilde{Z}^0 G_T(\vartheta)] = \lim_{n \to \infty} E[(1 - V_n) G_T(\vartheta I_{]\!]0, T_n]\!]})] = 0.$$

Moreover, each V_n can be written as $V_n = G_T(\xi^{(n)} I_{]\!]0, T_n]\!]})$ for some $\xi^{(n)} \in \Theta$, and so we deduce

$$E\left[\frac{dQ}{dP} \tilde{Z}^0\right] = \lim_{n \to \infty} E\left[\frac{dQ}{dP}(1 - V_n)\right] = 1$$

for every $Q \in \mathbb{P}_s(\Theta)$ and

$$\begin{aligned} E\left[(\tilde{Z}^0)^2\right] &= \lim_{n \to \infty} E\left[(1 - V_n)\left(1 - G_T\left(\xi^{(n)} I_{]\!]0, T_n]\!]}\right)\right)\right] \\ &= \lim_{n \to \infty} E[1 - V_n] = E[\tilde{Z}^0]. \end{aligned}$$

The same arguments as in the proof of Proposition 8 now show that $\tilde{P}$ with density

$$\frac{d\tilde{P}}{dP} := \frac{\tilde{Z}^0}{E[\tilde{Z}^0]}$$

is well defined, in $\mathbb{P}_s(\Theta)$ and variance-optimal, and since $\tilde{Z}^0 \geq 0$ P-a.s., this completes the proof. $\square$

An earlier version of this paper conjectured that $\tilde{P}$ is in fact equivalent to P if X is continuous and satisfies SC. In the meantime, this has been proved by Delbaen and Schachermayer (1994) under the natural additional assumption that

(3.13) there exists a probability measure $Q \approx P$ with $dQ/dP \in \mathscr{L}^2(P)$ such that X is a local Q-martingale.

This allows us in turn to give an existence result for the adjustment process β. As an aside, we remark that (3.13) already implies SC if X is continuous; see Theorem 1 of Schweizer (1995b). For sufficient conditions for (3.13), see also Stricker (1990).

In order to prove the next result, we need some notation. If $X \in \mathscr{S}^2_{\mathrm{loc}}(P)$ satisfies condition SC, we can define an exponential local martingale by $\hat{Z} := \mathscr{E}(-\int \hat{\lambda}\, dM)$. It is easy to check that $\hat{Z}X$ is a local P-martingale, and by the same kind of argument as in the proof of Lemma 12, so is $\hat{Z}G(\vartheta)$ for every $\vartheta \in \Theta$. If $\hat{K}_T$ is P-a.s. bounded, $\hat{Z}$ is even in $\mathscr{M}^2(P)$ by Theorem II.2 of Lepingle and Mémin (1978). In that case, we can define a signed measure $\hat{P}$ by setting

$$\frac{d\hat{P}}{dP} := \hat{Z}_T = \mathscr{E}\left(-\int \hat{\lambda}\, dM\right)_T, \tag{3.14}$$

and $\hat{P}$ is then in $\mathbb{P}_s(\Theta)$. This signed measure $\hat{P}$ is the so-called *minimal signed local martingale measure* for X, introduced in Föllmer and Schweizer (1991) and subsequently studied and used by several authors.

THEOREM 14. *Assume that X is continuous and satisfies the structure condition SC. If $\hat{K}_T$ is P-a.s. bounded, then there exists an adjustment process β for X.*

PROOF. Since $\hat{K}_T$ is bounded, (3.14) defines a signed measure whose density with respect to P is in $\mathscr{L}^2(P)$. Since X is continuous, $\hat{P}$ is in fact equivalent to P, and so (3.13) is satisfied with $Q = \hat{P}$. By Theorem 1.3 of Delbaen and Schachermayer (1994), this implies that $\tilde{P}$ is equivalent to P. Due to Theorem 2.4 of Monat and Stricker (1994), $G_T(\Theta)$ is closed since $\hat{K}_T$ is bounded, and so the assertion follows from Corollary 11. $\square$

REMARKS.

1. Actually, the boundedness assumption on $\hat{K}_T$ in Theorem 14 is unnecessarily strong. It is clear from the proof that β exists as soon as X is continuous, (3.13) is satisfied and $G_T(\Theta)$ is closed. For conditions guaranteeing these assumptions, see Delbaen, Monat, Schachermayer, Schweizer and Stricker (1995).

2. In view of Theorem 9, Theorem 14 also provides an existence result for the backward stochastic differential equation (3.6). It would be interesting to see a direct proof of that result.

To conclude this section, we now briefly discuss the question of when the variance-optimal $\tilde{P}$ coincides with $\hat{P}$. This also gives an alternative approach to the construction of the adjustment process β in some cases. We know from part (c) of Lemma 1 that $\hat{P} = \tilde{P}$ if $\hat{Z}_T$ can be represented as the sum of a constant and a stochastic integral of X with an integrand from Θ. For instance, this is possible if X is given by

$$X_t = W_t + \int_0^t \mu_s \, ds$$

with a one-dimensional Brownian motion W and a bounded process μ which is adapted to the augmentation of the filtration $\mathbb{F}^X$ generated by X; see Section 6.3 of Schweizer (1994). Another class of examples follows.

Example 1. Suppose that $X \in \mathscr{S}^2_{\mathrm{loc}}(P)$ satisfies the structure condition SC. If $\hat{K}$ is *continuous* and $\hat{K}_T$ is *deterministic*, then $\beta := \hat{\lambda}$ is an adjustment process for X and $\hat{P}$ is variance-optimal. In fact, continuity of $\hat{K}$ implies

$$\left[\int \hat{\lambda}\, dM, \int \hat{\lambda}^{\mathrm{tr}}\, dA\right] = \left[\int \hat{\lambda}\, dM, \hat{K}\right] = 0,$$

hence

$$\mathscr{E}^{\hat{\lambda}} = \mathscr{E}\left(-\int \hat{\lambda}\, dX\right) = \mathscr{E}\left(-\int \hat{\lambda}\, dM\right)\mathscr{E}(-\hat{K}) = \hat{Z}e^{-\hat{K}}, \tag{3.15}$$

and so $\beta := \hat{\lambda}$ satisfies (3.3) because $\hat{K}_T$ is deterministic and $\hat{P}$ is in $\mathbb{P}_s(\Theta)$. Note that here is the only place where we use the assumption that $\hat{K}_T$ is deterministic. Moreover, (3.15) and (3.12) yield

$$\begin{aligned}\int_0^T \overline{\beta}_s^{\mathrm{tr}} \sigma_s \overline{\beta}_s \, dB_s &\le \sup_{0 \le s \le T} |\hat{Z}_s|^2 \int_0^T \hat{\lambda}_s^{\mathrm{tr}} \sigma_s \hat{\lambda}_s \, dB_s \\ &= \hat{K}_T \sup_{0 \le s \le T} |\hat{Z}_s|^2 \in \mathscr{L}^1(P)\end{aligned}$$

since $\hat{K}_T$ is bounded and $\hat{Z} \in \mathscr{M}^2(P)$, and so $\hat{\lambda}\mathscr{E}^{\hat{\lambda}}_-$ is in Θ by part (a) of Lemma 12. This proves the assertions by Proposition 8 and thus ends Example 1.

Example 2. Suppose that $X \in \mathscr{S}^2_{\mathrm{loc}}(P)$ satisfies the structure condition SC. If the entire process $\hat{K}$ is *deterministic* (but not necessarily continuous), then $\beta := \hat{\lambda}$ is again an adjustment process for X and $\hat{P}$ is variance-optimal. In fact, the second assertion is proved in Theorem 8 of Schweizer (1995b) by an argument completely different from the one in Example 1, and the first claim then follows as in Example 1.

4. Examples and applications. This section contains several examples and applications of the concepts introduced so far. After illustrating various points by explicit examples, we use the variance-optimal signed Θ-martingale measure $\tilde{P}$ to solve some quadratic optimization problems related to (1.1), and we provide a feedback form expression for the optimal strategy ξ with the help of the adjustment process β and a certain backward stochastic differential equation.

4.1. *Some explicit examples.* The first example illustrates that, in general, $\tilde{P}$ is only a signed measure and differs from $\hat{P}$.

EXAMPLE 3. Suppose that $X_0 = 0$ and that ΔX_1 takes the values $+1, 0, -1$ with probability $\frac{1}{3}$. Given that $X_1 \neq +1$, ΔX_2 takes the values ± 1 with probability $\frac{1}{2}$ each. The conditional distribution of ΔX_2 given $X_1 = +1$ is denoted by ν, and we shall assume that

$$0 < \int_{-\infty}^{\infty} x^2 \nu(dx) < \infty. \tag{4.1}$$

The filtration $\mathbb{F}$ is generated by X. To simplify the notation, we denote the value of any $\mathscr{F}_1$-measurable random variable Y on the sets $\{X_1 = +1\}$, $\{X_1 = 0\}$ and $\{X_1 = -1\}$ by $Y^{(+)}$, $Y^{(0)}$ and $Y^{(-)}$, respectively. It is then easy to check that $\tilde{\lambda}_1 = \tilde{\lambda}_2^{(-)} = \tilde{\lambda}_2^{(0)} = 0$ and

$$\tilde{\lambda}_2^{(+)} = \frac{\int_{-\infty}^{\infty} x\nu(dx)}{\int_{-\infty}^{\infty} x^2\nu(dx)};$$

by (4.1) and Jensen's inequality, this is well defined and $\tilde{\lambda}_2^{(+)}\,\Delta A_2^{(+)} < 1$. In particular, $\hat{K}$ is bounded.

Next we compute the adjustment process β. By (2.1) and (2.15), $\beta_2 = \tilde{\lambda}_2$ and therefore

$$\begin{aligned}\beta_1 &= \frac{E\big[\Delta X_1\big(1 - \tilde{\lambda}_2\,\Delta A_2\big)\big]}{E\big[\Delta X_1^2\big(1 - \tilde{\lambda}_2\,\Delta A_2\big)\big]}\\ &= \frac{-\big(\int_{-\infty}^{\infty} x\nu(dx)\big)^2}{2\int_{-\infty}^{\infty} x^2\nu(dx) - \big(\int_{-\infty}^{\infty} x\nu(dx)\big)^2}\end{aligned}$$

by conditioning on $\mathscr{F}_1$ and using (2.4) and the structure of X. Thus the processes β and $\tilde{\lambda}$ are different as soon as

$$\int_{-\infty}^{\infty} x\nu(dx) \neq 0,$$

that is, whenever X is not a martingale. Furthermore, it is clear that

$$\tilde{Z}^0 = (1 - \beta_1\,\Delta X_1)(1 - \beta_2\,\Delta X_2)$$

will become negative with positive probability if supp ν is unbounded and X is not a martingale. This shows that $\tilde{P}$ will, in general, not be a measure, but only a signed measure.

In the special case where

$$\nu = \tfrac{1}{2}\big(\delta_{\{+2\}} + \delta_{\{-1\}}\big)$$

with $\delta_{\{x\}}$ denoting a unit mass at the point x, we obtain

$$\tilde{\lambda}_2^{(+)} = \frac{1}{5}, \qquad \tilde{\lambda}_2^{(+)}\,\Delta A_2^{(+)} = \frac{1}{10}, \qquad \beta_1 = -\frac{1}{19}.$$

By numbering the trajectories as ω_1 to ω_6, starting with $\omega_1 = \{\Delta X_1 = +1, \Delta X_2 = +2\}$, $\omega_2 = \{\Delta X_1 = +1, \Delta X_2 = -1\}$ and so on, we can write the random variable $\tilde{Z}^0$ as a vector:

$$\tilde{Z}^0 = \left(\frac{12}{19}, \frac{24}{19}, 1, 1, \frac{18}{19}, \frac{18}{19}\right).$$

Hence $E[\tilde{Z}^0] = 55/57$ and

$$\tilde{D} = \frac{\tilde{Z}^0}{E[\tilde{Z}^0]} = \left(\frac{36}{55}, \frac{72}{55}, \frac{57}{55}, \frac{57}{55}, \frac{54}{55}, \frac{54}{55}\right).$$

Similarly, we obtain

$$\hat{Z} = \left(\frac{2}{3}, \frac{4}{3}, 1, 1, 1, 1\right),$$

which shows that $\hat{Z}$ and $\tilde{D}$, hence also the measures $\hat{P}$ and $\tilde{P}$, do not agree. This ends Example 3.

EXAMPLE 4. There exists a square-integrable process $(X_k)_{k=0,1,2,3}$ such that

$$\prod_{j=1}^{2}\big(1 - \beta_j\,\Delta X_j\big) = 1 - G_2(\bar{\beta}) \notin \mathscr{L}^1(P) \tag{4.2}$$

so that $\bar{\beta}$ is not in Θ. Note that $1 - G_3(\bar{\beta}) \in \mathscr{L}^2(P)$ since $\tilde{P}$ exists. This counterexample to the question after Corollary 4 was provided by Schachermayer.

In a first step, fix $\varepsilon > 0$ and $M > 0$. We then construct a process $(Y_k)_{k=0,1,2}$ on a filtered probability space $(C, 2^C, \mathbb{G}, P)$ such that the unique martingale measure Q for Y satisfies

$$\left\| \frac{dQ}{dP} \right\|^2_{\mathscr{L}^2(P)} \le 1 + \varepsilon \tag{4.3}$$

and

$$\left\| E_Q\!\left[\frac{dQ}{dP} \,\middle|\, \mathscr{G}_1 \right] \right\|_{\mathscr{L}^1(P)} \ge M. \tag{4.4}$$

To do this, choose $C = \{c_1, c_2, c_3\}$,

$$P[\{c_1\}] = \delta^5, \qquad P[\{c_2\}] = \delta$$

for $\delta > 0$ small, and $\mathscr{G}_0$ trivial, $\mathscr{G}_1 = \sigma(\{c_3\})$ and $\mathscr{G}_2 = 2^C$. Let $Y_0 = 0$, $Y_1(c_1) = Y_1(c_2) > 0 > Y_1(c_3)$ and $\Delta Y_2(c_1) > 0 = \Delta Y_2(c_3) > \Delta Y_2(c_2)$ so that the filtration $\mathbb{G}$ is generated by Y. It is clear that Y has a unique equivalent martingale measure Q, and we can choose the values of Y_1, Y_2 in such a way that

$$Q[\{c_1\}] = Q[\{c_2\}] = \delta^3.$$

This implies that

$$\left\| \frac{dQ}{dP} \right\|^2_{\mathscr{L}^2(P)} = \delta^5\delta^{-4} + \delta\delta^4 + \frac{(1 - 2\delta^3)^2}{1 - \delta^5 - \delta} \le 1 + \varepsilon$$

for δ small enough. On the other hand,

$$E_Q\left[\frac{dQ}{dP} \middle| \mathscr{G}_1 \right](c_1) = \frac{1}{Q[\{c_1, c_2\}]} E_Q\left[\frac{dQ}{dP} I_{\{c_1, c_2\}} \right]$$
$$= \frac{\delta^3\delta^{-2} + \delta^3\delta^2}{2\delta^3} \ge \frac{1}{2}\delta^{-2}$$

yields

$$\left\| E_Q\left[\frac{dQ}{dP} \middle| \mathscr{G}_1 \right] \right\|_{\mathscr{L}^1(P)} \ge (\delta^5 + \delta)\frac{1}{2}\delta^{-2} \ge \frac{1}{2}\delta^{-1} \ge M$$

for δ small enough.

To construct X, take now $\varepsilon_n = 2^{-n}$, $M_n = 2^n$ and apply the first step to obtain a sequence $(C_n, \mathbb{G}^n, P_n, Y^n, Q_n)$. Define Ω as the disjoint union of the sets C_n, $X_0 = X_1 = 0$ and

$$X_k = \sum_{n=1}^{\infty} \lambda_n Y^n_{k-1} I_{C_n} \quad \text{for } k = 2, 3$$

for arbitrary numbers $\lambda_n \ne 0$. (For suitable λ_n, X even remains bounded.) Take $\mathscr{F}_0$ trivial, $\mathscr{F}_1 = \sigma(C_n; n \in \mathbb{N})$, $\mathscr{F}_2 = \mathscr{F}_1 \vee \sigma(X_2)$ and $\mathscr{F}_3 = \mathscr{F}_2 \vee \sigma(X_3) = 2^{\Omega}$. Finally, take

$$P[\cdot] = \sum_{n=1}^{\infty} 2^{-n} P_n[\cdot \cap C_n].$$

Since $\lambda_n \ne 0$, any signed martingale measure Q for X is of the form

$$Q[\cdot] = \sum_{n=1}^{\infty} \mu_n Q_n[\cdot \cap C_n] \tag{4.5}$$

for some $\mu_n \ne 0$ with $\sum_{n=1}^{\infty} \mu_n = 1$. Since $P[C_n] = 2^{-n}$, we thus obtain

$$\left\| \frac{dQ}{dP} \right\|^2_{\mathscr{L}^2(P)} = \left\| \sum_{n=1}^{\infty} \mu_n 2^n \frac{dQ_n}{dP_n} I_{C_n} \right\|^2_{\mathscr{L}^2(P)}$$
$$= \sum_{n=1}^{\infty} \mu_n^2 2^n (1 + \gamma_n),$$

where

$$1 + \gamma_n := \left\| \frac{dQ_n}{dP_n} \right\|^2_{\mathscr{L}^2(P_n)} \le 1 + \varepsilon_n$$

by (4.3). By minimizing over (μ_n), we conclude that the variance-optimal measure $\tilde{P}$ is given by (4.5) with

$$\tilde{\mu}_n = \text{const.} \frac{1}{2^n(1 + \gamma_n)}.$$

Note that $\tilde{P}$ is equivalent to P since $\tilde{\mu}_n > 0$, and that $\tilde{\mu}_n$ is of order 2^{-n}. By (2.14),

$$1 - G_2(\bar{\beta}) = \text{const.}\, \tilde{E}\left[\frac{d\tilde{P}}{dP} \middle| \mathscr{F}_2 \right] = \text{const.} \sum_{n=1}^{\infty} I_{C_n} \tilde{\mu}_n 2^n \tilde{E}\left[\frac{dQ_n}{dP_n} \middle| \mathscr{F}_2 \right]$$

and since $P[C_n] = 2^{-n}$, we obtain

$$\begin{aligned} \|1 - G_2(\bar{\beta})\|_{\mathscr{L}^1(P)} &= \text{const.} \sum_{n=1}^{\infty} \tilde{\mu}_n \left\| E_{Q_n}\left[\frac{dQ_n}{dP_n} \middle| \mathscr{G}_1^n \right] \right\|_{\mathscr{L}^1(P_n)} \\ &\ge \text{const.} \sum_{n=1}^{\infty} \tilde{\mu}_n M_n = +\infty \end{aligned}$$

by (4.4). This proves (4.2) and thus ends Example 4.

REMARK. It follows from Lemma 2.2 of Delbaen and Schachermayer (1994) that $G(\bar{\beta})$ is in $\mathscr{M}^1(Q)$ for every $Q \in \mathbb{P}_s^2(X)$ which is equivalent to P; see also (2.14). The conclusion to be drawn from Example 4 is therefore that, in general, integrability properties of β or $\bar{\beta}$ should not be formulated with respect to P, but with respect to Q. This issue will be studied more carefully in the future.

4.2. *Some related optimization problems.* As a first application, we now use $\tilde{P}$ to solve several quadratic optimization problems related to (1.1). To that end, we consider the following auxiliary problem:

(4.6) Given $H \in \mathscr{L}^2(P)$ and $c \in \mathbb{R}$, minimize $E\left[(H - c - G_T(\vartheta))^2 \right]$ over all $\vartheta \in \Theta$.

Note that in contrast to (1.1), the initial capital c is prescribed in (4.6). Denote the solution of (4.6) by $\xi^{(c)}$ if it exists and recall that $\tilde{D} = d\tilde{P}/dP$ and that π is the projection in $\mathscr{L}^2(P)$ on $G_T(\Theta)^\perp$.

LEMMA 15. *Assume* (1.2), *and fix* $c \in \mathbb{R}$ *and* $H \in \mathscr{L}^2(P)$. *If* (4.6) *has a solution* $\xi^{(c)}$, *then*

$$E\left[H - c - G_T(\xi^{(c)}) \right] = \frac{\tilde{E}[H] - c}{E[\tilde{D}^2]} \tag{4.7}$$

and

$$E\Big[\big(H - c - G_T(\xi^{(c)})\big)^2\Big] = \frac{c^2 - 2c\tilde{E}[H]}{E[\tilde{D}^2]} + E\Big[\big(\pi(H)\big)^2\Big]. \tag{4.8}$$

PROOF. Let $\gamma := E[H - c - G_T(\xi^{(c)})]$. If $\gamma = 0$, the same argument as in Proposition 2 shows that $c = \tilde{E}[H]$ and so both sides of (4.7) equal 0. If $\gamma \neq 0$, then

$$\frac{dQ}{dP} := \frac{1}{\gamma}\big(H - c - G_T(\xi^{(c)})\big)$$

defines a signed Θ-martingale measure Q, since $\xi^{(c)}$ solves (4.6). By part (a) of Lemma 1, this implies

$$E[\tilde{D}^2] = \frac{1}{\gamma}\tilde{E}\big[H - c - G_T(\xi^{(c)})\big] = \frac{1}{\gamma}\big(\tilde{E}[H] - c\big)$$

and therefore (4.7). Since $H - c - G_T(\xi^{(c)})$ is in $G_T(\Theta)^{\perp}$,

$$\begin{aligned}
&E\Big[\big(H - c - G_T(\xi^{(c)})\big)^2\Big]\\
&\quad = E\Big[\big(H - c - G_T(\xi^{(c)})\big)\big(H - \pi(H) + \pi(H) - c - G_T(\xi^{(c)})\big)\Big]\\
&\quad = E\Big[\big(H - c - G_T(\xi^{(c)})\big)\big(\pi(H) - c\big)\Big]\\
&\quad = E\Big[\big(\pi(H)\big)^2\Big] - cE[\pi(H)] - c\frac{\tilde{E}[H] - c}{E[\tilde{D}^2]},
\end{aligned}$$

where the last step uses (4.7). However, part (b) of Lemma 1 shows that

$$\tilde{D} = E[\tilde{D}^2] + R \quad \text{for some } R \in G_T(\Theta)^{\perp\perp},$$

and so we get

$$\tilde{E}[H] = E\big[\tilde{D}\pi(H)\big] = E[\tilde{D}^2]E[\pi(H)],$$

since $\tilde{D} \in G_T(\Theta)^{\perp}$ and $H - \pi(H) \in G_T(\Theta)^{\perp\perp}$. Putting everything together yields (4.8) and thus completes the proof. □

Lemma 15 is an abstract version of Corollary 2.5 in Schweizer (1995a). As an immediate consequence, we obtain the following corollary:

COROLLARY 16. *Assume* (1.2) *and that* $G_T(\Theta)$ *is closed in* $\mathscr{L}^2(P)$, *and fix* $H \in \mathscr{L}^2(P)$. *Then*:

(a) $(\tilde{E}[H], \xi^{(\tilde{E}[H])})$ *solves* (1.1).
(b) $\xi^{(\tilde{E}[H])}$ *minimizes* $\mathrm{Var}[H - G_T(\vartheta)]$ *over all* $\vartheta \in \Theta$.
(c) *If* $E[\tilde{D}^2] \neq 1$, *the solution of*

Given $m \in \mathbb{R}$, *minimize* $\mathrm{Var}[H - G_T(\vartheta)]$ *over all* $\vartheta \in \Theta$
satisfying the constraint $E[H - G_T(\vartheta)] = m$

is given by $\xi^{(c_m)}$, *where*

$$c_m = \frac{mE[\tilde{D}^2] - \tilde{E}[H]}{E[\tilde{D}^2] - 1}.$$

PROOF. Since $G_T(\Theta)$ is closed in $\mathscr{L}^2(P)$, (4.6) has a solution $\xi^{(c)}$ for every $c \in \mathbb{R}$. Thanks to Lemma 15, (a) is now proved like Corollary 3.2, (b) like Corollary 3.4 and (c) like Corollary 3.6 in Schweizer (1995a). □

REMARKS.

1. In the framework of Section 3, Corollary 16 generalizes previous results of Schweizer (1994) where the solutions to these problems were only obtained under the assumption that the mean-variance tradeoff process $\hat{K}$ is deterministic. Note that this implies $\tilde{P} = \hat{P}$ according to Example 2.
2. The condition $E[\tilde{D}^2] \neq 1$ in (c) can equivalently be expressed as $1 \notin G_T(\Theta)^{\perp}$ which (up to integrability) amounts to saying that X is not a martingale. If $G_T(\Theta)^{\perp}$ does contain 1, the constraint $E[H - G_T(\vartheta)] = m$ can of course only be satisfied if $m = E[H]$.
3. For a thorough study of the closedness of $G_T(\Theta)$, see Delbaen, Monat, Schachermayer, Schweizer and Stricker (1995).

4.3. *A description of the optimal strategy*. To illustrate the usefulness of the adjustment process β, we now provide a description in *feedback form* of the solution $\xi^{(c)}$ of the optimization problem (4.6) in the case $\mathscr{T} = [0, T]$ of continuous time. Due to part (b) of Corollary 16, this also furnishes a description of the solution $\xi = \xi^{(\tilde{E}[H])}$ of the basic problem (1.1). We shall obtain $\xi^{(c)}$ as solution of the equation

$$\xi_t^{(c)} = \varrho_t - \beta_t\left(c + \int_0^{t-} \xi_s^{(c)}\, dX_s\right) = \varrho_t - \beta_t\big(c + G_{t-}(\xi^{(c)})\big). \tag{4.9}$$

This kind of result was already obtained in the case $\mathscr{T} = \{0, 1, \dots, T\}$ of finite discrete time by Schweizer (1995a). In particular, one can find there an explicit expression for ϱ in the discrete-time situation. We shall see that (4.9) still holds in continuous time, but ϱ has to be constructed as the solution of a certain backward stochastic differential equation.

Throughout this subsection, we shall assume that $\mathscr{T} = [0, T]$, X is a semimartingale with respect to P and $\mathbb{F}$, and Θ and $G_T(\vartheta)$ are given by (3.1) and (3.2), respectively. We also suppose that there exists an adjustment process β for X. Consider the following backward stochastic differential equation for $(\varrho, Z) \in L(X) \times \mathscr{S}^2$:

$$dZ_t = \varrho_t\, dX_t - Z_{t-}\beta_t\, dX_t, \qquad Z_T = H - \pi(H). \tag{4.10}$$

In (4.10), $H \in \mathscr{L}^2(\mathscr{F}_T, P)$ is fixed and π is as usual the projection in $\mathscr{L}^2(P)$ on $G_T(\Theta)^{\perp}$. Note that (4.10), hence also ϱ, does not depend on c.

PROPOSITION 17. *Assume that there exists an adjustment process β for X. If $(\varrho, Z) \in L(X) \times \mathscr{S}^2$ is a solution of* (4.10) *with Z_0 deterministic, then* (4.9) *defines a process $\xi^{(c)}$ in Θ for every $c \in \mathbb{R}$ and $\xi^{(c)}$ solves* (4.6).

PROOF.

Step 1. To show that there exists a process $\xi^{(c)} \in L(X)$ satisfying (4.9), we denote by V the solution of the stochastic differential equation

$$dV_t = (\varrho_t - c\beta_t)\, dX_t - V_{t-}\beta_t\, dX_t, \qquad V_0 = 0; \tag{4.11}$$

this exists and is unique by Theorem V.7 of Protter (1990). The process

$$\xi^{(c)} := \varrho - \beta(c + V_-) \tag{4.12}$$

is then in $L(X)$, and since $G := G(\xi^{(c)})$ satisfies

$$dG_t = \xi_t^{(c)}\, dX_t = \varrho_t\, dX_t - (c + V_{t-})\beta_t\, dX_t = dV_t, \qquad G_0 = 0 = V_0,$$

we conclude that $G(\xi^{(c)}) = V$. Inserting this into (4.12) shows that $\xi^{(c)}$ satisfies (4.9).

Step 2. To show that $\xi^{(c)}$ is in Θ, we introduce the process $Y := Z - V - c(1 - U)$, where $U = \mathscr{E}^{\beta}$ satisfies the backward stochastic differential equation (3.6). Combining (4.10), (4.11) and (3.6) shows that Y satisfies the stochastic differential equation

$$dY_t = -Y_{t-}\beta_t\, dX_t, \qquad Y_0 = Z_0$$

and therefore $Y = Z_0\mathscr{E}^{\beta} = Z_0 U$. Since Z_0 is deterministic, we conclude that Y is in $\mathscr{S}^2$ and so is $G(\xi^{(c)}) = V = Z - Y - c(1 - U)$; hence $\xi^{(c)}$ is in Θ.

Step 3. It remains to show that $\xi^{(c)}$ defined above solves (4.6). However, this follows immediately from the observation that

$$\begin{aligned} H - c - G_T(\xi^{(c)}) &= H - c - V_T \\ &= H - Z_T + Y_T - cU_T = \pi(H) + (Z_0 - c)\pi(1) \end{aligned}$$

is in $G_T(\Theta)^{\perp}$ due to Step 2, (4.10) and (3.6). Note that this uses again that Z_0 is deterministic. □

Somewhat surprisingly, Proposition 17 can be used to establish a *uniqueness* result for the backward stochastic differential equation (4.10).

THEOREM 18. *Assume that there exists an adjustment process for X. Suppose that X is in $\mathscr{S}^2_{\mathrm{loc}}(P)$ and satisfies* (3.10). *If either X satisfies SC and $\hat{K}_T$ is P-a.s. bounded or* (3.13) *is satisfied, then there is at most one solution $(\varrho, Z) \in L(X) \times \mathscr{S}^2$ to* (4.10) *with Z_0 deterministic.*

PROOF.

Step 1. We remark first that each of the two hypotheses implies that the mapping $\vartheta \mapsto G_T(\vartheta)$ is injective from Θ into $\mathscr{L}^2(P)$. In fact, this is immediate in the first case, since boundedness of $\hat{K}_T$ implies that $\Theta = L^2(M)$ by Lemma 12 and that the norms $\|\vartheta\|_{L^2(M)}$ and $\|G_T(\vartheta)\|_{\mathscr{L}^2(P)}$ are equivalent by

Theorem 2.3 of Monat and Stricker (1994). If (3.13) is satisfied, $G(\vartheta)$ is in $\mathscr{M}_0^1(Q)$ for every $\vartheta \in \Theta$, so $G_T(\vartheta) = 0$ P-a.s. implies that $G(\vartheta) = \int \vartheta\, dM + \int \vartheta^{\mathrm{tr}}\, dA$ is indistinguishable from 0. By the uniqueness of the canonical decomposition, we then conclude that $\vartheta = 0$ in $L^2(M) \cap L^2(A)$.

Step 2. Now suppose that (ϱ^i, Z^i) are solutions in $L(X) \times \mathscr{S}^2$ to (4.10) with Z_0^i deterministic for $i = 1, 2$. If we set $\zeta := \varrho^1 - \varrho^2$ and $Y := Z^1 - Z^2$, then $(\zeta, Y) \in L(X) \times \mathscr{S}^2$ satisfies the backward stochastic differential equation

$$dY_t = \zeta_t\, dX_t - Y_{t-} \beta_t\, dX_t, \qquad Y_T = 0, \tag{4.13}$$

and Y_0 is deterministic. By Proposition 17, the process ψ defined by

$$\psi = \zeta - \beta G_-(\psi) \tag{4.14}$$

is therefore in Θ and solves

$$\text{minimize } E\Big[\big(G_T(\vartheta)\big)^2\Big] \text{ over all } \vartheta \in \Theta.$$

This implies that $G_T(\psi) = 0$ P-a.s., hence $\psi = 0$ in Θ by Step 1, and we conclude from (4.14) that $\zeta = 0$. By (4.13) and (3.6), Y_T is therefore given by $Y_T = Y_0 \mathscr{E}_T^\beta = Y_0 U_T = Y_0 \pi(1)$. Since $Y_T = 0$, we must have $Y_0 = 0$, because $\pi(1)$ cannot be P-a.s. equal to 0 by the standing assumption (1.2). Again from (4.13), we obtain $Y = Y_0 \mathscr{E}^\beta = 0$, and this completes the proof. □

Let us now turn to *existence* results for the backward stochastic differential equation (4.10).

PROPOSITION 19. *Assume that there exists an adjustment process β for X. If $H - \pi(H)$ is in $G_T(\Theta)$, then* (4.10) *has a solution $(\varrho, Z) \in L(X) \times \mathscr{S}^2$ with Z_0 deterministic.*

PROOF. By assumption, there exists $\vartheta \in \Theta$ with $H - \pi(H) = G_T(\vartheta)$. We claim that $\varrho := \vartheta + \beta G_-(\vartheta)$ and $Z := G(\vartheta)$ provide a solution to (4.10) with the desired properties. In fact, $\vartheta \in \Theta$ implies that (ϱ, Z) is in $L(X) \times \mathscr{S}^2$, $Z_0 = 0$ is deterministic, Z satisfies

$$dZ_t = \vartheta_t\, dX_t = \varrho_t\, dX_t - G_{t-}(\vartheta) \beta_t\, dX_t = \varrho_t\, dX_t - Z_{t-} \beta_t\, dX_t$$

and $Z_T = G_T(\vartheta) = H - \pi(H)$. □

THEOREM 20. *Assume that there exists an adjustment process β for X. Then $G_T(\Theta)$ is closed in $\mathscr{L}^2(P)$ if and only if the backward stochastic differential equation* (4.10) *has a solution $(\varrho, Z) \in L(X) \times \mathscr{S}^2$ with Z_0 deterministic for every $H \in \mathscr{L}^2(\mathscr{F}_T, P)$.*

PROOF. If $G_T(\Theta)$ is closed, then $H - \pi(H)$ is in $G_T(\Theta)$ for every H; hence the "only if" part follows from Proposition 19. Conversely, closedness of

$G_T(\Theta)$ clearly follows if the problem

$$\text{minimize } E\Big[\big(H - G_T(\vartheta)\big)^2\Big] \text{ over all } \vartheta \in \Theta$$

has a solution in Θ for every $H \in \mathscr{L}^2(\mathscr{F}_T, P)$ and so the "if" part is a consequence of Proposition 17. $\square$

REMARK. It would be interesting to see a direct argument for existence and/or uniqueness of the solution of the backward stochastic differential equation (4.10). In particular, this might provide a more concrete characterization for the closedness of $G_T(\Theta)$.

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REFERENCES

ANSEL, J.-P. and STRICKER, C. (1992). Lois de martingale, densités et décomposition de Föllmer–Schweizer. *Ann. Inst. H. Poincaré* **28** 375–392.

BARRON, E. N. and JENSEN, R. (1990). A stochastic control approach to the pricing of options. *Math. Oper. Res.* **15** 49–79.

BLACK, F. and SCHOLES, M. (1973). The pricing of options and corporate liabilities. *J. Political Economy* **81** 637–654.

BOULEAU, N. and LAMBERTON, D. (1989). Residual risks and hedging strategies in Markovian markets. *Stochastic Process. Appl.* **33** 131–150.

CVITANIĆ, J. and KARATZAS, I. (1993). Hedging contingent claims with constrained portfolios. *Ann. Appl. Probab.* **3** 652–681.

DAVIS, M. H. A. (1994). A general option pricing formula. Preprint, Imperial College, London.

DELBAEN, F., MONAT, P., SCHACHERMAYER, W., SCHWEIZER, M. and STRICKER, C. (1995). Weighted norm inequalities and closedness of a space of stochastic integrals. Preprint, Univ. Franche-Comté, Besançon.

DELBAEN, F. and SCHACHERMAYER, W. (1994). The variance-optimal martingale measure for continuous processes. Preprint, Univ. Vienna.

DUFFIE, D. and RICHARDSON, H. R. (1991). Mean-variance hedging in continuous time. *Ann. Appl. Probab.* **1** 1–15.

EL KAROUI, N. and QUENEZ, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* **33** 29–66.

FÖLLMER, H. and SCHWEIZER, M. (1991). Hedging of contingent claims under incomplete information. In *Applied Stochastic Analysis* (M. H. A. Davis and R. J. Elliott, eds.) 389–414. Gordon and Breach, London.

FÖLLMER, H. and SONDERMANN, D. (1986). Hedging of non-redundant contingent claims. In *Contributions to Mathematical Economics* (W. Hildenbrand and A. Mas-Colell, eds.) 205–223.

HANSEN, L. P. and JAGANNATHAN, R. (1991). Implications of security market data for models of dynamic economies. *J. Political Economy* **99** 225–262.

HARRISON, J. M. and KREPS, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *J. Economic Theory* **20** 381–408.

HARRISON, J. M. and PLISKA, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** 215–260.

HARRISON, J. M. and PLISKA, S. R. (1983). A stochastic calculus model of continuous trading: complete markets. *Stochastic Process. Appl.* **15** 313–316.

HIPP, C. (1993). Hedging general claims. *Proceedings of the 3rd AFIR Colloquium, Rome*, **2** 603–613.

LEPINGLE, D. and MÉMIN, J. (1978). Sur l'intégrabilité uniforme des martingales exponentielles. *Z. Wahrsch. Verw. Gebiete* **42** 175–203.

MERTON, R. C. (1973). Theory of rational option pricing. *Bell J. Economics Manage. Sci.* **4** 141–183.

MONAT, P. and STRICKER, C. (1994). Fermeture de $G_T(\Theta)$ et de $\mathscr{L}^2(\mathscr{F}_0) + G_T(\Theta)$. *Séminaire de Probabilités XXVIII. Lecture Notes in Math.* **1583** 189–194. Springer, Berlin.

MONAT, P. and STRICKER, C. (1995). Föllmer–Schweizer decomposition and mean-variance hedging for general claims. *Ann. Probab.* **23** 605–628.

MÜLLER, S. M. (1985). *Arbitrage Pricing of Contingent Claims. Lecture Notes in Econom. and Math. Systems* **254**. Springer, Berlin.

PROTTER, P. (1990). *Stochastic Integration and Differential Equations. A New Approach.* Springer, Berlin.

SCHÄL, M. (1994). On quadratic cost criteria for option hedging. *Math. Oper. Res.* **19** 121–131.

SCHWEIZER, M. (1992). Mean-variance hedging for general claims. *Ann. Appl. Probab.* **2** 171–179.

SCHWEIZER, M. (1994). Approximating random variables by stochastic integrals. *Ann. Probab.* **22** 1536–1575.

SCHWEIZER, M. (1995a). Variance-optimal hedging in discrete time. *Math. Oper. Res.* **20** 1–32.

SCHWEIZER, M. (1995b). On the minimal martingale measure and the Föllmer–Schweizer decomposition. *Stochastic Anal. Appl.* **13** 573–599.

STRICKER, C. (1990). Arbitrage et lois de martingale. *Ann. Inst. H. Poincaré* **26** 451–460.

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